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A method to obtain new copulas from a given one

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Abstract. Given a strictly increasing continuous function φ from [0, 1] to [0, 1] and its pseudo-inverse $\varphi^{[-1]}$, conditions that φ must satisfy for $C_\varphi(x_1, \ldots, x_n) = \varphi^{[-1]}(C(\varphi(x_1), \ldots, \varphi(x_n)))$ to be a copula for any copula C are studied. Some basic properties of the copulas obtained in this way are analyzed and several examples of generator functions φ that can be used to construct copulas C_φ are presented. In this manner, a method to obtain from a given copula C a variety of new copulas is provided. This method generalizes that used to construct Archimedean copulas in which the original copula C is the product copula, and it is related with mixtures

Key words: Probability distributions with given marginals, copulas, Archimedean copulas, mixtures.

1 Introduction

An *n-copula* is the restriction to the unit *n*-cube $[0, 1]^n$ of a multivariate cumulative distribution function whose marginals are uniform on [0, 1], more precisely, an n-copula is a function $C : [0, 1]^n \to [0, 1]$ that satisfies:

(a) $C(x_1,\ldots, x_n) = 0$ if $x_i = 0$ for any $i = 1,\ldots, n$.
(b) $C(1,\ldots, 1, x_i, 1,\ldots, 1) = x_i$ for each $i = 1,\ldots, n$ and all $x_i \in [0, 1]$.
(c) C is n-increasing, i.e., the C-volume of every n-box is nonnegative. Conditions (a) and (b) are known as *boundary conditions* whereas condition (c) is known as *monotonicity*. Condition (c) means that for all n-box $B = [a_1, b_1] \times\ldots\times [a_n, b_n] \subseteq [0, 1]^n$ with $a_i < b_i$, $i = 1,\ldots, n$,

$$V_c(B) = \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\ldots+i_n} C(x_{1i_1}, \ldots, x_{ni_n}) \geq 0,$$

where $x_{i1} = a_i$, $x_{i2} = b_i$. If C has *n*th-order derivatives, then condition (c) is equivalent to $\frac{\partial^n}{\partial x_1 \ldots \partial x_n} C \geq 0$.

In the theory of copulas, there are three functions of particular importance: the *product n-copula* Π^n, the *Fréchet-Hoeffding upper bound* M^n and the *Fréchet-Hoeffding lower bound* W^n. If $(x_1, \ldots, x_n) \in [0, 1]^n$ we have $\Pi^n(x_1, \ldots, x_n) = x_1 \ldots x_n$, $M^n(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n)$, $W^n(x_1, \ldots, x_n) = \max(x_1 + \ldots + x_n - n + 1, 0)$ and $W^n(x_1, \ldots, x_n) \leq C(x_1, \ldots, x_n) \leq M^n(x_1, \ldots, x_n)$ for each n-copula C. The functions Π^n and M^n are n-copulas for all $n \geq 2$, whereas the function W^n is an n–copula only for $n = 2$. For simplicity we are going to write Π, M and W instead of Π^2, M^2 and W^2, respectively. We are also going to use the term copula as a synonym of 2–copula.

A. Sklar introduced n-copulas in response to a question posed by M. Fréchet. Sklar proved that if H is the joint distribution function of n random variables $X_1, \ldots, X_n$, and $F_1, \ldots, F_n$ are the distribution functions of $X_1, \ldots, X_n$, respectively, then there exists an n-copula *C* such that

$$H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)). \tag{1.1}$$

(See Schweizer and Sklar (1983) and Sklar (1959) for relevant details). The n-copula is uniquely determined on range(F_1) ×…× range(F_n), so that C can be thought of as a description of the way in which a joint distribution function is related to its one-dimensional marginals.

Because of (1.1), if we have a collection of n-copulas then we automatically have a collection of n-dimensional distributions with whatever one-dimensional marginal distributions we desire. This fact is useful in modeling and simulation. Moreover, *n*-copulas are invariant under strictly increasing transformations of the random variables (see e.g. Schweizer and Sklar (1983), Theorem 6.5.6., p. 91), so they can be used in nonparametric statistic. Therefore, it is very important in statistics to have a great variety of n-copulas.

Different methods of constructing *n*-copulas have been proposed (see e.g. Nelsen (1999), Chapter 3 and Chapter 4). One of them yields an important class of n-copulas called Archimedean n-copulas. These n-copulas are of the form

$$C(x_1, \ldots, x_n) = \lambda^{[-1]}(\lambda(x_1)) + \ldots + \lambda(x_n)), \tag{1.2}$$

where $\lambda : [0, 1] \to [0, \infty]$ is a continuous strictly decreasing function such that $\lambda(1) = 0$ and $\lambda^{[-1]}$ is the pseudo-inverse of λ given by

$$\lambda^{[-1]}(t) = \begin{cases} \lambda^{-1}(t) & \text{if } 0 \leq t \leq \lambda(0) \\ 0 & \text{if } \lambda(0) \leq t \leq \infty \end{cases}. \tag{1.3}$$

The function λ that appears in (1.2) is known as an *additive generator* of C. Setting $\varphi(t) = \exp(-\lambda(t))$ and $\varphi^{[-1]}(t) = \lambda^{[-1]}(-\ln t)$, (1.2) can be written as

$$C(x_1, \ldots, x_n) = \varphi^{[-1]}(\varphi(x_1) \ldots \varphi(x_n)). \tag{1.4}$$

The function φ that appears in (1.4) is known as a *multiplicative generator* of C. The study of Archimedean n-copulas is fundamentally done using (1.2).

In the bivariate case, Mikusinski and Taylor (1999) considered an extension of (1.2). They studied copulas of the form $C(x, y) = \lambda^{-1}(\lambda(x) \oplus \lambda(y))$, where $\lambda : [0, 1] \to [0, a]$ is a strictly decreasing continuous surjection, $a \in [0, \infty]$, λ^{-1} is the inverse of λ and $\oplus$ denotes a continuous associative operation in [0, a].

In this paper another generalization of Archimedean n-copulas is considered. Note that equality (1.4) can be written as $C(x_1,\ldots, x_n) = \varphi^{[-1]}(\prod^n(\varphi(x_1),\ldots, \varphi(x_n)))$. Then it is possible to generalize (1.4) if we substitute the product n-copula $\prod^n$ by any arbitrary n-copula C, that is, we can consider n-copulas of the form

$$C_\varphi(x_1,\ldots,x_n) = \varphi^{[-1]}C(\varphi(x_1)\ldots\varphi(x_n)), \tag{1.5}$$

where C is an arbitrary n-copula. In fact this construction has been just considered by Genest and Rivest (2001). These authors studied some properties of the multivariate probability integral transformation for n-copulas given by (1.5) when φ is a strictly increasing, differentiable bijection on [0, 1]. If $\varphi(t) = t^{1/k}$ for some integer k, then C_φ is the copula associated with the componentwise maxima $\mathbf{X}_{k1} = \max(X_{11},\ldots, X_{k1}),\ldots, \mathbf{X}_{kn} = \max(X_{1n},\ldots, X_{kn})$ of a random sample $(X_{11} ,\ldots, X_{1n}) ,\ldots, (X_{k1},\ldots, X_{kn})$ from some arbitrary distribution with underlying copula C (see e.g. Galambos (1987), Theorem 5.2.1, p. 293).

The paper is organized as follows. In section 2, sufficient conditions on the function φ for C_φ, given by (1.5) with $n = 2$, to be a copula for any copula C are stated. Several examples of generator functions φ and some simple properties that permit us to construct new generators from others are presented. Then, in section 3, some basic properties of the copulas obtained in this way are studied. In section 4, the multivariate case is considered. Finally, in section 5, the method proposed in this paper is related with mixtures of n-copulas.

2. Conditions for C_φ to be a copula in the bivariate case

In this section we state sufficient conditions for C_φ, given by (1.5) with $n = 2$, to be a copula for any copula C. In order to fix ideas about how φ must be, we first observe what happens when we consider Archimedean copulas. When $n = 2$, the conditions that λ must satisfies for C given by (1.2) to be a copula appear in the next two sections; their proofs can be found in Nelsen (1999), pp. 90–91.

Lemma 2.1. *Let C be defined by* (1.2) *with* $n = 2$, *where* $\lambda : [0, 1] \to [0, \infty]$ *is a continuous strictly decreasing function such that* $\lambda(1) = 0$ *and* $\lambda^{[-1]}$ *is the pseudo-inverse of* λ. *Then C satisfies the boundary conditions (a) and (b) for a copula.*

Theorem 2.2. *Let* $\lambda : [0, 1] \to [0, \infty]$ *be a continuous strictly decreasing function, such that* $\lambda(1) = 0$ *and let* $\lambda^{[-1]}$ *be the pseudo-inverse of* λ. *Then C, defined by (1.2) with* $n = 2$, *is a copula if and only if* λ *is convex.*

The multiplicative generator φ and the additive generator λ are related by $\varphi(t) = \exp(-\lambda(t))$. So, by Lemma 2.1., $\varphi : [0, 1] \to [0, 1]$ is a continuous strictly increasing function such that $\varphi(1) = 1$. Taking into account this, we consider, as a first thing, that φ is a function with these characteristics. In this case the pseudo-inverse of φ is given by

$$\varphi^{[-1]}(\mathrm{t}) = \begin{cases} 0 & \text{if } 0 \leq \mathrm{t} \ \leq \varphi(0) \\ \varphi^{-1}(\mathrm{t}) & \text{if } \varphi(0) \ \leq \mathrm{t} \leq 1 \end{cases} \tag{2.1}$$

We see that $\varphi^{[-1]}$ is increasing, moreover, is strictly increasing on $[\varphi(0), 1]$. We have that $\varphi^{[-1]}(\varphi(\mathrm{t})) = \mathrm{t}$ and

$$\varphi\left(\varphi^{[-1]}(\mathrm{t})\right) = \begin{cases} \varphi(0) & \text{if } 0 \leq \mathrm{t} \leq \varphi(0) \\ \mathrm{t} & \text{if } \varphi(0) \leq \mathrm{t} \leq 1 \end{cases}. \tag{2.2}$$

Note that φ is continuous and strictly increasing on [0, 1] if and only if φ is increasing and a bijection from [0, 1] to $[\varphi(0), 1]$. Also, from the definition of $\varphi^{[-1]}$, we have

$$\text{If } 0 \leq \mathrm{t} \leq 1, \text{ then } \varphi(\varphi^{[-1]}(\mathrm{t})) = \max\ (\varphi(0), \mathrm{t}) \geq \mathrm{t}. \tag{2.3}$$

Theorem 2.3. *Let C be a copula, let $\varphi : [0, 1] \to [0, 1]$ be a continuous strictly increasing function such that $\varphi(1) = 1$, and let $\varphi^{[-1]}$ be the pseudo-inverse of φ defined by (2.1). Then C_φ, given by (1.5) with n = 2, satisfies the boundary conditions (a) and (b) for a copula.*

Proof. Consider $y \in [0, 1]$. Since $0 \leq C(\varphi(0),\varphi(y)) \leq \varphi(0)$ then $C_\varphi(0, y) = \varphi^{[-1]}(C(\varphi(0),\varphi(y))) = 0$. On the other hand, $C_\varphi(1, y) = \varphi^{[-1]}(C(\varphi(1),\varphi(y))) = \varphi^{[-1]}(\varphi(y)) = y$. In a similar manner it can be proved that $C_\varphi(x, 0) = 0$ and $C_\varphi(x, 1) = x$ for all $x \in [0, 1]$. ∎

Let C be the copula defined by $C(x, y) = x\,y\,(1 - (1 - x)(1 - y))$ for $x, y \in [0, 1]$ (this is a member of the Farlie-Gumbel-Morgenstern family of copulas, see e.g. Nelsen (1999), p. 68) and the function φ defined by $\varphi(t) = t^2$. In this case, C_φ is not a 2-increasing function. To see this we can consider the rectangle $B = [6/7,1] \times [6/7, 1]$. The C_φ-volume of this rectangle is $-\frac{5}{7} + \frac{216}{2401}\sqrt{62} < 0$. This example shows that the requirement of φ being a continuous strictly increasing function such that $\varphi(1) = 1$, it is not sufficient to assure that C_φ is 2-increasing for any copula C. The following lemma will help us to prove that a sufficient condition for C_φ to be 2-increasing for any copula C is that φ be concave. To prove it, we need the following proposition

Proposition 2.4. *Let I be a nonempty interval of R. A function ϕ* from I to R is convex if and only if, for all *x_1, x_2, y_1, y_2 in I such that $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, we have that $\frac{\phi(x_2)-\phi(x_1)}{x_2-x_1} \leq \frac{\phi(y_2)-\phi(y_1)}{y_2-y_1}$*

The proof of this proposition can be found in Pecaric, Proschan and Tong (1992), p. 2.

Lemma 2.5. *Let I be a nonempty interval of R and let ϕ be an increasing convex function from I to R. Then $\phi(t_4) - \phi(t_3) - \phi(t_2) + \phi(t_1) \geq 0$ for each $t_1, t_2, t_3, t_4 \in I$ such that $t_1 \leq t_2 \leq t_4$, $t_1 \leq t_3 \leq t_4$ and $t_4 - t_3 - t_2 + t_1 \geq 0$.*

Proof. Suppose that $t_1, t_2, t_3, t_4 \in I$, $t_1 \leq t_2 \leq t_4$, $t_1 \leq t_3 \leq t_4$ and $t_4 - t_3 - t_2 + t_1 \geq 0$. Since $t_1 \leq t_3$, then $t_1 + t_4 - t_3 \leq t_4$. Since ϕ is a convex function, from Proposition 2.4. we obtain that $\phi(t_1 + t_4 - t_3)$ –- $\phi(t_1) \leq \phi(t_4) - \phi(t_3)$. We have that $t_4 - t_3 - t_2 + t_1 \geq 0$ and ϕ is increasing, so $\phi(t_2) \leq \phi(t_1 + t_4 - t_3)$. These two inequalities that ϕ satisfies, yield the conclusion of the lemma. ∎

Now we are in conditions to state and prove the main result of this section.

Theorem 2.6. *Let C be a copula. Let* $\varphi : [0, 1] \to [0, 1]$ *be a continuous strictly increasing concave function such that* $\varphi(1) = 1$, *and let* $\varphi^{[-1]}$ *be the pseudo-inverse of* φ defined by (2.1). *Then* C_φ, *given by* (1.5) *with* $n = 2$, *is a copula.*

Proof. By virtue of Theorem 2.3., to see that C_φ is a copula we need only prove that C_φ is 2-increasing. Also note that φ is concave if and only if $\varphi^{[-1]}$ is convex. Now, let x_1, x_2, y_1 and y_2 be such that $0 \leq x_1 \leq x_2 \leq 1$ and $0 \leq y_1 \leq y_2 \leq 1$. Since φ is increasing then $0 \leq \varphi(x_1) \leq \varphi(x_2) \leq 1$ and $0 \leq \varphi(y_1) \leq \varphi(y_2) \leq 1$. Let $t_4 = C(\varphi(x_2),\varphi(y_2))$, $t_3 = C(\varphi(x_1), \varphi(y_2))$, $t_2 = C(\varphi(x_2), \varphi(y_1))$ and $t_1 = C(\varphi(x_1), \varphi(y_1))$. Using that C is a copula and consequently C is 2-increasing, we have that $t_4 - t_3 - t_2 + t_1 \geq 0$. Since C is increasing in each variable, we also have $t_1 \leq t_2 \leq t_4$ and $t_1 \leq t_3 \leq t_4$. Thus, applying Lemma 2.5. to $\varphi^{[-1]}$, we obtain that $C_\varphi(x_2, y_2) - C_\varphi(x_1, y_2) - C_\varphi(x_2, y_1) + C_\varphi(x_1, y_1) = \varphi^{[-1]}(t_4) - \varphi^{[-1]}(t_3) - \varphi^{[-1]}(t_2) + \varphi^{[-1]}(t_1) \geq 0$. This shows that C_φ is 2-increasing. ∎

Denote by ϑ the set of concave continuous strictly increasing functions $\varphi : [0, 1] \to [0, 1]$ such that $\varphi(1) = 1$ and with ϑ^* the set of function in ϑ such that $\varphi(0) = 0$. Each $\varphi \in \vartheta$ satisfies $id_{[0,1]} \leq \varphi \leq c_{(1]}$ and $c_{\{1\}} \leq \varphi^{[-1]} \leq id_{[0,1]}$ where $id_{[0,1]}$ is the identity function on [0, 1] and with c_A we denote the characteristic function of the set A, i. e., $c_A(t) = 1$ if $t \in A$ and $c_A(t) = 0$ otherwise. We now present some properties that enable us to construct new generators φ from others.

Proposition 2.7. *If* $\varphi_1, \varphi_2 \in \vartheta$ $(\in \vartheta^*)$, $\varphi \in \vartheta^*$, *then:*

1. $\varphi_1 \circ \varphi_2 \in \vartheta$ $(\in \vartheta^*)$.
2. $\varphi_1\varphi_2 \in \vartheta$ $(\in \vartheta^*)$.
3. $t\,\varphi_1 + (1 - t)\,\varphi_2 \in \vartheta$ $(\in \vartheta^*)$ for $0 \leq t \leq 1$.
4. $\min(\varphi_1, \varphi_2) \in \vartheta$ $(\in \vartheta^*)$.
5. $1 - \varphi^{-1}(1 - t) \in \vartheta^*$.

We are only going to prove part 2. of the above proposition. The proofs of the other parts are very simple.

Proof of part 2. of proposition 2.7. Suppose that $\varphi_1, \varphi_2 \in \vartheta$ $(\in \vartheta^*)$. If φ_1, φ_2 are twice differentiable on (0, 1), then it is easy to see that $\varphi_1\varphi_2 \in \vartheta$ $(\in \vartheta^*)$. For the general case, consider two sequence of functions in ϑ of class C^∞, $\{\varphi_{1k}\}$, $\{\varphi_{2k}\}$, such that $\{\varphi_{1k}\}$ converges pointwise to φ_1 and $\{\varphi_{2k}\}$ converges pointwise to φ_2 (for the existence of such a sequence, see e.g. Hiriart-Urruty and Lemaréchal (1993), Proposition 2.2.3., p. 12). Since ϑ is closed under pointwise limits and for each k, $\varphi_{1k}\varphi_{2k} \in \vartheta$, then $\varphi_1\varphi_2 \in \vartheta$. Clearly, if we also have that $\varphi_1, \varphi_2 \in \vartheta^*$, then $\varphi_1\varphi_2 \in \vartheta^*$. ∎

In table 1 we list several families of functions φ_r in ϑ (or in ϑ^*), along with the pseudo-inverses $\varphi_r^{[-1]}$, the range of the parameter r, and some special and limiting cases. We specially consider functions of the form φ_r

Table 1. Examples of generators functions φ_r, along with their pseudo–inverses $\varphi_r^{[-1]}$, the range of the parameter r, and some special and limiting cases. The numbers r_1, r_2, r_3 and r_4 that appears next are given by:

r_1 is the only positive real number for which $\left(\frac{r_1}{r_1+1}\right)^{r_1+1} + 1 - r_1 = 0$ and $r_1 \cong 1.26743951639433$.

r_2 is the only positive real number for which $r_2\,(\exp(-r_2)-1) = -2$ and $r_2 \cong 2.23864583460628$.

r_3 is the only positive real number for which $r_3\,(\exp(r_3)-1) = 2$ and $r_3 \cong 1.06009031989321$.

r_4 is the only positive real number for which $r_4\,(\left(\frac{1}{r_4+1}\right)^{1/r_4+1} - 1) + 1 = 0$ and $r_4 \cong 1.29745228892048$

	$\varphi_r(t)$	$\varphi_r^{[-1]}(t)$	$r \in$	Limiting and special cases
1	$(1\text{-}r)t + r$	$\begin{cases} 0 & \text{if } 0 \le t \le r \\ \frac{t-r}{1-r} & \text{if } r \le t \le 1 \end{cases}$	[0,1)	$\varphi_0 = id_{[0,1]}, \varphi_1 = c_{[0,1]}$
2	$t^{1/r}$	t^r	$[1,\propto)$	$\varphi_1 = id_{[0,1]}, \varphi_\propto = c_{[0,1]}$
3	$\frac{(1-r)t}{1-rt}$	$\frac{t}{1-r(1-t)}$	$(-\propto,0]$	$\varphi_0 = id_{[0,1]}, \varphi_{-\propto} = c_{[0,1]}$
4	$\frac{t}{1-r(1-t)}$	$\frac{(1-r)t}{1--rt}$	[0,1)	$\varphi_0 = id_{[0,1]}, \varphi_1 = c_{[0,1]}$
5	$\frac{e^{rt}-1}{e^r-1}$	$\mathrm{Ln}((e^r-1)t+1)^{1/r}$	$(-\propto,0)$	$\varphi_0 = id_{[0,1]}, \varphi_{-\propto} = c_{[0,1]}$
6	$\ln((e^r-1)t+1)^{1/r}$	$\frac{e^{rt}-1}{e^r-1}$	$(0, \propto)$	$\varphi_0 = id_{[0,1]}, \varphi_\propto = c_{[0,1]}$
7	$\exp\left(\frac{t-1}{1+(r-1)t}\right)$	$\begin{cases} 0 & \text{if } 0 \le t \le \exp(-1) \\ \frac{\ln t+1}{(1-r)\ln t+1} & \text{if } \exp(-1) \le t \le 1 \end{cases}$	$[2, \propto)$	$\varphi_\infty(t)\begin{cases} \exp(-1) & \text{if } t=0 \\ 1 & \text{if } 0 < t \le 1 \end{cases}$
8	$\begin{cases} 0 & \text{if } t=0 \\ \frac{1}{1-r\ln t} & \text{if } 0 \le t \le 1 \end{cases}$	$\begin{cases} 0 & \text{if } t=0 \\ \exp\left(\frac{t-1}{rt}\right) & \text{if } 0 < t \le 1 \end{cases}$	$(0,\frac{1}{2}]$	$\varphi_0 = c_{(0,1]}$
9	$\frac{t^r}{2-t^r}$	$\left(\frac{2t}{1+t}\right)^{1/r}$	$(0,\frac{1}{3}]$	$\varphi_0 = c_{(0,1]}$
10	$\left(\frac{2t}{1+t}\right)^{1/r}$	$\frac{t^r}{2-t^r}$	$[1, \propto)$	$\varphi_\propto = c_{(0,1]}$
11	$\frac{1}{2-t^r}$	$\begin{cases} 0 & \text{if } 0 \le t \le 12 \\ \left(\frac{2t-1}{t}\right)^{1/r} & \text{if } 12 \le t \le 1 \end{cases}$	$(0,\frac{1}{3}]$	$\varphi_\infty(t)\begin{cases} \frac{1}{2} & \text{if } t=0 \\ 1 & \text{if } 0 < t \le 1 \end{cases}$
12	$\begin{cases} 0 & \text{if } t=0 \\ \exp(1-(1-\ln t)^r) & \text{if } 0 < t \le 1 \end{cases}$	$\begin{cases} 0 & \text{if } t=0 \\ \exp(1-(1-\ln t)^{1/r}) & \text{if } 0 < t \le 1 \end{cases}$	(0,1]	$\varphi_0 = c_{[0,1]}, \varphi_1 = id_{[0,1]}$

13	$\begin{cases} 0 & \text{if } t=0 \\ \exp(1-(1-\ln t)^{1/r}) & \text{if } 0<t\leq 1 \end{cases}$	$\begin{cases} 0 & \text{if } t=0 \\ \exp(1-(1-\ln t)^{r}) & \text{if } 0<t\leq 1 \end{cases}$	$[1, \propto)$	$\varphi_{\alpha}=c_{[0,1]}, \varphi_1=id_{[0,1]}$
14	$\exp(-(1-t^{1/r})^{r})$	$\begin{cases} 0 & \text{if } 0\leq t\leq \exp(-1) \\ (1-(1-\ln t)^{1/r})^{r} & \text{if } \exp(-1)\leq t\leq 1 \end{cases}$	$[r_1, \propto)$	$\varphi_{\infty}(t)\begin{cases} \exp(-1) & \text{if } t=0 \\ 1 & \text{if } 0<t\leq 1 \end{cases}$
15	$\begin{cases} 0 & \text{if } t=0 \\ \frac{b(t)+\sqrt{b(t)^2+4r^2}}{2} & \text{if } 0<t\leq 1 \end{cases}$ Where $b(t)=\ln t+1-r$	$\begin{cases} 0 & \text{if } t=0 \\ \exp((1-t)(\frac{r}{t}+1)) & \text{if } 0<t\leq 1 \end{cases}$	$[\frac{27}{64}, \propto)$	$\varphi_{\propto}=c_{(0,1]}$
16	$\frac{(1+t)^{-r}-1}{2^{-r}-1}$	$(1+(2^{-r}-1)t)^{-1/r}-1$	$[-1,\alpha)\{0\}$	$\varphi_{-1}=id_{[0,1]}, \varphi_{\propto}=c_{[0,1]}$
17	$(1+(2^{-r}-1)t)^{-1/r}-1$	$\frac{(1+t)^{-r}-1}{2^{-r}-1}$	$(-\propto,-1]$	$\varphi_{-1}=id_{[0,1]}, \varphi_{-\propto}=c_{[0,1]}$
18	$\begin{cases} \frac{1}{\exp(\exp(\frac{r}{t-1}))} & \text{if } 0\leq t<1 \\ 1 & \text{if } t=1 \end{cases}$	$\begin{cases} 0 & \text{if } 0\leq t\leq 1 \exp(\exp(-r)) \\ \frac{r}{\ln(\ln\frac{1}{t})}+1 & \text{if } 1(\exp(\exp(r)))\leq t\leq 1 \end{cases}$	$[r_2, \propto)$	$\varphi_{\infty}(t)\begin{cases} \frac{1}{\exp(\exp(-r))} & \text{if } t=0 \\ 1 & \text{if } 0<t\leq=1 \end{cases}$
19	$\begin{cases} 0 & \text{if } t=0 \\ \frac{r}{\ln(\exp(r)-\ln t)} & \text{if } 0<t\leq 1 \end{cases}$	$\begin{cases} 0 & \text{if } t=0 \\ \exp(\exp(r)-\exp(\frac{r}{t})) & \text{if } 0<t\leq 1 \end{cases}$	$[r_3, \propto)$	$\varphi_{\infty}=c_{(0,1]}$
20	$\begin{cases} 0 & \text{if } t=0 \\ (\ln(e-\ln t))^{1/r} & \text{if } 0<t\leq 1 \end{cases}$	$\exp(e-\exp(t^{r}))$	$(-\propto, \frac{1}{2-e}]$	$\varphi_{-\infty}=c_{(0,1]}$
21	$\exp((1-(1-t)^{r})^{1/r}-1)$	$\begin{cases} 0 & \text{if } 0\leq t\leq \exp(-1) \\ 1-(1-(1+\ln t)^{r})^{1/r} & \text{if } \exp(-1)\leq t\leq 1 \end{cases}$	$[r_4, \propto)$	$\varphi_{\infty}(t)=\begin{cases} \exp(-1) & \text{if } t=0 \\ 1 & \text{if } 0<t\leq 1 \end{cases}$
22	$\sin\frac{\pi}{2} t$	$\frac{2}{\pi}\arcsin t$		
23	$\arctan(x \tan 1)$	$\frac{\tan t}{\tan 1}$		
24	$\frac{\ln(1+rt)}{\ln(1+r)}$	$\frac{(1+r)^{t}-1}{r}$	$(0, \propto)$	$\varphi=c_{(0,1]}$

$= \exp(-\lambda_r)$ where λ_r is an additive generator of an Archimedean copula (for examples of additive generators see e.g. Nelsen (1999) pp. 94–96).

3 Some basic properties of $C_\varphi(x\ y)$

In this section we are going to study some properties of copulas C_φ. It is important to note that the concavity of φ is not required to prove some of the results.

We begin observing that in defining C_φ we only use the values of C on $[\varphi(0), 1]^2$, and consequently, if $C = D$ on $[\varphi(0), 1]^2$ then $C_\varphi = D_\varphi$. Indeed, considering the set $E(C,\varphi(0)) = \{(x, y) \in [\varphi(0), 1]^2 : C(x, y) > \varphi(0)\}$, we have $E(C, \varphi(0)) = E(D,\varphi(0))$ and $C = D$ on $E(C, \varphi(0))$, if and only if $C_\varphi = D_\varphi$.

Bivariate Archimedean copulas are symmetric, associative and satisfies $C(x, x) < x$ for all $x \in (0, 1)$. Moreover, this two last properties characterize Archimedean copulas (see e.g. Nelsen (1999), Theorem 4.1.6., p. 93). For copulas C_φ we have:

Theorem 3.1 *Let $\varphi \in \vartheta$ and let C be a copula. If C is symmetric (or associative, or $C(x, x) < x$ for all $x \in (0,1)$) then C_φ is symmetric (or associative, or $C_\varphi(x,x) < x$ for all $x \in (0,1)$). If $\varphi \in \vartheta^*$ then the converse is also true.*

Proof. Suppose that C is associative. Choose $x, y, z \in [0, 1]$. If $\varphi(0) \le C(\varphi(x),\varphi(y))$ then, by (2.2), $\varphi\ \varphi^{[-1]}(C(\varphi(x), \varphi(y))) = C(\varphi(x), \varphi(y))$. Thus,

$$\begin{aligned} C_\varphi(C_\varphi(x,y),z) &= \varphi^{[-1]}(C(\varphi\varphi^{[-1]}C(\varphi(x),\varphi(y)),\varphi(z))) \\ &= \varphi^{[-1]}(C(C(\varphi(x),\varphi(y)),\varphi(z))). \end{aligned} \tag{3.1}$$

If $C(\varphi(x), \varphi(y)) \le \varphi(0)$, we have on the one hand, $C_\varphi(x, y) = 0$. So $C_\varphi(C_\varphi(x, y), z) = 0$. On the other hand, $C(C(\varphi(x), \varphi(y)), \varphi(z)) \le C(\varphi(0), \varphi(z)) \le \varphi(0)$. Then $\varphi^{[-1]}(C(C(\varphi(x), \varphi(y)), \varphi(z))) = 0$. Consequently, in this case it is also valid the equality (3.1). In a similar manner it can be proved that

$$C_\varphi(x, C_\varphi(y,z)) = \varphi^{[-1]}(C(\varphi(x), C(\varphi(y), \varphi(z)))). \tag{3.2}$$

Since C is associative, from (3.1) and (3.2) we obtain $C_\varphi(C_\varphi(x, y), z) = C_\varphi(x,C_\varphi(y, z))$. This shows that if C is associative then C_φ is associative. The proofs of the other statements are elemental. ∎

Remark 3.2. *If C_φ is symmetric and* $\varphi(0) \neq 0$ we can not conclude that C is symmetric. For example, take $\alpha, \beta \in (0, 1)$ such that $\alpha \le \beta - \alpha$ and the shuffle of M, $C_{\alpha\beta}$ given by

$$C_{\alpha\beta}(x,y) = \begin{cases} M(x+\alpha-\beta, y) & \text{if}\,(x,y) \in [\beta-\alpha,\beta] \times [0,\alpha] \\ M(x+y-\beta, 0) & \text{if}\,(x,y) \in [0,\beta-\alpha] \times [0,\beta]\ \text{or}\,(x,y) \in [0,\beta] \times [\alpha,\beta]; \\ M(x,y) & \text{if}\,(x,y) \in [\beta,1] \times [0,1]\ \text{or}\,(x,y) \in [0,1] \times [\beta,1] \end{cases}$$

(for a definition of shuffle of M, see e.g. Nelsen (1999), p. 59). We have $C_{\alpha\beta}(\beta - \frac{\alpha}{2}, \frac{\alpha}{2}) = \frac{\alpha}{2}$ and $C_{\alpha\beta}(\frac{\alpha}{2}, \beta - \frac{\alpha}{2}) = 0$. Thus, $C_{\alpha\beta}$ is not symmetric, whereas, if $\varphi \in \vartheta$ verify that $\varphi(0) = \beta$, then $(C_{\alpha\beta})_\varphi = M$ is symmetric.

Remark 3.3. Let C be a non associative copula, $0 < \alpha < 1$ and let C_α be the ordinal sum of {C, M} with respect to {[0, α], [α, 1]},

$$C_\alpha(x,y) = \begin{cases} \alpha C\left(\frac{x}{\alpha},\frac{y}{\alpha}\right) & \text{if}(\ x,y\ ) \in[\ 0,\alpha]2 \\ M(x,y) & \text{otherwise} \end{cases};$$

(for a definition of ordinal sum, see Nelsen (1999), p. 66). Since C is not associative, there exist u, v, w ∈ [0, 1] such that

$$C(C(u,v),w) \neq C(u,C(v,w)) \tag{3.3}$$

Let $x, y, z \in [0, \alpha]$ be such that $\alpha\, u = x, \alpha\, v = y, \alpha\, w = z$. We have $C_\alpha(x, y) = \alpha C(\frac{x}{\alpha},\frac{y}{\alpha}) \in [0, \alpha]$, so $C_\alpha(C_\alpha(x, y), z) = \alpha\, C(C\,(\frac{x}{\alpha},\frac{y}{\alpha}),\frac{z}{\alpha}) = \alpha\, C(C(u, v), w)$. In a similar way we can see that $C_\alpha(x, C_\alpha(y, z)) = \alpha\, C(\frac{x}{\alpha}, C(\frac{y}{\alpha},\frac{z}{\alpha})) = \alpha C(u,C(v,w))$. Thus, by (3.3), $C_\alpha(C_\alpha(x, y), z) \neq C_\alpha(x, C_\alpha(y, z))$, and then C_α is not associative. On the other hand, if $\varphi \in \vartheta$ is such that $\varphi(0) = \alpha$, $(C_\alpha)_\varphi((C_\alpha)_\varphi(x,y), z) = M(M(x, y), z) = M(x, M(y, z)) = (C_\alpha)_\varphi(x, (C_\alpha)_\varphi(y, z))$ for each $x, y, z \in [0, 1]$. Thus, $(C_\alpha)_\varphi$ is associative. This example shows that, when $\varphi(0) \neq 0$, from the fact that C_φ is associative we can not infer that C is associative.

Remark 3.4. When $\varphi(0) \neq 0$, if $C_\varphi(x, x) < x$ for all $x \in (0, 1)$ we can not deduce that $C(x, x) < x$ for all $x \in (0, 1)$. To see this, consider the shuffle of M given by

$$C(x,y) = \begin{cases} \max(\mathrm{x}+\mathrm{y}-1,\varphi(0)) & \text{if}\,(\mathrm{x,y}) \in[\varphi(0),1]2 \\ M(x,y) & \text{otherwise} \end{cases}.$$

In this case, if $x \in (0, 1)$, then $C_\varphi(x, x) = \varphi^{[-1]}(C(\varphi(x), \varphi(y))) = \varphi^{-1}(\max(2\varphi(x)$ - $1, \varphi(0))) < \varphi^{-1}(\varphi(x)) = x$, and if $x \in [0, \varphi(0)]$, then $C(x, x) = x$.

Corollary 3.5. *Let* $\varphi \in \vartheta$ and let C be a copula. If C is Archimedean, then C_φ is Archimedean. If $\varphi \in \vartheta^*$ then the converse is also true.

It follows immediately from theorem 3.1. and the fact that a copula C is Archimedean if and only if C is associative and C(x, x) < x for all x ∈ (0, 1).

Remark 3.6. If $\varphi(0) \neq 0$ and C_φ is Archimedean, we can not conclude that C is Archimedean. To see this, consider the copula C of Remark 3.4. This copula is the ordinal sum of $\{M, W\}$ with respect to $\{[0, \varphi(0)], [\varphi(0), 1]\}$. C is not Archimedean because $C(x, x) = x$ for $x \in [0, \varphi(0)]$. On the other hand, C is the ordinal sum of two associative copulas and then it is associative (see e.g. Schweizer B and Sklar A (1983), Theorem 5.2.5., p. 57). So, from theorem 3.1., C_φ is associative. We also have $C_\varphi(x, x) < x$ for each $x \in (0, 1)$. Consequently, C_φ is Archimedean.

The following lemma, which proof follows from (1.3) and (2.1), will be useful next.

Lemma 3.7. *Let* φ and γ be continuous strictly increasing functions from [0, 1] to [0, 1] *with* $\varphi(1) = \gamma(1) = 1$*, and let* λ *be a continuous strictly decreasing function from* [0,1] to $[0, \infty]$ *with* $\lambda(1) = 0$. *Then* $(\gamma \mathrm{o} \varphi)^{[-1]} = \varphi^{[-1]} \mathrm{o} \gamma^{[-1]}$ and $(\lambda\varphi)^{[-1]}\mathrm{o} = \varphi^{[-1]}\mathrm{o}\lambda^{[-1]}$.

If φ, $\gamma \in \vartheta$ and C is a copula then, using Lemma 3.7., $C_{\varphi o \gamma} = (C_\varphi)_\gamma$. From Corollary 3.5. and Lemma 3.7., it follows that if C is an Archimedean copula with additive generator λ then C_φ is an Archimedean copula with additive generator $\lambda o \varphi$. Reciprocally, if $\varphi \in \vartheta^*$ and C_φ is an Archimedean copula with additive generator λ then C is an Archimedean copula with additive generator $\lambda o \varphi^{-1}$. We also have, by Lemma 3.7. and Theorem 2.2., the following corollary.

Corollary 3.8. *Let* $\lambda : [0, 1] \to [0, \infty]$ *be a continuous strictly decreasing function such that* $\lambda(1) = 0$ *and C given by* (1.2) *with* $n = 2$*. Then* C_φ is a copula if and only if $\lambda o \varphi$ is convex.

Remark 3.9. If $\varphi : [0, 1] \to [0, 1]$ is a continuous strictly increasing function with $\varphi(1) = 1$ and C is an Archimedean copula with additive generator λ, we have two conditions to assure that C_φ is 2-increasing: from Theorem 2.6., φ concave and from Corollary 3.8., $\lambda o \varphi$ convex. Clearly, if φ is concave then $\lambda o \varphi$ is convex. The converse is not true. To see this, consider the functions $\lambda(t) = -ln\ t$ (the additive generator of Π) and $\varphi(t) = exp(-(1 - t)^2)$. We have that $\lambda o \varphi$ is convex whereas φ is not concave.

Given any continuous strictly increasing function $\varphi : [0, 1] \to [0, 1]$ such that $\varphi(1) = 1$ and any copula C, from Theorem 2.6. we have that a sufficient condition for C_φ to be 2-increasing is that φ be concave. When $C = W$ this condition is also necessary.

Theorem 3.10. *Let* $\varphi : [0, 1] \to [0, 1]$ *be a continuous strictly increasing function such that* $\varphi(1) = 1$*. Then* W_φ *is a copula if and only if* φ *is concave.*

Proof. W is an Archimedean copula with additive generator $\lambda(t) = 1 - t$. So, by Corollary 3.8., W_φ is a copula if and only if $1 - \varphi(t)$ is convex. ■

It is worth mentioning that copula M is invariant under the transformation considered in this paper, i. e., if $\varphi : [0, 1] \to [0, 1]$ is a continuous strictly increasing function then $M_\varphi = M$.

Following appear some results concerning uniform convergence of sequences of the form $\{\varphi_n^{[-1]}(C(\varphi_n(x), \varphi_n(y)))\}$ and $\{\varphi^{[-1]}(C_n(\varphi(x), \varphi(y)))\}$. This will be useful in the future to analyze limiting cases of different families of copulas C_φ. We recall that any n–copula C satisfies the Lipschitz condition

$$|C(x_1, \ldots, x_n) - C(y_1, \ldots, y_n)| \leq |x_1 - y_1| + \ldots + |x_n - y_n|, \tag{3.4}$$

whence any collection of n-copulas is equicontinuous (see e.g. Nelsen (1999), Theorem 2.10.7., p. 40). Consequently, any sequence of n-copulas $\{C_n\}$ that converges pointwise to C, converge uniformly to C (see e.g. Rudin (1976), Theorem 7.25, p. 158). Taking into account this, any of the following statements remain valid if the sequences of copulas converge pointwise instead of converge uniformly.

Theorem 3.11. *Let C and* $\{C_n\}$ be copulas, and let $\varphi \in \vartheta$. If $\{C_n\}$ converges uniformly to C then $\{(C_n)_\varphi\}$ converges uniformly to C_φ.

Proof. Let $\varepsilon > 0$. Since $\varphi^{[-1]}$ is continuous on [0, 1], it is uniformly continuous there. So, there exists $\delta > 0$ such that

$$|\varphi^{[-1]}(x) - \varphi^{[-1]}(y)| < \epsilon \tag{3.5}$$

for all $x, y \in [0, 1]$ with $|x - y| < \delta$. Since $\{C_n\}$ converges uniformly to C, then there exists $n_0 \in N$ such that

$$|C_n(x, y) - C(x, y)| < \delta \tag{3.6}$$

for all $n \geq n_0$ and all $x, y \in [0, 1]$.
From (3.5) and (3.6) it results that

$$|\varphi^{[-1]}(C_n(\varphi(x), \varphi(y))) - \varphi^{[-1]}(C(\varphi(x), \varphi(y)))| < \epsilon$$

for all $n \geq n_0$ and all $x, y \in [0, 1]$.
This shows that $\{(C_n)_\varphi\}$ converges uniformly to C_φ.

Theorem 3.12. *Let C be a copula. Let* $\{\varphi_n\}$ in ϑ such that $\{\varphi_n\}$ converges uniformly to f and $\{\varphi_n^{[-1]}\}$ *converges uniformly to g. Then* $\{C_{\varphi_n}\}$ *converges uniformly to g(C(f(x), f(y)))*.

Proof. Let $\varepsilon > 0$. Since $\{\varphi_n^{[-1]}\}$ converges uniformly to g, then $\{\varphi_n^{[-1]}\}$ is an equicontinuous family (see e.g. Rudin (1976), Theorem 7.24, p. 157). Thus, there exists $\delta > 0$ such that

$$|\varphi_n^{[-1]}(s) - \varphi_n^{[-1]}(t)| < \varepsilon/2 \tag{3.7}$$

for all $n \in N$ and all $s, t \in [0, 1]$ with $|s - t| < \delta$. Since $\{\varphi_n\}$ converges uniformly to f, then there exists $N_1 \in N$ such that

$$|\varphi_n(x) - f(x)| < \delta/2$$

for all $n \geq N_1$ and all $x \in [0, 1]$. Then, using (3.4),

$$|C(\varphi_n(x), \varphi_n(y)) - C(f(x), f(y))| \leq |\varphi_n(x) - f(x)| + |\varphi_n(y) - f(y)| < \delta \tag{3.8}$$

for all $n \geq N_1$ and all $x, y \in [0, 1]$. From (3.7) and (3.8),

$$|\varphi_n^{[-1]}(C(\varphi_n(x), \varphi_n(y))) - \varphi_n^{[-1]}(C(f(x), f(y)))| < \epsilon/2 \tag{3.9}$$

for all $n \geq N_1$ and all $x, y \in [0, 1]$. Since $\{\varphi_n^{[-1]}\}$ converges uniformly to g, then there exists $N_2 \in N$ such that

$$|\varphi_n^{[-1]}(C(f(x), f(y))) - g(C(f(x), f(y)))| < \epsilon/2 \tag{3.10}$$

for all $n \geq N_2$ and all $x, y \in [0, 1]$. From (3.9) and (3.10),

$$\begin{aligned}
&|\varphi_n^{[-1]}(C(\varphi_n(x), \varphi_n(y))) - g(C(f(x), f(y)))| \\
&\quad \leq |\varphi_n^{[-1]}(C(\varphi_n(x), \varphi_n(y))) - \varphi_n^{[} - 1](C(f(x), f(y)))| \\
&\quad + |\varphi_n^{[-1]}(C(f(x), f(y))) - g(C(f(x), f(y)))| \\
&\quad < \epsilon/2 + \epsilon/2 = \epsilon
\end{aligned}$$

for all $n \geq max(N_1, N_2)$ and all $x, y \in [0, 1]$. Hence $\{C_{\varphi_n}\}$ converges uniformly to $g(C(f(x), f(y)))$.

We recall that given two copulas C and D, we say that C is smaller than D (or D is larger than C), and write $C \prec D$ (or $D \succ C$) if $C(x, y) \le D(x, y)$ for all x, y in [0, 1]. It is easy to see that if $C \prec D$ then $C_\varphi \prec D_\varphi$. Conversely, if $C_\varphi \prec D_\varphi$ then $C \prec D$ on $\{(x, y) \in [\varphi(0), 1]^2 : C(x, y) \ge \varphi(0)\}$.

Definition 3.13. *Let* $C : [0, 1]^2 \to [0, 1]$. *A function* $f : [0, 1] \to [0, 1]$ *it said to be supra–C if* $C(f(x), f(y)) \le f(C(x, y))$ *for all* $x, y \in [0, 1]$.

The following theorem is similar to Theorem 4.4.2., p. 109, in Nelsen (1999).

Theorem 3.14. *Let* $\varphi_1, \varphi_2 \in \vartheta$ *and let C be a copula. Then* $C_{\varphi_1} \prec C_{\varphi 2}$ *if and only if* $\varphi_1 \circ \varphi_2^{[-1]}$ *is supra–C.*

Proof. Let $f = \varphi_1 \circ \varphi_2^{[-1]}$. From (1.5), $C_{\varphi_1} \prec C_{\varphi_2}$ if and only if

$$\varphi_1^{[-1]}(C(\varphi_1(x), \varphi_1(y))) \le \varphi_2^{[-1]}(C(\varphi_2(x), \varphi_2(y))). \tag{3.11}$$

for all $x, y \in [0, 1]$. Let $s, t \in [\varphi_2(0), 1]$, and let $x, y \in [0, 1]$ be such that $s = \varphi_2(x)$ and $t = \varphi_2(y)$. From (3.11), $\varphi_1^{[-1]}(C(f(s), f(t))) \le \varphi_2^{[-1]}(C(s, t))$ for all $s, t \in [\varphi_2(0),1]$. If $s < \varphi_2(0)$ or $t < \varphi_2(0)$, then $\varphi_1^{[-1]}(C(f(s),f(t))) = \varphi_2^{[-1]}(C(s, t)) = 0$. Thus, $C_{\varphi_1} \prec C_{\varphi_2}$ if and only if

$$\varphi_1^{[-1]}(C(f(s), f(t))) \le \varphi_2^{[-1]}(C(s, t)) \tag{3.12}$$

for all $s, t \in [0, 1]$. Suppose now that $C_{\varphi_1} \prec C_{\varphi_2}$. Applying φ_1 to both sides of (3.12), taking into account that φ_1 is an increasing function and using (2.3), we obtain

$$C(f(s), f(t)) \le f(C(s, t)) \tag{3.13}$$

for all $s, t \in [0, 1]$. This shows that f is supra-C. Conversely, if f is supra–C, applying $\varphi_1^{[-1]}$ to both sides of (3.13) we obtain the inequality (3.12). Thus, $C_{\varphi_1} \prec C_{\varphi_2}$. ∎

4 The multivariate case

We begin this section with a theorem similar to Theorem 2.3.

Theorem 4.1. *Let C be an n-copula, let* $\varphi : [0, 1] \to [0, 1]$ *be a continuous strictly increasing function such that* $\varphi(1) = 1$, *and let* $\varphi^{[-1]}$ *be the pseudo-inverse of* φ *defined by (2.1). Then* C_φ, *given by* (1.5), *satisfies the boundary conditions (a) and (b) for an n-copula.*

If $n > 2$, the requirement of φ being a continuous strictly increasing concave function such that $\varphi(1) = 1$, it is not sufficient to assure that C_φ, given by (1.5), is n-increasing. To see this, consider the mixture of extremal 3-copulas,

$$C(x, y, z) = \frac{1}{2} W(x, M(y, z)) + \frac{1}{2} W(y, M(x, z)); \tag{4.1}$$

(for a definition of extremal n-copulas see e.g. Tiit (1996)). If $\varphi(t) = t^{1.5}$, the C_φ-volume of the 3-box $B = [27/64,\ 1] \times [27/64,\ 1] \times [8/27,\ 27/64]$ is approximately equal to -0.0194. So C_φ is not 3-increasing.

As we did in the bivariate case, in order to fix ideas about how φ must be, we first observe what happens when we consider Archimedean n–copulas. In this case $\lambda^{[-1]}$ has derivatives that alternate in sign.

Definition 4.2. *Let n be a nonnegative integer. A function g(t) is completely monotonic (D) of order n on an interval J if it is continuous there and has derivatives up to and including the nth which alternate in sign, i. e., if it satisfies* $(-1)^{\mathrm{k}}\, g^{(\mathrm{k})}(t) \geq 0$ *for all t in the interior of J and k* $= 0, 1, \ldots, n$. *A function g(t) is completely monotonic (D) on an interval J if it is completely monotonic (D) of order n on J for all nonnegative integers n.*

It is worth mentioning that if the pseudo-inverse $\lambda^{[-1]}$ of an Archimedean generator λ is completely monotonic (D), it must be positive on $[0, \infty)$, i. e., λ is strict and $\lambda^{[-1]} = \lambda^{-1}$.

The following theorem, that appears in Kimberling (1974), gives necessary and sufficient conditions for a strict generator λ to generate Archimedean n-copulas for all $n \geq 2$. See also Schweizer and Sklar (1983), Theorem 6.3.6., p. 88.

Theorem 4.3. *Let* $\lambda : [0, 1] \to [0, \infty]$ *be a continuous strictly decreasing function such that* $\lambda(0) = \infty$ *and* $\lambda(1) = 0$, *and let* λ^{-1} *be the inverse of* λ. *Then* C_λ *defined by (1.2) is an n–copula for all* $n \geq 2$ *if and only if* λ^{-1} *is completely monotonic (D) on* $[0, \infty)$.

Indeed the arguments in Kimberling (1974) can be used to partially extend theorem 4.3:

Theorem 4.4. *Let* $\lambda : [0, 1] \to [0, \infty]$ *be a continuous strictly decreasing function such that* $\lambda(0) = \infty$ and $\lambda(1) = 0$, *and let* $\lambda^{[-1]}$ *be the pseudo-inverse of* λ. *If* $\lambda^{[-1]}$ *is completely monotonic (D) of order m on* $[0, \infty)$ *for some* $m \geq 2$, then C_λ *defined by (1.2) is an n-copula for* $2 \leq n \leq m$.

Taking into account theorem 4.4 and theorem 2.6., it is natural to think that for $n > 2$ we need that $\varphi^{[-1]}$ has nonnegative derivatives up to and including the nth to assure that C_φ is an n–copula.

Definition 4.5. *Let n be a nonnegative integer. A function g(t) is absolutely monotonic (D) of order n on an interval J if it is continuous there and has nonnegative derivatives up to and including the nth, i. e., if it satisfies* $g^{(\mathrm{k})}(t) \geq 0$ *for all t in the interior of J and k* $= 0, 1, \ldots, n$. *A function g(t) is absolutely monotonic (D) on an interval J if it is absolutely monotonic (D) of order n on J for all nonnegative integers n.*

In Feller (1961) several properties of absolutely monotonic functions can be found.

The following fact will be useful next.

Proposition 4.6. *Every n-copula C is the uniform limit of a sequence $\{C_k\}$ of n-copulas which has nth-order derivatives.*

Proof. Let C be an n–copula and $k \in N$. If C has nth-order derivatives, then we can choose $C_k = C$. Otherwise, consider the multivariate Bernstein polynomial associate with C given by

$$C_k(x1,\ldots,x_n) = \sum_{i_1,\ldots,i_n=1}^{k} C(\frac{i_1}{k},\ldots,\frac{i_n}{k})\binom{k}{i_1}x_1^{i_1}(1-x_1)^{k-i_1}\ldots\binom{k}{i_n}x_n^{i_n}(1-x_n)^{k-i_n}.$$

The function C_k has nth-order derivatives and it is easy to verify that it is an n–copula. Also $\{C_k\}$ converges uniformly to C (see e.g. Lorentz (1986), p. 51). ■

Now we are in conditions to state and prove the main result of this section.

Theorem 4.7. *Let C be an n-copula. Let* $\varphi : [0, 1] \to [0, 1]$ *be a continuous strictly increasing function such that* $\varphi(1) = 1$, *and let* $\varphi^{[-1]}$ *be the pseudo-inverse of* φ *defined by* (2.1). *If* $\varphi^{[-1]}$ *is absolutely monotonic (D) of order n on* [0, 1], *then* C_φ *given by (1.5) is an n-copula.*

Proof. By virtue of Theorem 4.1., to see that C_φ is an n–copula we need only prove that C_φ is n-increasing. Let $(a_1,\ldots, a_n)$, $(b_1,\ldots, b_n) \in [0, 1]^n$, with $a_i < b_i$, $i = 1,\ldots, n$. Set $B = [a_1, b_1] \times\ldots\times [a_n, b_n]$. Since φ is strictly increasing, then $\varphi(a_i) < \varphi(b_i)$, $i = 1,\ldots, n$. Set $B' = [\varphi(a_1), \varphi(b_1)] \times\ldots\times [\varphi(a_n), \varphi(b_n)]$. Let $\{C_k\}$ be a sequence of n-copulas that converge uniformly to C and such that for each k, C_k has nth-order derivatives (this sequence exists by Proposition 4.6.). Each function $\varphi^{[-1]}\circ C_k$ has nth-order derivatives given by

$$\frac{\partial^n}{\partial x_1\ldots\partial x_n}(\varphi^{[-1]}\circ C_k) = \sum_{j=1}^{n}((\varphi^{[-1]})^{(j)}\circ C_k)\sum_{p\in p_j}\prod_{S\in p}(C_k)S,$$

where P_j denotes the set of partitions of $\{1,\ldots,n\}$ in j sets and $(C_k)_S = \frac{\partial^m}{\partial x_{i_1}\ldots\partial x_{i_m}}C_k$ with $S = \{i_1,\ldots,i_m\}$. Since C_k is an n-copula and $\varphi^{[-1]}$ is absolutely monotonic (D) of order n on [0, 1], the second hand of the above equality is nonnegative. Thus, $\frac{\partial^n}{\partial x_1\ldots\partial x_n}(\varphi^{[-1]}\circ C_k) \geq 0$, and consequently, $\varphi^{[-1]}\circ C_k$ is n-increasing. In particular, $V_{\varphi^{[-1]}\circ C_k}(B') \geq 0$. Since $\{C_k\}$ converge uniformly to C, it results that $\{V_{\varphi^{[-1]}\circ C_k}(B')\}$ converge to $V_{\varphi^{[-1]}\circ C}(B')$. Then $V_{C_\varphi}(B) = V_{\varphi^{[-1]}\circ C}(B') \geq 0$. This shows that C_φ is n-increasing. ■

In the sequel, we will denote by ϑ_n the set of continuous strictly increasing functions $\varphi : [0, 1] \to [0, 1]$ with $\varphi(1) = 1$, such that $\varphi^{[-1]}$ is absolutely monotonic (D) of order n on [0, 1], and with ϑ_n^* the set of functions φ in ϑ_n such that $\varphi(0) = 0$. Clearly, $\vartheta_n \subset \vartheta_{n-1}$ and $\vartheta_2 \subset \vartheta$. It is important to note that part 1. of proposition 2.7. remains true if we replace ϑ by ϑ_n. In section 3. we saw several examples of generators functions φ_r in ϑ_2. We limit us to mention that the functions of family 2. of table 1 are in ϑ_n^* for all $r \in IN$, they are in ϑ_n^* for $r \in R$, $r \geq n$-1, and the functions 22 and 23 of table 1 are in ϑ_n^*.

As the reader can easy note, some of the properties presented in section 3. have a direct multivariate extension. In particular, suppose that C is an Archimedean n–copula with additive generator λ such that $\lambda^{[-1]}$ is completely monotonic (D) of order n on $[0, \infty)$. Since $C_\varphi(x_1,\ldots, x_n) = (\lambda o \varphi)^{[-1]}((\lambda o \varphi)(x_1) + \ldots + (\lambda \circ \varphi)(X_n))$, then we have two conditions to assure that C_φ is an n–copula. By Theorem 4.4., $\varphi^{[-1]} o \lambda^{[-1]}$ completely monotonic (D) of order n on $[0, \infty)$ and, by Theorem 4.7., $\varphi^{[-1]}$ absolutely monotonic (D) of order n on $[0, 1]$. Clearly, if $\varphi^{[-1]}$ is absolutely monotonic (D) of order n on $[0, 1]$, then $\varphi^{[-1]} o \lambda^{[-1]}$ is completely monotonic (D) of order n on $[0, \infty)$. But the converse is not true. To see this, consider the functions $\lambda(t) = (-ln\ t)^2$ and $\varphi(t) = exp(-(-ln\ t)^2)$. In this case, $\varphi^{-1} \circ \lambda^{-1}(t) = exp(-t^{1/4})$ is completely monotonic (D) on $[0, \infty)$, whereas $\varphi^{-1}(t) = exp(-(-ln\ t)^{1/2})$ is not absolutely monotonic (D) of order 2 on $[0, 1]$ (φ^{-1} is not convex on $[0, 1]$).

5 Mixtures and n-copulas C_φ

Let Λ be an univariate distribution function such that $\Lambda(0) = 0$ and let Ψ be the Laplace transform of Λ, i. e., $\Psi(\mathrm{t}) = \int_0^\infty \exp(-r\ t)\ d\Lambda(r)$ for $t \geq 0$. The function Ψ is completely monotonic on $[0, \infty)$ with $\Psi(0) = 1$ (see e.g. Feller (1966), Theorem 1., p. 415).

Consider the distribution function $\exp(-\Psi^{-1})$ and an arbitrary n–copula C. Then the function

$$\int_0^\infty [C(\exp(-\psi^{-1}(x_1)),\ldots,\exp(-\psi^{-1}(x_n)))]^r d\wedge(r)$$
$$= \psi(-\ln(C(\exp(-\psi^{-1}(x_1)),\ldots,\exp(-\psi^{-1}(x_n))))) \tag{5.1}$$

is the mixture of $\{[C(exp(-\Psi^{-1}),\ldots, \exp(-\Psi^{-1}))]^r\}_{r \geq 0}$ with respect to Λ. If $C = \Pi$, then the function given by (5.1) can be written as $\Psi(\Psi^{-1}(x_1) + \ldots + \Psi^{-1}(x_n))$, and then it is an Archimedean n–copula. If $\varphi(t) = \exp(-\Psi^{-1})$, then the function given by (5.1) coincides with C_φ.

Suppose now that $\varphi \in \vartheta_n^*$ and φ^{-1} is absolutely monotonic (D) on $[0, 1]$. Since $exp(-t)$ is completely monotonic (D) on $[0, \infty)$, then $\Psi(t) = \varphi^{-1}(exp(-t))$ is completely monotonic (D) on $[0, \infty)$. We also have $\Psi(0) = 1$, consequently Ψ is the Laplace transform of certain univariate distribution function Λ such that $\Lambda(0) = 0$ (see e.g. Feller (1966), Theorem 1., p. 415). Since the function given by (5.1) coincides with C_φ, we can use Theorem 4.7. to conclude that for this Ψ it defines an n–copula for any n–copula C. The previous result is not necessary valid for other functions Ψ. For example, consider the completely monotonic (D) function $\Psi(t) = exp(-t^{1/2})$. In this case, the function given by (5.1) is not an n–copula for any n–copula C. To see this, let C be the 3-copula given by (4.1). The C_φ-volume of the 3-box $B = [1/4, 1/2] \times [3/4, 1] \times [1/4, 1/2]$ is approximately equal to -0.02. So C_φ is not 3-increasing. In general, for (5.1) to be an n–copula (concretely, to be n-increasing) C must be max-infinitely divisible: C^r n–copula for all $r \geq 0$. For this reason n-copulas given by (5.1) are known as mixtures of max-infinitely divisible n-copulas. The sufficient condition that it is considered to assure that C is max-infinitely divisible, is that $ln\ C$ posses nonnegative partial derivatives up to and including the nth (see e.g. Joe and Hu (1996), Theorem 4.1., p. 253).

6 Conclusion

We have found sufficient conditions under which a function φ generates an n–copula C_φ via $C_\varphi(x_1,\ldots, x_n) = \varphi^{[-1]}(C(\varphi(x_1),\ldots, \varphi(x_n)))$, for any n–copula C. This result permits us to construct new n-copulas from others. We have also presented examples of generator functions φ and we have explored some basic properties of n-copulas C_φ. Clearly, it is necessary to do a deeper theoretical study of the behavior of n-copulas C_φ and to compare it with that of the original n–copula C. In particular, the relation between the construction considered in this paper and mixtures could be useful to front this study.

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