Metrika (2004) 59: 305–313
DOI 10.1007/s001840300287

Metrika

Bivariate semi-Pareto minification processes

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Received July 2002

Abstract. Marshall-Olkin bivariate semi-Pareto distribution (MO-BSP) and Marshall-Olkin bivariate Pareto distribution (MO-BP) are introduced and studied. AR(1) and AR(k) time series models are developed with minification structure having MO-BSP stationary marginal distribution. Various characterizations are investigated.

Key words: MO-BSP distribution; Autoregressive minification processes of order 1 and k; Characterizations.

1 Introduction

The study of first order autoregressive processes having minification structure began with the work of Tavares (1980) and subsequently many authors developed autoregressive minification processes having various marginal distributions. Lewis and McKenzie (1991) define a first order autoregressive minification process as a sequence having the general structure

$$X_n = \begin{cases} kX_{n-1} & \text{with probability } p \\ k\min(X_{n-1}, \in_n) & \text{with probability } 1-p \end{cases} \tag{1.1}$$

where $\{\in_n\}$ is an innovation process of independent and identically distributed random variables chosen to ensure that $\{X_n\}$ is a stationary Markov process with a specified marginal distribution function $F_X(x)$.

Another form of minification process is having the structure

$$X_n = \begin{cases} k \in_n & \text{with probability } p \\ k\min(X_{n-1}, \in_n) & \text{with probability } 1-p \end{cases} \tag{1.2}$$

Because of the structure of the process, $\{X_n\}$ as defined above is called a minification process. (For details see Yeh et al. (1988), Arnold and Robertson

(1989) and Pillai et al. (1995)). Minification processes find applications for modeling time series data from hydrological as well as socio-economic contexts. Block et al. (1988) introduced additive first order autoregressive processes with bivariate exponential and geometric stationary marginal distributions and studied their properties. Dewald et al. (1989) introduced an additive first order autoregressive bivariate exponential process.

In many socio-economic contexts, the data are usually multivariate in nature. Several components like income, expenditure, area of land holdings etc. are taken into consideration. In such cases the data are often skewed and show a tendency to follow heavy tailed distributions. The role of Pareto law in modeling data on income, stock price fluctuations, insurance risks, business failures etc. is well known. Thus a wide variety of socio-economic data have distributions which are heavy tailed and reasonably well fitted by Pareto or generalized Pareto distribution. In reliability contexts also bivariate Pareto distributions are found to be appropriate models. Sankaran and Nair (1993) discuss the application of bivariate Pareto distributions in reliability theory. Alice and Jose (2002) introduced and studied the univariate Marshall-Olkin Pareto processes. In this paper, AR (1) and AR (k) time series models useful in generating first order and k^{th} order autoregressive minification processes having a specified stationary bivariate marginal distribution are introduced and studied.

In the present paper we consider a new family of distributions introduced by Marshall-Olkin (1997) and similar to those introduced by Pillai, Jose and Jayakumar (1995). In section 2, we introduce the Marshall-Olkin bivariate semi-Pareto distribution as a generalization of the bivariate semi-Pareto distribution of Balakrishna and Jayakumar (1997). In section 3, we consider the Marshall-Olkin bivariate Pareto (MO-BP) distribution in detail. Some characteristic properties of MO-BSP distribution are obtained in section 4. In section 5, we construct a bivariate semi-Pareto AR (1) model having MO-BSP stationary distribution. We generalize it to the k^{th} order autoregressive model in section 6. The model developed here is analogous to the model introduced by Lawrance and Lewis (1982) where the role of addition is taken by minimization.

2 Marshall-Olkin bivariate semi-Pareto distributions

Now we consider the bivariate semi-Pareto distribution and distributions related to it, which are generally used for modelling socio-economic data. A random vector (X, Y) is said to have the bivariate semi-Pareto distribution with parameters β_1, β_2 and ρ if its survival function is of the form

$$\bar{F}(x, y) = \frac{1}{1 + \psi(x,y)} \tag{2.1}$$

where $\psi(x,y)$ satisfies the functional equation

$$\rho\psi(x,y) = \psi\left(\rho^{1/\beta_1}x, \rho^{1/\beta_2}y\right). \tag{2.2}$$

The equation is true for all $x \geq 0,\ y \geq 0$ and particular ρ, β_1, β_2 where $0 < \rho < 1$; $\beta_1 > 0$, $\beta_2 > 0$. Also $\psi(x,y)$ is a monotonically increasing function in both x and y satisfying

$$\lim_{x\to 0}\lim_{y\to 0}\psi(x,y)=0 \text{ and } \lim_{x\to\infty}\lim_{y\to\infty}\psi(x,y)=\infty.$$

Writing $\rho = \exp(z)$, $x = \exp(u)$, $y = \exp(v)$, $a = 1/\beta_1$, $b = 1/\beta_2$ and taking logarithms, equation (2.2) can be written as

$$z + H(u, v) = H(u + az, v + bz).$$

This functional equation is a special case of the general equation (4) in pp. 310 in Aczel (1966).

The solution of the functional equation (2.2) is given by

$$\psi(x,y) = x^{\beta_1}h(x) + y^{\beta_2}h(y),$$

where h(x) and h(y) are the periodic functions in $\log x$ and $\log y$ with periods $\frac{2\pi\beta_1}{-\log\rho}$ and $\frac{2\pi\beta_2}{-\log\rho}$ respectively. (See Kagan et al. (1973)). It can be verified that the univariate marginal distributions of X and Y are the univariate semi-Pareto distribution of Pillai (1991), by taking $\psi(x) \equiv \psi(x,0)$ and $\psi(y) \equiv \psi(0,y)$.

Marshall and Olkin (1997) considered a bivariate extension of a family of distributions as follows.

Let (X, Y) be a random vector with joint survival function $\bar{F}(x,y)$. Then

$$\bar{G}(x,y) = \frac{\alpha\bar{F}(x,y)}{1-(1-\alpha)\bar{F}(x,y)}; \quad x, y \geq 0, \ \ 0 < \alpha < 1, \tag{2.3}$$

is a proper bivariate survival function. The family of distributions of the form (2.3) shall be called Marshall-Olkin bivariate family of distributions.

From (2.3) we can see that the new survival function is

$$\bar{G}(x, y; \alpha) = \frac{1}{1+\frac{1}{\alpha}\psi(x,y)}; \quad x, y \geq 0, \ \ 0 < \alpha < 1, \tag{2.4}$$

which we shall refer to as Marshall-Olkin bivariate semi-Pareto distribution denoted by MO-BSP. Similar bivariate distributions can be developed by considering bivariate Weibull and exponential survival functions. For example a bivariate semi-Weibull distribution has a survival function of the form $\bar{F}(x, y) = \exp(-\psi(x,y))$, where $\psi(x,y)$ satisfies the conditions specified above. Then the Marshall-Olkin Bivariate Semi-Weibull distribution has the survival function given by

$$\bar{G}(x,y;\alpha) = \frac{\alpha e^{-\psi(x,y)}}{1-(1-\alpha)e^{-\psi(x,y)}}. \tag{2.6}$$

Theorem 2.1. *Let* $\{(X_i, Y_i), i \geq 1\}$ *be a bivariate sequence of non-negative random vectors independently and identically distributed as Marshall-Olkin Bivariate Semi-Pareto, then* $Z_n = ((n/\alpha)^{1/\beta_1}\min(X_1, X_2, \ldots, X_n), (n/\alpha)^{1/\beta_2}\min(Y_1, Y_2, \ldots, Y_n))$*;* $\beta_1 > 0$*,* $\beta_2 > 0$*,* $n > 1$*,* $n > \alpha$ *is asymptotically distributed as bivariate semi-Weibull as n goes to infinity.*

Proof. If (X, Y) is distributed as MO-BSP, then from (2.4) we have

$$\bar{G}(x,\ y) = \frac{1}{1 + \frac{1}{\alpha}\psi(x,y)},$$

$$F^{-}_{Z^{(n)}}(x,y) = P\Big[(n/\alpha)^{1/\beta_1}\min(X_1, X_2, \ldots, X_n) > x, (n/\alpha)^{1/\beta_2}\min(Y_1, Y_2, \ldots, Y_n) > y\Big]$$

$$= \left(\bar{G}\left((n/\alpha)^{-1/\beta_1}x, (n/\alpha)^{-1/\beta_2}y\right)\right)^n$$

$$= \left(\frac{1}{1 + \frac{\psi(x,y)}{n}}\right)^n$$

which tends to $e^{-\psi(x,y)}$ as n goes to infinity.

This establishes the theorem. As a corollary we have the following result. ■

Corollary 2.1. *If* $\{(X_i, Y_i), i \geq 1\}$ *be a sequence of bivariate non–negative random vectors identically and independently distributed as Marshall-Olkin bivariate Pareto, then* $Z^{(n)} = ((n/\alpha)^{1/\beta_1}\min(X_1, X_2, \ldots, X_n), (n/\alpha)^{1/\beta_2}\min(Y_1, Y_2, \ldots, Y_n))$*;* $\beta_1, \beta_2 > 0$*,* $n > 1$*,* $n > \alpha$ *is asymptotically distributed as bivariate Weibull as n goes to infinity.*

Now we consider the extension of the concept of domain of attraction to the multivariate set up given by Marshall and Olkin (1983) and discussed in detail by Castillo (1988).

Definition 2.1. For $\mathbf{a}, \mathbf{b}, \mathbf{x} \in R^k$ write, $\mathbf{a} + \mathbf{bx}$ to denote the vector $(a_1 + b_1x_1, \ldots, a_k + b_kx_k)$ where $\mathbf{a} = (a_1, a_2, \ldots, a_k)$, $\mathbf{b} = (b_1, b_2, \ldots, b_k)$ and $\mathbf{x} = (x_1, x_2, \ldots, x_k)$. Let $X^{(1)}, X^{(2)}, \ldots$ be a sequence of independent k-dimensional random vectors with common distribution G and let $W_j^{(n)} = \min_{1<i\leq n} X_j^{(i)}$; $j = 1, 2, \ldots, k$. We say G is in the domain of attraction of F for minimum if there exists sequences of vectors $\{\mathbf{a}^{(n)}\}$ and $\{\mathbf{b}^{(n)} > \mathbf{0}\}$ such that $\lim_{n\to\infty} \overset{-(n)}{G}\left(\mathbf{a}^{(n)} + \mathbf{b}^{(n)}x\right) = \bar{F}(x)$ for all $\mathbf{x}$.

Now we have the following result.

Theorem 2.2. *The Marshall-Olkin bivariate semi-Pareto distribution (MO-BSP) is in the domain of attraction for minimum of the bivariate semi-Weibull distribution.*

Proof. We have

$$\bar{G}(x,y) = \frac{1}{1 + \frac{1}{\alpha}\psi(x,y)}.$$

Taking $\mathbf{a}^{(n)} = (0, 0)$ and $\mathbf{b}^{(n)} = ((n/\alpha)^{-1/\beta_1}), (n/\alpha)^{-1/\beta_2})$ and $\mathbf{x} = (x,y)$ we have

$$\bar{G}^{(n)}\left(a^{(n)}+b^{(n)}x\right)=\left(\bar{G}\left((n/\alpha)^{-1/\beta_1}x,(n/\alpha)^{-1/\beta_2}y\right)\right)^n$$
$$=\left(\frac{1}{1+\frac{\psi(x,y)}{n}}\right)^n.$$

Therefore

$$\lim_{n\to\infty}\bar{G}^{(n)}\left(a^{(n)}+b^{(n)}x\right)=e^{-\psi(x,y)}$$

which is the survival function of the bivariate semi-Weibull family of distributions ■

3 Marshall-Olkin bivariate Pareto distribution

In this section we consider the special case namely bivariate Pareto distribution having the survival function

$$\bar{F}(x,y)=\frac{1}{1+x^{\beta_1}+y^{\beta_2}};\quad x,y\geq 0,\ \beta_1,\beta_2>0.$$

It may be noted that if $\bar{F}(x,y)$ is a survival function, then $\bar{F}(x^\alpha,y^\beta)$ is also a survival function.

From (2.3) the new survival function is

$$\bar{G}(x,y)=\frac{1}{1+\frac{1}{\alpha}(x^{\beta_1}+y^{\beta_2})};\quad x,y\geq 0,\ \beta_1>0,\ \beta_2>0,\ 0<\alpha<1,$$

which is known as Marshall-Olkin bivariate Pareto(MO-BP) distribution.

The density function is given by

$$g(x,y)=2\beta_1\beta_2x^{\beta_1-1}y^{\beta_2-1}\frac{1}{\alpha^2}\left(1+(1/\alpha)(x^{\beta_1}+y^{\beta_2})\right)^{-3};$$
$$x,y\geq 0,\ \beta_1>0,\ \beta_2>0,\ 0<\alpha<1.$$

The marginal distributions of X and Y are

$$g(x)=\frac{\beta_1}{\alpha}x^{\beta_1-1}\left(1+(1/\alpha)x^{\beta_1}\right)^{-2};\quad x\geq 0,\ \beta_1>0,\ 0<\alpha<1.$$

and

$$g(y)=\frac{\beta_2}{\alpha}y^{\beta_2-1}\left(1+(1/\alpha)y^{\beta_2}\right)^{-2};\ y\geq 0,\ \beta_2>0,\ 0<\alpha<1.$$

$$E(X_i^r)=\left(\alpha^{1/\beta_i}\right)^rB\left(1+\frac{r}{\beta_i},1-\frac{r}{\beta_i}\right);\ r<\beta_i$$

$$V(X_i)=\alpha^{2/\beta_i}\left\{\Gamma(1+(2/\beta_i))\Gamma(1-(2/\beta_i))-(\Gamma(1+(1/\beta_i))\Gamma(1-(1/\beta_i)))^2\right\};$$
$$\text{if }(\beta_i>2).$$

$$E(XY)=2\alpha^{\frac{1}{\beta_1}+\frac{1}{\beta_2}-2}B\left(\frac{1}{\beta_1}+1,2-\frac{1}{\beta_1}\right)B\left(\frac{1}{\beta_2}+1,1-\frac{1}{\beta_1}-\frac{1}{\beta_2}\right).$$

Using these the correlation between X and Y can be obtained.

4 Characterizations

Now we have the following characterizations of the MO-BP distribution.

Theorem 4.1. *Let N be a geometric r.v. with parameter p such that* $P\{N = n\} = pq^{n-1},\ n = 1, 2, \ldots, 0 < p < 1, q = 1 - p$. *Consider a sequence* $\{(X_i, Y_i), i \geq 1\}$ *of independently and identically distributed random vectors with common survival function* $\bar{F}(x,y)$, *where N and* (X_i, Y_i) *are independent for all* $i \geq 1$. *Let* $U_N = \min_{1 \leq i \leq N} X_i$ *and* $V_N = \min_{1 \leq i \leq N} Y_i$. *Then the random vectors* (U_N, V_N) *are distributed as MO-BSP if and only if* (X_i, Y_i) *have the bivariate semi-Pareto distribution.*

Proof. Consider

$$\begin{aligned}\bar{S}(x,y) &= P[U_N > x, V_N > y] \\ &= \sum_{n=1}^{\infty} [\bar{F}(x,\ y)]^n pq^{n-1} \\ &= p\bar{F}(x,\ y)/[1 - (1 - p)\bar{F}(x,\ y)].\end{aligned}$$

Let $\bar{F}(x,\ y) = 1/\{1 + \psi(x,y)\}$, which is the survival function of bivariate semi-Pareto.

Substituting this in the above equation we have

$$\bar{S}(x,\ y) = p/\{p + \psi(x,\ y)\} = \bar{G}(x,\ y), \tag{4.1}$$

which is the survival function of MO-BSP.

Conversely suppose that

$$\bar{S}(x,\ y) = p/\{p + \psi(x,y)\}.$$

Then

$$p\bar{F}(x,\ y)/[1 - (1 - p)\bar{F}(x,\ y)] = p/\{p + \psi(x,\ y)\}$$

This yields

$$\bar{F}(x,\ y) = 1/\{1 + \psi(x,y)\}.$$

Hence the proof is complete.

Now we shall establish another characterization of the MO-BSP distribution.

Let $\{N_k, k \geq 1\}$ be a sequence of geometric random variables with parameters p_k, $0 \leq p_k < 1$.

Define

$$\begin{aligned}\bar{F}_k(x,y) &= P(U_{N_{k-1}} > x, V_{N_{k-1}} > y), \quad k = 2, 3 \ldots \\ &= p_{k-1}\bar{F}_{k-1}(x,y)\})/\{1 - (1 - p_{k-1})\bar{F}_{k-1}(x,y)\}\end{aligned} \tag{4.2}$$

Here we refer $\bar{F}_k$ as the survival function of the geometric p_{k-1} minimum of independent and identically distributed random vectors with $\bar{F}_{k-1}$ as the common survival function.

Theorem 4.2. *Let* $\{(X_i, Y_i), i \geq 1\}$ *be a sequence of independent and identically distributed non-negative random vectors with common survival function* $\bar{G}(x,y)$. *Define* $\bar{G}_1 = \bar{G}$ *and* $\bar{F}_k$ *as the survival function of the geometric* p_{k-1} *minimum of independent and identically distributed random vectors with common survival function* $\bar{F}_{k-1}, k = 2, 3 \ldots$. *Then*

$$\bar{F}_k(x,\ y) = \bar{G}(x,\ y) \tag{4.3}$$

if and only if (X_i, Y_i) has MO-BSP distribution.

Proof. By definition, the survival function $\bar{F}_k$ satisfies the equation (4.2). We have

$$\bar{G}(x,\ y) = \frac{1}{1 + \frac{1}{p}\psi(x,y)} = \frac{1}{1 + \phi(x,y)},$$

where $\phi(x,y)$ is a monotonically increasing function in both x and y $(x \geq 0, y \geq 0)$ and

$$\lim_{x\to 0}\lim_{y\to 0}\phi(x,y) = 0 \text{ and } \quad \lim_{x\to\infty}\lim_{y\to\infty}\phi(x,y) = \infty.$$

Hence we can write, $\bar{G}_k(x,\ y) = \frac{1}{1+\phi_k(x,y)}$; $k = 1, 2, \ldots$

Substituting this in (4.2), we get

$$\phi_k(x,y) = \frac{\phi_{k-1}(x,y)}{p_{k-1}}, \quad k = 2, 3, 4 \ldots$$

Recursively using this relation we have

$$\phi_k(x,y) = \frac{\phi_1(x,y)}{p_1 p_2 \cdots p_{k-1}}, \quad \text{since } G_1 = G \text{ implies that } \phi_1 = \phi.$$

This implies that

$$\phi_k(x,y) = \frac{\phi_1(x,y)}{p_1 p_2 \cdots p_{k-1}} \tag{4.4}$$

Hence

$$\bar{F}_k(x,\ y) = \bar{G}(x,\ y)$$

This proves the sufficiency part.

Conversely assume that equation (5.3) is true. By the hypothesis of the theorem equation (4.4) follows.

Thus equation (4.3) and equation (4.4) together lead to the equation

$$\left\{1 + \frac{1}{p_1 p_2 \cdots p_{k-1}}\phi_1[(x,y)]\right\}^{-1} = \bar{G}(x,y)$$
$$= \frac{1}{1 + \phi(x,y)}$$

This implies that

$$\phi(x,y) = \frac{\phi_1(x,y)}{p_1 p_2 \cdots p_{k-1}}.$$

Hence the proof is complete. ■

5 Marshall-Olkin bivariate semi-Pareto AR (1) model

Now we construct a first order autoregressive time series model with MO-BSP distribution as stationary marginal distribution.

Theorem 5.1. *Consider a bivariate autoregressive minification process* $\{(X_n, Y_n)\}$ *having the structure*

$$X_n = \begin{cases} U_n & \text{w.p. } p \\ \min(X_{n-1}, U_n) & \text{w.p. } 1-p \end{cases}$$
$$Y_n = \begin{cases} V_n & \text{w.p } p \\ \min(Y_{n-1}, V_n) & \text{w.p. } 1-p \end{cases} \tag{5.1}$$

where $\{(U_n, V_n)\}$ *are the innovations, which are independent of* $\{(X_{n-k}, Y_{n-k})\}$ *for* $k = 1, 2, \ldots, n$. *Then* $\{(X_n, Y_n)\}$ *has stationary marginal distribution as MO-BSP if and only if* $\{(U_n, V_n)\}$ *is jointly distributed as bivariate semi-Pareto distribution.*

Proof. From (5.1) we have

$$\bar{F}_{Xn,Yn}(x,y) = p\bar{G}_{Un,Vn}(x,y) + (1-p)\bar{F}_{Xn-1,Yn-1}(x,y) \cdot \bar{G}_{Un,Vn}(x,y) \tag{5.2}$$

Under stationarity we get

$$\bar{F}_{X,Y}(x,y) = p\bar{G}_{U,V}(x,y)/\{1-(1-p)\bar{G}_{U,V}(x,y)\}$$

If we take

$$\bar{G}_{U,V}(x,y) = 1/\{1+\psi(x,y)\},$$

then $\bar{F}_{X,Y}(x, y) = p/\{p + \psi(x, y)\}$
which is the survival function of MO-BSP.

Conversely if we take

$$\bar{F}_{X,Y}(x, y) = p/\{p + \psi(x,y)\}$$

it is easy to show that $G_{U,V}(x, y)$ is distributed as bivariate semi-Pareto distribution and the process is stationary. In order to establish stationarity we proceed as follows.

Assume $\{(X_{n-1}, Y_{n-1})\} \stackrel{d}{=} \text{MO-BSP}$ and $\{(U_n, V_n)\} \stackrel{d}{=} \text{BSP}$.
Then from (5.2)

$$\bar{F}_{Xn,Yn}(x, y) = p/\{p + \psi(x, y)\}.$$

This establishes that $\{(X_n, Y_n)\}$ is distributed as MO-BSP. Even if (X_0, Y_0) is arbitrary, it is easy to establish that $\{(X_n, Y_n)\}$ is stationary and is asymptotically marginally distributed as MO-BSP. ■

6 Generalisation to the k-th order Autoregressive Model

In this section we extend the results to develop a k-th order bivariate autoregressive model $\{(X_n, Y_n)\}$ having the structure. This leads to the following theorem.

Theorem 6.1. *Consider an AR(k) model with structure*

$$X_n = \begin{cases} U_n & \text{w.p. } p_0 \\ \min(X_{n-1}, U_n) & \text{w.p. } p_1 \\ \min(X_{n-2}, U_n) & \text{w.p. } p_2 \\ \min(X_{n-k}, U_n) & \text{w.p. } p_k \end{cases}$$
$$Y_n = \begin{cases} V_n & \text{w.p. } p_0 \\ \min(Y_{n-1}, V_n) & \text{w.p. } p_1 \\ \min(Y_{n-2}, V_n) & \text{w.p. } p_2 \\ \min(Y_{n-k}, V_n) & \text{w.p. } p_k \end{cases} \tag{6.1}$$

where $0 < p_i < 1$, $\sum_{i=1}^{k} p_i = 1 - p_0$. Then $\{(X_n, Y_n)\}$ has stationary marginal distribution as MO-BSP if and only if $\{(U_n, V_n)\}$ is jointly distributed as bivariate semi-Pareto distribution.

Remark 6.1. Theorems 4.1, 4.2, 5.1 and 6.1 can be extended to Marshall-Olkin bivariate Pareto distribution.

Acknowledgements. The authors thank the Editor and the referee for their valuable suggestions which led to an improved version of the original paper. The first author is grateful to the University Grants Commission of India for the support under Teacher Fellowship Scheme.

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