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Marginal distributions of sequential and generalized order statistics

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Abstract. Expressions for marginal distribution functions of sequential order statistics and generalized order statistics are presented without any restrictions imposed on the model parameters. The results are related to the relevation transform, to the distribution of the product of Beta distributed random variables, and to Meijer's *G*-functions. Some selected applications in the areas of moments, conditional distributions, recurrence relations, and reliability properties are shown.

Key words: Order statistics; Generalized order statistics; Sequential order statistics; Record values; Distribution theory; Meijer's *G*-function; Recurrence relations; Reliability properties

1 Introduction

Sequential order statistics have been introduced in Kamps (1995a) as an extension of (ordinary) order statistics. They serve as a model for k-out-of-n structures, which takes into account effects of failures of components on the remaining working units. These influences may be due to damages caused by failures or to an increased stress on the active components. In contrast to the commonly used model of (ordinary) order statistics, this approach enables a more flexible description of such effects, since the lifetime distribution of the remaining components may change after the failure of some component.

In order to illustrate the notion of sequential order statistics we introduce them first intuitively by means of a triangular scheme of random variables. In line r we consider $n - r + 1$ random variables indicating that $r - 1$ components previously failed. A formal definition of sequential order statistics is given in Section 2.

Let $F_1, \ldots, F_n$ be continuous distribution functions and let $z_{1,n}^{(1)} \leq z_{1,n-1}^{(2)} \leq \cdots \leq z_{1,2}^{(n-1)}$ be real numbers. Consider a triangular scheme $(Z_j^{(i)})_{1 \leq i \leq n, 1 \leq j \leq n-i+1}$ of random variables. The i-th failure time is modelled by the minimum

$$X_*^{(i)} = \min\{Z_1^{(i)}, \ldots, Z_{n-i+1}^{(i)}\}, \quad 1 \leq i \leq n,$$

where $Z_1^{(i)}, \ldots, Z_{n-i+1}^{(i)}$ are assumed to be conditionally independent given the previous failure time $X_*^{(i-1)} = z_{1,n-i+2}^{(i-1)}$.

$Z_1^{(i)}, \ldots, Z_{n-i+1}^{(i)}$ are supposed to be distributed according to the left truncated distribution function

$$\frac{F_i(\cdot) - F_i(z_{1,n-i+2}^{(i-1)})}{1 - F_i(z_{1,n-i+2}^{(i-1)})}, \quad 1 \leq i \leq n, \quad z_{1,n+1}^{(0)} = -\infty,$$

which is given by F_i truncated at the occurrence time $z_{1,n-i+2}^{(i-1)}$ of the $(i-1)$-th failure in the system.

Ordinary order statistics describing (ordinary) k-out-of-n systems are contained in the model of sequential order statistics by the specific choice $F_1 = \ldots = F_n$. In this sense ordinary k-out-of-n systems can be viewed as particular sequential k-out-of-n systems. For more details we refer to Kamps (1995a, Chap. I.1) and Cramer and Kamps (1996, 2001b).

The model of sequential order statistics is closely connected to several other models of ordered random variables (see Table 1.1). In its general form the model coincides with Pfeifer's record model (cf. Pfeifer 1982a, b) in the distribution theoretical sense [cf. Kamps (1995a), p. 29]. The specific choice of distribution functions

$$F_i(t) = 1 - (1 - F(t))^{\alpha_i}, \quad t \in \mathbb{R}, \quad 1 \leq i \leq n,$$

with a distribution function F and positive real numbers $\alpha_1, \ldots, \alpha_n$, leads to the model of generalized order statistics with parameters $\gamma_r = (n - r + 1)\alpha_r$,

Table 1.1. Models of ordered random variables and their correspondence (for details see Cramer and Kamps 2001b).

	$\gamma_n = k$	γ_r $(1 \leq r \leq n-1)$	m_r $(1 \leq r \leq n-1)$
sequential order statistics	α_n	$(n-r+1)\alpha_r$	$(n-r+1)\alpha_r - (n-r)\alpha_{r+1} - 1$
generalized order statistics	k	$k + n - r + \sum_{j=r}^{n-1} m_j$	m_r
ordinary order statistics	1	$n - r + 1$	0
progressive type II censored order statistics	$R_n + 1$	$N - r + 1 - \sum_{i=1}^{r-1} R_i$ $= n - r + 1 + \sum_{i=r}^{n} R_i$	$R_r \quad (\in \mathbb{N}_0)$
record values	1	1	-1
Pfeifer's record values	β_n	β_r	$\beta_r - \beta_{r+1} - 1$

$1 \leq r \leq n$. They serve as a unified approach to a variety of models of ordered random variables (see Kamps, 1995a,b, 1999), such as ordinary order statistics, order statistics with nonintegral sample size (i.e., fractional order statistics, cf. Stigler 1977, Rohatgi and Saleh 1988), progressively type II censored order statistics (cf. Balakrishnan and Aggarwala 2000), record values (cf. Arnold et al. 1998), kth record values (cf. Dziubdziela and Kopociński 1976), and k_n records from nonidentical distributions.

The concept of generalized order statistics enables a common approach to structural similarities and analogies. Known results in submodels can be subsumed, generalized, and integrated within a general framework. Well-known distributional and inferential properties of ordinary order statistics and record values turn out to remain valid for generalized order statistics (cf. Kamps 1995a, Cramer and Kamps 2001b). Thus, the concept of generalized order statistics provides a large class of models with many interesting and useful properties for both the description and the analysis of practical problems.

Due to this reason, the question arises whether the distribution theory of sequential order statistics as well as their properties can be obtained by analogy with ordinary order statistics, which have been extensively investigated in the literature (cf. David 1981; Arnold et al. 1992, Balakrishnan and Rao, 1998a,b, Goldie and Maller 1999).

Although some work has been done in this area, representations for the marginal distributions of generalized order statistics (and of sequential order statistics) are only available in particular cases. For instance, results for ordinary order statistics and record values are well-known. Moreover, Nasri-Roudsari (1996) proved that the marginal distribution of fractional order statistics can be written in terms of the incomplete Beta function ratio. In terms of generalized order statistics we may choose the positive parameters γ_i, $1 \leq i \leq n$, according to $\gamma_i - \gamma_{i+1} - 1 = m_i$ with $m_1 = \cdots = m_{r-1}$ w.r.t. the r-th marginal distribution to obtain useful results (see Kamps 1995a). Supposing that the parameters $\gamma_r = (n - r + 1)\alpha_r$, $1 \leq r \leq n$, are pairwise different, i.e.,

$$\gamma_i \neq \gamma_j \quad \text{for all} \quad 1 \leq i, \quad j \leq n, \quad i \neq j, \tag{1}$$

Kamps and Cramer (2001) present explicit expressions. They can be applied, e.g., in the model of progressive type II censoring.

In this paper, we present in Section 2 a simple iterative, modified definition of sequential order statistics. This leads to expressions for marginal distributions of sequential order statistics in terms of the so-called relevation transform (cf. Krakowski, 1973). In Section 3, these results are applied to derive marginal distributions of generalized order statistics. It turns out that generalized order statistics are related to the product of independent Beta random variables. This observation enables us to obtain a representation of the distribution function in terms of a so-called Meijer's G-function. Section 4 contains applications of the results in the areas of recurrence relations, bounds on moments, and reliability properties.

2 Sequential order statistics

Let F be a distribution function and let its quantile function $F^{-1} : [0, 1] \to \mathbb{R}$ be defined by

$$F^{-1}(y) = \inf\{x : F(x) \geq y\}, \quad y \in (0,1),$$

and $F^{-1}(0) = \lim_{y\to 0+} F^{-1}(y)$, $F^{-1}(1) = \lim_{y\to 1-} F^{-1}(y)$.

Sequential order statistics have been introduced as follows (cf. Kamps, 1995a, Chap. I.1):

Definition. 2.1 Let $F_1, \ldots, F_n$ be distribution functions with $F_1^{-1}(1) \leq \cdots \leq F_n^{-1}(1)$, and let $(Y_j^{(r)})_{1\leq r\leq n, 1\leq j\leq n-r+1}$ be independent random variables with $Y_j^{(r)} \sim F_r$, $1 \leq j \leq n-r+1$, $1 \leq r \leq n$.

Let $X_j^{(1)} = Y_j^{(1)}$, $1 \leq j \leq n$, $X_*^{(1)} = \min\{X_1^{(1)}, \ldots, X_n^{(1)}\}$, and, for $2 \leq r \leq n$, let

$$X_j^{(r)} = F_r^{-1}\{F_r(Y_j^{(r)})[1 - F_r(X_*^{(r-1)})] + F_r(X_*^{(r-1)})\},$$
$$X_*^{(r)} = \min_{1\leq j\leq n-r+1} X_j^{(r)}.$$

Then, the random variables $X_*^{(1)}, \ldots, X_*^{(n)}$ are called sequential order statistics (based on $F_1, \ldots, F_n$).

Formally, there is no necessity of imposing further assumptions on $F_1, \ldots, F_n$. However, in view of interpretation and handling, continuity of the distribution functions is useful.

The definition of sequential order statistics simplifies due to the fact that the function F_r and its quantile function F_r^{-1} are increasing. Let $\overline{F}_r^{-1} = F_r^{-1}(1-\cdot)$ and $\overline{F}_r = 1 - F_r$ be the survival function of the distribution function F_r. For $r \geq 2$, we obtain

$$\begin{aligned}
X_*^{(r)} &= \min_{1\leq j\leq n-r+1} F_r^{-1}\{F_r(Y_j^{(r)})[1 - F_r(X_*^{(r-1)})] + F_r(X_*^{(r-1)})\} \\
&= F_r^{-1}\{F_r(\min_{1\leq j\leq n-r+1} Y_j^{(r)})[1 - F_r(X_*^{(r-1)})] + F_r(X_*^{(r-1)})\} \\
&= F_r^{-1}\{F_r(Z^{(r)})[1 - F_r(X_*^{(r-1)})] + F_r(X_*^{(r-1)})\} \\
&= F_r^{-1}\{1 - \overline{F}_r(Z^{(r)})\overline{F}_r(X_*^{(r-1)})\} \\
&= \overline{F}_r^{-1}\{\overline{F}_r(Z^{(r)})\overline{F}_r(X_*^{(r-1)})\},
\end{aligned}$$

where $Z^{(r)} = \min_{1\leq j\leq n-r+1} Y_j^{(r)}$. The distribution function of $Z^{(r)}$ is given by $1 - \overline{F}_r^{n-r+1}$ and, if F_r is continuous, $F_r(Z^{(r)})$ is distributed as $U_{1,n-r+1}$ which is the minimum of $n-r+1$ iid random variables with a standard uniform distribution. Hence, $\overline{F}_r(Z^{(r)})$ is distributed as $U_{n-r+1,n-r+1}$ which is Beta distributed with parameters $n-r+1$ and 1.

Defining $X_*^{(r)}$ via

$$X_*^{(r)} = \overline{F}_r^{-1}\{V_r\overline{F}_r(X_*^{(r-1)})\}, \quad V_r \sim \mathrm{Beta(n-r+1, 1)}, 2 \leq \mathrm{r} \leq \mathrm{n}, \tag{2}$$

one may drop the continuity assumption to obtain a modified definition of sequential order statistics. In the case of continuity of $F_1, \ldots, F_n$, it coincides with Definition 2.1.

Definition 2.2. Let $F_1, \ldots, F_n$ be distribution functions with $F_1^{-1}(1) \leq \cdots \leq F_n^{-1}(1)$, and let $V_1, \ldots, V_n$ be independent random variables with $V_r \sim \mathrm{Beta(n-r+1, 1)}$, $1 \leq r \leq n$.

Then the random variables

$$X_*^{(r)} = F_r^{-1}(X^{(r)}) \quad \text{with}\, X^{(r)} = 1 - V_r \overline{F}_r(X_*^{(r-1)}), \quad 1 \le r \le n, \qquad X_*^{(0)} = -\infty,$$

are called sequential order statistics (based on $F_1, \dots, F_n$).

Other classes of random variables may be introduced by applying the defining equation of Definition 2.2 to random variables $V_1, \dots, V_n$ with other distributions.

The preceding recursion can be visualized as in Figure 1, where F_r is assumed to be strictly increasing and continuous such that $F_r^{-1}(1 - \cdot)$ is the inverse function of $\overline{F}_r$ ($\overline{F}_r^{-1}$ is just a notation otherwise).

By the recursive definition of sequential order statistics it is directly seen that $X_*^{(1)}, \dots, X_*^{(n)}$ form a Markov chain with transition probabilities ($r \ge 2$)

$$P(X_*^{(r)} \le t | X_*^{(r-1)} = x) = 1 - \left(\frac{\overline{F}_r(t)}{\overline{F}_r(x)} \right)^{n-r+1}, \quad x \le t,\ F_r(x) < 1$$

(cf. Kamps, 1995a, p. 29).

This knowledge enables us to establish a formula for the joint distribution of the first r sequential order statistics.

Theorem 2.3. *Let $X_*^{(1)}, \dots, X_*^{(n)}$ be sequential order statistics based on absolutely continuous distribution functions $F_1, \dots, F_n$ with density functions $f_1, \dots, f_n$.*

Then, the joint density of the r first sequential order statistics $X_*^{(1)}, \dots, X_*^{(r)}$ is given by

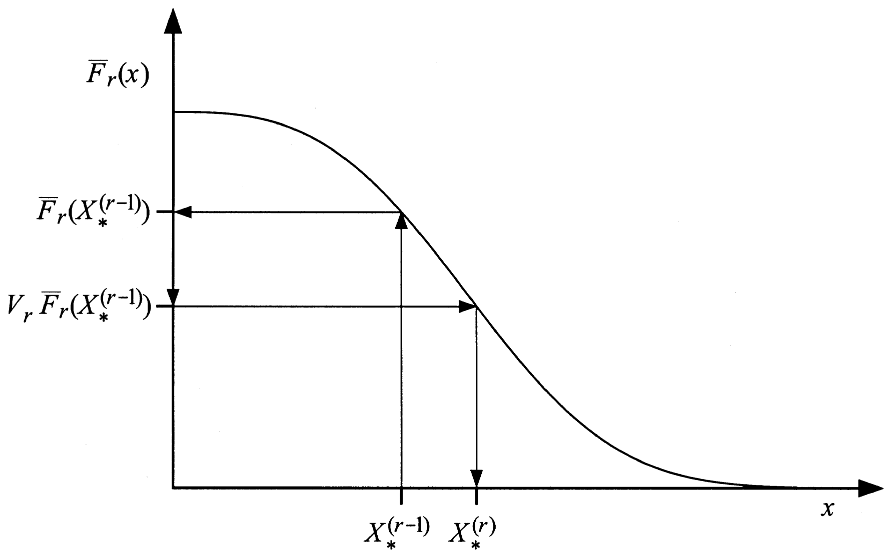

Fig. 1. Recursive construction of sequential order statistics.

$$f_{X_*^{(1)},\dots,X_*^{(r)}}(x_1,\dots,x_r)$$
$$=\frac{n!}{(n-r)!}f_1(x_1)\overline{F}_1^{n-1}(x_1)\prod_{j=2}^{r}\left[\frac{f_j(x_j)}{\overline{F}_j(x_{j-1})}\cdot\left(\frac{\overline{F}_j(x_j)}{\overline{F}_j(x_{j-1})}\right)^{n-j}\right],x_1<\cdots<x_r,\ 1\le r\le n,$$

where $\prod_{j=2}^{1}[\cdots]=1$.

Proof. Let $r\geq 2$. From the Markov chain property we conclude by an induction argument and the definition of $X_*^{(1)}$ that

$$f_{X_*^{(1)},\dots,X_*^{(r)}}(x_1,\dots,x_r)=f_{X_*^{(1)},\dots,X_*^{(r-1)}}(x_1,\dots,x_{r-1})f_{X_*^{(r)}|X_*^{(r-1)}}(x_r|x_{r-1})$$
$$=f_{X_*^{(1)},\dots,X_*^{(r-1)}}(x_1,\dots,x_{r-1})\cdot\frac{f_r(x_r)}{\overline{F}_r(x_{r-1})}\cdot(n-r+1)\left(\frac{\overline{F}_r(x_r)}{\overline{F}_r(x_{r-1})}\right)^{n-r}.$$

■

By choosing $F_1=\cdots=F_r=F$, say, we obtain the joint density of ordinary order statistics $X_{1,n},\dots,X_{r,n}$ based on F.

2.1 Representations of marginal distribution functions

Applying Definition 2.2 we obtain the following recursion formula for the marginal distribution functions $F_{*,1},\dots,F_{*,n}$ of the sequential order statistics $X_*^{(1)},\dots,X_*^{(n)}$.

Lemma 2.4.

$$F_{*,1}(t)=1-(1-F_1(t))^n,$$
$$F_{*,r}(t)=\begin{cases}F_{*,r-1}(t)-\overline{F}_r^{n-r+1}(t)\int_{-\infty}^{t}\frac{1}{\overline{F}_r^{n-r+1}(z)}dF_{*,r-1}(z), & \text{if } F_r(t)<1,\\ 1, & \text{if } F_r(t)=1,\end{cases}\quad 2\le r\le n.$$

Proof. The representation of $F_{*,1}$ is obvious. The derivation of the distribution functions for $r\geq 2$ proceeds as follows. Let $t\in\mathbb{R}$ with $\overline{F}_r(t)>0$ such that $\overline{F}_r(z)>0$ for $z<t$.

$$\begin{aligned}F_{*,r}(t)&=P(X_*^{(r)}\le t)\\&=P(V_r\overline{F}_r(X_*^{(r-1)})\ge\overline{F}_r(t))\\&=\int_{-\infty}^{\infty}P(V_r\overline{F}_r(X_*^{(r-1)})\ge\overline{F}_r(t)|X_*^{(r-1)}=z)\,dF_{*,r-1}(z)\\&=\int_{-\infty}^{\infty}P(V_r\overline{F}_r(z)\ge\overline{F}_r(t))\,dF_{*,r-1}(z)\\&=\int_{-\infty}^{t}P\left(V_r\ge\frac{\overline{F}_r(t)}{\overline{F}_r(z)}\right)dF_{*,r-1}(z)\end{aligned}$$

$$= \int_{-\infty}^{t} \left\{ 1 - \left(\frac{\overline{F}_r(t)}{\overline{F}_r(z)} \right)^{n-r+1} \right\} d F_{*,r-1}(z)$$

$$= F_{*,r-1}(t) - \overline{F}_r^{n-r+1}(t) \int_{-\infty}^{t} \frac{1}{\overline{F}_r^{n-r+1}(z)} d F_{*,r-1}(z).$$

For $\overline{F}_r(t) = 0$ we obtain $F_{*,r}(t) = P(\overline{F}_r(X_*^{(r)}) \geq 0) = 1$. ∎

Remark 2.5. *From the preceding lemma we conclude that the corresponding distribution functions of sequential order statistics can be viewed as relevation transforms which were introduced by Krakowski (1973). The relevation transform $\overline{F}\#\overline{G}$ of survival functions $\overline{F}$ and $\overline{G}$ is defined by the Lebesgue-Stieltjes integral*

$$(\overline{F}\#\overline{G})(t) = \overline{F}(t) - \int_{-\infty}^{t} \frac{\overline{G}(t)}{\overline{G}(u)} d\overline{F}(u), \quad t \in \mathbb{R},$$

(*cf. Lau and Prakasa Rao 1990*). *In view of Lemma* 2.4, *we obtain the representation*

$$\overline{F}_{*,r}(t) = \overline{F}_{*,r-1}(t) - \int_{-\infty}^{t} \frac{\overline{F}_r^{n-r+1}(t)}{\overline{F}_r^{n-r+1}(z)} d\overline{F}_{*,r-1}(z), \quad t \in \mathbb{R}.$$

Hence, we can write the survival function of the r-th sequential order statistic as relevation transform

$$\overline{F}_{*,r} = \overline{F}_{*,r-1}\#\overline{F}_r^{n-r+1}. \tag{3}$$

Lemma 2.4. leads to a simple representation of the density function of the r-th sequential order statistic.

Corollary 2.6. *Let F_r be an absolutely continuous distribution function with density function f_r. The distribution function of the $(r-1)$-th sequential order statistic $F_{*,r-1}$ is assumed to be absolutely continuous with density function $f_{*,r-1}$.*

Then, the density function of the r-th *sequential order statistic is given by*

$$f_{*,r}(t) = (n-r+1)\frac{f_r(t)}{\overline{F}_r(t)}\{\overline{F}_{*,r}(t) - \overline{F}_{*,r-1}(t)\}, \quad t \in \mathbb{R}. \tag{4}$$

The above relation is well known in the case of ordinary order statistics (see David and Shu, 1978).

3 Generalized order statistics

In this section we consider a particular choice of the distribution functions F_r. Namely, let

$$F_r(t) = 1 - (1 - F(t))^{\gamma_r/(n-r+1)}, \quad r = 1, \ldots, n, \tag{5}$$

with a continuous distribution function F and positive numbers $\gamma_1, \ldots, \gamma_n$. In this specific setup sequential order statistics can be viewed as generalized order statistics and vice versa (cf. Kamps, 1995a, p. 56). Many well known models of ordered random variables are included (see Table 1.1) Let $\tilde{\gamma}_r = \gamma_r/(n - r + 1)$. Then the quantile function F_r^{-1} of F_r can be written in terms of F^{-1} as

$$F_r^{-1}(z) = F^{-1}(1 - (1 - z)^{1/\tilde{\gamma}_r}).$$

In the situation of the preceding definition and by using the representation for F_r^{-1}, sequential order statistics based on distribution functions (5) are defined as follows ($r \geq 2$):

$$\begin{aligned}
X_*^{(r)} &= F_r^{-1}[1 - V_r \overline{F}_r(X_*^{(r-1)})] \\
&= F_r^{-1}[1 - V_r \overline{F}^{\tilde{\gamma}_r}(X_*^{(r-1)})] \\
&= F^{-1}[1 - V_r^{1/\tilde{\gamma}_r} \overline{F}(X_*^{(r-1)})] \\
&= \overline{F}^{-1}[V_r^{1/\tilde{\gamma}_r} \overline{F}(X_*^{(r-1)})].
\end{aligned}$$

Hence, sequential order statistics based on a distribution function F and numbers $\gamma_1, \ldots, \gamma_n$ are iteratively defined by ($r \geq 2$)

$$X_*^{(r)} = \overline{F}^{-1}[V_r^{1/\tilde{\gamma}_r} \overline{F}(X_*^{(r-1)})]. \tag{6}$$

In terms of the relevation transform we obtain the recurrence relation

$$\overline{F}_{*,1} = \overline{F}^{\gamma_1}, \quad \overline{F}_{*,r} = \overline{F}_{*,r-1} \# \overline{F}_r^{n-r+1} = \overline{F}_{*,r-1} \# \overline{F}^{\gamma_r}, \quad r \geq 2. \tag{7}$$

for the marginal distribution functions of generalized order statistics. If we choose $\gamma_j = 1$, $j = 1, \ldots, n$, the recursion reads $\overline{F}_{*,r} = \overline{F}_{*,r-1} \# \overline{F}$. This result is well-known in the setting of record values. In the case of ordinary order statistics, the parameters are given by $\gamma_r = n - r + 1$, $1 \leq r \leq n$, so that $\overline{F}_{*,r} = \overline{F}_{*,r-1} \# \overline{F}^{n-r+1}$. Hence, the survival function $\overline{F}_{r,n}$ of an ordinary order statistic can be viewed as a relevation transform, too.

In case of an underlying exponential distribution $F(t) = 1 - \exp(-t)$, $t \geq 0$, we obtain the representation

$$X_*^{(r)} = \sum_{j=1}^{r} Z^{(j)}$$

of sequential order statistics (cf. Cramer and Kamps, 2001a), where $Z^{(1)}, \ldots, Z^{(r)}$ are independent and $Z^{(j)}$ has an exponential distribution with parameter γ_j, $1 \leq j \leq r$. Hence, generalized order statistics from exponential distributions are sums of independent random variables with possibly non-identical exponential distributions. The distribution of $(Z^{(1)}, Z^{(1)} + Z^{(2)}, \ldots, \sum_{j=1}^{n} Z^{(j)})$ is the same as that of order statistics from a Weinman multivariate exponential distribution (cf. Cramer and Kamps, 1997).

Applying recursion (6), we obtain the following theorem.

Theorem 3.1. *Let* $X_*^{(1)}, \ldots, X_*^{(n)}$ *be sequential order statistics based on a continuous distribution function* F *and positive numbers* $\gamma_1, \ldots, \gamma_n$. *Let* $W_1 = V_1^{1/\widetilde{\gamma}_r}$ *and, for* $r \geq 2$, $W_r = V_r^{1/\tilde{\gamma}_r} \overline{F}(X_*^{(r-1)})$ *with* $\widetilde{\gamma}_r = \gamma_1/(n - r + 1)$ *and* V_r *as in* (2) *and* (6).

Then, $W_r = \prod_{j=1}^{r} B_j$, $r \geq 1$, $X_*^{(0)} = -\infty$, *and*

$$X_*^{(r)} = \overline{F}^{-1}\left(\prod_{j=1}^{r} B_j\right), \tag{8}$$

where $B_r = V_r^{1/\tilde{\gamma}_r}$, $1 \leq r \leq n$, *are independent power-function distributed random variables with parameters* $\gamma_1, \ldots, \gamma_n$ and 1.

Proof. Since $V_r \sim \mathrm{Beta}(n - r + 1, 1)$ the random variable B_r is power-function distributed with parameter γ_r, $1 \leq r \leq n$.

With the notations of the theorem, relation (6) yields $X_*^{(r)} = \overline{F}^{-1}(W_r)$. Since F is supposed to be continuous we obtain directly from (6)

$$W_r = B_r W_{r-1}, \quad r \geq 1, \quad W_0 = 1,$$

with $B_r = V_r^{1/\tilde{\gamma}_r} \sim \mathrm{Beta}(\gamma_r, 1)$. This leads directly to the representation

$$W_r = \prod_{j=1}^{r} B_j, \quad r \geq 1. \qquad \blacksquare$$

Remark 3.2. *Theorem 3.1 in combination with (7) shows that the distribution of the product of Beta random variables can be seen as relevation transform. Moreover, this product representation illustrates that the distribution of* $X_*^{(r)}$ *does not depend on the ordering of the parameters* $\gamma_1, \ldots, \gamma_r$ (*see also* (8)). *This can also be deduced from a result for the relevation transform given by Krakowski (1973, p. 110), stating that* $\overline{F}\#\overline{G} = \overline{G}\#\overline{F}$ *iff* $\overline{G} = \overline{F}^a$ *for some positive number a.*

Theorem 3.1 allows us to derive directly a result concerning the behaviour of normalized spacings from generalized order statistics if the underlying distribution is an exponential one. From (8) we obtain the equation $X_*^{(r)} = \sum_{j=1}^{r}(-\ln B_j)$ leading to

$$\gamma_r(X_*^{(r)} - X_*^{(r-1)}) = -\gamma_r \ln B_r \sim \mathrm{Exp}(1)$$

(see Kamps, 1995a, Theorem 3.3.5). References in particular cases, e.g., ordinary order statistics and record values, are provided by Kamps (1995a, p. 80/1) and Sukhatme (1937) (cf. Rényi, 1953).

Example 3.3. Subsequently, we consider two particular cases of Theorem 3.1.

1. In the case $\gamma_j = k$, $j = 1, \ldots, n$, generalized order statistics coincide in the distribution theoretical sense with k-records introduced by Dziubdziela and Kopociński (1976) (see also Kamps, 1995a, p. 52). If we assume that F belongs to a standard uniform distribution, the preceding theorem yields that

$$1 - X_*^{(r)} = \prod_{j=1}^{r} B_j \quad \text{with } B_j \sim \mathrm{Beta}(k, 1).$$

This result is similar to the one derived by Rider (1955) and Rahman (1964). They illustrate that the probability density function of the product $X_1^{(k)} \cdots X_r^{(k)}$ of r independent random variables which are maxima of k independent and standard uniformly distributed random variables is given by

$$f_k(t) = \frac{k^r}{\Gamma(r)} t^{k-1}(-\ln t)^{r-1}, \quad t \in (0,1),$$

(see also David, 1981, p. 27). Hence, a simple linear transformation of sequential order statistics from rectangular distributions, i.e., $1 - X_*^{(r)}$, can be seen as such a product. The density function of the r-th k-record from a rectangular distribution is given by

$$\widetilde{f}_k(t) = \frac{k^r}{\Gamma(r)} (1-t)^{k-1}(-\ln(1-t))^{r-1}, \quad t \in (0,1).$$

Applying the quantile transformation, we arrive at the result of Dziubdziela and Kopociński (1976).

2. Nasri-Roudsari (1996) presents a representation for the distribution function of the r-th generalized order statistic $X_*^{(r)}$, $1 \le r \le n$, when the numbers $\gamma_1, \ldots, \gamma_n$ have a specific structure, i.e., $\gamma_j = k + (n-j)(m+1)$, $k > 0$, $m > -1$, $j = 1, \ldots, n$. In order to prove this result Nasri-Roudsari (1996) makes use of some identities of hypergeometric functions. We obtain the result directly from Theorem 3.1 which yields in this particular setting

$$W_r = \prod_{j=1}^{r} B_j, \quad B_j \sim \text{Beta}(k + (n-j)(m+1), 1).$$

Introducing the notations $\kappa = k/(m+1)$ and $C_j = B_j^{m+1}$, we obtain

$$W_r = \left[\prod_{j=1}^{r} C_j\right]^{1/(m+1)}, \quad C_j \sim \text{Beta}(\kappa + n - j, 1).$$

Using a result of Rao (1949) (see also Jambunathan 1954, Kotlarski 1962, Fan 1991, and Johnson et al. 1995, p. 257) we get $\prod_{j=1}^{r} C_j \sim \text{Beta}(\kappa + n - r, r)$. Hence, the distribution function of W_r is given by the incomplete Beta function ratio

$$\begin{aligned} F_{W_r}(t) &= I_{t^{m+1}}(\kappa + n - r, r) = 1 - I_{1-t^{m+1}}(r, \kappa + n - r) \\ &= \frac{1}{\mathrm{B}(r, \kappa + n - r)} \int_{1-t^{m+1}}^{1} z^{r-1}(1-z)^{\kappa+n-r-1} dz, \quad t \in (0,1). \end{aligned}$$

Applying the quantile transformation F^{-1} to $1 - W_r$, we obtain the representation derived in Nasri-Roudsari (1996).

Mathai and Saxena (1973, Section 5.2 and Chapter VI) point out the appearance of products of independent Beta variates in a number of statistical problems. The respective density function can be expressed in terms of a particular Meijer's G-function (see also Mathai, 1993, Section 2.3.4).

Thus, Theorem 3.1 allows us to derive the distribution function of the r-th generalized order statistic via some Meijer's G-function which is defined by the Mellin-Barnes type integral

$$\mathrm{G}_{p,q}^{m,n}\left[s \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right] = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - z) \prod_{j=1}^{n} \Gamma(1 - a_j + z)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + z) \prod_{j=n+1}^{p} \Gamma(a_j - z)} s^z dz, \ |s| < 1. \tag{9}$$

where L is an appropriately chosen integration path. For detailed accounts on Meijer's G-functions, we refer to Erdélyi et al. (1953, Chapters 5.3–5.6), Luke (1969, Chapters V/VI), Mathai and Saxena (1973) and Mathai (1993). The relation to products of Beta distributed random variables is discussed in Springer (1979) and Mathai (1993). In our setup, the integrand of the Meijer's G-function cancels down and we obtain

$$\mathsf{G}^{r,0}_{r,r}\left[s\left|\begin{matrix}\gamma_1,\ldots,\gamma_r\\ \gamma_1-1,\ldots,\gamma_r-1\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L\frac{s^z}{\prod_{j=1}^r(\gamma_j-1-z)}dz. \tag{10}$$

Since $g(z)=s^z$, $z\in\mathbb{C}$, is a holomorphic function on the complex plane the preceding integral can be evaluated by summing up the residues at γ_j-1, $j=1,\ldots,r$, where two or more of these values may coincide.

From these results and relation 8, an integral representation of the distribution function $F_{*,r}$ results

$$F_{*,r}(t)=1-\left(\prod_{j=1}^r\gamma_j\right)\int_0^{\overline{F}(t)}\mathsf{G}^{r,0}_{r,r}\left[s\left|\begin{matrix}\gamma_1,\ldots,\gamma_r\\ \gamma_1-1,\ldots,\gamma_r-1\end{matrix}\right.\right]ds,\quad t\in\mathbb{R}. \tag{11}$$

If the underlying distribution function F is absolutely continuous the density function of the r-th generalized order statistic is given by

$$f_{*,r}(t)=\left(\prod_{j=1}^r\gamma_j\right)\mathsf{G}^{r,0}_{r,r}\left[\overline{F}(t)\left|\begin{matrix}\gamma_1,\ldots,\gamma_r\\ \gamma_1-1,\ldots,\gamma_r-1\end{matrix}\right.\right]f(t),\quad t\in\mathbb{R}.$$

Although the preceding representation is useful to derive some properties of the distribution function $F_{*,r}$ (see next section), it contains complex integration with respect to an integration path. Applying (11), a closed form representation for the distribution function of the r-th generalized order statistic can be derived. The proof is carried out similar to the one given in Springer and Thompson (1970) for Beta distributions with integral parameters a_j and b_j, and is therefore omitted. In order to simplify the notation let, without loss of generality,

$$\gamma_1=\cdots=\gamma_{d_1}<\gamma_{d_1+1}=\cdots=\gamma_{d_1+d_2}<\cdots<\gamma_{d_1+d_2+\cdots+d_{\ell-1}+1}=\cdots=\gamma_{d_1+d_2+\cdots+d_\ell}$$

with an integer $\ell\in\{1,\ldots,r\}$ and $\delta_j=\gamma_{d_1+d_2+\cdots+d_j}$, $j=1,\ldots,\ell$. Then, $\delta_1<\cdots<\delta_\ell$ and d_i denotes the multiplicity of δ_i in the sequence $(\gamma_1,\ldots,\gamma_r)$.

Theorem 3.4 1. *Let* $\ell=1$, *i.e.,* $\gamma_1=\cdots=\gamma_r$.

Then,

$$F_{*,r}(t)=1-\frac{1}{(r-1)!}\Gamma\big(r,-\gamma_1\ln\overline{F}(t)\big),$$

where

$$\Gamma(r,z)=\int_z^\infty y^{r-1}\exp(-y)dy$$

denotes the incomplete gamma function.

2. *Let* $\ell\geq 2$. *The distribution function* $F_{*,r}$ *of the* r-th *generalized order statistic is given by*

$$F_{*,r}(t) = 1 - \left(\prod_{j=1}^{r} \gamma_j\right) \sum_{v=1}^{\ell} \sum_{j=0}^{d_v-1} \frac{K_{vj}}{\delta_v^{d_v-j}(d_v-1-j)!j!} \Gamma(d_v - j, -\delta_v \ln \overline{F}(t)),$$

where $K_{v0} = \prod_{q=1,q\neq v}^{\ell} (\delta_q - \delta_v)^{-d_q}$,

$$K_{vj} = \sum_{p=0}^{j-1} \sum_{q=1,q\neq v}^{\ell} (-1)^{p+1} \binom{j-1}{p} \frac{p!d_q}{(\delta_q - \delta_v)^{p+1}} K_{v,j-1-p}, \quad j \geq 1. \tag{12}$$

Remark 3.5. 1. *Considering the transformation* $-\ln W_r$, *we obtain that*

$$-\ln W_r = -\ln \prod_{j=1}^{r} B_j = \sum_{j=1}^{r} (-\ln B_j)$$

is the sum of independent and exponentially distributed random variables with parameters $\gamma_1, \ldots, \gamma_r > 0$. *Supposing that all parameters* $\gamma_1, \ldots, \gamma_r$ *are different, the following representation of the distribution function of* $X_*^{(r)} = \overline{F}^{-1}(W_r)$ *holds:*

$$F_{*,r}(t) = 1 - (-1)^{r-1} \left(\prod_{j=1}^{r} \gamma_j\right) \sum_{j=1}^{r} \gamma_j^{-1} \left(\prod_{v=1,v\neq j}^{r} (\gamma_j - \gamma_v)^{-1}\right) \overline{F}^{\gamma_j}(t), \quad t \in \mathbb{R}$$

(*cf. Likeš, 1967; Kamps, 1990; Kamps and Cramer, 2001). This type of distribution is called hyperexponential distribution. A review on this topic including various applications of hyperexponential distributions is provided by Botta et al. (1987). If some of the* γ'*s are equal, the distribution of* $-\ln W_r$ *can be found in Scheuer* (1988) *who proves the result by Laplace transforms.*

2. *Similar distributional problems arise in the study of the likelihood ratio criterion for testing regression coefficients under a normal distribution assumption* (*cf. Anderson, 1984, pp. 298–308*).
3. *Representations for the distribution of the product of independent Beta random variables were derived by Lomnicki (1969). For recent results on closed form expressions of the cumulative distribution function as well as the density function for general parameters we refer to Dennis III* (1994).

4 Applications

Some selected applications of the above theory are shown in this section. The results are well known in particular submodels of generalized order statistics.

4.1 Moments and conditional distributions of generalized order statistics

The first result is an immediate consequence of Theorem 3.1. It can be utilized to obtain lower (upper) bounds for the expectation of the r-th generalized order statistic (cf. Cramer et al., 2002a, b).

Theorem 4.1. *The moments of order* α *of the* r*-th generalized order statistic are given by*

$$E(X_*^{(r)})^\alpha = \left(\prod_{j=1}^{r}\gamma_j\right)\int_0^1 (F^{-1}(s))^\alpha \mathrm{G}_{r,r}^{r,0}\left[1-s\,\middle|\,\begin{matrix}\gamma_1,\ldots,\gamma_r\\ \gamma_1-1,\ldots,\gamma_r-1\end{matrix}\right]ds,$$

where $(F^{-1}(s))^\alpha$, $s \in (0,1)$, *is assumed to be well defined.*

The following theorem extends and simplifies a result for generalized order statistics given by Kamps (1995a, p. 159). In case of ordinary order statistics, the result is given in Blom (1958, p. 68) and Arnold and Balakrishnan (1989, p. 58). The result for progressively type II censored order statistics can be found in Balakrishnan et al. (2001).

Theorem 4.2. *Let* $\alpha > 0$, $\overline{F}^{-1}(0) \geq 0$, *and* $(\overline{F}^{-1})^\alpha$ *be convex (concave) on* $(0,1)$. *Then, the lower (upper) bound*

$$E(X_*^{(r)})^\alpha \overset{(\leq)}{\geq} \left[\overline{F}^{-1}\left(\prod_{j=1}^{r}\frac{\gamma_j}{\gamma_j+1}\right)\right]^\alpha$$

holds.

Proof. From (8), we obtain by Jensen's inequality

$$E(X_*^{(r)})^\alpha = E\left[\overline{F}^{-1}\left(\prod_{j=1}^{r}B_j\right)\right]^\alpha \overset{(\leq)}{\geq} \left[\overline{F}^{-1}\left(E\left(\prod_{j=1}^{r}B_j\right)\right)\right]^\alpha.$$

Since the random variables B_j, $1 \leq j \leq r$, are independent and Beta$(\gamma_j, 1)$-distributed the right hand side simplifies to $E(\prod_{j=1}^r B_j) = \prod_{j=1}^r E(B_j)$ $= \prod_{j=1}^r \frac{\gamma_j}{\gamma_j+1}$.

Another application of Theorem 3.1 leads to a generalization of a result which is well known for ordinary order statistics (cf. Arnold et al., 1992, Theorem 2.4.1, p. 23).

Theorem 4.3 *Let* $x \in \mathbb{R}$, $1 \leq s < r$ *and let* $X_*^{(r)}$, $X_*^{(s)}$ *be generalized order statistics based on a continuous distribution function F and parameters* $\gamma_1, \ldots, \gamma_r$.

Then, the distribution $P^{X_*^{(r)}|X_*^{(s)}=x}$ *coincides with the distribution of a generalized order statistic based on a distribution function* F_x *and parameters* $\gamma_{s+1}, \ldots, \gamma_r$. *The distribution function* F_x *is obtained from F by truncation on the left at x, i.e.,*

$$F_x(t) = \left(1 - \frac{\overline{F}(t)}{\overline{F}(x)}\right)1_{[x,\infty)}(t), \quad t \in \mathbb{R}.$$

4.2 Recurrence relations for distribution functions of generalized order statistics

First we deduce a recurrence relation for distribution functions of generalized order statistics, and, thus, for density functions and for moments as well. In the particular case of order statistics ($\gamma_j = n - j + 1$, $1 \leq j \leq n$) the relation is well-known and is nothing but a recurrence relation for the incomplete Beta function (cf. David 1981, p. 46/7). For specific choices of the parameters, we refer to Kamps (1995a, Chap. I.3) and to Kamps and Cramer (2001). Formulae for the derivative of a G-function (Mathai, 1993, Properties 2.14 and 2.15, p. 94) together with a formula on dimension reduction (Mathai 1993, Property 2.2) imply

$$(\gamma_r - \gamma_1)\mathsf{G}^{r,0}_{r,r}\left[z\middle|\begin{matrix}\gamma_1,\dots,\gamma_r\\ \gamma_1-1,\dots,\gamma_r-1\end{matrix}\right] = \mathsf{G}^{r-1,0}_{r-1,r-1}\left[z\middle|\begin{matrix}\gamma_1,\dots,\gamma_{r-1}\\ \gamma_1-1,\dots,\gamma_{r-1}-1\end{matrix}\right] - \mathsf{G}^{r-1,0}_{r-1,r-1}\left[z\middle|\begin{matrix}\gamma_2,\dots,\gamma_r\\ \gamma_2-1,\dots,\gamma_r-1\end{matrix}\right].$$

Applying (11), we obtain

Theorem 4.4 *Let $F^{(\gamma_1,\dots,\gamma_s)}_{*,s}$ denote the distribution function of a generalized order statistic based on a continuous distribution function F and positive parameters $\gamma_1,\dots,\gamma_s$.*

Then,

$$(\gamma_r - \gamma_1)F^{(\gamma_1,\dots,\gamma_r)}_{*,r}(t) = \gamma_r F^{(\gamma_1,\dots,\gamma_{r-1})}_{*,r-1}(t) - \gamma_1 F^{(\gamma_2,\dots,\gamma_r)}_{*,r-1}(t), \quad t\in\mathbb{R}.$$

4.3 Reliability properties of generalized order statistics

A well-known result for ordinary order statistics states that the IFR-property (increasing failure rate) is preserved under the ordering operation (cf. Barlow and Proschan, 1965, p. 38). We extend this result to the case of generalized order statistics.

Let F be a distribution function with $F(0) = 0$. Then, F is said to have the IFR-property iff the ratio

$$\frac{F(t+x) - F(t)}{\overline{F}(t)}$$

is increasing in t for $x > 0$, $t \geq 0$ such that $F(t) < 1$.

It should be mentioned that IFR distribution functions are absolutely continuous and, thus, continuous on the set $\{t \in \mathbb{R} : F(t) < 1$ (cf. Barlow and Proschan, 1975, p. 77). If the right endpoint $\omega(F)$ of the support of F is finite then the distribution function F may have a jump at $\omega(F)$.

Theorem 4.5. *Let F be an IFR-distribution function and let $F_{*,r}$ be the distribution function of the r-th generalized order statistic based on F and parameters $\gamma_1,\dots,\gamma_r$.*

Then, $F_{*,r}$ is an IFR-distribution.

Proof. Let $F^{(r)}$ denote the distribution function of the convolution of r independent, exponentially distributed random variables with parameters $\gamma_1,\dots,\gamma_r > 0$. The exponential distribution has the IFR-property, and, thus, $F^{(r)}$ is IFR (cf. Barlow and Proschan, 1965, p. 36). From (8), we obtain

$$F_{*,r} = F^{(r)} \circ (-\ln \overline{F}).$$

According to Theorem 4.1 of Barlow and Proschan (1965, p. 25), $F_{*,r}$ is an IFR distribution iff $\ln \overline{F}_{*,r}$ is concave on $T = \{t \geq 0 : F_{*,r}(t) < 1\}$.

A Meijer's G-function is a nonnegative analytic function (cf. Erdélyi et al., 1953, p. 208), and, thus, it is a continuous function. Equation (11) and the continuity yield $F_{*,r}(t) < 1$ provided that $F(t) < 1$. Hence, $T = \{t \geq 0 : F(t) < 1\}$.

By assumption, F is IFR so that $\ln \overline{F}$ is concave on T, i.e., $H = -\ln \overline{F}$ is convex on T. Moreover, $H_r = -\ln \overline{F^{(r)}}$ is convex and increasing. Summing up, $H_r \circ H$ is a convex function on T (see Rockafellar, 1970, Theorem 5.1, p. 32). Therefore, $-H_r \circ H = (\ln \overline{F^{(r)}}) \circ (-\ln \overline{F}) = \ln \overline{F}_{*,r}$ is concave on T, proving that $F_{*,r}$ is an IFR-distribution. ■

Remark 4.6. *As mentioned before, the result of Theorem* 4.5 *has been proved by Barlow and Proschan (1965) in the case of ordinary order statistics. Gupta and Kirmani (1988) establish the result for record values. Kamps (1995a, Remark 1.6, p. 172) points out that the result remains true for generalized order statistics with a certain restriction imposed on the parameters* $\gamma_1, \gamma_2, \ldots$

The IFR-property is often given in terms of the hazard rate

$$\lambda(t) = \frac{g(t)}{\overline{G}(t)}$$

with some distribution function G and its density function g, i.e., λ is an increasing function in t.

Remark 4.7. *The hazard rate* $\lambda_{*,r}$ *of the* r*-th generalized order statistic* $X_*^{(r)}$ *based on an absolutely continuous distribution function* F *with density function* f *is given by*

$$\lambda_{*,r}(t) = \lambda(t) h_r(\overline{F}(t)), \quad t \in \mathbb{R},$$

where $\lambda(t) = \frac{f(t)}{\overline{F}(t)}$ *denotes the hazard rate function of* F. h_r *is defined by*

$$h_r(z) = \frac{z \mathrm{G}_{r,r}^{r,0}\left[z \,\middle|\, \begin{matrix} \gamma_1, \ldots, \gamma_r \\ \gamma_1 - 1, \ldots, \gamma_r - 1 \end{matrix}\right]}{\int_0^z \mathrm{G}_{r,r}^{r,0}\left[s \,\middle|\, \begin{matrix} \gamma_1, \ldots, \gamma_r \\ \gamma_1 - 1, \ldots, \gamma_r - 1 \end{matrix}\right] ds}, \quad z \in (0, 1).$$

From Theorem 4.5 *we know that the* r*-th generalized order statistic is IFR if the underlying distribution is a standard exponential one. In this setting, we have* $\lambda(t) = 1$ *and* $\lambda_{*,r}(t) = h_r(\mathrm{e}^{-t})$ *which is increasing in* t *because of the IFR-property. Hence,* $h_r(z)$ *is decreasing in* z.

The next theorem proves that the hazard rates of the r-th and the $(r-1)$-th generalized order statistics are ordered according to $\lambda_{*,r-1}(t) \geq \lambda_{*,r}(t)$. It extends a result of Baxter (1982, p. 326) who has considered record values in terms of the relevation transform.

Theorem 4.8. *Let* $X_*^{(r)}$ *and* $X_*^{(r-1)}$ *be generalized order statistics based on an absolutely continuous distribution function* F *with density function* f *and parameters* $\gamma_1, \ldots, \gamma_r$.

The hazard rates of the r-th and the $(r-1)$-th generalized order statistics are ordered according to $\lambda_{*,r-1}(t) \geq \lambda_{*,r}(t)$, $0 \leq t < F^{-1}(1)$.

Proof. By applying relation (4) and noticing that $\overline{F}_r(t) = \overline{F}^{\gamma_r/(n-r+1)}(t)$, we find

$$\lambda_{*,r}(t) = \frac{f_{*,r}(t)}{\overline{F}_{*,r}(t)} = (n-r+1)\frac{f_r(t)}{\overline{F}_r(t)}\left(1 - \frac{\overline{F}_{*,r-1}(t)}{\overline{F}_{*,r}(t)}\right)$$

$$= \lambda(t)\left(\gamma_r - \frac{\int_0^{\overline{F}(t)} \mathsf{G}^{r-1,0}_{r-1,r-1}\left[s\,\middle|\,\begin{matrix}\gamma_1,\ldots,\gamma_{r-1}\\ \gamma_1-1,\ldots,\gamma_{r-1}-1\end{matrix}\right]ds}{\int_0^{\overline{F}(t)} \mathsf{G}^{r,0}_{r,r}\left[s\,\middle|\,\begin{matrix}\gamma_1,\ldots,\gamma_r\\ \gamma_1-1,\ldots,\gamma_r-1\end{matrix}\right]ds}\right).$$

On the other hand, we have the relation $\lambda_{*,r}(t) = \lambda(t)h_r(\overline{F}(t))$. Moreover, h_r is a decreasing function such that $h_r'(z) \leq 0$, $z \in (0,1)$. Hence, we conclude from the above alternative representation of h_r that

$$\mathsf{G}^{r-1,0}_{r-1,r-1}\left[z\,\middle|\,\begin{matrix}\gamma_1,\ldots,\gamma_{r-1}\\ \gamma_1-1,\ldots,\gamma_{r-1}-1\end{matrix}\right]\int_0^z \mathsf{G}^{r,0}_{r,r}\left[s\,\middle|\,\begin{matrix}\gamma_1,\ldots,\gamma_r\\ \gamma_1-1,\ldots,\gamma_r-1\end{matrix}\right]ds$$

$$-\mathsf{G}^{r,0}_{r,r}\left[z\,\middle|\,\begin{matrix}\gamma_1,\ldots,\gamma_r\\ \gamma_1-1,\ldots,\gamma_r-1\end{matrix}\right]\int_0^z \mathsf{G}^{r-1,0}_{r-1,r-1}\left[s\,\middle|\,\begin{matrix}\gamma_1,\ldots,\gamma_{r-1}\\ \gamma_1-1,\ldots,\gamma_{r-1}-1\end{matrix}\right]ds \geq 0$$

or, equivalently, $h_{r-1}(z) \geq h_r(z)$. Subsuming the preceding results, we obtain

$$\lambda_{*,r}(t) = \lambda(t)h_r(\overline{F}(t)) \leq \lambda(t)h_{r-1}(\overline{F}(t)) = \lambda_{*,r-1}(t). \qquad \blacksquare$$

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