Metrika (2003) 58: 85–97

Metrika

Generalized trapezoidal distributions

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Revised: April 2002

Abstract. We present a construction and basic properties of a class of continuous distributions of an arbitrary form defined on a compact (bounded) set by concatenating in a continuous manner three probability density functions with bounded support using a modified mixture technique. These three distributions may represent growth, stability and decline stages of a physical or mental phenomenon.

Key words: Mixtures of Distributions, Risk Analysis, Applied Physics

1. Introduction

Trapezoidal distributions have been advocated in risk analysis problems by Pouliquen (1970) and more recently by Powell and Wilson (1997). They have also found application as membership functions in fuzzy set theory (see, e.g. Chen and Hwang (1992)). Our interest in trapezoidal distributions and their modifications stems mainly from the conviction that many physical processes in nature and human body and mind (over time) reflect the form of the trapezoidal distribution. In this context, trapezoidal distributions have been used in the screening and detection of cancer (see, e.g. Flehinger and Kimmel, (1987) and Brown (1999)).

Trapezoidal distributions seem to be appropriate for modeling the duration and the form of a phenomenon which may be represented by three stages. The first stage can be viewed as a growth-stage, the second corresponds to a relative stability and the third represents a decline (decay). These distributions however are restricted since the growth and decay (in the first and third stages) are limited in the trapezoidal case to linear forms and the second stage represents complete stability rather than a possible mild incline or decline. The trapezoidal probability density function is of the form

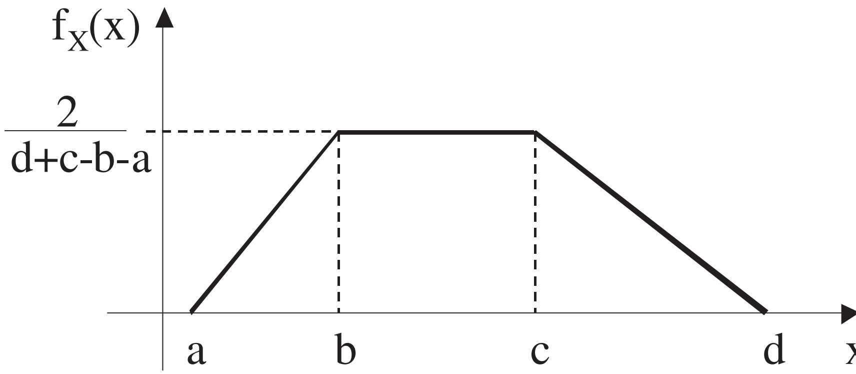

Fig. 1. Probability Density Function of a Trapezoidal Distribution

$$f(x\,|\,a,b,c,d)=\begin{cases} u\left(\frac{x-a}{b-a}\right) & a\leq x<b \\ u & b\leq x<c \\ u\left(\frac{d-x}{d-c}\right) & c\leq x<d \\ 0 & \text{elsewhere} \end{cases} \tag{1}$$

where $a\leq b\leq c\leq d$ and $u=2(d+c-b-a)^{-1}$. The name "trapezoidal" reflects the shape of a graph of the probability density function (See Figure 1). Triangular and uniform distributions are special cases of the trapezoidal family.

Another domain for applications of the trapezoidal distribution is the applied physics arena (see, e.g. Davis and Sorenson (1969), Nakao and Iwaki (2000), Sentenac et al. (2000), Straaijer and De Jager (2000)). Specifically, in the context of nuclear engineering, uniform and trapezoidal distribution have been assumed as models for observed axial distributions for burnup credit calculations (see, Wagner and DeHart (2000) and Neuber (2000) for a comprehensive description). These distributions are important to burnup credit criticality safety analyses for pressurized-water-reactor (PWR) fuel. Figure 2, adapted from Wagner and DeHart (2000), depicts the actual data and axial burnup distributions for two profiles of normalized burnup versus percent axial height (using interpolation between observed data points). The uniform distribution has been shown to be only conservative for low burnups, not when burnup increases (see, Wagner and DeHart (2000)). The use of trapezoidal distributions tend to result in conservative analyses (see, Neuber (2000)). The modeling of axial burnup distributions has been recognized as an important and timely research area in nuclear engineering (See, Parks et al. (2000)).

In the case of the distribution given by (1) both the growth and decay stages are linear and the density at b and c is

$$f_X(b)=f_X(c)\equiv 2(d+c-b-a)^{-1}, \tag{2}$$

where $a\leq b\leq c\leq d$. We shall strive for a continuous generalization of the trapezoidal distribution where the growth and decay may exhibit a nonlinear convex or concave behavior and the densities $f_X(b)$ and $f_X(c)$ do not neces-

	Normalized Burnup	
Axial Height %	Profile 1	Profile 2
2.78%	0.652	0.649
8.33%	0.967	1.044
13.89%	1.074	1.208
19.44%	1.103	1.215
25.00%	1.108	1.214
30.56%	1.106	1.208
36.11%	1.102	1.197
41.69%	1.097	1.189
47.22%	1.094	1.188
57.80%	1.094	1.192
58.33%	1.095	1.195
63.89%	1.096	1.190
69.44%	1.095	1.156
75.00%	1.086	1.022
80.56%	1.059	0.756
86.11%	0.971	0.614
91.67%	0.738	0.481
97.22%	0.462	0.284

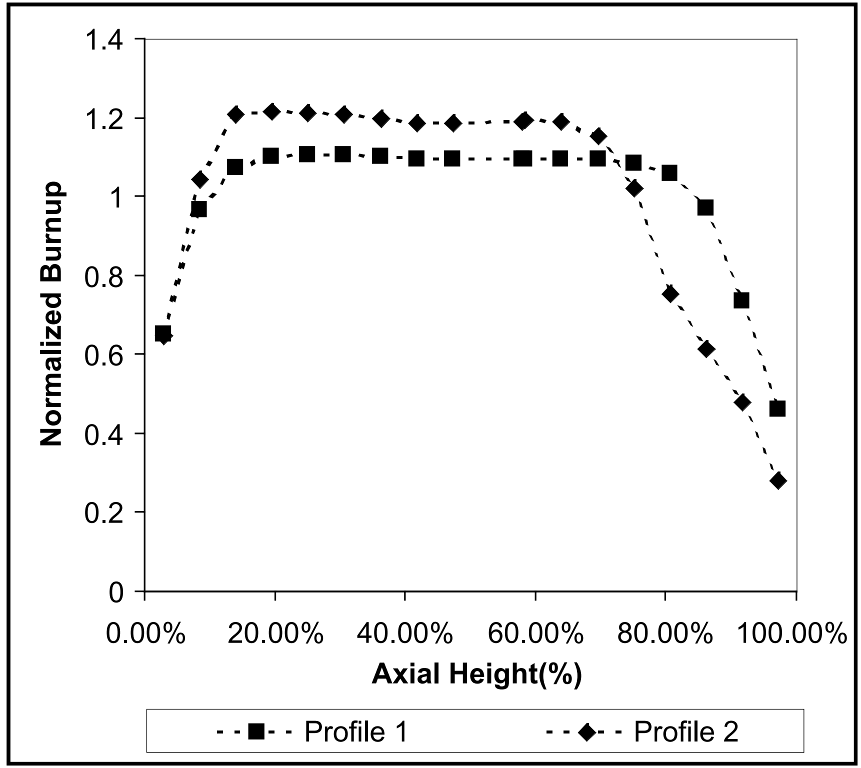

Fig. 2. Two Typical Profiles of Observed Axial Distributions in PWR

sarily have to take the same value. Rather, a *boundary ratio parameter* α is introduced such that $f_X(b) = \alpha f_X(c)$. Generalized trapezoidal distributions herein inherit the four basic trapezoidal parameters a, b, c and d and need, for complete specification, two additional parameters n_1 and n_3 specifying the growth rate and decay rate in the first and third stage of the distribution, in addition to the boundary ratio parameter α. An advantage of the generalized trapezoidal distribution is in its flexibility which allows us inter alia to appropriately mimic the great variety of the growth and decay behaviors.

In Section 2, the functional form of the generalized trapezoidal distribution is derived to be

$$f_X(x \mid a, b, c, d, n_1, n_3, \alpha)$$
$$= \begin{cases} \frac{2\alpha^2 n_1 n_3}{2\alpha(b-a)n_3 + (\alpha+1)(c-b)n_1 n_3 + 2(d-c)n_1} \left(\frac{x-a}{b-a}\right)^{n_1-1} & a \le x < b \\ \frac{2 n_1 n_3}{2\alpha(b-a)n_3 + (\alpha+1)(c-b)n_1 n_3 + 2(d-c)n_1} \left\{(\alpha-1)\frac{c-x}{c-b} + 1\right\} & b \le x < c \\ \frac{2\alpha n_1 n_3}{2\alpha(b-a)n_3 + (\alpha+1)(c-b)n_1 n_3 + 2(d-c)n_1} \left(\frac{d-x}{d-c}\right)^{n_3-1} & c \le x < d \\ 0 & \text{elsewhere} \end{cases} \tag{3}$$

where $n_1 > 0$, $n_3 > 0$, $\alpha > 0$ and $a < b < c < d$. Expression (3) is constructed using a mixture of three densities f_{X_1}, f_{X_2}, f_{X_3},

$$f_X(x) = \begin{cases} \sum_{i=1}^{3} \pi_i f_{X_i}(x) & a \le x < d \\ 0 & \text{elsewhere} \end{cases} \tag{4}$$

where $\sum_{i=1}^{3} \pi_i = 1$, $\pi_i > 0$, with

$$f_{X_1}(x \mid a, b, n_1) = \left(\frac{n_1}{b-a}\right)\left(\frac{x-a}{b-a}\right)^{n_1-1}, \quad a \le x < b, n_1 > 0, \tag{5}$$

$$f_{X_2}(x \mid b, c, \alpha) = \frac{2}{(\alpha+1)(c-b)^2}\{(1-\alpha)x + \alpha c - b\}, \quad b \le x \le c, \alpha > 0, \tag{6}$$

$$f_{X_3}(x \mid c, d, n_3) = \left(\frac{n_3}{d-c}\right)\left(\frac{d-x}{d-c}\right)^{n_3-1}, \quad c \le x < d, n_3 > 0. \tag{7}$$

Note that, the density function in the second stage is restricted to a linear form such that $f_{X_2}(b \mid b, c, \alpha) = \alpha f_{X_2}(c \mid b, c, \alpha)$. For $0 < \alpha < 1$ ($\alpha > 1$) the density in (6) exhibits an inclining (declining) behavior. For $\alpha = 1$, (7) reduces to a uniform density on $[b, c]$. Figure 3 depicts two members in the generalized trapezoidal family that *closely* follow the axial distribution profiles in Figure 2. From Figure 3 it can be concluded that the density function of the generalized trapezoidal distribution (cf. (3)) may well be geared towards modeling axial distribution profiles. Note especially Case B, where the decline in the central part is closely tracked. Applications to reliability and risk analysis may also become more appropriate by replacing the linear parts with a power function.

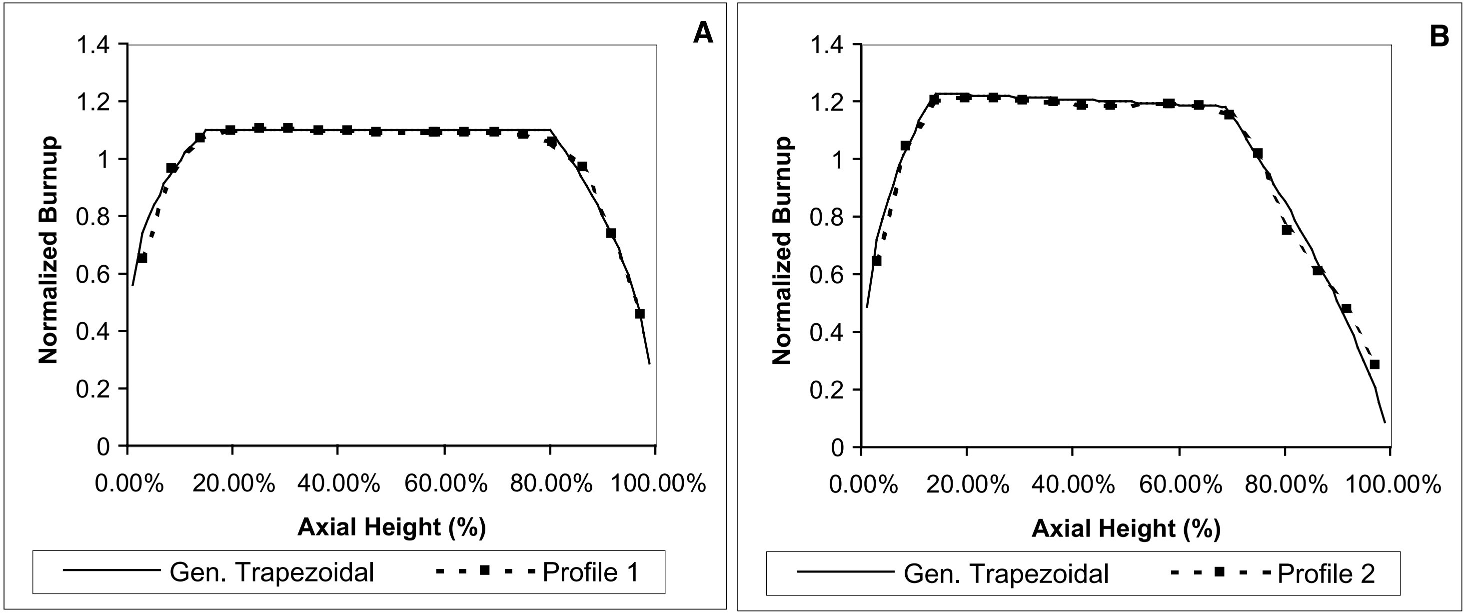

Fig. 3. Generalized Trapezoidal approximation of Axial Distributions depicted in Figure 2; A: $a = 0$, $b = 0.15$, $c = 0.8$, $d = 1$, $n_1 = 1.25$, $n_3 = 1.45$, $\alpha = 1$; B: $a = 0$, $b = 0.14$, $c = 0.69$, $d = 1$, $n_1 = 1.35$, $n_3 = 1.75$, $\alpha = 1.04$.

Some additional examples of generalized trapezoidal distributions will be presented in Section 2. In Section 3 the "mixing" behavior in (4) will be studied for some limiting cases. In Section 4 we shall briefly discuss some basic properties associated with (3). Concluding remarks are provided in Section 5.

2. Construction of probability density function

Our approach towards constructing the desired distribution requires to specify: (1) the ends and beginnings of the three stages (a, b, c, d), (2) the growth behavior of the first stage (parameter n_1), (3) the decay behavior of the third stage (parameter n_3) and (4) the relative likelihood of capabilities at the end of the growth stage $[a, b]$ and at the beginning of the decay stage $[c, d]$, namely the boundary ratio parameter

$$\alpha = f_X(b)/f_X(c). \tag{8}$$

To allow for nonlinear growth and decay the probability density functions (5) and (7) are chosen for X_1 and X_3 in (4), respectively. The density function at the second stage will be restricted to the linear form given by (6) satisfying (as previously noted)

$$f_{X_2}(b \mid b, c, \alpha) = \alpha f_{X_2}(c \mid b, c, \alpha). \tag{9}$$

The main challenge in the construction is to select the remaining mixing probabilities π_1, π_2, π_3 in (4) so that the overall density function in (4) be continuous. This turns out to be a nontrivial problem.

Proposition: *The probability density function given by (3) follows from expressions (4), (5), (6), (7) and (8) utilizing mixture probabilities*

$$\begin{cases} \pi_1 = \frac{2\alpha(b-a)n_3}{2\alpha(b-a)n_3+(\alpha+1)(c-b)n_1n_3+2(d-c)n_1} \\ \pi_2 = \frac{(\alpha+1)(c-b)n_1n_3}{2\alpha(b-a)n_3+(\alpha+1)(c-b)n_1n_3+2(d-c)n_1} \\ \pi_3 = \frac{2(d-c)n_1}{2\alpha(b-a)n_3+(\alpha+1)(c-b)n_1n_3+2(d-c)n_1}, \end{cases} \tag{10}$$

where $a < b < c < d$, $n_1 > 0$, $n_3 > 0$, $\alpha > 0$ *and the probability density function given by (3) is continuous.*

Proof: Utilizing (4), (5), (6) and (7) the density function of the proposed generalized trapezoidal distribution given by (4) can be rewritten as

$$f_X(x|\boldsymbol{\Theta}) = \begin{cases} \pi_1 f_{X_1}(x \mid a, b, n_1) & a \le x < b \\ \pi_2 f_{X_2}(x \mid b, c, \alpha) & b \le x < c \\ \pi_3 f_{X_3}(x \mid c, d, n_3) & c \le x < d \\ 0 & \text{elsewhere}, \end{cases} \tag{11}$$

where $\boldsymbol{\Theta} = (a, b, c, d, n_1, n_3, \alpha)$, $\pi_i > 0$, $i = 1, 2, 3$,

$$a < b < c < d, \quad \sum_{i=1}^{3} \pi_i = 1, \quad n_1 > 0, n_3 > 0, \alpha > 0. \tag{12}$$

It will be convenient to write the mixture weights π_i, $i = 1, 2, 3$, in the form

$$\pi_1 = \beta p, \quad \pi_2 = (1-\beta), \quad \pi_3 = \beta(1-p) \tag{13}$$

where $0 < \beta < 1$, $0 < p < 1$. This implies that $\pi_i > 0$, $i = 1, \ldots, 3$ and

$$\sum_{i=1}^{3} \pi_i = \beta p + (1-\beta) + \beta(1-p) = 1. \tag{14}$$

From (8), utilizing (5), (7), (11) and (13), we have

$$\alpha = \frac{f_X^-(b|\boldsymbol{\Theta})}{f_X^+(c|\boldsymbol{\Theta})} = \frac{\beta p f_{X_1}(b \mid a, b, n_1)}{\beta(1-p) f_{X_3}(c \mid c, d, n_3)} = \frac{p(d-c)n_1}{(1-p)(b-a)n_3}, \tag{15}$$

where $f_X^-(b|\boldsymbol{\Theta}) = \lim_{x \uparrow b} f_X(x|\boldsymbol{\Theta})$ and $f_X^+(c|\boldsymbol{\Theta}) = \lim_{x \downarrow c} f_X(x|\boldsymbol{\Theta})$, yielding

$$p = \frac{(b-a)n_3\alpha}{(d-c)n_1 + (b-a)n_3\alpha}. \tag{16}$$

Observe that p does not depend on β. Also the stipulations $a < b < c < d$, $n_1 > 0$, $n_3 > 0$ and $\alpha > 0$ imply $0 < p < 1$.

Continuity of (11) at b will follow from the requirement that

$$f_X^-(b|\boldsymbol{\Theta}) = f_X^+(b|\boldsymbol{\Theta}), \tag{17}$$

implying with (13) and (11) that

$$\beta p f_{X_1}(b \mid a, b, n_1) = (1-\beta) f_{X_2}(b \mid b, c, \alpha). \tag{18}$$

Utilizing (5), (6), (18) and (16) we obtain

$$\beta = \frac{2(d-c)n_1 + 2\alpha(b-a)n_3}{2(d-c)n_1 + (\alpha+1)(c-b)n_1 n_3 + 2\alpha(b-a)n_3}. \tag{19}$$

From $a < b < c < d$, $n_1 > 0$, $n_3 > 0$ and $\alpha > 0$ it follows that $0 < \beta < 1$. The choice of β in (19) assures continuity of $f_X(\cdot|\boldsymbol{\Theta})$ (cf. (11)) at b. Utilizing (18), (9) and (15), it follows that

$$\beta(1-p) f_{X_3}(c \mid c, d, n_3) = (1-\beta) f_{X_2}(c \mid b, c, \alpha). \tag{20}$$

The continuity of $f_X(\cdot|\boldsymbol{\Theta})$ (cf. (11)) at c is implied by (13) and (20). Substituting (16) and (19) into (13) we arrive at

$$\begin{cases} \pi_1 = \frac{2\alpha(b-a)n_3}{2\alpha(b-a)n_3+(\alpha+1)(c-b)n_1n_3+2(d-c)n_1} \\ \pi_2 = \frac{(\alpha+1)(c-b)n_1n_3}{2\alpha(b-a)n_3+(\alpha+1)(c-b)n_1n_3+2(d-c)n_1} \\ \pi_3 = \frac{2(d-c)n_1}{2\alpha(b-a)n_3+(\alpha+1)(c-b)n_1n_3+2(d-c)n_1}. \end{cases} \tag{21}$$

Finally, substitution of (5), (6), (7) and (21) into (11), yields (3). □

The conditions in (5) and (7) stipulated $n_1 > 0$ and $n_3 > 0$. To adhere to the truly "trapezoidal" shape one may restrict $n_1 > 1$ and $n_3 > 1$ in the first and third stages. In case $0 < n_1 < 1$, $0 < n_3 < 1$ the first stage reflects decay and the third expresses growth of the density $f_X(x|\boldsymbol{\Theta})$ given by (3) resulting in a "bathtub" shape rather than a trapezoidal shape for the combined density. Figure 4 displays different shapes of generalized trapezoidal distributions.

The graphs in Figure 4 alternate between the three cases $0 < \alpha < 1$, $\alpha = 1$ and $\alpha > 1$. Substituting $n_1 = n_3 = 2$ and $\alpha = 1$ into (3) we arrive at the trapezoidal distribution given by (1).

We note in passing that $f_{X_2}(\cdot)$ can be taken to be a conditional Two Sided Power density (see, Van Dorp and Kotz (2002)) on $[a, d]$ truncated to $[b, c]$ (rather than the linear form in (6)), which results in an extension of the trapezoidal distribution (cf. (1)) permitting mild oscillation in the central stage.

3. Mixing behavior

Some insight about the mixing behavior for generalized trapezoidal distributions in (3) can be gained by studying limiting behavior of the mixing probabilities in (10). From (19) we have $\beta = (1 + G)^{-1}$, where

$$G = \frac{(\alpha+1)(c-b)}{\frac{2(d-c)}{n_3} + \frac{2\alpha(b-a)}{n_1}}. \tag{22}$$

Since $a < b < c < d$, $n_1 > 0$, $n_3 > 0$ and $\alpha > 0$ we have $G > 0$ and the largest (least) β corresponds to least (greatest) G. As $n_1 \to \infty$ and $n_3 \to \infty$, $G \to \infty$ and therefore $\beta \downarrow 0$ (limiting "least" case). Hence, from $\pi_2 = 1 - \beta$ (cf. (13)) it follows that no probability mass is attributed to the first and last stages in the limit when $n_1 \to \infty$ and $n_3 \to \infty$ and (3) converges to $f_{X_2}(x \mid b, c, \alpha)$ (cf. (6)). As $n_1 \downarrow 0$ and $n_3 \downarrow 0$, $G \downarrow 0$ and $\beta \uparrow 1$ (limiting "greatest" case). Hence, from $\pi_2 = 1 - \beta$ (cf. (13)) it follows that all the probability mass is attributed to the first and last stages in the limit as $n_1 \downarrow 0$ and $n_3 \downarrow 0$ (cf. (5) and (7)).

From (21) it follows that letting $n_1 \downarrow 0$ and $n_3 \downarrow 0$ while keeping $\frac{n_1}{n_3} = \mathscr{C}$ (constant) we have

$$\pi_1 \to \frac{2\alpha(b-a)}{2\alpha(b-a) + 2(d-c)\mathscr{C}}, \quad \pi_3 \to \frac{2(d-c)\mathscr{C}}{2\alpha(b-a) + 2(d-c)\mathscr{C}}. \tag{23}$$

It is easy to verify that as $n_1 \downarrow 0$ and $n_3 \downarrow 0$, the density $f_{X_1}(x \mid a, b, n_1)$ converges to a single point mass of 1 at a and the density $f_{X_3}(x \mid c, d, n_3)$ converges to a single point mass of 1 at d. Thus, (3) converges to a transformed Bernoulli

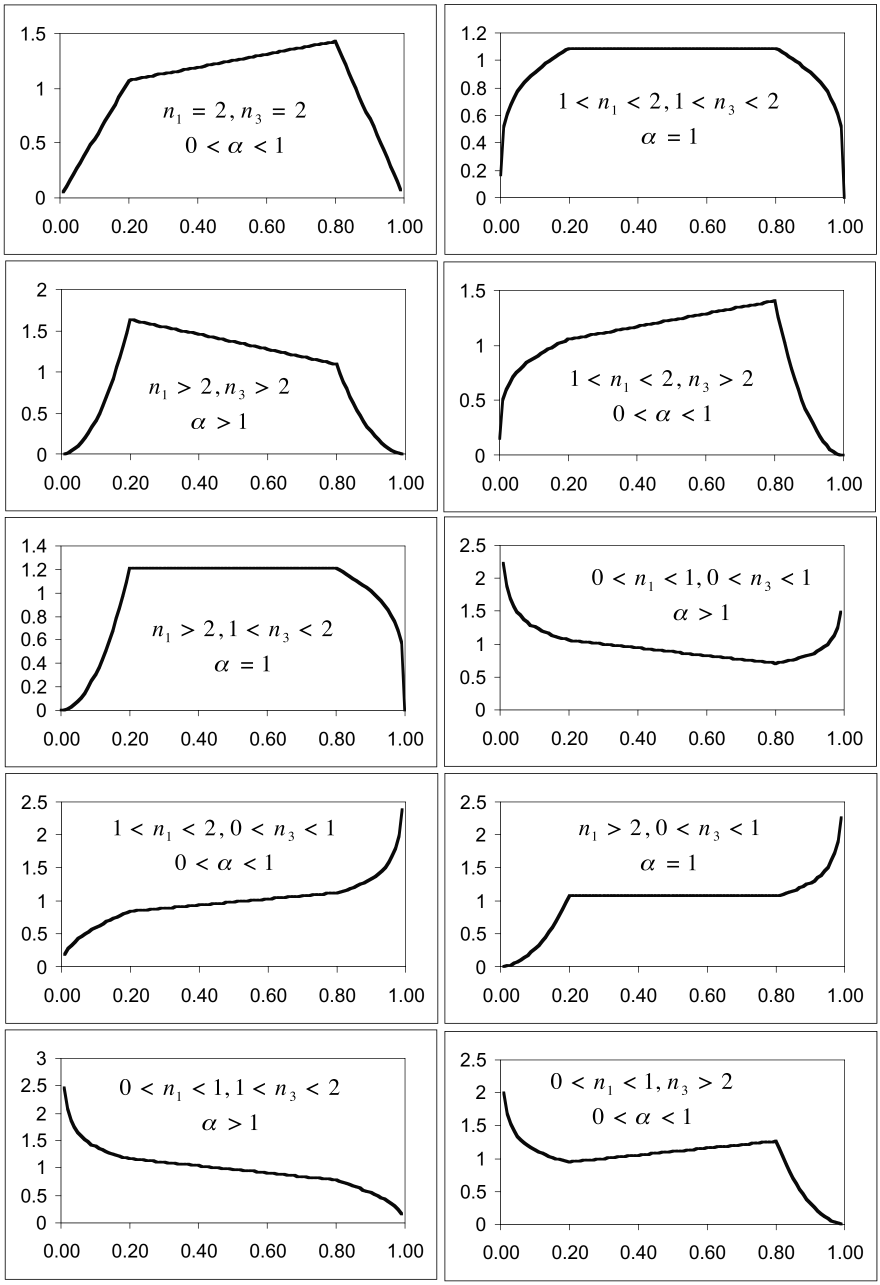

Fig. 4. Generalized Trapezoidal Distributions.

distribution assigning the limiting probability π_1 in (23) to a and limiting probability π_3 in (23) to d.

Letting $n_1 \downarrow 0$ and keeping n_3 fixed, it follows from (21) that in this case $\pi_1 \uparrow 1$, $\pi_2 \downarrow 0$ and $\pi_3 \downarrow 0$. Hence, all the probability mass is attributed to the first stage. It is easy to verify that when $n_1 \downarrow 0$ the density $f_{X_1}(x \mid a, b, n_1)$ converges to a single point mass of 1 at a. Vice versa, letting $n_1 \to \infty$ and keeping n_3 fixed, we have $\pi_1 \downarrow 0$. Here no probability mass is attributed to the first stage and

$$\pi_2 \to \frac{(\alpha+1)(c-b)n_3}{(\alpha+1)(c-b)n_3 + 2(d-c)}, \quad \pi_3 \to \frac{2(d-c)}{(\alpha+1)(c-b)n_3 + 2(d-c)}. \tag{24}$$

Consequently, (3) reduces to a mixture of $f_{X_2}(x \mid \alpha, b, c)$ and $f_{X_3}(x \mid c, d, n_3)$ assigning the limiting probability π_2 in (24) to the first density and the limiting probability π_3 in (24) to the second density. Analogous conclusions can be drawn letting $n_3 \downarrow 0$, keeping n_1 fixed.

4. Basic properties

In the sections below we shall briefly investigate the cumulative distribution and the moments of the generalized trapezoidal type distributions.

4.1. Cumulative distribution function

The cdf associated with $X \sim f_X(x|\boldsymbol{\Theta})$ in (3) can be derived using (11), (10), (5), (6) and (7) yielding

$$F_X(x|\boldsymbol{\Theta}) = \begin{cases} 0 & x < a \\ \frac{2\alpha(b-a)n_3}{2\alpha(b-a)n_3+(\alpha+1)(c-b)n_1n_3+2(d-c)n_1}\left(\frac{x-a}{b-a}\right)^{n_1} & a \le x < b \\ \frac{2\alpha(b-a)n_3+2(x-b)n_1n_3\left\{1+\frac{(\alpha-1)}{2}\frac{(2c-b-x)}{(c-b)}\right\}}{2\alpha(b-a)n_3+(\alpha+1)(c-b)n_1n_3+2(d-c)n_1} & b \le x < c, \\ 1-\frac{2(d-c)n_1}{2\alpha(b-a)n_3+(\alpha+1)(c-b)n_1n_3+2(d-c)n_1}\left(\frac{d-x}{d-c}\right)^{n_3} & c \le x < d \\ 1 & x > d \end{cases} \tag{25}$$

Setting $n_1 = n_3 = 2$ and $\alpha = 1$ in (25) yields

$$F_X(x \mid a, b, c, d) = \begin{cases} 0 & x < a \\ \frac{(b-a)}{d+c-b-a}\left(\frac{x-a}{b-a}\right)^2 & a \le x < b \\ \frac{(b-a)+2(x-b)}{d+c-b-a} & b \le x < c, \\ 1-\frac{(d-c)}{d+c-b-a}\left(\frac{d-x}{d-c}\right)^2 & c \le x < d \\ 1 & x > d \end{cases} \tag{26}$$

which is recognized as the cdf of the standard trapezoidal density given by (1).

4.2. Moments

Utilizing (4) and (13), the k-th moment of $X \sim f_X(x|\boldsymbol{\Theta})$ (cf. (3)) may be derived as

$$E[X^k|\boldsymbol{\Theta}] = \beta p E[X_1^k \,|\, a, b, n_1] + (1-\beta)E[X_2^k \,|\, b, c, \alpha]$$
$$+ \beta(1-p)E[X_3^k \,|\, c, d, n_3], \tag{27}$$

where β, p are given by (19) and (16), respectively. The pdf's of X_1, X_2 and X_3 are defined in (5), (6) and (7), respectively. Numerical calculations of k-th moment $E[X^k|\boldsymbol{\Theta}]$ given by (27) are quite straightforward employing the current advances in computer technology. Deriving a closed form for the expression of $E[X^k|\boldsymbol{\Theta}]$ for $X \sim f_X(x|\boldsymbol{\Theta})$ (cf. (3)) in its general form, although tedious, does not present intrinsic difficulties. We shall conclude by providing closed form expressions for the first and second moments of a generalized trapezoidal variable X.

From (5) and (7) we obtain that

$$E[X_1 \,|\, a, b, n_1] = \frac{a + n_1 b}{n_1 + 1}, \quad E[X_3 \,|\, c, d, n_3] = \frac{n_3 c + d}{n_3 + 1}. \tag{28}$$

Utilizing (6), yields

$$E[X_2 \,|\, b, c, \alpha] = \frac{-\frac{2}{3}(\alpha - 1)(c^3 - b^3) + (\alpha c - b)(c^2 - b^2)}{(c-b)^2(\alpha + 1)} \tag{29}$$

Hence from (27) (setting $k = 1$), (28), (29), (16) and (19) we have

$$E[X|\boldsymbol{\Theta}]$$
$$= \frac{2\alpha(b-a)n_3\left(\frac{a+n_1 b}{n_1+1}\right) - n_1 n_3\left(\frac{\frac{2}{3}(\alpha-1)(c^3-b^3)-(\alpha c-b)(c^2-b^2)}{(c-b)}\right) + 2(d-c)n_1\left(\frac{n_3 c+d}{n_1+1}\right)}{2\alpha(b-a)n_3 + (\alpha+1)(c-b)n_1 n_3 + 2(d-c)n_1}. \tag{30}$$

Analogously, from (5), (6) and (7) we obtain

$$E[X_1^2 \,|\, a, b, n_1] = \frac{2a^2 + 2n_1 ab + n_1(n_1+1)b^2}{(n_1+2)(n_1+1)}$$
$$E[X_2^2 \,|\, b, c, \alpha] = \frac{-\frac{1}{2}(\alpha - 1)(c^4 - b^4) + \frac{2}{3}(\alpha c - b)(c^3 - b^3)}{(c-b)^2(\alpha+1)} \tag{31}$$
$$E[X_3^2 \,|\, c, d, n_3] = \frac{2d^2 + 2n_3 cd + n_3(n_3+1)c^2}{(n_3+2)(n_3+1)}.$$

Using (31), the second moment $E[X^2|\boldsymbol{\Theta}]$ now follows from (27) (setting $k = 2$), (16) and (19) to be

$$E[X^2|\boldsymbol{\Theta}] = \frac{2\alpha(b-a)n_3}{2\alpha(b-a)n_3 + (\alpha+1)(c-b)n_1n_3 + 2(d-c)n_1} \times \left(\frac{2a^2 + 2n_1ab + n_1(n_1+1)b^2}{(n_1+2)(n_1+1)}\right) - \frac{n_1n_3}{2\alpha(b-a)n_3 + (\alpha+1)(c-b)n_1n_3 + 2(d-c)n_1} \times \left(\frac{\frac{1}{2}(\alpha-1)(c^4-b^4) - \frac{2}{3}(\alpha c - b)(c^3-b^3)}{(c-b)}\right) + \frac{2(d-c)n_1}{2\alpha(b-a)n_3 + (\alpha+1)(c-b)n_1n_3 + 2(d-c)n_1} \times \left(\frac{n_3(n_3+1)c^2 + 2n_3cd + 2d^2}{(n_3+2)(n_3+1)}\right). \tag{32}$$

The variance of a generalized trapezoidal variable X may be calculated utilizing (31) and (32). Setting $n_1 = n_3 = 2$ and $\alpha = 1$ in (31) and in (32), we have the elegant formulas

$$E[X \mid a,b,c,d] = \frac{(b-a)(a+2b) - 3(b^2-c^2) + (d-c)(2c+d)}{3(d+c-b-a)}, \tag{33}$$

$$E[X^2 \mid a,b,c,d] = \frac{(b-a)}{(d+c-b-a)}\left(\frac{1}{6}(a+b)^2 + \frac{1}{3}b^2\right) + \frac{1}{(d+c-b-a)}\left(\frac{2}{3}(c^3-b^3)\right) + \frac{(d-c)}{(d+c-b-a)} \times \left(\frac{1}{3}c^2 + \frac{1}{6}(c+d)^2\right), \tag{34}$$

for the first and second moment of the standard trapezoidal density given by (1).

5. Concluding remarks

In the course of the construction some interesting features emerged which may be worthy of specific mention. Firstly the structure of these distributions – although formally a mixture of three components – differs from the commonly encountered mixtures in two aspects: (i) the mixing parameters are of a special form (a product of two quantities (cf. (13)) each performing a function needed to properly link the three components in (11) and (ii) the components represent different distributions each capable of taking a variety of forms. Next, while classical continuous distributions are characterized by the property that continuity is generated by means of a mathematical function that forces a special form of the distribution, here continuity is generated by linking appropriately the three relevant parts of the distribution rendering an additional flexibility. We have attempted to demonstrate a method of constructing versatile and flexible family of continuous distributions on a compact set. The procedure depends on the values of the parameters of the constituent distributions and provides an example of a new form of a mixture consisting of nonlinear components. The family has transparent physical interpretation and potential applications in engineering, behavioral and medical sciences.

6. Acknowledgment

The authors are indebted to the Editor of *Metrika* and the referee for their most valuable comments and suggestions which improved the content and streamlined the presentation.

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