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Failure rate of the minimum and maximum of a multivariate normal distribution

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Abstract. It is well known that if the parent distribution has a nonnegative support and has increasing failure rate (IFR), then all the order statistics have IFR. The result is not necessarily true in the case of bivariate distributions with dependent structures. In this paper we consider a multivariate normal distribution and prove that, the distributions of the minimum and maximum retain the IFR property.

Key words: Maximum, minimum, failure rate, monotonicity

1. Introduction

The distributions of the maximum and minimum of p random variables $X_1, X_2, \dots, X_p$ play an important role in various statistical applications. For example in the competing risk survival analysis due to p causes, $X_1, X_2, \dots, X_p$ are not observed but $T_1 = \min(X_1, X_2, \dots, X_p)$ is the observable time of death. Similarly in reliability studies, $T_1 = \min(X_1, X_2, \dots, X_p)$ is observable if the components are arranged in the series system and $T_2 = \max(X_1, X_2, \dots, X_p)$ is observed if the components are arranged in the parallel system.

In the case of independent and identically distributed random variables from a distribution $F(\cdot), T_1$, and T_2 constitute order statistics for a random sample of size p from a distribution $F(\cdot)$. In reliability theory literature, it is well known that if the parent distribution has a nonnegative support and has increasing failure rate (IFR), then all the order statistics have IFR, see for example Barlow and Proschan (1981) and a recent monograph by Kamps (1995).

Nagaraja and Baggs (1996) have studied the order statistics of bivariate exponential random variables and noted that even if the marginal distribution is IFR, it is not necessary that T_1, and T_2 have IFR. For example, for Raftery's

(1984) bivariate exponential distribution, the marginals are exponential and yet the failure rate of T_1 is non-monotonic for certain values of the parameters.

In this paper we consider a random variable $(X_1, X_2, \ldots, X_p)$ having a multivariate normal distribution and so the marginals have IFR. We are interested in the monotonicity of the failure rates of T_1 and T_2. In section 2, we obtain the distributions of T_1 and T_2 for the bivariate normal distribution and in section 3 we prove that both T_1 and T_2 have IFR . These results are similar to the ones in the independent case. The three dimensional and the general p dimensional case are discussed in sections 4 and 5.

2. Distributions of T_1 and T_2

Consider a random variable (X_1, X_2) having a bivariate normal distribution, BVN $(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$, having pdf

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \left\{ \left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\frac{(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 \right\} \right]. \tag{2.1}$$

Let $T_1 = \min(X_1, X_2)$ and $T_2 = \max(X_1, X_2)$. Let us also denote by $n(x|\mu, \sigma^2)$ a normal density at x with mean μ and variance σ^2.

2.1. Distribution of T_1

Suppose (X_1, X_2) is a bivariate random vector with pdf $f(x, y)$. Then

$$P(T_1 > t) = \int_t^\infty \int_t^\infty f(x, y)\, dx\, dy$$

Therefore, the pdf of T_1 can be written as

$$f_{T_1}(t) = f_{X_2}(t) P(X_1 > t \,|\, X_2 = t) + f_{X_1}(t) P(X_2 > t \,|\, X_1 = t). \tag{2.2}$$

This gives us a general formula for expressing the pdf of T_1 in terms of the survival functions of the conditionals. Thus in our case, the pdf of T_1 is given by

$$f_{T_1}(t) = n(t_1|\mu_1, \sigma_1^2)\left[1 - \Phi\left(\frac{t - m_1(t)}{\sigma_2\sqrt{1-\rho^2}}\right)\right] + n(t|\mu_2, \sigma_2^2)\left[1 - \Phi\left(\frac{t - m_2(t)}{\sigma_1\sqrt{1-\rho^2}}\right)\right], \tag{2.3}$$

where

$$m_2(t) = E(X_1|X_2 = t) = \mu_1 + \frac{\rho\sigma_1}{\sigma_2}(t - \mu_2),$$

and

$$m_1(t) = E(X_2|X_1 = t) = \mu_2 + \frac{\rho\sigma_2}{\sigma_1}(t - \mu_1).$$

Remark 1. Note that the pdf of T_1 is not a mixture of two normal densities.

2.2. *Distribution of* T_2

Proceeding as before, the pdf of T_2 is given by

$$f_{T_2}(t) = f_{X_1}(t)P(X_2 < t \mid X_1 = t) + f_{X_2}(t)P(X_1 < t \mid X_2 = t). \tag{2.4}$$

This gives us a general formula for expressing the pdf of T_2 in terms of the survival functions of the conditionals. Thus in our case, the pdf of T_2 is given by

$$f_{T_2}(t) = n(t|\mu_1, \sigma_1^2)\Phi\left(\frac{t - m_1(t)}{\sigma_2\sqrt{1-\rho^2}}\right) \tag{2.5}$$

$$+ n(t|\mu_2, \sigma_2^2)\Phi\left(\frac{t - m_2(t)}{\sigma_1\sqrt{1-\rho^2}}\right). \tag{2.6}$$

Remark 2. Kella (1986) has obtained the Laplace-Stieltjes transform (LST) of the distribution of T_2 in a very complicated way. From the LST, it is not very easy to recover the pdf.

3. Failure rates of T_1 and T_2

Let X be a random variable with pdf $f_X(t)$ and distribution function $F_X(t)$. Then the failure rate of X is defined by

$$h(t) = \frac{f_X(t)}{1 - F_X(t)} = \frac{d}{dt}(-\ln S_X(t)),$$

where $S_X(t) = 1 - F_X(t)$.

X is said to have increasing failure rate (IFR) if $h(t)$ is increasing. Likewise we define decreasing failure rate (DFR) distributions. It is well known that if X is normally distributed, then X has IFR distribution. We are interested in determining the monotonicity of the failure rates of T_1 and T_2.

In most practical applications, the failure rate is quite complicated and so the straight derivative method is very complex. In such cases, we work with the density function and use the following procedure. Let $f_X(t)$ be the pdf of X. Define

$$\eta_X(t) = -f'_X(t)/f_X(t) = -\frac{d}{dt}\ln f_X(t).$$

If $\eta'_X(t) > 0$ for all t, then X has IFR distribution, and if $\eta'_X(t) < 0$ for all t, then X has DFR distribution. For details, see Glaser (1980). We now determine the monotonicity of the failure rate of T_1.

In our case, even the expression for $\eta_X(t)$ is quite involved to yield an analytic solution of the problem. So we proceed as follows:

Let us denote by $h(t)$ the failure rate of T_1 with survival function $S_{T_1}(t)$. Then $h(t) = -\frac{d}{dt}\ln S(t,t)$, where $S(t_1, t_2)$ is the survival function of (X_1, X_2). Define

$$h_i(t_1, t_2) = -\frac{\partial}{\partial t_i}\ln S(t_1, t_2), \quad i = 1, 2.$$

Then $h_1(t_1, t_2)$ and $h_2(t_1, t_2)$ are called the hazard components of the hazard gradient defined by Johnson and Kotz (1975). $h_1(t_1, t_2)$ represents the hazard rate or the failure rate of the conditional distribution of X_1 given $X_2 > t_2$. Likewise $h_2(t_1, t_2)$. It can be verified that

$$h(t) = h_1(t,t) + h_2(t,t). \tag{3.1}$$

Note that $h_i(t,t)$, $i = 1, 2$ is proportional to the failure rate $h_i^*(t,t)$ of the conditional distribution of T_1 given $X_1 < X_2 (X_2 < X_1)$. In the context of competing risks $h_i(t,t)$ describe the (instantaneous) rate of dying from cause i when both the causes are acting simultaneously, see Gupta (1979) and Elandt-Johnson and Johnson (1980).

Then the probability density functions $f_i^*(t)$ of the conditional distributions are given by

$$f_i^*(t) = \frac{1}{\pi_i} h_i(t,t) S_{T_1}(t) \quad i = 1, 2, \tag{3.2}$$

where $S_{T_1}(u)$ is given by

$$S_{T_1}(u) = \exp\left\{-\int_0^u \sum_{i=1}^{2} h_i(x,x)\,dx\right\} \tag{3.3}$$

and π_i is a proper constant of proportionality. For more details and applications, see Gaynor et. al. (1993).

Because of (3.1) and the proportionality mentioned before, the monotonicity of $h(t)$ can be established if $f_1^*(t)$ and $f_2^*(t)$ fulfill the criteria mentioned before.

In our case it can be verified that

$$h_1(t_1, t_2) = \frac{\phi\left(\frac{t_1 - \mu_1}{\sigma_1}\right)\left[1 - \Phi\left\{\frac{t_2 - \mu_2}{\sigma_2} - \rho\left(\frac{t_1 - \mu_1}{\sigma_1}\right)\right\}\right]}{S(t_1, t_2)}. \tag{3.4}$$

This gives

$$f_1^*(t) = \frac{1}{\pi_1}\left\{1 - \Phi\left(\frac{t\left(\frac{1}{\sigma_2} - \frac{\rho}{\sigma_1}\right)}{\sqrt{1-\rho^2}} + \frac{\frac{\rho\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma_2}}{\sqrt{1-\rho^2}}\right)\right\}\phi\left(\frac{t-\mu_1}{\sigma_1}\right), \qquad (3.5)$$

where $\phi(x)$ and $\Phi(x)$ are the pdf and the cumulative distribution function of a standard normal, respectively. This gives

$$\begin{aligned}\eta_1^*(t) &= -\frac{d}{dt}\ln f_1^*(t)\\ &= -\frac{d}{dt}\ln\phi\left(\frac{t-\mu_1}{\sigma_1}\right) - \frac{d}{dt}\ln\{1-\Phi(at+b)\},\end{aligned} \qquad (3.6)$$

where

$$a = \frac{\frac{1}{\sigma_2} - \frac{\rho_1}{\sigma_1}}{\sqrt{1-\rho^2}}, \quad \text{and} \quad b = \frac{\frac{\rho\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma_2}}{\sqrt{1-\rho^2}}$$

or

$$\begin{aligned}\eta_1^*(t) &= -\frac{1}{\sigma_1}\frac{\phi'\left(\frac{t-\mu_1}{\sigma_1}\right)}{\phi\left(\frac{t-\mu_1}{\sigma_1}\right)} + a\frac{\phi(at+b)}{1-\Phi(at+b)}\\ &= \frac{t-\mu_1}{\sigma_1^2} + ar(at+b),\end{aligned}$$

where $r(.)$ is the failure rate of a standard normal. This gives

$$\eta_1^{*'}(t) = \frac{1}{\sigma_1^2} + a^2 r'(at+b) > 0,$$

since the failure rate of a normally distributed random variable is increasing. From this we conclude that $h_1(t,t)$ is increasing. Similarly $h_2(t,t)$ is increasing i.e. T_1 has IFR.

Remark 3. The proof above shows that the values of $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$ are not playing any important role in proving the IFR property. Hence, it could be sufficient to take $\mu_1 = \mu_2 = 0$ and $\sigma_1^2 = \sigma_2^2 = 1$.

3.1. Failure Rate of T_2

Note that $T_2 = \max(X_1, X_2) = -\min(-X_1, -X_2)$. Let $T_1^* = \min(-X_1, -X_2)$. Since (X_1, X_2) has a bivariate normal distribution, $(-X_1, -X_2)$ also has a

bivariate normal distribution. This means that the failure rate of T_1^* is increasing. So $\eta_{T_1}^{*'} > 0$ for all t. Now T_2 has the same distribution as $-T_1^*$. Therefore
$f_{T_2}(t) = f_{T_1}^*(-t)$.

$$\eta_{T_2}(t) = -\frac{f'_{T2}(t)}{f_{T_2}(t)} = \frac{f'_{T_1^*}(-t)}{f_{T_1^*}(-t)} = -\eta_{T_1}^*(-t).$$

This gives

$$\eta'_{T_2}(t) = \eta_{T_1}^{*'}(-t) > 0.$$

Hence T_2 has IFR.

4. Three dimensional case

As seen in the two dimensional case, it will be sufficient to consider $X = (X_1, X_2, X_3)$ having a trivariate normal distribution with correlation matrix R given by

$$R = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix}$$

Then $h(t) = -\frac{d}{dt} \ln S(t, t, t)$, where $S(t_1, t_2, t_3)$ is the survival function of (X_1, X_2, X_3). Define

$$h_i(t_1, t_2, t_3) = -\frac{\partial}{\partial t_i} \ln S(t_1, t_2, t_3), \quad i = 1, 2, 3.$$

Then

$$h(t) = h_1(t, t, t) + h_2(t, t, t) + h_3(t, t, t). \tag{4.1}$$

Interpreting as before, $h_1(t, t, t)$ is the hazard rate of the conditional distribution of X_1 given $X_2 > t, X_3 > t$, evaluated at the point (t, t, t). Likewise $h_2(t, t, t)$ and $h_3(t, t, t)$. We shall now find expressions for these functions. Now

$$\begin{aligned} h_1(t_1, t_2, t_3) &= -\frac{\partial}{\partial t_1} \ln S(t_1, t_2, t_3) \\ &= -\frac{\partial}{\partial t_1} S(t_1, t_2, t_3) / S(t_1, t_2, t_3). \end{aligned}$$

So

$$h_1(t, t, t) = -\frac{\frac{\partial}{\partial t_1} S(t_1, t_2, t_3)|_{t_1=t_2=t_3=t}}{S(t, t, t)}.$$

This gives

$$f_1^*(t) = -\frac{\partial}{\partial t_1} S(t_1, t_2, t_3)|_{t_1=t_2=t_3=t}/\pi_1.$$

Now

$$-\frac{\partial}{\partial t_1} \ln S(t_1, t_2, t_3) = \frac{1}{(2\pi)^{3/2}|R|^{1/2}} \int_{t_3}^{\infty} \int_{t_2}^{\infty} I_2 \, dz_2 \, dz_3, \tag{4.2}$$

where

$$I_2 = \exp\left\{-\frac{1}{2|R|}(c_{11}t_1^2 + c_{22}z_2^2 + c_{33}z_3^2 + 2c_{23}z_2z_3 + 2c_{12}t_1z_2 + 2c_{13}t_1z_3)\right\}$$

and

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

is the matrix of cofactors of R. Letting $r(x, y) = (x - \rho_{12}t_1)^2 c_{22} + (y - \rho_{13}t_1)^2 c_{33} + 2c_{23}(x - \rho_{12}t_1)(y - \rho_{13}t_1)$, it can be seen that (4.2) is of the form

$$\frac{e^{-t_1^2/2}}{(2\pi)^{3/2}|R|^{1/2}} \int_{t_3}^{\infty} \int_{t_2}^{\infty} \exp\left\{-\frac{r(z_2, z_3)}{2|R|}\right\} dz_2 \, dz_3$$

$$= \frac{\phi(t_1)}{2\pi|R|^{1/2}} \int_{t_3}^{\infty} \int_{t_2}^{\infty} \exp\left\{-\frac{r(z_2, z_3)}{2|R|}\right\} dz_2 \, dz_3. \tag{4.3}$$

Note that the integrand in the above expression is proportional to the conditional pdf of $(X_2, X_3)'$ given $X_1 = t_1$ with mean $(\rho_{12}t_1, \rho_{13}t_1)'$ and covariance matrix given by $\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$, where $\Sigma_{11}, \Sigma_{12}, \Sigma_{22}$ and Σ_{21} are the partitioning matrices of R shown below.

$$R = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix} = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}.$$

By the transformation

$$V = \begin{bmatrix} v_2 \\ v_3 \end{bmatrix} = D_2 \begin{bmatrix} \dfrac{z_2 - \rho_{12}t_1}{\sqrt{1 - \rho_{12}^2}} \\ \dfrac{z_3 - \rho_{13}t_1}{\sqrt{1 - \rho_{13}^2}} \end{bmatrix},$$

where $D_2 = \text{diag}(\sqrt{1-\rho_{12}^2}, \sqrt{1-\rho_{13}^2})$, (4.3) will reduce to

$$\phi(t_1)\int_{a_3(t_1)}^{\infty}\int_{a_2(t_1)}^{\infty}\frac{1}{2\pi|K_2|^{1/2}}e^{-(1/2)y'K_2^{-1}y}\,dy = \phi(t_1)S_{K_2}(a_2(t_1), a_3(t_1)), \quad (4.4)$$

where $S_{K_2}(a_2(t_1), a_3(t_1))$ is the survival function of a standard bivariate normal with correlation matrix K_2 at the point $(a_2(t_1), a_3(t_1))$, where

$$a_2(t_1) = \frac{t_2 - \rho_{12}t_1}{\sqrt{1-\rho_{12}^2}}, a_3(t_1) = \frac{t_3 - \rho_{13}t_1}{\sqrt{1-\rho_{13}^2},}$$

and

$$K_2 = \begin{bmatrix} 1 & k_{12} \\ k_{12} & 1 \end{bmatrix} = D_2(\Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12})D_2'.$$

Now

$$\begin{aligned} f_1^*(t) &= -\frac{\partial}{\partial t_1}S(t_1, t_2, t_3)|_{t_1=t_2=t_3=t}/\pi_1 \\ &= \phi(t)S_{K_2}(a_2(t), a_3(t))/\pi_1. \end{aligned}$$

This gives

$$\eta_1^*(t) = -\frac{d}{dt}\ln f_1^*(t) = -\frac{d}{dt}\ln[\phi(t)S_{K_2}(b_2t, b_3t)],$$

where $b_2 = \sqrt{\frac{1-\rho_{12}}{1+\rho_{12}}}$ and $b_3 = \sqrt{\frac{1-\rho_{13}}{1+\rho_{13}}}$.

Or

$$\begin{aligned} \eta_1^*(t) &= t - \frac{d}{dt}\ln S_{K_2}(b_2t, b_3t) \\ &= t - \frac{\frac{d}{dt}\int_{b_3t}^{\infty}\int_{b_2t}^{\infty}\frac{1}{2\pi\sqrt{1-k_{12}^2}}e^{-(1/(2(1-k_{12}^2)))[y_1^2 - 2k_{12}y_1y_2 + y_2^2]}\,dy_1\,dy_2}{S_{k_2}(b_2t, b_3t)} \\ &= t + \frac{b_2\phi(b_2t)\left[1-\Phi\left(\frac{b_3t - k_{12}b_2t}{\sqrt{1-k_{12}^2}}\right)\right] + b_3\phi(b_3t)\left[1-\Phi\left(\frac{b_2t - k_{12}b_3t}{\sqrt{1-k_{12}^2}}\right)\right]}{S_{k_2}(b_2t, b_3t)} \\ &= t + b_2g_1(b_2t, b_3t) + b_3g_2(b_2t, b_3t), \end{aligned} \quad (4.5)$$

where $g_i(b_2t, b_3t)$, $i = 1, 2$ is the ith component of the hazard gradient of a standard bivariate normal with correlation k_{12}. Thus

$$\eta_1^{*'}(t) = 1 + b_2 \frac{d}{dt} g_1(b_2t, b_3t) + b_3 \frac{d}{dt} g_2(b_2t, b_3t).$$

Proceeding as before, it can be verified that $\frac{d}{dt} g_1(b_2t, b_3t) > 0$ and $\frac{d}{dt} g_2(b_2t, b_3t) > 0$. Hence $\eta_1^{*'}(t) > 0$. This implies that $\frac{d}{dt} h_1(t, t, t) > 0$. Likewise $\frac{d}{dt} h_2(t, t, t) > 0$ and $\frac{d}{dt} h_3(t, t, t) > 0$. Hence $\frac{d}{dt} h(t) > 0$.

5. General case

Suppose $X = (X_1, X_2, \ldots, X_p)'$ have a multivariate normal distribution with correlation matrix R given as follows:

$$\mathrm{R} = [\rho_{ij}]_{p \times p}, \rho_{ij} = 1 \quad \text{if } i = j.$$

Then $h(t) = -\frac{d}{dt} \ln S(t, t, \ldots, t)$, where $S(t_1, t_2, \ldots, t_p)$ is the survival function of $(X_1, X_2, \ldots, X_p)$. As before

$$h(t) = \sum_{i=1}^{p} h_i(t, t, \ldots, t).$$

The density corresponding to $h_1(t, t, \ldots, t)$ is given by

$$f_1^*(t) = -\frac{\partial}{\partial t_1} S(t_1, t_2, \ldots, t_p)|_{t_1 = t_2 = \cdots t_p = t} / \pi_1.$$

Proceeding as before

$$-\frac{\partial}{\partial t_1} S(t_1, t_2, \ldots, t_p) = \phi(t_1) S_{k_{p-1}}(a_2(t_1), a_3(t_1), \ldots, a_p(t_1)),$$

where $S_{k_{p-1}}(a_2(t_1), a_3(t_1), \ldots, a_p(t_1))$ is the survival function of a standard $(p-1)$ dimensional normal variable with correlation matrix K_{p-1} at the point $(a_2(t_1), a_3(t_1), \ldots a_p(t_1))$, where $a_i(t_1) = \frac{t_i - \rho_{1i} t_1}{\sqrt{1 - \rho_{1i}^2}}$, $i = 2, 3, \ldots, p$. The correlation matrix K_{p-1} is given below:

$$K_{p-1} = D_{p-1} \left(\sum_{p-1, p-1} - \sum_{p-1, 1} \sum_{1,1}^{-1} \sum_{1, p-1} \right) D'_{p-1},$$

$$D_{p-1} = diag \left(\frac{1}{\sqrt{1 - \rho_{12}^2}}, \frac{1}{\sqrt{1 - \rho_{13}^2}}, \ldots, \frac{1}{\sqrt{1 - \rho_{1p}^2}} \right),$$

and

$$\sum_{p-1,p-1}, \sum_{p-1,1}, \sum_{1,1}, \sum_{1,p-1}$$

are the partitions of R defined by

$$\mathrm{R} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \cdot & \cdot & \cdot & \rho_{1p} \\ \rho_{12} & 1 & \rho_{23} & \cdot & \cdot & \cdot & \rho_{2p} \\ \rho_{13} & \rho_{23} & 1 & \cdot & \cdot & \cdot & \rho_{3p} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \rho_{1p} & \rho_{2p} & \rho_{3p} & \cdot & \cdot & \cdot & 1 \end{bmatrix} = \begin{bmatrix} \sum_{1,1} & \sum_{1,p-1} \\ \sum_{p-1,1} & \sum_{p-1,p-1} \end{bmatrix}$$

Now

$$f_1^*(t) = \phi(t) S_{k_{p-1}}(a_2(t), a_3(t), \dots a_{p-1}(t))/\pi_1$$

This gives

$$\eta_1^*(t) = -\frac{d}{dt} \ln f_1^*(t) = -\frac{d}{dt} \ln[\phi(t) S_{k_{p-1}}(b_2 t, b_3 t, \dots, b_{p-1} t)],$$

where

$$b_i = \sqrt{\frac{1-\rho_{1i}}{1+\rho_{1i}}}, \quad i = 2, 3, \dots p,$$

or

$$\eta_1^*(t) = t - \frac{d}{dt} \ln S_{k_{p-1}}(b_2 t, b_3 t, \dots, b_{p-1} t)$$

$$= t + \sum_{i=2}^{p} b_i g_{i-1}(b_2 t, b_3 t, \dots, b_{p-1} t),$$

where $g_1(b_2 t, b_3 t, \dots, b_{p-1} t)$ is the first component of the hazard gradient of a standard $(p-1)$ dimensional normal variable with correlation matrix K_{p-1}. Thus,

$$\eta_1^{*'}(t) = 1 + \sum_{i=2}^{p} b_i \frac{d}{dt} g_{i-1}(b_2 t, b_3 t, \dots, b_{p-1} t).$$

Using the induction argument,

$$\frac{d}{dt} g_{i-1}(b_2 t, b_3 t, \dots b_{p-1} t) > 0, \quad i = 2, 3, \dots, p.$$

Therefore, $\eta_1^{*'}(t) > 0$. Hence $\frac{d}{dt} h(t) > 0$. This completes the proof.

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