Metrika (2024) 87:133–153
https://doi.org/10.1007/s00184-023-00905-5

Comparison of extreme order statistics from two sets of heterogeneous dependent random variables under random shocks

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Received: 5 February 2022 / Accepted: 3 April 2023 / Published online: 26 April 2023

Abstract
In this paper, we consider two k-out-of-n systems comprising heterogeneous dependent components under random shocks, with an Archimedean copula. We then provide sufficient conditions on the distributions of components' lifetimes and the generator of the Archimedean copula and on the random shocks for comparing the lifetimes of two systems with respect to the usual stochastic order. Finally, we present some examples to illustrate the established results.

Keywords Usual stochastic order · Random shocks · Majorization · Archimedean survival copula

1 Introduction

In reliability theory, a system of n components is said to be a k-out-of-n system if it functions as long as at least k of the n components are functioning. The kth order statistic, $X_{k:n}$, arising from random variables $X_1, \ldots, X_n$ then corresponds to the lifetime of a $(n-k+1)$-out-of-n system. In particular, $X_{n:n}$ and $X_{1:n}$ represent the lifetimes of parallel and series systems, respectively. For comprehensive discussions on various properties and general results on stochastic comparisons of k-out-of-n system when the components are independent and identically (or non-identically) distributed, one may refer to Balakrishnan and Rao (1998a, b), Fang and Zhang (2010), Khaledi and Kochar (2000), David and Nagaraja (2003), Dykstra et al. (1997), Khaledi and Kochar (2006), Ding et al. (2013), Amini-Seresht et al. (2016), Balakrishnan and Zhao

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(2013), Esna-Ashari et al. (2022, 2023) and Kochar and Xu (2007a, b). In practical situations, however, the components may not be independent and there may be a structural dependence among components' lifetimes; for instance, different shocks or a common shock impinging on the components of the system, thus impacting the lifetime of the system. Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a dependent and non-identically distributed random vector with $X_i \sim F(\cdot; \lambda_i)$, where $F(\cdot; \lambda_i)$ denotes the distribution function of X_i and $\lambda_i > 0$ is the model parameter of X_i, for $i = 1, \ldots, n$. Let us denote the survival function and density function of X_i, for $i = 1, \ldots, n$, by $\bar{F}(\cdot; \lambda_i) = 1 - F(\cdot; \lambda_i)$ and $f(\cdot; \lambda_i)$, respectively. Furthermore, let $I_{p_1}, \ldots, I_{p_n}$ be independent Bernoulli random variables, independent by of X_i's, with $\mathbb{E}[I_{p_i}] = p_i$, $i = 1, \ldots, n$. Here, $I_{p_1}, \ldots, I_{p_n}$ are used to indicate whether the components of the system, with random lifetimes $X_1, \ldots, X_n$, have actually failed or not. Specifically, if $I_{p_i} = 1$, then the component i survives from random shocks; otherwise, $I_{p_i} = 0$ if component i fails due to the shocks, for $i = 1, \ldots, n$. We can then define $I_{p_1}X_1, \ldots, I_{p_n}X_n$ as the components' lifetimes subject to random shocks. Denote by $X_i^{p_i} := I_{p_i}X_i$, $i = 1, \ldots, n$. Thus, we have a product of two random variables here, one is X with distribution function $F(x)$ and another is an independent Bernoulli random variable with probability p_i. Then, the distribution of the product random variable, for any $i = 1, \ldots, n$, will be $\bar{F}_{X_i^{p_i}}(x) = p_i \bar{F}(x)$. Thus,

$$F_{X_i^{p_i}}(x) = \begin{cases} F(x) & \text{with probability } p_i \\ 1 & \text{with probability } 1 - p_i \end{cases}$$

By conditioning and unconditioning argument, we get $F_{X_i^{p_i}}(x) = 1 - p_i + p_i F(x)$. What this says is that when x goes to 0, we have a positive mass of $1 - p_i$, and consequently the rest of the distribution needs to have a total probability of p_i. The corresponding survival function of $X_i^{p_i}$ is $\bar{F}_{X_i^{p_i}}(x) = p_i \bar{F}(x)$. This also captures the practical scenario of claim distribution. If a claim is made, the distribution function for the amount of claim would be $F(x)$ and it occurs with probability p_i, but a claim need not be made at all and this occurs with probability $1 - p_i$.

A number of researchers have discussed stochastic comparisons of order statistics when the components of the system are dependent; one may refer to Li and Fang (2015), Mesfioui et al. (2017), Li et al. (2016), Fang et al. (2018), and Torrado and Navarro (2021). Recently, Zhang et al. (2018) considered stochastic comparisons of fail-safe systems under random shocks and provided sufficient conditions on components' lifetimes and their survival probabilities from random shocks for comparing the lifetimes of two fail-safe systems with respect to various stochastic orders. Balakrishnan et al. (2018), under this framework, studied the ordering property between $X_{n:n}^{\boldsymbol{p}}$ and $X_{n:n}^{\boldsymbol{q}}$, corresponding to the largest claim amounts arising from two sets of heterogeneous portfolios, in terms of the usual stochastic ordering, where $X_{n:n}^{\boldsymbol{p}}$ and $X_{n:n}^{\boldsymbol{q}}$ are the nth order statistics from the vectors $\boldsymbol{X}^{\boldsymbol{p}}$ and $\boldsymbol{X}^{\boldsymbol{q}}$, respectively.

Most of the existing results in the literature compare the ifetimes of coherent systems in which the components of the system are independent and at fixed time t whether they are up/down. Hence, it will naturally be of interest to consider the situation when the components in the system experience random shocks with probability p_i, $i = 1, \ldots, n$.

A natural question is then how the random shocks affect the lifetime of the system. We use majorization order to evaluate the influence of the random shocks on the systems' lifetime and then compare two systems with respect to the usual stochastic order. Here, in this work, we consider the lifetime of two systems with distributional parameters under random shocks, and then present sufficient conditions for comparing them with respect to the usual stochastic order.

Let $(X_1, \ldots, X_n)$ be a random vector of component lifetimes, having joint distribution $\boldsymbol{F}$, survival function $\bar{\boldsymbol{F}}$ and marginal distributions F_i, $i = 1, \ldots, n$. Then, the function $\mathbb{C} : [0,1]^n \to [0,1]$ is said to be the copula of $(X_1, \ldots, X_n)$ if

$$\boldsymbol{F}(x_1, \ldots, x_n) = \mathbb{C}(F_1(x_1), \ldots, F_n(x_n)), \quad \text{for all } (x_1, \ldots, x_n) \in \mathbb{R}^+.$$

If $\boldsymbol{F}$ continuous, then the copula $\mathbb{C}$ is unique and is given by

$$\mathbb{C}(u_1, \ldots, u_n) = \boldsymbol{F}(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)), \quad \text{for } u_1, \ldots, u_n \in (0,1),$$

where F_i^{-1} denotes the inverse of the distribution function of X_i. For an elaborate treatment on copulas and their applications, one may refer to Nelsen (2006).

Likewise, a survival copula associated with a multivariate distribution function $\boldsymbol{F}$ is given by

$$\bar{\boldsymbol{F}}(x_1, \ldots, x_n) = \bar{\mathbb{C}}(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n)), \quad \text{for all } (x_1, \ldots, x_n) \in \mathbb{R}^+,$$

where $\bar{\mathbb{C}} : [0,1]^n \to [0,1]$ is the survival copula with uniform marginals. Also, when the variables are continuous, $\bar{\mathbb{C}}$ is unique. One interesting class of copulas is the Archimedean copula class, and they have been used widely in reliability theory and actuarial science due to their mathematical tractability and their wide range of dependence. For a decreasing and continuous function $\phi : [0, \infty) \to [0,1]$ such that $\phi(0) = 1$ and $\phi(\infty) = 0$, with $\psi = \phi^{-1}$ being the pseudo-inverse, the copula is said to be Archimedean if it can be expressed as

$$\mathbb{C}(u_1, \ldots, u_n) = \phi(\psi(u_1) + \cdots + \psi(u_n)), \quad \text{for all} \quad u_i \in [0,1], i = 1, \ldots, n,$$

where ϕ is referred to as the generator of the Archimedean copula $\mathbb{C}$ if $(-1)^k \phi^{[k]}(x) \geq 0$, for $k = 0, \ldots, n-2$, and $(-1)^{n-2}\phi^{[n-2]}(x)$ is decreasing and convex. Here, $\phi^{[k]}(x)$ denotes the kth derivative of the function $\phi(x)$ with respect to x.

The rest of this paper is organized as follows. In Sect. 2, we introduce some notations, basic concepts, and some well-known results. In Sect. 3, we discuss stochastic comparisons of order statistics under Archimedean copulas with respect to the usual stochastic order. In Sect. 4, we present some numerical examples to illustrate the established results. Finally, in Sect. 5, we provide a summary of the work carried out here and make some concluding comments.

2 Preliminaries

Let us first recall some notions and useful lemmas that are key to the main results established in the sequel. Throughout the paper, the terms *increasing* and *decreasing* mean *non-decreasing* and *non-increasing*, respectively. Further, we use the notations $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = (0, \infty)$,

$$\begin{aligned} &\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n), \quad \boldsymbol{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*), \quad \boldsymbol{h}(\boldsymbol{u}) = (h(u_1), \ldots, h(u_n)), \\ &\mathfrak{D} = \{\boldsymbol{\lambda} : \lambda_1 \geq \cdots \geq \lambda_n\}, \quad \mathfrak{D}_+ = \{\boldsymbol{\lambda} : \lambda_1 \geq \cdots \geq \lambda_n > 0\}, \\ &\mathfrak{I} = \{\boldsymbol{\lambda} : \lambda_1 \leq \cdots \leq \lambda_n\}, \quad \mathfrak{I}_+ = \{\boldsymbol{\lambda} : 0 < \lambda_1 \leq \cdots \leq \lambda_n\}. \end{aligned}$$

Definition 2.1 For two random variables X and Y with distribution functions F and G and survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, respectively, X is said to be smaller than Y in the usual stochastic order (denoted by $X \leq_{\text{st}} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all $x \in \mathbb{R}_+$, or equivalently, $\mathbb{E}[\phi(X)] \leq [\geq]\mathbb{E}[\phi(Y)]$ for any increasing [decreasing] function $\phi : \mathbb{R} \to \mathbb{R}$.

For comprehensive discussions on stochastic orders and their applications, one may refer to Müller and Stoyan (2002) and Shaked and Shanthikumar (2007).

Majorization is a pre-ordering on vectors with all the components in the vectors being in an increasing order. Then, the concepts of majorization of vectors and Schur-concavity (Schur-convexity) of functions are as follows.

Definition 2.2 Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two n-dimensional real vectors and $x_{(1)} \leq \ldots \leq x_{(n)}$ and $y_{(1)} \leq \cdots \leq y_{(n)}$ be increasing arrangements of their components, respectively. Then:

- $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$, denoted by $\boldsymbol{x} \overset{m}{\preceq} \boldsymbol{y}$, if $\sum_{i=1}^{k} y_{(i)} \leq \sum_{i=1}^{k} x_{(i)}$, for $k = 1, \ldots, n-1$, and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$;
- $\boldsymbol{x}$ is said to be weakly supermajorized by $\boldsymbol{y}$, denoted by $\boldsymbol{x} \overset{w}{\preceq} \boldsymbol{y}$, if $\sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)}$, for $k = 1, \ldots, n$;
- $\boldsymbol{x}$ is said to be weakly submajorized by $\boldsymbol{y}$, denoted by $\boldsymbol{x} \preceq_w \boldsymbol{y}$, if $\sum_{i=k}^{n} x_{(i)} \leq \sum_{i=k}^{n} y_{(i)}$, for $k = 1, \ldots, n$.

It is known that $\boldsymbol{x} \overset{m}{\preceq} \boldsymbol{y}$ implies $\boldsymbol{x} \overset{w}{\preceq} (\preceq_w)\boldsymbol{y}$, while the reverse is not true in general. For more details on majorization, weak majorization and their applications, one may refer to Marshall et al. (2011),

Lemma 2.3 (Marshall et al. 2011, Theorem 5.A.2) *If an increasing function h is concave (convex), then $\boldsymbol{x} \overset{w}{\preceq} (\preceq_w)\boldsymbol{y}$ implies $\boldsymbol{h}(\boldsymbol{x}) \overset{w}{\preceq} (\preceq_w)\boldsymbol{h}(\boldsymbol{y})$.*

The following lemmas are useful for establishing the results in the subsequent sections.

Lemma 2.4 (Marshall et al. 2011, Theorem 3.A.4) *Let a permutation-symmetric function $\Phi : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. Then, necessary and sufficient* *conditions for* Φ *to be Schur-convex (concave) on* $\mathbb{R}^n$ *are* Φ *is symmetric on* $\mathbb{R}^n$*, and for all* $i \neq j$,

$$(z_i - z_j)\left(\frac{\partial \Phi(\boldsymbol{z})}{\partial z_i} - \frac{\partial \Phi(\boldsymbol{z})}{\partial z_j}\right) \geq (\leq)\ 0 \quad \textit{for all } \boldsymbol{z} \in \mathbb{R}^n.$$

Lemma 2.5 *For a real-valued function* $\Phi : \mathbb{C} \subseteq \mathbb{R}^n \to \mathbb{R}$, $\boldsymbol{x} \overset{m}{\preceq} \boldsymbol{y}$ *implies*

$$\Phi(\boldsymbol{x}) \leq (\geq)\ \Phi(\boldsymbol{y})$$

if and only if Φ *is Schur-convex (concave) on* $\mathbb{R}^n$.

Lemma 2.6 (Marshall et al. 2011, Theorem 3.A.8) *For a real-valued function* $\Phi : \mathbb{C} \subseteq \mathbb{R}^n \to \mathbb{R}$, $\boldsymbol{x} \overset{w}{\preceq} (\preceq_w) \boldsymbol{y}$ *implies*

$$\Phi(\boldsymbol{x}) \leq \Phi(\boldsymbol{y})$$

if and only if Φ *is decreasing (increasing) and Schur-convex on* $\mathbb{R}^n$.

3 Main results

We consider here the usual stochastic order between parallel systems and obtain some conditions on $F(\cdot; \lambda_i)$, $i = 1, \ldots, n$, and the generator function of the Archimedean copula. The established result shows that more heterogeneity among the random shocks in terms of weakly supmajorization order results in larger lifetimes of parallel systems.

Theorem 3.1 *Let* $\boldsymbol{X^p} = (I_{p_1}X_1, \ldots, I_{p_n}X_n)$ *and* $\boldsymbol{X^q} = (I_{q_1}X_1, \ldots, I_{q_n}X_n)$ *be dependent and non-identically distributed random vectors with* X_i*'s having the Archimedean copula with generator* ϕ *and* $I_{p_1}, \ldots, I_{p_n}$ *and* $I_{q_1}, \ldots, I_{q_n}$ *being independent Bernoulli random variables with parameters* $\boldsymbol{p} = (p_1, \ldots, p_n)$ *and* $\boldsymbol{q} = (q_1, \ldots, q_n)$*, respectively, that are independent of* X_i*'s. Let*

(i) $u\psi'(1-u)$ *be increasing in* $u \in (0, 1)$,
(ii) $\bar{F}(\cdot; \lambda)$ *be increasing in* λ,
(iii) $h(u)$ *and* $uh'(u)$ *be increasing functions in* $u \in [0, 1]$.

Then, for $\boldsymbol{h}(\boldsymbol{p}) \in \mathfrak{D}^+$ *and* $\boldsymbol{\lambda} \in \mathfrak{D}^+$*, we have*

$$\boldsymbol{h}(\boldsymbol{p}) \overset{m}{\preceq} \boldsymbol{h}(\boldsymbol{q}) \Rightarrow X^{\boldsymbol{p}}_{n:n} \leq_{st} X^{\boldsymbol{q}}_{n:n},$$

where $X^{\boldsymbol{p}}_{n:n}$ *and* $X^{\boldsymbol{q}}_{n:n}$ *are the nth order statistics from the vectors* $\boldsymbol{X^p}$ *and* $\boldsymbol{X^q}$*, respectively.*

Proof The survival function of $X^{\boldsymbol{p}}_{n:n}$ is

$$\bar{F}_{X^{\boldsymbol{p}}_{n:n}}(x) = 1 - \phi\left(\sum_{j=1}^{n} \psi(1 - p_j \bar{F}(x; \lambda_j))\right).$$

Let us set $h(p_i) = u_i$, for $u_i \in [0, 1]$, so that $p_i = h^{-1}(u_i)$ for $u_i \in [0, 1]$. Then, the survival function $\bar{F}_{X^{\boldsymbol{p}}_{n:n}}(x)$ can be rewritten as

$$\bar{F}_{X^{\boldsymbol{p}}_{n:n}}(x) = 1 - \phi\left(\sum_{j=1}^{n} \psi(1 - h^{-1}(u_j)\bar{F}(x; \lambda_j))\right).$$

Now, taking derivative of $\bar{F}_{X^{\boldsymbol{p}}_{n:n}}$ with respect to u_i, we obtain

$$\begin{aligned}\frac{\partial \bar{F}_{X^{\boldsymbol{p}}_{n:n}}(x; \boldsymbol{\lambda})}{\partial u_i} &= \bar{F}(x; \lambda_i)\frac{\partial h^{-1}(u_i)}{\partial u_i}\psi'(1 - h^{-1}(u_i)\bar{F}(x, \lambda_i))\phi' \\ &\times \left(\sum_{j=1}^{n} \psi(1 - h^{-1}(u_j)\bar{F}(x; \lambda_j))\right).\end{aligned}$$

Thus, according to Lemma 2.5, it is sufficient to show that $\bar{F}_{X^{\boldsymbol{p}}_{n:n}}$ is schur-convex to obtain the desired result. So, as ϕ' is non-positive function, it is enough to show that, for $1 < i \leq j \leq n$,

$$\begin{aligned}\Delta_1 :&= \bar{F}(x; \lambda_j)\frac{\partial h^{-1}(u_j)}{\partial u_j}.\psi'(1 - h^{-1}(u_j)\bar{F}(x; \lambda_j)) \\ &- \bar{F}(x; \lambda_i)\frac{\partial h^{-1}(u_i)}{\partial u_i}.\psi'(1 - h^{-1}(u_i)\bar{F}(x, \lambda_i)) \\ &= \frac{1}{h^{-1}(u_j)h'(h^{-1}(u_j))}h^{-1}(u_j)\bar{F}(x; \lambda_j)\psi'(1 - h^{-1}(u_j)\bar{F}(x; \lambda_j)) \\ &- \frac{1}{h^{-1}(u_i)h'(h^{-1}(u_i))}h^{-1}(u_i)\bar{F}(x; \lambda_i)\psi'(1 - h^{-1}(u_i)\bar{F}(x; \lambda_i))\end{aligned}$$

is non-positive. From Assumptions (i) and (ii), we see that

$$\begin{aligned}\Delta_1 \leq &\left(\frac{1}{h^{-1}(u_j)h'(h^{-1}(u_j))} - \frac{1}{h^{-1}(u_i)h'(h^{-1}(u_i))}\right) \\ &h^{-1}(u_i)\bar{F}(x; \lambda_i)\psi'(1 - h^{-1}(u_i)\bar{F}(x; \lambda_i)) \leq 0,\end{aligned}$$

where the second inequality follows from Assumption (iii) and the fact that ψ' is a non-positive function. Hence, from Lemma 2.4, the required result follows. □

Remark 3.2 It is important to observe that in reliability and actuarial applications, the parameters $p_1, \ldots, p_n$ (for example) involved in the model would correspond specifically to the probabilities of the n component in the system surviving the random shocks experienced and the probabilities of the n customers in the portfolio making the insurance claims, respectively. Hence, for the purpose of reducing the dimensionality of this parameter vector $\boldsymbol{p}$, it would be convenient to use a link function while modelling; for example, $h(p) = p^\theta$ for $\theta \in \mathbb{R}^+$ and $h(p) = \frac{1-e^{-\theta p}}{1-e^{-\theta}}$ for $\theta \in (0, 1]$ could be two such choices. It is easy to verify that these link functions do satisfy condition (iii) in Theorem 3.1 as $h(u)$ and $uh'(u)$ are both increasing functions in $u \in [0, 1]$. In fact, we could more generally consider any distribution function in [0, 1] and suitably scale in to make such choices for link functions.

In the especial case when $\phi(x) = e^{-x}$, it is well known that the Archimedean copula reduces to the independence copula; in this case, Balakrishnan et al. (2018) considered two sets of heterogeneous independent samples and obtained some conditions involving the survival function of X_i^p and the function h and the weakly supermajorization order between $\boldsymbol{h}(\boldsymbol{p})$ and $\boldsymbol{h}(\boldsymbol{q})$ to establish the usual stochastic ordering between $X_{n:n}^{\boldsymbol{p}}$ and $X_{n:n}^{\boldsymbol{q}}$. Moreover, for the case of common $\boldsymbol{p}$, Balakrishnan et al. (2018) also obtained some sufficient conditions for comparing $X_{n:n}^{\boldsymbol{p}}$ and $X_{n:n}^{\boldsymbol{q}}$ based on the usual stochastic ordering; but, as will be seen later, their results are different from the ones estabilished here, because the conditions assumed here are different from these assumed in Theorems 3.1 and 3.2 of Balakrishnan et al. (2018).

Proposition 3.3 *Let $\boldsymbol{X^p} = (I_{p_1}X_1, \ldots, I_{p_n}X_n)$ and $\boldsymbol{X^q} = (I_{q_1}X_1, \ldots, I_{q_n}X_n)$ be dependent and non-identically distributed random vectors with X_i's having an Archimedean copula with generator ϕ and $I_{p_1}, \ldots, I_{p_n}$ and $I_{q_1}, \ldots, I_{q_n}$ being independent Bernoulli random variables with parameters, $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$, respectively, that are independent of X_i's. Let*

(i) *ϕ be a log-concave function,*
(ii) *$\bar{F}(\cdot; \lambda)$ be increasing in λ,*
(iii) *$h(u)$ and $uh'(u)$ be increasing functions in $u \in [0, 1]$.*

Then, for $\boldsymbol{h}(\boldsymbol{p}) \in \mathfrak{I}^+$ and $\boldsymbol{\lambda} \in \mathfrak{I}^+$, we have

$$\boldsymbol{h}(\boldsymbol{p}) \preceq_w \boldsymbol{h}(\boldsymbol{q}) \Rightarrow X_{n:n}^{\boldsymbol{p}} \geq_{st} X_{n:n}^{\boldsymbol{q}},$$

where $X_{n:n}^{\boldsymbol{p}}$ and $X_{n:n}^{\boldsymbol{q}}$ are the nth order statistics from $\boldsymbol{X^p}$ and $\boldsymbol{X^q}$, respectively.

Proof As in the proof of Theorem 3.1, the partial derivative of $\bar{F}_{X_{n:n}^{\boldsymbol{p}}}(x)$ with respect to u_j is given by

$$\frac{\partial \bar{F}_{X_{n:n}^{\boldsymbol{p}}}(x; \boldsymbol{\lambda})}{\partial u_j} = \bar{F}(x; \lambda_j)\frac{\partial h^{-1}(u_j)}{\partial u_j}\psi'(1 - h^{-1}(u_j)\bar{F}(x, \lambda_j))\phi' \left(\sum_{j=1}^{n} \psi(1 - h^{-1}(u_j)\bar{F}(x; \lambda_j))\right)$$

$$= \frac{1}{h^{-1}(u_j)h'(h^{-1}(u_j))} \frac{h^{-1}(u_j)\bar{F}(x;\lambda_j)}{1-h^{-1}(u_j)\bar{F}(x;\lambda_j)} \frac{\phi\left(\psi(1-h^{-1}(u_j)\bar{F}(x;\lambda_j))\right)}{\phi'\left(\psi(1-h^{-1}(u_j)\bar{F}(x;\lambda_j))\right)}$$
$$\times \phi'\left(\sum_{j=1}^{n}\psi(1-h^{-1}(u_j)\bar{F}(x;\lambda_j))\right).$$

From the fact that ϕ' is a non-positive function and from the assumption that $h(p)$ is increasing function in p, it follows that $\bar{F}_{X^{\boldsymbol{p}}_{n:n}}(x;\boldsymbol{\lambda})$ is an increasing function. To prove its Schur-concavity, from Lemma 2.6 and the fact that ϕ' is a non-positive function, it is sufficient to show that, for all $i \leq j \in \mathfrak{I}^+$,

$$\frac{\partial \bar{F}_{X^{\boldsymbol{p}}_{n:n}}(x;\boldsymbol{\lambda})}{\partial u_j} \overset{sgn}{=} \frac{1}{h^{-1}(u_j)h'(h^{-1}(u_j))} \frac{h^{-1}(u_j)\bar{F}(x;\lambda_j)}{1-h^{-1}(u_j)\bar{F}(x;\lambda_j)} \frac{\phi\left(\psi(1-h^{-1}(u_j)\bar{F}(x;\lambda_j))\right)}{\phi'\left(\psi(1-h^{-1}(u_j)\bar{F}(x;\lambda_j))\right)}, \tag{1}$$

is an increasing function in u_j. Note that from Assumption (ii), it follows that the first part of (1) is decreasing and from the fact that $\frac{\bar{F}}{F}$ is a decreasing function, it follows that $\frac{\bar{F}_{X_i^{p_i}}(x)}{F_{X_i^{p_i}}(x)} = \frac{h^{-1}(u_j)\bar{F}(x;\lambda_j)}{1-h^{-1}(u_j)\bar{F}(x;\lambda_j)}$ is a decreasing function. Assumption (i) implies the increasing property of $\frac{\phi(x)}{\phi'(x)}$ and on the other hand, from Assumptions (ii) and (iii) and the fact that ψ is a decreasing function, we have

$$\frac{\phi\left(\psi(1-h^{-1}(u_j)\bar{F}(x;\lambda_j))\right)}{\phi'\left(\psi(1-h^{-1}(u_j)\bar{F}(x;\lambda_j))\right)}$$

to be increasing and a non-positive function with respect to u_j. Hence, the decreasing property of the first and second parts of (1) follow, implying that $\frac{\partial \bar{F}_{X^{\boldsymbol{p}}_{n:n}}(x;\boldsymbol{\lambda})}{\partial u_j}$ is increasing in u_j, as required. □

It should be mentioned that the condition "ϕ be a log-concave function" in Proposition 3.3 holds for many Archimedean copulas. For example, the Gumbel-Hougaard copula with generator

$$\psi(x) = e^{\frac{1}{\theta}(1-e^x)}, \quad 0 < \theta < 1,$$

possesses this property. As the majorization order implies both weakly supermajorization and submajorization orders, we may wonder whether there is a relationship between Condition (i) in Proposition 3.3 and Condition (i) in Theorem 3.1. In Condition (i) in Proposition 3.3, we have $\frac{\phi(x)}{\phi'(x)}$ to be increasing in x; on the other hand, from Condition (i) in Theorem 3.1, we see that

$$x\psi'(1-x) \overset{sgn}{=} \frac{x}{1-x}\frac{\phi(\psi(1-x))}{\phi'(\psi(1-x))}. \tag{2}$$

We that the first part of (2) is nonnegative and increasing in x while the second part is non-positive and increasing function in x. Consequently, the product of the first and second parts of (2) may not be increasing, and thus there exists a difference between the two results.

In the following theorem, we consider two parallel systems under the same random shocks, but with two different lifetime distributions, and then discuss the usual stochastic order between the systems.

Theorem 3.4 *Let $\boldsymbol{X}^{\boldsymbol{p}} = (I_{p_1}X_1, \ldots, I_{p_n}X_n)$ and $\boldsymbol{Y}^{\boldsymbol{p}} = (I_{p_1}Y_1, \ldots, I_{p_n}Y_n)$ be dependent and non-identically distributed random vectors with X_i's and Y_i's having the same Archimedean copula with generator ϕ and distribution parameters λ_i's and λ_i^*'s, respectively, and $I_{p_1}, \ldots, I_{p_n}$ being independent Bernoulli random variables with parameters $\boldsymbol{p} = (p_1, \ldots, p_n)$, that are independent of X_i's and Y_i's. Let*

(i) *$u\psi'(1-u)$ be increasing in $u \in (0, 1)$,*
(ii) *$\bar{F}(\cdot; \lambda)$ be increasing and log-concave in λ.*

Then, for $\boldsymbol{\lambda} \in \mathfrak{D}^+$, we have

$$\boldsymbol{\lambda} \overset{m}{\preceq} \boldsymbol{\lambda}^* \Rightarrow Y^{\boldsymbol{p}}_{n:n} \le_{st} X^{\boldsymbol{p}}_{n:n},$$

where $X^{\boldsymbol{p}}_{n:n}$ and $Y^{\boldsymbol{q}}_{n:n}$ are the nth order statistics from $\boldsymbol{X}^{\boldsymbol{p}}$ and $\boldsymbol{Y}^{\boldsymbol{p}}$, respectively.

Proof As in the proof of Theorem 3.1, we have

$$\frac{\partial \bar{F}_{X^{\boldsymbol{p}}_{n:n}}(x; \boldsymbol{\lambda})}{\partial \lambda_i} = p_i \frac{\partial \bar{F}(x; \lambda_i)}{\partial \lambda_i} \psi'(1 - p_i \bar{F}(x, \lambda_i))\phi' \left(\sum_{j=1}^{n} \psi(1 - p_j \bar{F}(x; \lambda_j)) \right). \tag{3}$$

So, we find that, for $1 \le i \le j \le n$,

$$\begin{aligned}
&\frac{\partial \bar{F}_{X^{\boldsymbol{p}}_{n:n}}(x; \boldsymbol{\lambda})}{\partial \lambda_i} - \frac{\partial \bar{F}_{X^{\boldsymbol{p}}_{n:n}}(x; \boldsymbol{\lambda})}{\partial \lambda_j} \\
&= \left[p_i \frac{\partial \bar{F}(x; \lambda_i)}{\partial \lambda_i} \psi'(1 - p_i \bar{F}(x, \lambda_i)) - p_j \frac{\partial \bar{F}(x; \lambda_j)}{\partial \lambda_j} \psi'(1 - p_j \bar{F}(x, \lambda_j)) \right] \\
&\quad \times \phi' \left(\sum_{j=1}^{n} \psi(1 - p_j \bar{F}(x; \lambda_j)) \right) \\
&=: \Delta_2 \phi' \left(\sum_{j=1}^{n} \psi(1 - p_j \bar{F}(x; \lambda_j)) \right),
\end{aligned}$$

where

$$\Delta_2 = p_i \frac{\partial \bar{F}(x; \lambda_i)}{\partial \lambda_i} \psi'(1 - p_i \bar{F}(x, \lambda_i)) - p_j \frac{\partial \bar{F}(x; \lambda_j)}{\partial \lambda_j} \psi'(1 - p_j \bar{F}(x, \lambda_j)).$$

Thus according to Lemma 2.5, it is sufficient to show that Δ_2 is non-negative. For this, we see that

$$\Delta_2 = \frac{\frac{\partial \bar{F}(x;\lambda_i)}{\partial \lambda_i}}{\bar{F}(x,\lambda_i)} p_i \bar{F}(x,\lambda_i)\psi'(1-p_i\bar{F}(x,\lambda_i)) - \frac{\frac{\partial \bar{F}(x;\lambda_j)}{\partial \lambda_j}}{\bar{F}(x,\lambda_j)} p_j \bar{F}(x,\lambda_j)\psi'(1-p_j\bar{F}(x,\lambda_j))$$
$$\geq \left[\frac{\frac{\partial \bar{F}(x;\lambda_i)}{\partial \lambda_i}}{\bar{F}(x,\lambda_i)} - \frac{\frac{\partial \bar{F}(x;\lambda_j)}{\partial \lambda_j}}{\bar{F}(x,\lambda_j)}\right] p_j \bar{F}(x,\lambda_j)\psi'(1-p_j\bar{F}(x,\lambda_j)) \geq 0.$$

In the above, the first inequality follows from Assumption (i) while the second inequality follows from Assumption (ii) and the fact that ψ' is a non-positive function. We thus find that $\bar{F}_{X^{\boldsymbol{p}}_{n:n}}(x)$ is schur-concave. Hence, the theorem. □

It should be mentioned here that, in Theorem 3.2 of Balakrishnan et al. (2018) some sufficient conditions have been provided in the sense of the usual stochastic ordering for comparing $X^{\boldsymbol{p}}_{n:n}$, and $Y^{\boldsymbol{p}}_{n:n}$ with independent non-negative random variables. They especially showed that if $\bar{F}(\cdot;\lambda)$ is decreasing and convex and $h:[0,1]\to\mathbb{R}_{++}$ is a differentiable and decreasing function, then $\boldsymbol{\lambda} \overset{w}{\preceq} \boldsymbol{\lambda}^* \Rightarrow X^{\boldsymbol{p}}_{n:n} \leq_{st} Y^{\boldsymbol{p}}_{n:n}$. Now, since majorization order implies weakly supermajorization order, we can ask a question whether the conditions in Theorem 3.4 and the conditions in Theorem 3.2 of Balakrishnan et al. (2018) could overlap. However, in Theorem 3.4, contrary to Theorem 3.2 of Balakrishnan et al. (2018), there is no restriction on the function h and moreover the conditions on $\bar{F}(\cdot;\lambda)$ are also different in both theorems. Balakrishnan et al. (2018) in Theorem 3.1 have been provide sufficient conditions for the usual stochastic ordering to hold between $X^{\boldsymbol{p}}_{n:n}$, and $Y^{\boldsymbol{q}}_{n:n}$ with independent non-negative random variables. They showed that if $h(u)$ is a differentiable and strictly decreasing convex function and $\bar{F}(\cdot;\lambda)$ is decreasing in λ, then $\boldsymbol{p} \overset{w}{\preceq} \boldsymbol{q} \Rightarrow X^{\boldsymbol{p}}_{n:n} \leq_{st} Y^{\boldsymbol{q}}_{n:n}$. Now, using the same argument in the above and comparing Theorem 3.1 of Balakrishnan et al. (2018) and Theorem 3.1 here, we see that the conditions on $h(u)$ and $\bar{F}(\cdot;\lambda)$ are different in both theorems, therefore, the different conditions are required to derive the comparisons results in both theorems.

Theorem 3.5 *Let $\boldsymbol{X^p} = (I_{p_1}X_1,\ldots,I_{p_n}X_n)$ and $\boldsymbol{X^q} = (I_{q_1}X_1,\ldots,I_{q_n}X_n)$ be dependent and non-identically distributed random vectors with X_i's having the Archimedean copula with generator ϕ and $I_{p_1},\ldots,I_{p_n}$ and $I_{q_1},\ldots,I_{q_n}$ being independent Bernoulli random variables with parameters $\boldsymbol{p}=(p_1,\ldots,p_n)$ and $\boldsymbol{q} = (q_1,\ldots,q_n)$, respectively, that are independent of X_i's. Let*

(i) *$u\psi'(1-u)$ be increasing in $u\in(0,1)$,*
(ii) *$\bar{F}(\cdot;\lambda)$ be decreasing in λ,*
(iii) *$h(u)$ and $uh'(u)$ be decreasing functions in $u\in[0,1]$.*

Then, for $\boldsymbol{h}(\boldsymbol{u}) \in \mathfrak{D}^+$ and $\boldsymbol{\lambda} \in \mathfrak{D}^+$, we have

$$\boldsymbol{h}(\boldsymbol{p}) \overset{w}{\preceq} \boldsymbol{h}(\boldsymbol{q}) \Rightarrow X^{\boldsymbol{p}}_{n-1:n} \leq_{st} X^{\boldsymbol{q}}_{n-1:n},$$

where $X^{\boldsymbol{p}}_{n-1:n}$ and $X^{\boldsymbol{q}}_{n-1:n}$ are the $(n-1)$th order statistics from $\boldsymbol{X}^{\boldsymbol{p}}$ and $\boldsymbol{X}^{\boldsymbol{q}}$, respectively.

Proof The distribution function of $X^{\boldsymbol{p}}_{n-1:n}$ arising from $\boldsymbol{X}^{\boldsymbol{p}}$ can be expressed as

$$\begin{aligned}
F_{X^{\boldsymbol{p}}_{n-1:n}}(x;\boldsymbol{\lambda}) &= \sum_{i=1}^{n} P(\text{all } X_j \leq x,\ j \neq i) - (n-1)P(X_1 \leq x, \ldots, X_n \leq x) \\
&= \sum_{i=1}^{n} \phi\left(\sum_{j=1}^{n} \psi(1 - p_j \bar{F}(x;\lambda_j)) - \psi(1 - p_i \bar{F}(x;\lambda_i))\right) \\
&\quad -(n-1)\phi\left(\sum_{j=1}^{n} \psi(1 - p_j \bar{F}(x;\lambda_j))\right) \\
&= \sum_{i=1}^{n} \phi\left(\sum_{j=1}^{n} \psi(1 - h^{-1}(u_j)\bar{F}(x;\lambda_j)) - \psi(1 - h^{-1}(u_i)\bar{F}(x;\lambda_i))\right) \\
&\quad -(n-1)\phi\left(\sum_{j=1}^{n} \psi(1 - h^{-1}(u_j)\bar{F}(x;\lambda_j))\right),
\end{aligned}$$

where $h(p_i) = u_i$, $i = 1, \ldots, n$, and h^{-1} is the inverse function of h. Now, taking derivative of $F_{X^{\boldsymbol{p}}_{n-1:n}}$ with respect to u_1, we get

$$\begin{aligned}
\frac{\partial F_{X^{\boldsymbol{p}}_{n-1:n}}(x;\boldsymbol{\lambda})}{\partial u_1} &= \bar{F}(x;\lambda_i)\frac{\partial h^{-1}(u_1)}{\partial u_1}\psi'(1 - h^{-1}(u_1)\bar{F}(x,\lambda_1)) \\
&\quad \left(\sum_{k=2}^{n} \phi'\left(A_k(x)\right) - (n-1)\phi'\left(B(x)\right)\right),
\end{aligned}$$

where, for all $k = 1, \ldots, n$,

$$A_k(x) := \sum_{j=1}^{n} \psi(1 - h^{-1}(u_j)\bar{F}(x;\lambda_j)) - \psi(1 - h^{-1}(u_k)\bar{F}(x;\lambda_k))$$

and

$$B(x) = \sum_{j=1}^{n} \psi(1 - h^{-1}(u_j)\bar{F}(x;\lambda_j)).$$

Using the fact that $0 \le A_k(x) \le B(x)$, $k = 1, \dots, n$, we get $\phi'(A_k(x)) \le \phi'(B(x))$. Now, by using the fact that ψ' is a non-positive function and also Assumption (iii), we observe that $\frac{\partial F_{X^{\boldsymbol{p}}_{n-1:n}}(x;\boldsymbol{\lambda})}{\partial u_i} \ge 0$, which in turn implies that $\bar{F}_{X^{\boldsymbol{p}}_{n-1:n}}(x;\boldsymbol{\lambda})$ is a decreasing function in u_i, $i = 1, \dots, n$. So, according to Lemma 2.6, it is enough to show that the function $F_{X^{\boldsymbol{p}}_{n-1:n}}(x)$ is Schur-concave, for obtaining the desired result. For this purpose, for $1 \le i < j \le n$ and $\boldsymbol{u}, \boldsymbol{\lambda} \in \mathfrak{D}_+$, we show that

$$\frac{\partial F_{X^{\boldsymbol{p}}_{n-1:n}}(x;\boldsymbol{\lambda})}{\partial u_i}$$

is decreasing in u_i. We first note that

$$\begin{aligned}
&\left(\sum_{k=2}^{n} \phi'(A_k(x)) - (n-1)\phi'(B(x))\right) - \left(\sum_{k\neq 2}^{n} \phi'(A_k(x)) - (n-1)\phi'(B(x))\right)\\
&\quad = \phi'(A_2(x)) - \phi'(A_1(x))\\
&\quad = \psi(1 - h^{-1}(u_1)\bar{F}(x;\lambda_1)) - \psi(1 - h^{-1}(u_2)\bar{F}(x;\lambda_2))\\
&\quad \le 0, \qquad (4)
\end{aligned}$$

where the inequality in (4) follows from the assumptions that $h(p)$ is decreasing and $\bar{F}(\cdot;\lambda)(x)$ is decreasing in λ. So, we first have

$$\begin{aligned}
\frac{\partial F_{X^{\boldsymbol{p}}_{n-1:n}}(x;\boldsymbol{\lambda})}{\partial u_1} - \frac{\partial F_{X^{\boldsymbol{p}}_{n-1:n}}(x;\boldsymbol{\lambda})}{\partial u_2} &= \bar{F}(x;\lambda_1)\frac{\partial h^{-1}(u_1)}{\partial u_1}\psi'(1 - h^{-1}(u_1)\bar{F}(x,\lambda_1))\\
&\quad \times \left(\sum_{k=2}^{n} \phi'(A_k(x)) - (n-1)\phi'(B(x))\right)\\
&\quad - \bar{F}(x;\lambda_2)\frac{\partial h^{-1}(u_2)}{\partial u_2}\psi'(1 - h^{-1}(u_2)\bar{F}(x,\lambda_2))\\
&\quad \times \left(\sum_{k\neq 1}^{n} \phi'(A_k(x)) - (n-1)\phi'(B(x))\right)\\
&\le \Bigg[\bar{F}(x;\lambda_1)\frac{\partial h^{-1}(u_1)}{\partial u_1}\psi'(1 - h^{-1}(u_1)\bar{F}(x,\lambda_1))\\
&\quad - \bar{F}(x;\lambda_2)\frac{\partial h^{-1}(u_2)}{\partial u_2}\psi'(1 - h^{-1}(u_2)\bar{F}(x,\lambda_2))\Bigg]\\
&\quad \times \left(\sum_{k=2}^{n} \phi'(A_k(x)) - (n-1)\phi'(B(x))\right)\\
&= h_{1,2}(x)\left(\sum_{k=2}^{n} \phi'(A_k(x)) - (n-1)\phi'(B(x))\right), \qquad (5)
\end{aligned}$$

where the inequality follows from (4) and

$$h_{1,2}(x) = \bar{F}(x;\lambda_1)\frac{\partial h^{-1}(u_1)}{\partial u_1}\psi'(1-h^{-1}(u_1)\bar{F}(x,\lambda_1)) \\ -\bar{F}(x;\lambda_2)\frac{\partial h^{-1}(u_2)}{\partial u_2}\psi'(1-h^{-1}(u_2)\bar{F}(x,\lambda_2)).$$

Note that, due to the fact that $\phi'(A_i(x)) \leq \phi'(B(x))$, the sign of (5) is equivalent to the sign of $h_{i,j}(x)$. Now, to obtain the desired result, it is enough to show that $h_{i,j}(x)$ is non-negative. We have

$$\begin{aligned} h_{1,2}(x) &= \frac{\frac{\partial h^{-1}(u_1)}{\partial u_1}}{h^{-1}(u_1)}h^{-1}(u_1)\bar{F}(x;\lambda_i)\psi'(1-h^{-1}(u_1)\bar{F}(x,\lambda_1)) \\ &\quad -\frac{\frac{\partial h^{-1}(u_2)}{\partial u_2}}{h^{-1}(u_2)}h^{-1}(u_2)\bar{F}(x;\lambda_2)\psi'(1-h^{-1}(u_2)\bar{F}(x,\lambda_2)) \\ &= \frac{1}{h^{-1}(u_1)h'\left(h^{-1}(u_1)\right)}h^{-1}(u_1)\bar{F}(x;\lambda_1)\psi'(1-h^{-1}(u_1)\bar{F}(x,\lambda_1)) \\ &\quad -\frac{1}{h^{-1}(u_2)h'\left(h^{-1}(u_2)\right)}h^{-1}(u_2)\bar{F}(x;\lambda_2)\psi'(1-h^{-1}(u_2)\bar{F}(x,\lambda_2)), \end{aligned}$$

from which, we readily see that

$$\frac{1}{h^{-1}(u_1)h'\left(h^{-1}(u_1)\right)} \leq \frac{1}{h^{-1}(u_2)h'\left(h^{-1}(u_2)\right)} \leq 0. \tag{6}$$

Using the fact that ψ' is negative and also Assumptions (i) and (ii), we get

$$\begin{aligned} &h^{-1}(u_1)\bar{F}(x;\lambda_1)\psi'(1-h^{-1}(u_1)\bar{F}(x,\lambda_1)) \\ &\quad\leq h^{-1}(u_2)\bar{F}(x;\lambda_2)\psi'(1-h^{-1}(u_2)\bar{F}(x,\lambda_2)) \leq 0. \end{aligned} \tag{7}$$

Upon combining (6) and (7), we see that $h_{1,2}(x)$ is non-negative. From these observations, according to Lemma 2.4, it follows that $F_{X^{\boldsymbol{p}}_{n-1:n}}(x)$ is Schur-concave. This, in turn, guarantees that $\bar{F}_{X^{\boldsymbol{p}}_{n-1:n}}(x)$ is Schur-convex. Then, the desired result follows by Lemma 2.6. □

Remark 3.6 As mentioned earlier following Theorem 3.1, it is necessary to use some link function for the model parameter $\boldsymbol{p}$. Hence, as we require $h(u)$ to be a decreasing function in u, it would be convenient to consider any survival function in [0, 1] and suitably scale it to make some choices for link functions. For example, using an exponential survival function, we could come up with $h(p) = \frac{e^{-\theta p}}{1-e^{-\theta}}$ for $\theta \in \mathbb{R}^+$ as such a choice for link function

The following theorem is concerning series systems and it provides sufficient conditions for comparing the lifetimes of to series systems with respect to the usual stochastic order.

Theorem 3.7 *Let $\boldsymbol{X^p} = (I_{p_1}X_1, \ldots, I_{p_n}X_n)$ and $\boldsymbol{X^q} = (I_{q_1}X_1, \ldots, I_{q_n}X_n)$ be dependent and non-identically distributed random vectors with X_i's having the Archimedean copula with generator ϕ, and $I_{p_1}, \ldots, I_{p_n}$ and $I_{q_1}, \ldots, I_{q_n}$ being independent Bernoulli random variables with parameters $\boldsymbol{p} = (p_1, \ldots, p_n)$ and $\boldsymbol{q} = (q_1, \ldots, q_n)$, respectively, that are independent of X_i's. Let*

(i) *$u\psi'(u)$ be increasing in $u \in (0, 1)$,*
(ii) *$\bar{F}(\cdot; \lambda)$ be decreasing in λ,*
(iii) *$h(u)$ be a decreasing function and $uh'(u)$ be a decreasing function in $u \in [0, 1]$.*

Then, for $\boldsymbol{h}(\boldsymbol{u}) \in \mathfrak{D}^+$ and $\boldsymbol{\lambda} \in \mathfrak{D}^+$, we have

$$\boldsymbol{h}(\boldsymbol{p}) \overset{w}{\preceq} \boldsymbol{h}(\boldsymbol{q}) \Rightarrow X^{\boldsymbol{p}}_{1:n} \leq_{st} X^{\boldsymbol{q}}_{1:n},$$

where $X^{\boldsymbol{p}}_{1:n}$ and $X^{\boldsymbol{q}}_{1:n}$ are the smallest order statistics from $\boldsymbol{X^p}$ and $\boldsymbol{X^q}$, respectively.

Proof The distribution function of $X^{\boldsymbol{p}}_{1:n}$ arising from $\boldsymbol{X^p}$ can be expressed as

$$F_{X^{\boldsymbol{p}}_{1:n}}(x; \boldsymbol{\lambda}) = 1 - \phi\left(\sum_{i=1}^{n} \psi\left(p_i \bar{F}(x; \lambda_i)\right)\right).$$

Let us set $u_i = h(p_i)$, for $i = 1, \ldots, n$. Then, for establishing the desired result, according to Lemma 2.6, it is enough to show that the function $F_{X^{\boldsymbol{p}}_{1:n}}(x; \boldsymbol{\lambda})$ is increasing and Schur-concave with respect to u_i. For this purpose, by taking derivative with respect to u_i, we get

$$\begin{aligned}\frac{\partial F_{X^{\boldsymbol{p}}_{1:n}}(x; \boldsymbol{\lambda})}{\partial u_i} &= -\bar{F}(x; \lambda_i)\frac{\partial h^{-1}(u_i)}{\partial u_i}\psi'(h^{-1}(u_i)\bar{F}(x, \lambda_i))\phi' \\ &\quad \times \left(\sum_{i=1}^{n} \psi\left(h^{-1}(u_i)\bar{F}(x; \lambda_i)\right)\right),\end{aligned}$$

which is observed to be a non-negative function. This, in turn, implies that $F_{X^{\boldsymbol{p}}_{1:n}}(x; \boldsymbol{\lambda})$ is a increasing function with respect to u_i. As in the proof of Theorem 3.5, we obtain, for $1 \leq i < j \leq n$ and $\boldsymbol{u}, \boldsymbol{\lambda} \in \mathfrak{D}_+$, that

$$\frac{\partial F_{X^{\boldsymbol{p}}_{1:n}}(x; \boldsymbol{\lambda})}{\partial u_i} - \frac{\partial F_{X^{\boldsymbol{p}}_{1:n}}(x; \boldsymbol{\lambda})}{\partial u_j} = \eta_{i,j}(x)\phi'\left(\sum_{i=1}^{n} \psi\left(h^{-1}(u_i)\bar{F}(x; \lambda_i)\right)\right)$$

is a non-positive function, where

$$\eta_{i,j}(x) = \bar{F}(x; \lambda_j)\frac{\partial h^{-1}(u_j)}{\partial u_j}\psi'(h^{-1}(u_j)\bar{F}(x, \lambda_j))$$

$$
\begin{aligned}
&-\bar{F}(x;\lambda_i)\frac{\partial h^{-1}(u_i)}{\partial u_i}\psi'(h^{-1}(u_i)\bar{F}(x,\lambda_i))\\
=&\frac{1}{h^{-1}(u_j)h'\left(h^{-1}(u_j)\right)}h^{-1}(u_j)\bar{F}(x;\lambda_j)\psi'(h^{-1}(u_j)\bar{F}(x,\lambda_j))\\
&-\frac{1}{h^{-1}(u_i)h'\left(h^{-1}(u_i)\right)}h^{-1}(u_i)\bar{F}(x;\lambda_i)\psi'(h^{-1}(u_i)\bar{F}(x,\lambda_i)).
\end{aligned}
$$

As ϕ' is a non-positive function, it is sufficient to show that $\eta_{i,j}(x)$ is a non-negative function. From Assumptions (i) and (ii), for all $1 \leq i < j \leq n$ and $\boldsymbol{u}, \boldsymbol{\lambda} \in \mathfrak{D}_+$, we obtain

$$
\begin{aligned}
&h^{-1}(u_i)\bar{F}(x;\lambda_i)\psi'\left(h^{-1}(u_i)\bar{F}(x,\lambda_i)\right)\\
&\leq h^{-1}(u_j)\bar{F}(x;\lambda_j)\psi'\left(h^{-1}(u_j)\bar{F}(x,\lambda_j)\right) \leq 0,
\end{aligned}
\tag{8}
$$

and from Assumption (iii), we have

$$
\frac{1}{h^{-1}(u_i)h'\left(h^{-1}(u_i)\right)} \leq \frac{1}{h^{-1}(u_j)h'\left(h^{-1}(u_j)\right)} \leq 0. \tag{9}
$$

Upon combining (8) and (9), we see that $\eta_{i,j}(x)$ is non-positive. From these observations, according to Lemma 2.4, we observe that $F_{X^{\boldsymbol{p}}_{n-1:n}}(x)$ is Schur-concave. This, in turn, guarantees that $\bar{F}_{X^{\boldsymbol{p}}_{n-1:n}}$ is Schur-convex. Then, the desired result follows by Lemma 2.6. □

It is important to mention that the condition "$x\psi'(x)$ be an increasing function" in Theorem 3.7 is quite general and holds for many known Archimedean copulas. For example, we can consider

- the Clayton copula with generator $\phi(x) = (1+\theta x)^{-\frac{1}{\theta}}$ for $\theta \geq 1$. Then, $\psi(x) = \frac{1}{\theta}(t^{-\theta} - 1)$ with $\psi'(x) = -x^{-\theta-1}$, and so

$$
\frac{d}{dx}\left(x\psi'(x)\right) = \frac{1}{x^{\theta+1}},
$$

  which reveals that $x\psi'(x)$ is an increasing function;
- the Gumbel copula with generator function $\phi(x) = e^{-x^{\frac{1}{\theta}}}$ for $\theta \in [0, \infty)$. Then, it follows that

$$
\frac{d}{dx}(x\psi'(x)) = \theta(\theta-1)\frac{(-\ln(x))^{\theta-1}}{x},
$$

  which reveals that $x\psi'(x)$ is increasing, for $\theta \geq 1$;

- the Ali–Mikhail–Haq copula with generator function $\phi(x) = \frac{1-\theta}{e^{\theta}-\theta}$ for $\theta \in [-1, 1]$. Then, we can see that

$$\frac{d}{dx}(x\psi'(x)) = \frac{\theta(\theta-1)}{(1-\theta(1-x))^2},$$

which reveals that $x\psi'(x)$ is increasing, for $\theta \in (0, 1)$.

Theorem 3.8 *Let $\boldsymbol{X}^{\boldsymbol{p}} = (I_{p_1}X_1, \ldots, I_{p_n}X_n)$ and $\boldsymbol{Y}^{\boldsymbol{p}} = (I_{p_1}Y_1, \ldots, I_{p_n}Y_n)$ be dependent and non-identically distributed random vectors with X_i's and Y_i's having the same Archimedean copula with generator ϕ and distribution parameters λ_i's and λ_i^*'s, respectively, and $I_{p_1}, \ldots, I_{p_n}$ being independent Bernoulli random variables with parameters $\boldsymbol{p} = (p_1, \ldots, p_n)$, that are independent of X_i's and Y_i's. Let*

(i) *ϕ be a log-concave function,*
(ii) *$\bar{F}(\cdot; \lambda)$ be increasing in λ and a log–concave function.*

Then, for $\boldsymbol{p} \in \mathfrak{D}^+$ and $\boldsymbol{\lambda} \in \mathfrak{D}^+$, we have

$$\boldsymbol{\lambda} \preceq_w \boldsymbol{\lambda}^* \Rightarrow X_{1:n}^{\boldsymbol{p}} \leq_{st} X_{1:n}^{\boldsymbol{q}},$$

where $X_{1:n}^{\boldsymbol{p}}$ and $X_{1:n}^{\boldsymbol{q}}$ are the smallest order statistics from $\boldsymbol{X}^{\boldsymbol{p}}$ and $\boldsymbol{X}^{\boldsymbol{q}}$, respectively.

Proof The distribution function of $X_{1:n}^{\boldsymbol{p}}$ arising from $\boldsymbol{X}^{\boldsymbol{p}}$ can be expressed as

$$F_{X_{1:n}^{\boldsymbol{p}}}(x; \boldsymbol{\lambda}) = 1 - \phi\left(\sum_{i=1}^{n} \psi\left(p_i \bar{F}(x; \lambda_i)\right)\right).$$

Then, we have

$$\frac{\partial F_{X_{n:n}^{\boldsymbol{p}}}(x; \boldsymbol{\lambda})}{\partial \lambda_i} = -p_i \frac{\partial \bar{F}(x; \lambda_i)}{\partial \lambda_i} \psi'(p_i \bar{F}(x, \lambda_i))\phi'\left(\sum_{j=1}^{n} \psi(p_j \bar{F}(x; \lambda_j))\right). \quad (10)$$

As $\phi'(u)$ and $\psi'(u)$ are non-positive functions, from Assumption (ii), we see that $\frac{\partial \bar{F}_{X_{n:n}^{\boldsymbol{p}}}(x;\boldsymbol{\lambda})}{\partial u_i} \leq 0$. This reveals that $F_{X_{n:n}^{\boldsymbol{p}}}(x; \boldsymbol{\lambda})$ is a decreasing function with respect to λ or, equivalently, $\bar{F}_{X_{n:n}^{\boldsymbol{p}}}(x; \boldsymbol{\lambda})$ is an increasing function in λ. Hence, to obtain the desired result from Lemma 2.5, it is enough to show that $F_{X_{n:n}^{\boldsymbol{p}}}(x; \boldsymbol{\lambda})$ is Schur-concave. For this purpose, (10) can be rewritten as

$$\frac{\partial F_{X_{n:n}^{\boldsymbol{p}}}(x; \boldsymbol{\lambda})}{\partial \lambda_i} = -\frac{\frac{\partial \bar{F}(x;\lambda_i)}{\partial \lambda_i}}{\bar{F}(x; \lambda_i)} \frac{\phi\left(\psi(p_i \bar{F}(x, \lambda_i))\right)}{\phi'\left(\psi(p_i \bar{F}(x, \lambda_i))\right)} \phi'\left(\sum_{j=1}^{n} \psi(p_j \bar{F}(x; \lambda_j))\right).$$

So, we find that, for $1 \le i \le j \le n$, $\boldsymbol{p} \in \mathfrak{D}^+$ and $\boldsymbol{\lambda} \in \mathfrak{D}^+$,

$$\begin{aligned}
&\frac{\partial \bar{F}_{X^{\boldsymbol{p}}_{n:n}}(x;\boldsymbol{\lambda})}{\partial \lambda_i} - \frac{\partial \bar{F}_{X^{\boldsymbol{p}}_{n:n}}(x;\boldsymbol{\lambda})}{\partial \lambda_j} \\
&= \frac{\frac{\partial \bar{F}(x;\lambda_j)}{\partial \lambda_j}}{\bar{F}(x;\lambda_j)} \frac{\phi\left(\psi(p_j \bar{F}(x,\lambda_j))\right)}{\phi'\left(\psi(p_j \bar{F}(x,\lambda_j))\right)} \phi'\left(\sum_{j=1}^{n} \psi(p_j \bar{F}(x;\lambda_j))\right) \\
&\quad - \frac{\frac{\partial \bar{F}(x;\lambda_i)}{\partial \lambda_i}}{\bar{F}(x;\lambda_i)} \frac{\phi\left(\psi(p_i \bar{F}(x,\lambda_i))\right)}{\phi'\left(\psi(p_i \bar{F}(x,\lambda_i))\right)} \phi'\left(\sum_{j=1}^{n} \psi(p_j \bar{F}(x;\lambda_j))\right) \\
&=: \varpi \phi'\left(\sum_{j=1}^{n} \psi(p_j \bar{F}(x;\lambda_j))\right),
\end{aligned}$$

where

$$\varpi = \frac{\frac{\partial \bar{F}(x;\lambda_j)}{\partial \lambda_j}}{\bar{F}(x;\lambda_j)} \frac{\phi\left(\psi(p_j \bar{F}(x,\lambda_j))\right)}{\phi'\left(\psi(p_j \bar{F}(x,\lambda_j))\right)} - \frac{\frac{\partial \bar{F}(x;\lambda_i)}{\partial \lambda_i}}{\bar{F}(x;\lambda_i)} \frac{\phi\left(\psi(p_i \bar{F}(x,\lambda_i))\right)}{\phi'\left(\psi(p_i \bar{F}(x,\lambda_i))\right)}.$$

As ϕ' is a non-positive function, it suffices to show that ϖ is non-negative. We now observe that

$$\varpi \ge \left[\frac{\frac{\partial \bar{F}(x;\lambda_j)}{\partial \lambda_j}}{\bar{F}(x;\lambda_j)} - \frac{\frac{\partial \bar{F}(x;\lambda_i)}{\partial \lambda_i}}{\bar{F}(x;\lambda_i)}\right] \frac{\phi\left(\psi(p_i \bar{F}(x,\lambda_i))\right)}{\phi'\left(\psi(p_i \bar{F}(x,\lambda_i))\right)} \ge 0,$$

where the first inequality follows from Assumption (i) implying the increasing property of $\frac{\phi(x)}{\phi'(x)}$ which the second inequality follows from Assumption (ii). This completes the proof of the theorem. □

4 Numerical examples

In this section, we present some numerical examples to illustrate the results established in the last section. It should be mentioned that some of the well known Archimedean copulas satisfy the condition that $u\psi'(1-u)$ is increasing or decreasing in u. For example, consider the Ali-Mikhail-Haq (AMH) copula with generator $\phi(u) = \frac{1-\theta}{e^u - \theta}$, for $\theta \in [-1, 1]$, for which $\psi(u) = \ln\left(\frac{1-\theta+\theta u}{u}\right)$. In this case, we find

$$u\psi'(1-u) = \frac{(1-\theta)u}{(1-u)\,(1-\theta+\theta(1-u))}$$

to be increasing in u for $\theta \in [0, 1)$, and decreasing in u for $\theta \in [-1, 0)$. As another example, let us consider the Clayton copula with generator inverse $\psi(u) = \frac{1}{\theta}(u^{-\theta} - 1)$

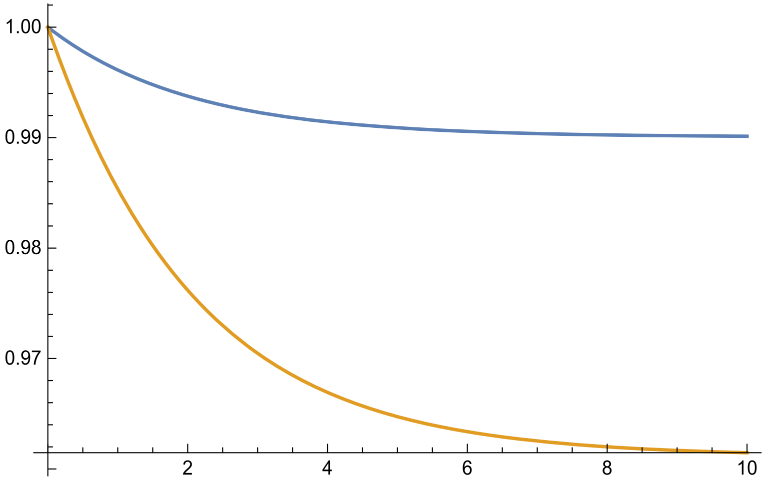

Fig. 1 Plots of survival functions of $X^{\boldsymbol{p}}_{3:3}$ (brown-down) and $Y^{\boldsymbol{q}}_{3:3}$ (blue-up) in Example 4.1 (colour figure online)

for $\theta \in [-1, \infty) - \{0\}$. In this case, we see that

$$u\psi'(1-u) = \frac{u}{(1-u)^{\theta+1}}$$

is increasing in u for all $\theta \in [-1, \infty]$.

Example 4.1 Consider the Clayton copula described by the generator $\psi_\theta(x) = \frac{1}{\theta}\left(\frac{1}{x^\theta} - 1\right)$, for $\theta > 0$, such that $x\psi'_\theta(1-x)$ is increasing in $x \in (0, 1)$. Suppose X_i and Y_i, $i = 1, 2, 3$, are exponential random variables with mean $\frac{1}{\lambda_i}$ linked by an Archimedean copula with generator function ψ_θ and scale parameters $(\lambda_1, \lambda_2, \lambda_3) = (5, 3, 2)$. Assume that $h(p) = \sqrt{p}$, $(p_1, p_2, p_3) = (0.49, 0.09, 0.04)$ and $(q_1, q_2, q_3) = (0.81, 0.04, 0.01)$. It is then easy to see that $\boldsymbol{h}(\boldsymbol{p}) \overset{m}{\preceq} \boldsymbol{h}(\boldsymbol{q})$. For $\theta = 3$, the survival functions of $X^{\boldsymbol{p}}_{3:3}$ and $Y^{\boldsymbol{q}}_{3:3}$ have been plotted in Fig. 1, which does show that $X^{\boldsymbol{p}}_{3:3} \leq_{st} Y^{\boldsymbol{q}}_{3:3}$, thus verifying the result in Theorem 3.1.

Example 4.2 Under the setup of Example 4.1, suppose X_i and Y_i, $i = 1, 2, 3$, are exponential random variables with mean $\frac{1}{\lambda_i}$, linked by an Archimedean copula with generator function ψ_θ and scale parameters $(\lambda_1, \lambda_2, \lambda_3) = (7, 3, 2)$ and $(\lambda_1^*, \lambda_2^*, \lambda_3^*) = (7, 4, 1)$, respectively. Assume that $(p_1, p_2, p_3) = (0.9, 0.5, 0.2)$. It is then easy to see that $(\lambda_1, \lambda_2, \lambda_3) \overset{m}{\preceq} (\lambda_1^*, \lambda_2^*, \lambda_3^*)$ and also that the conditions of Theorem 3.4 are satisfied. For $\theta = 4$, the survival functions of $X^{\boldsymbol{p}}_{3:3}$ and $Y^{\boldsymbol{q}}_{3:3}$ have been plotted in Fig. 2, which does show that $X^{\boldsymbol{p}}_{3:3} \leq_{st} Y^{\boldsymbol{q}}_{3:3}$, thus verifying the result in Theorem 3.4.

The next example provides an illustration of the result in Theorem 3.5.

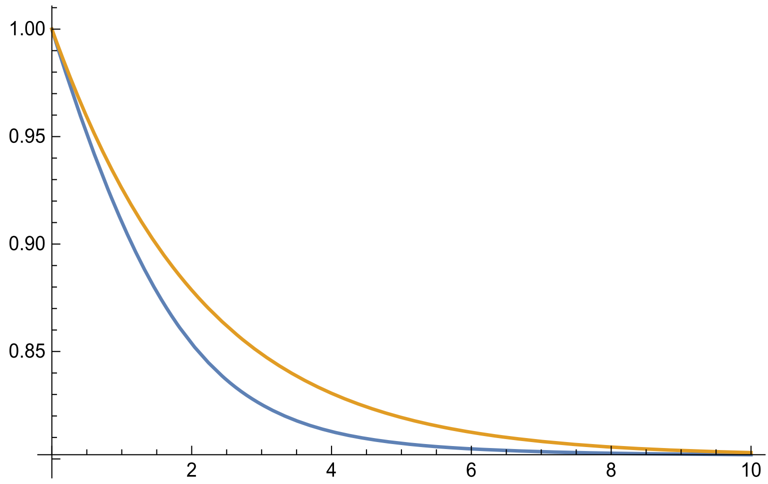

Fig. 2 Plots of survival functions of $X_{3:3}^{\boldsymbol{p}}$ (brown-up) and $Y_{3:3}^{\boldsymbol{q}}$ (blue-down) in Example 4.2 (colour figure online)

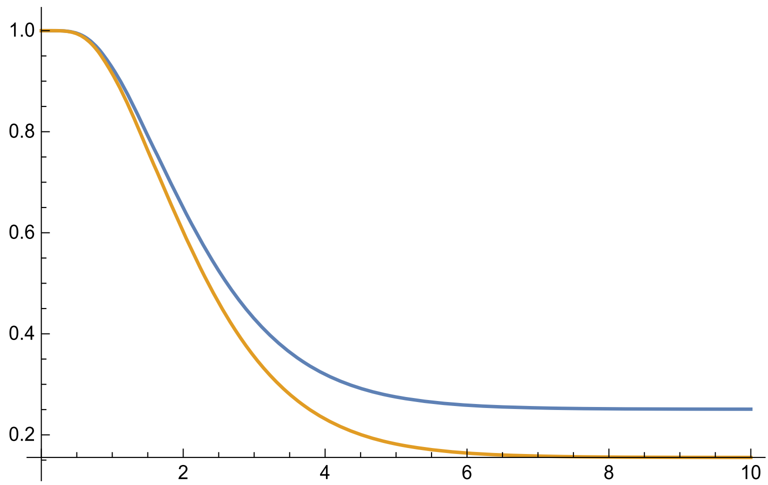

Fig. 3 Plots of survival functions of $X_{2:3}^{\boldsymbol{p}}$ (brown-down) and $Y_{2:3}^{\boldsymbol{q}}$ (blue-up) in Example 4.3 (colour figure online)

Example 4.3 Consider the multivariate AMH copula described by the generator $\psi(u) = \ln\left(\frac{1-\theta+\theta u}{u}\right)$, for $\theta \in (0, 1]$, such that $u\psi_\theta'(1-u)$ is increasing in $u \in (0, 1)$. Assume that X_i and Y_i, $i = 1, 2, 3$, are Gamma random variables with common shape parameter 2 and common scale parameter $\frac{1}{\lambda}$, linked by an AMH copula with generator function ψ_θ. Assume that $h(p) = e^{-p}$, $p \in [0, 1]$, $(h(p_1), h(p_2), h(p_3)) = (e^{-0.1}, e^{-0.2}, e^{-0.6})$ and $(h(q_1), h(q_2), h(q_3)) = (e^{-0.2}, e^{-0.3}, e^{-0.7})$. It is then easy to check that $(h(p_1), h(p_2), h(p_3)) \overset{w}{\preceq} (h(q_1), h(q_2), h(q_3))$. It can also be verified that the conditions of Theorem 3.5 are satisfied in this case. For $\theta = 0.8$, the survival functions of $X^{\boldsymbol{p}}_{2:3}$ and $Y^{\boldsymbol{q}}_{2:3}$ have been plotted in Fig. 3, which does show that $X^{\boldsymbol{p}}_{2:3} \leq_{st} Y^{\boldsymbol{q}}_{2:3}$, thus verifying the result of Theorem 3.5.

5 Concluding remarks

In this work, we have discussed some comparisons of two k-out-of-n systems comprising heterogeneous dependent components experiencing random shocks. For modelling the dependence between the components, we have used Archimedean copulas for the joint distribution of the component lifetimes. We have then provided some sufficient conditions on the distributions of components' lifetimes and the generator of the Archimedean copula and on the random shocks for comparing the lifetimes of two systems with respect to the usual stochastic order. We have also presented some illustrative examples. It will naturally be of interest to extend the results established here for some other stochastic orders such as hazard rate, reversed hazard rate, likelihood ratio and dispersive orders. We are currently working in this direction and hope to report the findings in a future paper.

Acknowledgements We express our sincere thanks to the Editor and the anonymous reviewers for their useful comments and suggestions on an earlier version of this manuscript which resulted in this improved version.

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