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On conditional residual lifetimes of coherent systems consisting of components with discrete lifetimes

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Abstract
In this paper, we consider a coherent system composed of components whose lifetimes are independent and identically discretely distributed random variables. We study several aging and stochastic properties of the conditional residual lifetime of the system under the condition that some of its components have failed by time t. Moreover, we compare the conditional residual lifetimes of two coherent systems by using various stochastic orders.

Keywords Coherent systems · Residual lifetime · Discrete lifetime distribution · Order statistics · Stochastic orders

Mathematics Subject Classification 90B25 · 60K10 · 60E15

1 Introduction

In technical systems, a coherent system is a very important structure, which works as long as a given selections of its elements work. Consider a system which consists of n two-state (i.e., working or failed) components. Let us denote by $\mathbf{y} = (y_1, \ldots, y_n) \in \{0, 1\}^n$ the state vector, where for each i, $y_i = 1$ if the ith component is functioning and $y_i = 0$ if it is not functioning. The structure function $\tau : \{0, 1\}^n \to \{0, 1\}$ is a mapping that associates those state vectors $\mathbf{y}$ for which the system operates with the value 1 and those vectors $\mathbf{y}$ for which the system fails with the value 0. Moreover, the system is said to be coherent when its structure function τ is increasing in each vector argument and each component of the system is relevant (that is, actually affects the working or failure of the system). The classical monograph here is Barlow and Proschan (1975).

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If the random variable (rv) $Y_i(t)$ represents the state of the ith component at time $t \geq 0$, where $Y_i(t) = 0$ ($Y_i(t) = 1$), $i = 1, \ldots, n$, means that the ith component has failed (is functioning) at time t, the system state at time t is $Y(t) = \tau(Y_1(t), \ldots, Y_n(t))$. Analogously we can define the component lifetimes as $X_i = \sup\{t \geq 0 : Y_i(t) = 1\}$, $i = 1, \ldots, n$, and the coherent system lifetime as $T = \sup\{t \geq 0 : Y(t) = 1\}$.

We assume that the component lifetimes $X_1, \ldots, X_n$ are independent and identically distributed (iid) discrete random variables with cumulative distribution function (cdf) $F(t) = \mathrm{P}(X_i \leq t)$, $i = 1, \ldots, n$, and take values in finite or infinite subsets of the set of non-negative integers.

The case when the component lifetimes are discretely distributed is more complicated than the continuous one because of the positive probability of ties between times of component failures. However, in various practical situations component might have discrete operation times e.g. when we consider systems in which the component lifetimes represent the numbers of turn-on and switch-off up to the failure or when the components of the system operate in discrete cycles. The systems with discrete operating components have been studied over the years, see Weiss (1962), Young (1970), Tank and Eryilmaz (2015), Dembińska (2018), Davies and Dembińska (2019), Dembińska and Goroncy (2020), Dembińska et al. (2021), Jasiński (2021). The systems with arbitrary lifetime distributions which can be discrete in particular were discussed by Navarro et al. (2008), Miziuła and Rychlik (2014) or Eryilmaz et al. (2016).

We recall the concept of minimal paths of a coherent system. It is useful in the efficient probability calculations. We say that $P \subset \{1, 2, \ldots, n\}$ is a path set of the system if it operates when all the elements with indices in P work. A path is said to be minimal if it does not contain any strict subset being a path set. Denoting by z the number of minimal path sets, the system lifetime T can be represented as

$$T = \max_{1 \leq j \leq z} \min_{p \in P_j} X_p, \tag{1}$$

where $P_1, \ldots, P_z$ are the minimal paths sets, see Barlow and Proschan (1975, p. 13). Under the assumption that the component lifetimes $X_1, \ldots, X_n$ are exchangeable (that is, for any permutation $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$, the random vector $(X_{j_1}, \ldots, X_{j_n})$ has the same distribution as $(X_1, \ldots, X_n)$), the existence of a vector $\mathbf{s} = (s_1, \ldots, s_n)$ such that

$$P(T > t) = \sum_{i=1}^{n} s_i \, \mathrm{P}(X_{i:n} > t),$$

where $X_{1:n} \leq \ldots \leq X_{n:n}$ are the order statistics of lifetimes $X_1, \ldots, X_n$, was proved by Navarro et al. (2008). We have $s_i \geq 0$, $i = 1, \ldots, n$, and $\sum_{i=1}^{n} s_i = 1$. They generalized the earlier results established by Samaniego (1985) and Navarro and Rychlik (2007). The vector $\mathbf{s}$ is called the Samaniego signature of a coherent system. It depends only on the structure of the system and is independent of the distribution of the component lifetimes. Assume that $\tilde{S}_i = \sum_{m=i+1}^{n} s_m$, $0 \leq i \leq n-1$. Using the concept of minimal paths, Jasiński (2021, formula (25)) proposed the following formula

$$\tilde{S}_i = \sum_{m=1}^{n-i}\left[\frac{\binom{n-i}{m}}{\binom{n}{m}}\sum_{j=1}^{z}(-1)^{j+1}\sum_{1\le k_1<\ldots<k_j\le z}\mathrm{I}\left(|\bigcup_{w=1}^{j}P_{k_w}|=m\right)\right], \quad (2)$$

$0 \le i \le n-1$, where $|\bigcup_{w=1}^{j} P_{k_w}|$ denotes the cardinality of each $\bigcup_{w=1}^{j} P_{k_w}$ and $\mathrm{I}(\cdot)$ stands for the indicator function, that is $\mathrm{I}(A) = 1$ if the event A occurs and $\mathrm{I}(A) = 0$ otherwise. The domain of $\mathrm{I}(\cdot)$ is a Boolean domain consisting of exactly two elements whose interpretations include false and true.

The k-out-of-n system, $k = 1, \ldots, n$, is a coherent system that functions as long as at least k of the components function. Thus $T = X_{n-k+1:n}$. For $k = n$ and $k = 1$ we have the series and the parallel systems, respectively. Here there are $\binom{n}{k}$ minimal path sets, namely, all of the sets consisting of exactly k components. The Samaniego signature of a k-out-of-n system has the form $\mathbf{s} = (0, \ldots, 0, s_{n-k+1}, 0, \ldots, 0)$ with $s_{n-k+1} = 1$. Maximum likelihood estimation based on discrete component lifetimes of a k-out-of-n system was considered by Dembińska and Jasiński (2021).

For the rv X and the event A with positive probability by $[X|A]$ we denote any rv having the same distribution as the conditional distribution of X given A, i.e., for any x,

$$P([X|A] > x) = P(X > x|A).$$

Furthermore, $a/0$ is taken to be equal to ∞ whenever $a > 0$.

In this paper we focus on the coherent systems with the Samaniego signature

$$\mathbf{s} = (0, \ldots, 0, s_k, s_{k+1}, \ldots, s_n), \text{ for } k = 2, \ldots, n \quad (3)$$

that is, at the $(k-1)$th failure, the system is still working with probability one. The assumption (3) holds throughout the whole paper except for Theorem 2. Then we concentrate on the following residual lifetime of the system

$$[T - t|X_{j:n} \le t < X_{k:n}], \quad 1 \le j \le k-1, \quad (4)$$

which describes the system lifetime after time t, given that, at time t, at least j $(j < k)$ elements have been broken but the kth failure has not occurred yet and so the system functions.

In Sect. 2, we obtain a mixture representation for the residual lifetime given in (4) and we study some of its aging properties. Then we use it to compare two coherent systems with identically distributed components and ordered signatures. We extend the respective results obtained by Goliforushani et al. (2012) and Parvardeh and Balakrishnan (2013) when the component lifetimes are continuously distributed.

Before proceeding to present the main result, we recall the definitions of various stochastic orders. Let X and Y be two discrete non-negative rvs with the distribution functions F and G, the reliability functions $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$ and the probability mass functions (pmfs) $f(x) = P(X = x)$ and $g(x) = P(Y = x)$, respectively. X is said to be less than Y in

i) the usual stochastic order (denoted by $X \leq_{st} Y$), if $\overline{F}(x) \leq \overline{G}(x)$ for all x,
ii) the hazard rate order (denoted by $X \leq_{hr} Y$), if $\overline{G}(x)/\overline{F}(x)$ is increasing in x for x such that $\overline{F}(x) > 0$ or $\overline{G}(x) > 0$,
iii) the likelihood ratio order (denoted by $X \leq_{lr} Y$), if $g(x)/f(x)$ is increasing in x on the union of the supports of X and Y.

It is well known that $X \leq_{lr} Y$ implies $X \leq_{hr} Y$ and $X \leq_{st} Y$.

Definition 1 Let X be a discrete non-negative rv. Then X is said to have increasing failure rate distribution (denoted by IFR distribution), if the function $f(x)/\overline{F}(x)$ is increasing in x on the support of X.

Throughout the paper decreasing (increasing) means non-increasing (non-decreasing). We also make use of the following notation. $\mathcal{P}$ stands for the set of all permutations $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$. Moreover, by $\stackrel{\mathrm{d}}{=}$ let us denote the equality in distribution.

2 Main results

Under a more general assumption that $X_1, \ldots, X_n$ are arbitrary dependent and not necessarily identically distributed discrete rvs, Dembińska (2018) obtained the formula for the survival function

$$P(X_{i:n} - t > x | X_{l:n} \leq t < X_{l+1:n}) = \frac{P(X_{i:n} > t + x, X_{l:n} \leq t < X_{l+1:n})}{P(X_{l:n} \leq t < X_{l+1:n})} \tag{5}$$

$$= \frac{\sum_{v=0}^{i-l-1} \binom{n-l}{v} \sum_{p \in \mathcal{P}} P\left({}^{p}D_{l,v}^{t,t+x}\right)}{\sum_{p \in \mathcal{P}} P\left({}^{p}H_l^t\right)}, \quad l < i \leq n, \tag{6}$$

where t is such that $P(X_{l:n} \leq t < X_{l+1:n}) > 0$ and

$$^{(j_1,\ldots,j_n)}D_{l,v}^{t,t+x} = \left(\bigcap_{r=1}^{l} \{X_{j_r} \leq t\}\right) \cap \left(\bigcap_{r=l+1}^{l+v} \{t < X_{j_r} \leq t + x\}\right) \cap \left(\bigcap_{r=l+v+1}^{n} \{X_{j_r} > t + x\}\right)$$

and

$$^{(j_1,\ldots,j_n)}H_l^t = \left(\bigcap_{r=1}^{l} \{X_{j_r} \leq t\}\right) \cap \left(\bigcap_{r=l+1}^{n} \{X_{j_r} > t\}\right).$$

It describes the lifetime of $(n-i+1)$-out-of-n system after time t, given that, at time t, exactly l elements are broken and the system functions. Next it was simplified depending on the structure between $X_1, \ldots, X_n$. In particular, if $X_1, \ldots, X_n$ are iid rvs with a cdf F and t is such that $P(X_{l:n} \leq t < X_{l+1:n}) > 0$, we have

$$P(X_{i:n} - t > x | X_{l:n} \leq t < X_{l+1:n}) = \alpha_{i,l}^{n}(t, x), \tag{7}$$

where

$$\alpha_{i,l}^{n}(t, x) = \sum_{v=0}^{i-l-1} \binom{n-l}{v} \left(\frac{F(t+x) - F(t)}{\overline{F}(t)} \right)^{v} \left(\frac{\overline{F}(t+x)}{\overline{F}(t)} \right)^{n-l-v}. \tag{8}$$

Using the result proposed by Dembińska (2018), we can rewrite (8) as

$$\begin{aligned} \alpha_{i,l}^{n}(t, x) &= \sum_{v=0}^{i-l-1} \binom{n-l}{v} (P(Z_1 \leq t+x))^{v} (P(Z_1 > t+x))^{n-l-v} \\ &= P(Z_{i-l:n-l} - t > x), \end{aligned}$$

where $Z_1, Z_2, \ldots, Z_{n-l}$ are iid rvs with cdf F_Z defined as

$$F_Z(z) = P(X_1 \leq z | X_1 > t) = \begin{cases} \frac{F(z)-F(t)}{\overline{F}(t)}, & \text{if } z > t, \\ 0, & \text{if } z \leq t. \end{cases} \tag{9}$$

Consequently, the function defined in (8) is the unconditional survival function of a $(n-i+1)$-out-of-$(n-l)$ system consisting of homogeneous elements with lifetimes $Z_1, Z_2, \ldots, Z_{n-l}$ having cdf F_Z given by (9). Thus for $l < i \leq n$, we have

$$(X_{i:n} - t | X_{l:n} \leq t < X_{l+1:n}) \stackrel{\mathrm{d}}{=} Z_{i-l:n-l}.$$

Remark 1 The above result readily yields

$$(X_{i:n} - t | X_{l:n} \leq t < X_{l+1:n}) \stackrel{\mathrm{d}}{=} (X_{i-l:n-l} - t | X_{1:n-l} > t).$$

Using Theorem 1.C.37 and Corollaries 1.C.38 and 1.C.39 proved by Shaked and Shanthikumar (2007), we have

$$X_{k-1:m-1} \leq_{lr} X_{k:m}, \quad k = 2, 3, \ldots, m$$

and

$$X_{k:m-1} \geq_{lr} X_{k:m}, \quad k = 1, 2, \ldots, m-1.$$

Hence

$$Z_{i-l:n-l} \leq_{lr} Z_{i+1-l:n-l+1} = Z_{i-(l-1):n-(l-1)}, \quad l < i \leq n,$$

where $Z_{i-l:n-l}$, $Z_{i-(l-1):n-(l-1)}$ are the order statistics corresponding to the respective iid rvs with the cdf defined in (9). Now since $Z_{i-l:n-l} \leq_{lr} Z_{i-(l-1):n-(l-1)}$ implies $Z_{i-l:n-l} \leq_{st} Z_{i-(l-1):n-(l-1)}$, we obtain the relation

$$\alpha^n_{i,l}(t,x) \leq \alpha^n_{i,l-1}(t,x). \tag{10}$$

Remark 2 For $m \leq l < i$, under the conditions that t is such that $P(X_{l:n} \leq t < X_{l+1:n}) > 0$ and $P(X_{m:n} \leq t < X_{m+1:n}) > 0$, we have

$$(X_{i:n} - t | X_{l:n} \leq t < X_{l+1:n}) \leq_{lr} (X_{i:n} - t | X_{m:n} \leq t < X_{m+1:n}).$$

Now, under the assumption that the component lifetimes are IFR, we show that the function given in (8) is a decreasing function of time t.

Lemma 1 *If X's are IFR, then for all $x > 0$ and $l < i \leq n$, $\alpha^n_{i,l}(t,x)$ defined in (8) is a decreasing function of t.*

Proof From (8) we have

$$\begin{aligned} \alpha^n_{i,l}(t,x) &= \sum_{v=0}^{i-l-1} \binom{n-l}{v} \left(1 - \frac{\overline{F}(t+x)}{\overline{F}(t)}\right)^v \left(\frac{\overline{F}(t+x)}{\overline{F}(t)}\right)^{n-l-v} \\ &= \int_{1-\overline{F}(t+x)/\overline{F}(t)}^1 \frac{(n-l)!}{(i-l-1)!(n-i)!} y^{i-l-1}(1-y)^{n-i}\, dy. \end{aligned} \tag{11}$$

We conclude from the assumption that the function $\overline{F}(t+x)/\overline{F}(t)$ is decreasing in t (see e.g. Lai and Xie 2007, p. 171 for discrete failure time models), which implies by (11) that $\alpha^n_{i,l}(t,x)$ is a decreasing function of t, which completes the proof. □

Remark 3 If X's are IFR and t is such that $P(X_{l:n} \leq t < X_{l+1:n}) > 0$, then using (7), (8) and Lemma 1, we obatin for all $x > 0$ and $l < i \leq n$, $P(X_{i:n} - t > x | X_{l:n} \leq t < X_{l+1:n})$ is a decreasing function of t.

In the same manner as the expression for the probability $P(X_{i:n} - t > x, X_{l:n} \leq t < X_{l+1:n})$, in the iid case we obtain

$$\begin{aligned} P(X_{i:n} - t > x, X_{j:n} \leq t < X_{k:n}) = &\sum_{v=j}^{k-1} \binom{n}{v} F^v(t) \sum_{u=0}^{i-1-v} \binom{n-v}{u} \\ &\cdot (F(t+x) - F(t))^u \, (\overline{F}(t+x))^{n-v-u}. \end{aligned} \tag{12}$$

We are now in a position to prove the extended version of Remark 3.

Theorem 1 *Let the discrete lifetimes $X_1, \ldots, X_n$ are iid rvs. If X's are IFR, and t is such that $P(X_{j:n} \le t < X_{k:n}) > 0$, then for all $x > 0$ and $1 \le j < k \le i \le n$, $P(X_{i:n} - t > x | X_{j:n} \le t < X_{k:n})$ is a decreasing function of t.*

Proof Note that using (12), we obtain

$$
\begin{aligned}
\gamma^n_{i,j,k}(t,x) &= P(X_{i:n} - t > x | X_{j:n} \le t < X_{k:n}) = \frac{\mathrm{P}(X_{i:n} - t > x, X_{j:n} \le t < X_{k:n})}{P(X_{j:n} \le t < X_{k:n})} \\
&= \frac{\sum_{v=j}^{k-1} \binom{n}{v} F^v(t) \sum_{u=0}^{i-1-v} \binom{n-v}{u} (F(t+x) - F(t))^u (\overline{F}(t+x))^{n-v-u}}{\sum_{m=j}^{k-1} \binom{n}{m} F^m(t) \overline{F}^{n-m}(t)} \\
&= \frac{\sum_{v=j}^{k-1} \binom{n}{v} \left(\frac{F(t)}{\overline{F}(t)}\right)^v \sum_{u=0}^{i-1-v} \binom{n-v}{u} \left(\frac{F(t+x)-F(t)}{\overline{F}(t)}\right)^u \left(\frac{\overline{F}(t+x)}{\overline{F}(t)}\right)^{n-v-u}}{\sum_{m=j}^{k-1} \binom{n}{m} \left(\frac{F(t)}{\overline{F}(t)}\right)^m}, \\
&= \frac{\sum_{v=j}^{k-1} \binom{n}{v} \phi^v(t) \alpha^n_{i,v}(t,x)}{\sum_{m=j}^{k-1} \binom{n}{m} \phi^m(t)},
\end{aligned}
$$

where $\phi(t) = \frac{F(t)}{\overline{F}(t)}$ is increasing in t, and $\alpha^n_{i,v}(t,x)$ defined in (8) is a decreasing function of t (cf. Lemma 1). We need to show that, for $t_1 < t_2$,

$$
\begin{aligned}
&\gamma^n_{i,j,k}(t_1,x) - \gamma^n_{i,j,k}(t_2,x) \\
&= \frac{\sum_{v=j}^{k-1} \sum_{m=j}^{k-1} \binom{n}{v}\binom{n}{m} \left[\alpha^n_{i,v}(t_1,x)\phi^v(t_1)\phi^m(t_2) - \alpha^n_{i,v}(t_2,x)\phi^v(t_2)\phi^m(t_1)\right]}{\sum_{m=j}^{k-1} \binom{n}{m}\phi^m(t_1) \sum_{m=j}^{k-1} \binom{n}{m}\phi^m(t_2)} \ge 0.
\end{aligned}
$$

Since the denominator is positive, it suffices to check the sign of the numerator. It can be rewritten as

$$
\begin{aligned}
&\sum_{v=j}^{k-1} \sum_{m=j}^{k-1} \binom{n}{v}\binom{n}{m} [\alpha^n_{i,v}(t_1,x) - \alpha^n_{i,v}(t_2,x)]\phi^v(t_1)\phi^m(t_2) \\
&\quad + \sum_{v=j}^{k-1} \sum_{m=j}^{k-1} \binom{n}{v}\binom{n}{m} [\alpha^n_{i,v}(t_2,x) - \alpha^n_{i,m}(t_2,x)]\phi^v(t_1)\phi^m(t_2). \qquad (13)
\end{aligned}
$$

We conclude from Lemma 1 that $\alpha^n_{i,v}(t_1,x) \ge \alpha^n_{i,v}(t_2,x)$, for $t_2 > t_1$ and all $x > 0$ and finally that the first term of (13) is non-negative. What is left is to show that the second one is non-negative too. It is seen to be equal to

$$\begin{aligned}
&\sum_{v=j}^{k-1}\left(\sum_{m=j}^{v-1}+\sum_{m=v+1}^{k-1}\right)\binom{n}{v}\binom{n}{m}[\alpha_{i,v}^{n}(t_2,x)-\alpha_{i,m}^{n}(t_2,x)]\phi^{v}(t_1)\phi^{m}(t_2)\\
&=\sum_{v=j}^{k-1}\sum_{m=j}^{v-1}\binom{n}{v}\binom{n}{m}[\alpha_{i,v}^{n}(t_2,x)-\alpha_{i,m}^{n}(t_2,x)]\phi^{v}(t_1)\phi^{m}(t_2)\\
&\quad-\sum_{v=j}^{k-1}\sum_{m=j}^{v-1}\binom{n}{v}\binom{n}{m}\left[\alpha_{i,v}^{n}(t_2,x)-\alpha_{i,m}^{n}(t_2,x)\right]\phi^{v}(t_2)\phi^{m}(t_1)\\
&=\sum_{v=j}^{k-1}\sum_{m=j}^{v-1}\binom{n}{v}\binom{n}{m}\left[\alpha_{i,v}^{n}(t_2,x)-\alpha_{i,m}^{n}(t_2,x)\right][\phi^{v}(t_1)\phi^{m}(t_2)-\phi^{v}(t_2)\phi^{m}(t_1)].
\end{aligned} \tag{14}$$

Since $\alpha_{i,v}^{n}(t_2,x) \le \alpha_{i,m}^{n}(t_2,x)$ for $m \le v < i$ (cf. (10)), we further obtain

$$\phi^{v}(t_1)\phi^{m}(t_2)-\phi^{v}(t_2)\phi^{m}(t_1)=\phi^{m}(t_2)\phi^{m}(t_1)\left[\phi^{v-m}(t_1)-\phi^{v-m}(t_2)\right]\le 0,$$

because the function ϕ is increasing in t. Thus we prove the non-negativity of (14). This completes the proof of the theorem. □

To extend Theorem 1 to the arbitrary coherent system we need the following mixture representation for the residual lifetime of the system. In the case when the cdf F is continuous, it has been already obtained by Parvardeh and Balakrishnan (2013, Theorem 1) by the use of Theorem 1 of Kochar et al. (1999). Below representation holds for the coherent system with arbitrary Samaniego signature $\mathbf{s}=(s_1,\ldots,s_n)$, $s_i \ge 0$, $i=1,\ldots,n$, and $\sum_{i=1}^{n} s_i = 1$.

Theorem 2 *Let T be the lifetime of a coherent system with n iid components whose discrete lifetimes are denoted by $X_1,\ldots,X_n$. Then, for $1 \le l < n$, we obtain*

$$\begin{aligned}
P(T-t>x|X_{l:n}\le t<X_{l+1:n}) &= \sum_{i=l+1}^{n} s_i\, P(X_{i:n}-t>x|X_{l:n}\le t<X_{l+1:n})\\
&= \sum_{i=l+1}^{n} s_i\, \alpha_{i,l}^{n}(t,x).
\end{aligned} \tag{15}$$

Proof Our purpose is to prove the equality

$$P(T-t>x, X_{l:n}\le t<X_{l+1:n}) = \sum_{i=l+1}^{n} s_i\, P(X_{i:n}-t>x, X_{l:n}\le t<X_{l+1:n}). \tag{16}$$

Using the formula for the numerator of (5) in the iid case, the right-hand side of (16) can be rewritten as

$$\sum_{i=l+1}^{n} s_i P\left(X_{i:n} > t + x, X_{l:n} \le t < X_{l+1:n}\right)$$
$$= \sum_{i=l+1}^{n} s_i \sum_{v=0}^{i-l-1} \binom{n}{l}\binom{n-l}{v}(F(t))^l (F(t+x) - F(t))^v (\overline{F}(t+x))^{n-l-v}. \quad (17)$$

Now by the representation (1), the left hand side of (16) has the form

$$P(T > t + x, X_{l:n} \le t < X_{l+1:n}) = P(X_{l:n} \le t < X_{l+1:n}, \max_{1\le j\le z} \min_{p\in P_j} X_p > t + x)$$
$$= P\left(X_{l:n} \le t < X_{l+1:n}, \bigcup_{j=1}^{z}\{\min_{p\in P_j} X_p > t + x\}\right)$$
$$= P\left(\bigcup_{j=1}^{z}\{X_{l:n} \le t < X_{l+1:n}, \min_{p\in P_j} X_p > t + x\}\right).$$

Further, using the inclusion-exclusion formula, we deduce that

$$P(X_{l:n} \le t < X_{l+1:n}, T > t + x)$$
$$= \sum_{j=1}^{z}(-1)^{j+1} \sum_{1\le k_1<\ldots<k_j\le z} P\left(X_{l:n} \le t < X_{l+1:n}, \bigcap_{w=1}^{j}\{\min_{p\in P_{k_w}} X_p > t + x\}\right)$$
$$= \sum_{j=1}^{z}(-1)^{j+1} \sum_{1\le k_1<\ldots<k_j\le z} P\left(X_{l:n} \le t < X_{l+1:n}, \min_{p\in P_{k_1}\cup\ldots\cup P_{k_j}} X_p > t + x\right)$$
$$= \sum_{j=1}^{z}(-1)^{j+1} \sum_{1\le k_1<\ldots<k_j\le z} P\left(X_{l:n} \le t < X_{l+1:n}, \bigcap_{p\in P_{k_1}\cup\ldots\cup P_{k_j}} \{X_p > t + x\}\right).$$

Notice that

$$P\left(X_{l:n} \le t < X_{l+1:n}, \bigcap_{p\in P_{k_1}\cup\ldots\cup P_{k_j}} \{X_p > t+x\}\right) = \sum_{v=0}^{n-l-1} \mathrm{I}\left(\left|\bigcup_{w=1}^{j} P_{k_w}\right| \le n-l-v\right)$$
$$\cdot P\left(\text{for all } p \in \bigcup_{w=1}^{j} P_{k_w}, X_p > t + x,\right.$$
$$\text{exactly } l \text{ of } X_p \text{ are } \le t,$$
$$\text{exactly } v \text{ of } X_p \text{ belong to } (t, t + x],$$

$$\text{and the rest } n - v - l - \Big|\bigcup_{w=1}^{j} P_{k_w}\Big| \text{ of } X_p \text{ are } > t + x \Bigg).$$

Consequently

$$\begin{aligned}
&P(X_{l:n} \le t < X_{l+1:n}, T > t + x) \\
&\quad = \sum_{v=0}^{n-l-1} \binom{v+l}{l} (F(t))^l (F(t+x) - F(t))^v (\overline{F}(t+x))^{n-l-v} \\
&\quad \cdot \sum_{j=1}^{z} (-1)^{j+1} \sum_{1 \le k_1 < \ldots < k_j \le z} \sum_{m=1}^{n-l-v} \mathrm{I}\left(\Big|\bigcup_{w=1}^{j} P_{k_w}\Big| = m \right) \binom{n-m}{v+l}.
\end{aligned}$$

Since $\binom{n-m}{v+l}/\binom{n}{v+l} = \binom{n-v-l}{m}/\binom{n}{m}$ and applying (2), we obtain

$$\begin{aligned}
&P(X_{l:n} \le t < X_{l+1:n}, T > t + x) \\
&\quad = \sum_{v=0}^{n-l-1} \binom{v+l}{l} \binom{n}{v+l} (F(t))^l (F(t+x) - F(t))^v (\overline{F}(t+x))^{n-l-v} \sum_{i=v+l+1}^{n} s_i,
\end{aligned}$$

which after changing the order of summation is equal to (17). This proves the theorem. □

Remark 4 Under the assumption of Theorem 2, if moreover the signature of the system has the form (3), we have for $1 \le j < k \le n$

$$\begin{aligned}
P(T - t > x | X_{j:n} \le t < X_{k:n}) &= \sum_{i=k}^{n} s_i \, P(X_{i:n} - t > x | X_{j:n} \le t < X_{k:n}) \\
&= \sum_{i=k}^{n} s_i \, \gamma_{i,j,k}^{n}(t, x).
\end{aligned} \tag{18}$$

Proof Note that

$$\begin{aligned}
P(T - t > x | X_{j:n} \le t < X_{k:n}) &= \frac{\sum_{l=j}^{k-1} P\left(T > t + x, X_{l:n} \le t < X_{l+1:n}\right)}{P\left(X_{j:n} \le t < X_{k:n}\right)} \\
&= \frac{\sum_{l=j}^{k-1} \sum_{i=l+1}^{n} s_i P\left(X_{i:n} > t + x, X_{l:n} \le t < X_{l+1:n}\right)}{P\left(X_{j:n} \le t < X_{k:n}\right)},
\end{aligned} \tag{19}$$

where the last equality follows from (15). Since we consider the coherent systems with the signatures of the form (3), the conditional probability given in (19) can be rewritten as

$$P(T-t>x|X_{j:n}\leq t<X_{k:n})=\frac{\sum_{l=j}^{k-1}\sum_{i=k}^{n}s_i P\left(X_{i:n}>t+x,X_{l:n}\leq t<X_{l+1:n}\right)}{P\left(X_{j:n}\leq t<X_{k:n}\right)}$$

$$=\frac{\sum_{i=k}^{n}s_i P\left(X_{i:n}>t+x,X_{j:n}\leq t<X_{k:n}\right)}{P\left(X_{j:n}\leq t<X_{k:n}\right)}$$

$$=\sum_{i=k}^{n}s_i\gamma^n_{i,j,k}(t,x),$$

which completes the proof. □

Theorem 3 *Consider a coherent system consisting of n elements whose discrete lifetimes $X_1,\ldots,X_n$ are assumed to be iid rvs. If X's are IFR and the signature of the system has the form (3), then for all $x>0$, $P(T-t>x|X_{j:n}\leq t<X_{k:n})$ is a decreasing function of t.*

Proof From Theorem 1, we know that for all $x>0$, $P(X_{i:n}-t>x|X_{j:n}\leq t<X_{k:n})$ is a decreasing function of t. Combining it with the representation given in (18), we get the desired conclusion. □

Theorem 4 *Let the discrete lifetimes $X_1,\ldots,X_n$ are iid rvs and t is such that $P(X_{j:n}\leq t<X_{k:n})>0$. Then for $j<k\leq i\leq m$, we have*

$$(X_{i:n}-t|X_{j:n}\leq t<X_{k:n})\leq_{hr}(X_{m:n}-t|X_{j:n}\leq t<X_{k:n}).$$

Proof Our aim is to show that the function

$$\frac{\gamma^n_{m,j,k}(t,x)}{\gamma^n_{i,j,k}(t,x)}=\frac{P(X_{m:n}-t>x,X_{j:n}\leq t<X_{k:n})}{P(X_{i:n}-t>x,X_{j:n}\leq t<X_{k:n})}$$

is increasing in x. Using the conditional probability given in (12) we have

$$\frac{\gamma^n_{m,j,k}(t,x)}{\gamma^n_{i,j,k}(t,x)}=\frac{\sum_{v=j}^{k-1}\binom{n}{v}\left(\frac{F(t)}{\overline{F}(t+x)}\right)^v\left[\sum_{u_1=0}^{m-1-v}\binom{n-v}{u_1}\left(\frac{F(t+x)-F(t)}{\overline{F}(t+x)}\right)^{u_1}\right]}{\sum_{v=j}^{k-1}\binom{n}{v}\left(\frac{F(t)}{\overline{F}(t+x)}\right)^v\left[\sum_{u_2=0}^{i-1-v}\binom{n-v}{u_2}\left(\frac{F(t+x)-F(t)}{\overline{F}(t+x)}\right)^{u_2}\right]}$$

$$= 1 + \frac{\sum_{v=j}^{k-1} \binom{n}{v} \left(\frac{F(t)}{\overline{F}(t+x)} \right)^v \left[\sum_{u_1=i-v}^{m-1-v} \binom{n-v}{u_1} \left(\frac{F(t+x)-F(t)}{\overline{F}(t+x)} \right)^{u_1} \right]}{\sum_{v=j}^{k-1} \left(\frac{F(t)}{\overline{F}(t+x)} \right)^v \left[\sum_{u_2=0}^{i-1-v} \binom{n-v}{u_2} \left(\frac{F(t+x)-F(t)}{\overline{F}(t+x)} \right)^{u_2} \right]}. \tag{20}$$

Note that to prove the desired property of the function in (20), it suffices to show the following inequality

$$\frac{\sum_{u_1=i-v}^{m-1-v} \binom{n-v}{u_1} \left(\frac{F(t+y)-F(t)}{\overline{F}(t+y)} \right)^{u_1}}{\sum_{u_1=i-v}^{m-1-v} \binom{n-v}{u_1} \left(\frac{F(t+x)-F(t)}{\overline{F}(t+x)} \right)^{u_1}} \geq \frac{\sum_{u_2=0}^{i-1-v} \binom{n-v}{u_2} \left(\frac{F(t+y)-F(t)}{\overline{F}(t+y)} \right)^{u_2}}{\sum_{u_2=0}^{i-1-v} \binom{n-v}{u_2} \left(\frac{F(t+x)-F(t)}{\overline{F}(t+x)} \right)^{u_2}}$$

for $y > x$. Considering the relation between the sums

$$\begin{aligned} &\sum_{u_1=i-v}^{m-1-v} \binom{n-v}{u_1} \left(\frac{F(t+y)-F(t)}{\overline{F}(t+y)} \right)^{u_1} \sum_{u_2=0}^{i-1-v} \binom{n-v}{u_2} \left(\frac{F(t+x)-F(t)}{\overline{F}(t+x)} \right)^{u_2} \\ &\quad \geq \sum_{u_1=i-v}^{m-1-v} \binom{n-v}{u_1} \left(\frac{F(t+x)-F(t)}{\overline{F}(t+x)} \right)^{u_1} \sum_{u_2=0}^{i-1-v} \binom{n-v}{u_2} \left(\frac{F(t+y)-F(t)}{\overline{F}(t+y)} \right)^{u_2}, \end{aligned} \tag{21}$$

which consist of the same number of summands, it suffices to compare their respective summands. For fixed $u_1 = i - l, \ldots, m - l - 1$ and $u_2 = 0, \ldots, i - l - 1$, we verify that

$$\begin{aligned} &\binom{n-v}{u_1} \binom{n-v}{u_2} \left(\frac{F(t+y)-F(t)}{\overline{F}(t+y)} \right)^{u_2} \left(\frac{F(t+x)-F(t)}{\overline{F}(t+x)} \right)^{u_2} \\ &\quad \cdot \left[\left(\frac{F(t+y)-F(t)}{\overline{F}(t+y)} \right)^{u_1-u_2} - \left(\frac{F(t+x)-F(t)}{\overline{F}(t+x)} \right)^{u_1-u_2} \right] \geq 0. \end{aligned} \tag{22}$$

Note that the expression in the brackets is positive because the function $(F(t+x) - F(t))/\overline{F}(t+x)$ is increasing in x for all t and $u_1 > u_2$. Thus it proves (22). If the same relation holds for each pair of summands, the inequality is inherited by the respective sums (cf. (21)), which completes the proof. □

As the consequence of the representation given in (18) and the mixture preservation results established by Shaked and Shanthikumar (2007, Theorems 1.A.6 and 1.B.14), we compare two coherent systems with identically distributed components and the same ordered signatures.

Theorem 5 *Let* $\mathbf{s}_i = (0, \ldots, 0, s_{i,k}, s_{i,k+1}, \ldots, s_{i,n})$*,* $i = 1, 2$*, be the signatures of two coherent systems* $T_1 = \tau_1(X_1, \ldots, X_n)$ *and* $T_2 = \tau_2(Y_1, \ldots, Y_n)$ *whose lifetimes* $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ *are iid rvs with a common discrete distribution function* F.

a) If $\mathbf{s}_1 \leq_{st} \mathbf{s}_2$ $(\sum_{j=i}^{n} s_{1,j} \leq \sum_{j=i}^{n} s_{2,j}, i = 2\ldots, n)$*, then* $[T_1 - t | X_{j:n} \leq t < X_{k:n}] \leq_{st} [T_2 - t | Y_{j:n} \leq t < Y_{k:n}]$.
b) If $\mathbf{s}_1 \leq_{hr} \mathbf{s}_2$*,* $(\sum_{j=i}^{n} s_{2,j} / \sum_{j=i}^{n} s_{1,j}$ *is increasing in* i*,* $i = 1, \ldots, n)$*, then* $[T_1 - t | X_{j:n} \leq t < X_{k:n}] \leq_{hr} [T_2 - t | Y_{j:n} \leq t < Y_{k:n}]$.

Remark 5 Notice that the proofs of Theorems 1–5 still go when we drop the assumption that $X_1, \ldots, X_n$ are the discrete rvs. Therefore they can be applied not only in the discrete case but also in the general situation of any non-degenerate distribution of component lifetimes.

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