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Some conditional reliability properties of *k*-out-of-*n* system composed of different types of components with discrete independent lifetimes

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Abstract
In this paper, we study reliability properties of k-out-of-n system consisting of l $(1 \leq l \leq n)$ different types of components with discrete, independent lifetimes. We obtain some conditional survival functions of lifetime of a used system. Next, we use them to calculate two conditional failure probabilities of k-out-of-n systems and show that they are equal to unconditional failure probability of a k-out-of-$(n-r)$ system, $r < n-k+1$. These results are extended versions of the respective ones existing in the literature.

Keywords Discrete lifetime distributions · k-out-of-n system · Not identically distributed random variables · Reliability theory

1 Introduction

A technical system has a k-out-of-n structure if it works when at least k of the n components operate. It fails if $n-k+1$ or more components fail. Two important particular cases of k-out-of-n systems are parallel and series ones corresponding to $k = 1$ and $k = n$. In the literature, many authors paid attention to the reliability and aging properties of k-out-of-n systems and their variants and extensions, see, for example Eryilmaz (2011, 2012, 2013), Navarro and Duarte (2017), Navarro et al. (2017), Misra and Francis (2018), Zhang et al. (2018), Balakrishnan et al. (2018) and Salehi et al. (2019). Most of these results have been restricted to the case when the component lifetimes are independent and identically distributed. However, in some practical situations systems might be composed of independent and nonidentical components. The most recent results in this direction are in Li and Chen (2004), Xu (2008), Sadegh (2008), Zhao et al. (2008), Gurler and Bairamov (2009), Kochar and Xu (2010), Salehi

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and Asadi (2010), Salehi et al. (2011) and Sutar and Naik-Nimbalkar (2019), under the assumption that the component lifetimes have absolutely continuous distributions.

The situation becomes more complicated in the case in which the parent distribution of the component lifetimes is discrete. This is so due to the presence of ties between components failures. This assumption might be more adequate for example when the component lifetimes represent the numbers of turn-on and switch-off up to the failure or when the system's elements operate in discrete cycles, or are exposed to shocks occurring in discrete times. Reliability properties of k-out-of-n systems composed of components which have discrete operation times have been considered by Weiss (1962), Young (1970), Tank and Eryilmaz (2015), Dembińska and Goroncy (2020) and Dembińska et al. (2021).

Dembińska (2018) established explicit expressions for unconditional and some conditional probabilities of a failure of a k-out-of-n system whose component lifetimes $X_1, \ldots, X_n$, are not necessarily independent nor identically distributed discrete variates. Let $X_{1:n} \leq \ldots \leq X_{n:n}$ stand for the order statistics corresponding to $X_1, \ldots, X_n$ and $T_{k,n}$ denote the lifetime of the k-out-of-n system. In particular, she obtained the formula describing the conditional probability that this system will break down at time t_j given the times of failures of its components which occurred up to time t_i:

$$\mathrm{P}\left(T_{k,n} = t_j | X_{1:n} = t_{i_1}, X_{2:n} = t_{i_2}, \ldots, X_{r:n} = t_{i_r}, X_{r+1:n} > t_i\right), \tag{1}$$

$r < n-k+1$ and $t_{i_1} \leq t_{i_2} \leq \ldots \leq t_{i_r} \leq t_i < t_j$, where $t_{i_1} \leq t_{i_2} \leq \ldots \leq t_{i_r} \leq t_i$ are such that $\mathrm{P}\left(X_{1:n} = t_{i_1}, X_{2:n} = t_{i_2}, \ldots, X_{r:n} = t_{i_r}, X_{r+1:n} > t_i\right) > 0$. She also considered the situation when at time t_i we registered a failure of a component of the k-out-of-n system and we observed that at this time exactly r components were broken, $r < n-k+1$. Then, repeating similar arguments as for (1) she computed the conditional probability that the system will fail to function at time $t_j > t_i$:

$$\mathrm{P}(T_{k,n} = t_j | X_{r:n} = t_i, X_{r+1:n} > t_i), \tag{2}$$

where t_i is such that $P(X_{r:n} = t_i, X_{r+1:n} > t_i) > 0$. Next, the probabilities (1) and (2) were applied to obtain the corresponding residual lifetimes of a used system. Under the assumption that $X_1, \ldots, X_n$ are identically distributed with common cumulative distribution function (cdf) F, she observed that the probability (1) does not depend on $t_{i_1} \leq t_{i_2} \leq \ldots \leq t_{i_r}$ and as well as the probability (2) is equal to unconditional probability that a k-out-of-$(n-r)$ system, consisting of homogeneous elements with lifetimes $Y_1, \ldots, Y_{n-r}$ having cdf given by

$$F^Y(x) = \mathrm{P}(X_i \leq x | X_i > t_i) = \begin{cases} \frac{F(x)-F(t_i)}{\overline{F}(t_i)}, & \text{if } x > t_i, \\ 0, & \text{if } x \leq t_i, \end{cases}$$

will brake down at time t_j $(t_j > t_i)$. Our aim is to extend these results by considering k-out-of-n systems with independent component lifetimes that are of l $(1 \leq l \leq n)$ different types and adding some extra information in the conditions of the probabilities (1) and (2) which concerns failures of these components. This is done in Sect. 2.

Throughout the paper we write $\mathrm{I}(\cdot)$ for the indicator function, that is $\mathrm{I}(x \in A) = 1$ if $x \in A$ and $\mathrm{I}(x \in A) = 0$ otherwise.

2 Main result

Consider a k-out-of-n system which is composed of n independently operating components. We assume that the lifetimes of the components, $X_1, X_2, \ldots, X_n$, are discrete random variables (rvs) of l $(1 \leq l \leq n)$ different types. There are exactly n_w rvs of type w having cdf F_w, $w = 1, \ldots, l$ (the cdfs F_w, $w = 1, \ldots, l$, are pairwise different and $n_1 + n_2 + \ldots + n_l = n$). Without loss of generality we can assume that

$$X_1, \ldots, X_{n_1} \sim F_1, \quad X_{n_1+1}, \ldots, X_{n_1+n_2} \sim F_2, \quad \ldots \quad X_{n_1+\ldots+n_{l-1}+1}, \ldots, X_n \sim F_l.$$

The k-out-of-n system functions as long as at least k of its n components function. It fails when the $(n-k+1)$-th component failure occurs. Thus the lifetime of the k-out-of-n system is $T_{k,n} = X_{n-k+1:n}$. The discrete case becomes more complicated than the continuous one due to possible ties between component failures with non-zero probability. In this case at the moment of the system failure the number of inoperative elements can be larger than $n-k+1$.

Let $\mathcal{T} = \{t_1, \ldots, t_N\}$, where $N \leq \infty$, be the union of the supports of F_w, $w = 1, \ldots, l$, and assume that $t_1 < t_2 < \ldots < t_N$. Next, if $X_i \sim F_w$, $w = 1, \ldots, l$, then $p_w(t) = \mathrm{P}_w(X_i = t)$, i.e. p_w is the probability mass function (pmf) corresponding to F_w, $F_w(t^-) = \mathrm{P}_w(X_i < t)$ and $\overline{F}_w(t) = 1 - F_w(t)$.

We need the following definition.

Definition 1 For a fixed $\omega \in \Omega$ we write $X_{j:n}(\omega) \rightsquigarrow F$ instead of $X_{h_j(\omega)} \sim F$, where the function h_j is defined as follows:

(i) if there is exactly one p such that $X_{j:n}(\omega) = X_p(\omega)$, then $h_j(\omega) = p$,

(ii) otherwise, if $X_{1:n}(\omega) = X_{j:n}(\omega)$ and $X_{j:n}(\omega) = X_{p_1}(\omega) = X_{p_2}(\omega) = \ldots = X_{p_m}(\omega)$, where $j \leq m$ and $1 \leq p_1 < p_2 < \ldots < p_m \leq n$, then $h_j(\omega) = p_j$; and if j_1 is the largest integer satisfying $X_{j_1:n}(\omega) < X_{j:n}(\omega)$ and $X_{j:n}(\omega) = X_{p_1}(\omega) = X_{p_2}(\omega) = \ldots = X_{p_m}(\omega)$, where $1 \leq p_1 < p_2 < \ldots < p_m \leq n$, then $h_j(\omega) = p_{j-j_1}$.

Now we are able to define

$$S_i = \left(S_i^{(1)}, \ldots, S_i^{(l)}\right) \quad \text{and} \quad G_i = \left(G_i^{(1)}, \ldots, G_i^{(l)}\right), \quad i = 1, \ldots, n,$$

where

$$\begin{aligned} S_i^{(w)} &= \#\{j \leq i : X_{j:n} \rightsquigarrow F_w\}, \\ G_i^{(w)} &= \#\{j \leq i : X_{j:n} = X_{i:n}, X_{j:n} \rightsquigarrow F_w\}, \quad w = 1, \ldots, l, \quad i = 1, \ldots, n. \end{aligned}$$

Observe that $S_i^{(w)}$ informs us how many of X_j's of type w are not greater than $X_{i:n}$ and $G_i^{(w)}$ limits to such of them which are equal to $X_{i:n}$. Moreover $S_i^{(l)} = i - \sum_{w=1}^{l-1} S_i^{(w)}$,

$i = 1, \ldots, n$. Definition 1 with the example of its application as well as the constructions of vectors S_i and G_i, $i = 1, \ldots, n$ were proposed by Jasiński (2020).

Firstly, we assume that at the moment $X_{r:n} = t_i$ we registered $g_r^{(w)}$ failures of components of type w, $w = 1, \ldots, l$ of a used k-out-of-n system. We also assume that at this time exactly $s_r^{(w)}$ elements of type w were broken, where $\sum_{w=1}^{l} s_r^{(w)} = r$ and $r < n - k + 1$. Then it is of interest to obtain the following conditional survival function of $T_{k,n}$

$$p_{k;r}(t_i + x | t_i, g_r, s_r) = \mathrm{P}\left(T_{k,n} > t_i + x | X_{r:n} = t_i, X_{r+1:n} > t_i, G_r = g_r, S_r = s_r\right),$$

where t_i, $g_r = \left(g_r^{(1)}, \ldots, g_r^{(l)}\right)$ and $s_r = \left(s_r^{(1)}, \ldots, s_r^{(l)}\right)$, are chosen so that the probability $p^*_{k;r}(t_i, g_r, s_r)$ of $\{X_{r:n} = t_i, X_{r+1:n} > t_i, G_r = g_r, S_r = s_r\}$ is not equal to 0.

We begin with an observation that

$$p^*_{k;r}(t_i, g_r, s_r) = \prod_{w=1}^{l} \mathrm{P}\left(A^{t_i}_{g_r^{(w)}, s_r^{(w)}}\right), \tag{3}$$

where

$$\begin{aligned} A^{t_i}_{g_r^{(w)}, s_r^{(w)}} = \big\{&\text{exactly } s_r^{(w)} - g_r^{(w)} \text{ of } X_i \sim F_w \text{ are } < t_i, \\ &\text{exactly } g_r^{(w)} \text{ of } X_i \sim F_w \text{ are } = t_i, \\ &\text{and the rest } n_w - s_r^{(w)} \text{ of } X_i \sim F_w \text{ are } > t_i\big\}. \end{aligned}$$

It follows immediately that

$$\begin{aligned} \mathrm{P}\left(A^{t_i}_{g_r^{(w)}, s_r^{(w)}}\right) = &\frac{n_w!}{\left(s_r^{(w)} - g_r^{(w)}\right)! g_r^{(w)}! \left(n_w - s_r^{(w)}\right)!} \left(F_w(t_i^-)\right)^{s_r^{(w)} - g_r^{(w)}} \\ &\cdot \left(p_w(t_i)\right)^{g_r^{(w)}} \left(\overline{F}_w(t_i)\right)^{n_w - s_r^{(w)}}. \end{aligned} \tag{4}$$

Now we determine the probability

$$\begin{aligned} p^{**}_{k;r}(t_i, t_i + x, g_r, s_r) &= \mathrm{P}\left(X_{r:n} = t_i, X_{r+1:n} > t_i, T_{k,n} > t_i + x, G_r = g_r, S_r = s_r\right) \\ &= \mathrm{P}\left(X_{r:n} = t_i, X_{r+1:n} > t_i, X_{n-k+1:n} > t_j, G_r = g_r, S_r = s_r\right), \end{aligned}$$

where

$$t_j \in \mathcal{T} \text{ is such that } t_i + x \in [t_j, t_{j+1}). \tag{5}$$

For abbreviation, let

$$\tilde{v}_0 = 0, \quad \tilde{v}_w = \sum_{j=1}^{w} v_j, \quad \delta_w(v) = \min\{n_w - s_r^{(w)}, v - \tilde{v}_{w-1}\}, \quad w = 1, \ldots, l-1. \tag{6}$$

Then

$$p_{k;r}^{**}(t_i, t_i + x, g_r, s_r) = \sum_{v=0}^{n-r-k} \left(\prod_{w=1}^{l-1} \sum_{v_w=0}^{\delta_w(v)} \mathrm{P}\left(B_{g_r^{(w)}, s_r^{(w)}, v_w}^{t_i, t_j} \right) \right) \cdot \left(\mathrm{P}\left(B_{g_r^{(l)}, s_r^{(l)}, v - \tilde{v}_{l-1}}^{t_i, t_j} \right) \mathrm{I}(v - \tilde{v}_{l-1} \le n_l - s_r^{(l)}) \right), \tag{7}$$

where

$$\begin{aligned} B_{g_r^{(w)}, s_r^{(w)}, v_w}^{t_i, t_j} = \big\{ & \text{exactly } s_r^{(w)} - g_r^{(w)} \text{ of } X_i \sim F_w \text{ are } < t_i, \\ & \text{exactly } g_r^{(w)} \text{ of } X_i \sim F_w \text{ are } = t_i, \\ & \text{exactly } v_w \text{ of } X_i \sim F_w \text{ are } \in (t_i, t_j], \\ & \text{and the rest } n_w - s_r^{(w)} - v_w \text{ of } X_i \sim F_w \text{ are } > t_j \big\}. \end{aligned} \tag{8}$$

and $B_{g_r^{(l)}, s_r^{(l)}, v - \tilde{v}_{l-1}}^{t_i, t_j}$ is given by (8) with $w = l$ and v_w replaced by $v - \tilde{v}_{l-1}$. Denoting

$$f_w(t_i, t_j, u) = \binom{n_w - s_r^{(w)}}{u} \left(F_w(t_j) - F_w(t_i) \right)^u \left(\overline{F}_w(t_j) \right)^{n_w - s_r^{(w)} - u}, \tag{9}$$

we have

$$\mathrm{P}\left(B_{g_r^{(w)}, s_r^{(w)}, v_w}^{t_i, t_j} \right) = \frac{n_w!}{\left(s_r^{(w)} - g_r^{(w)} \right)! g_r^{(w)}! \left(n_w - s_r^{(w)} \right)!} \left(F_w(t_i^-) \right)^{s_r^{(w)} - g_r^{(w)}} \cdot (p_w(t_i))^{g_r^{(w)}} f_w(t_i, t_j, v_w). \tag{10}$$

Similarly, we obtain $\mathrm{P}\left(B^{t_i,t_j}_{g_r^{(l)},s_r^{(l)},v-\tilde{v}_{l-1}}\right)$. Now combining (3) with (4) and (7) with (10), after simple algebra, we derive the desired conditional probability as follows

$$
\begin{aligned}
p_{k;r}(t_i+x|t_i,g_r,s_r) &= \frac{p^{**}_{k;r}(t_i,t_i+x,g_r,s_r)}{p^{*}_{k;r}(t_i,g_r,s_r)} \\
&= \frac{\sum\limits_{v=0}^{n-r-k}\left(\prod\limits_{w=1}^{l-1}\sum\limits_{v_w=0}^{\delta_w(v)} f_w(t_i,t_j,v_w)\right)\left(f_l(t_i,t_j,v-\tilde{v}_{l-1})\,\mathrm{I}(v-\tilde{v}_{l-1}\le n_l-s_r^{(l)})\right)}{\prod\limits_{w=1}^{l}\left(\overline{F}_w(t_i)\right)^{n_w-s_r^{(w)}}} \\
&= \sum_{v=0}^{n-r-k}\left[\prod_{w=1}^{l-1}\sum_{v_w=0}^{\delta_w(v)}\binom{n_w-s_r^{(w)}}{v_w}\left(\frac{F_w(t_j)-F_w(t_i)}{\overline{F}_w(t_i)}\right)^{v_w}\left(\frac{\overline{F}_w(t_j)}{\overline{F}_w(t_i)}\right)^{n_w-s_r^{(w)}-v_w}\right] \\
&\quad\cdot\binom{n_l-s_r^{(l)}}{v-\tilde{v}_{l-1}}\left(\frac{F_l(t_j)-F_l(t_i)}{\overline{F}_l(t_i)}\right)^{v-\tilde{v}_{l-1}}\left(\frac{\overline{F}_l(t_j)}{\overline{F}_l(t_i)}\right)^{n_l-s_r^{(l)}-v+\tilde{v}_{l-1}} \\
&\quad\cdot\mathrm{I}(v-\tilde{v}_{l-1}\le n_l-s_r^{(l)}).
\end{aligned}
\tag{11}
$$

Since

$$
\begin{aligned}
\mathrm{P}\left(T_{k,n}=t_j|X_{r:n}\right. &= t_i, X_{r+1:n}>t_i, G_r=g_r, S_r=s_r) \\
&= p_{k;r}(t_{j-1}|t_i,g_r,s_r)-p_{k;r}(t_j|t_i,g_r,s_r),
\end{aligned}
$$

applying (11), we immediately get the conditional probability that the system will fail to function at time t_j.

Notice that

$$
\begin{aligned}
&p_{k;r}(t_i+x|t_i,g_r,s_r) \\
&\quad=\sum_{v=0}^{n-r-k}\left[\prod_{w=1}^{l-1}\sum_{v_w=0}^{\delta_w(v)}\binom{n_w-s_r^{(w)}}{v_w}\left(F_w^Y(t_j)\right)^{v_w}\left(\overline{F}_w^Y(t_j)\right)^{n_w-s_r^{(w)}-v_w}\right] \\
&\qquad\cdot\binom{n_l-s_r^{(l)}}{v-\tilde{v}_{l-1}}\left(F_l^Y(t_j)\right)^{v-\tilde{v}_{l-1}}\left(\overline{F}_l^Y(t_j)\right)^{n_l-s_r^{(l)}-v+\tilde{v}_{l-1}}\mathrm{I}(v-\tilde{v}_{l-1}\le n_l-s_r^{(l)}) \\
&\quad=\mathrm{P}(Y_{n-r-k+1:n-r}>t_j),
\end{aligned}
$$

where $Y_1,\ldots,Y_{n-r}$ are independent rvs of possibly l different types having cdfs F_w^Y, $w=1,\ldots,l$, given by

$$
F_w^Y(y)=\mathrm{P}_w(X_i\le y|X_i>t_i)=\begin{cases}\frac{F_w(y)-F_w(t_i)}{\overline{F}_w(t_i)}, & \text{if } y>t_i,\\ 0, & \text{if } y\le t_i.\end{cases}
\tag{12}
$$

More precisely, there are exactly $n_w-s_r^{(w)}$ rvs of type w having cdf F_w^Y, $w=1,\ldots,l$, where $\sum_{w=1}^{l} n_w-s_r^{(w)}=n-r$.

Because the survival function uniquely determines the distribution, this proves the following theorem.

Theorem 1 *Under the above assumptions and notation, for* $r < n-k+1$ *the conditional distribution of* $T_{k,n}$ *given* $X_{r:n} = t_i$, $X_{r+1:n} > t_i$, $G_r = g_r$, $S_r = s_r$

(i) does not depend on $g_r^{(1)}, \ldots, g_r^{(l)}$,
(ii) is just equal to the unconditional distribution of $T_{k,n-r}^Y$ *the lifetime of k-out-of-*$(n-r)$ *system consisting of components with independent lifetimes* $Y_1, \ldots, Y_{n-r}$, *where there are exactly* $n_w - s_r^{(w)}$ *of* Y_i*'s of type* w *having cdf* F_w^Y, $w = 1, \ldots, l$, *defined in (12).*

Now for $r < n-k+1$ we are interested in finding the following conditional survival function of $T_{k,n}$

$$
\begin{aligned}
p_{k;r}(t_i + x | t_i, s_1, \ldots, s_n) = \mathrm{P}\big(&T_{k,n} > t_i + x | X_{1:n} = t_{i_1}, \ldots, \\
&X_{r:n} = t_{i_r}, X_{r+1:n} > t_i, S_1 = s_1, \ldots, S_r = s_r\big),
\end{aligned}
$$

where $t_{i_1} \leq \ldots \leq t_{i_r} \leq t_i$ and $s_j = \left(s_j^{(1)}, \ldots, s_j^{(l)}\right)$, $j = 1, \ldots, r$, are chosen so that the probability of $\{X_{1:n} = t_{i_1}, \ldots, X_{r:n} = t_{i_r}, X_{r+1:n} > t_i, S_1 = s_1, \ldots, S_r = s_r\}$ is not equal to 0. Here $t_{i_1}, \ldots, t_{i_r}$ are the ordered failures times of the components of the system which occurred up to time t_i. Moreover, there are exactly $s_j^{(w)}$ elements of type w, $w = 1, \ldots, l$, that were broken at the time t_{i_j}, $j = 1, \ldots, r$.

Using the concept of tie-runs proposed by Gan and Bain (1995), let us assume that $t_{i_1} \leq t_{i_2} \leq \ldots \leq t_{i_r}$ have m tie-runs with lengths $z_1, z_2, \ldots, z_m$ $(z_1 + \ldots + z_m = r)$, i.e.

$$
t_{i_1} = \ldots = t_{i_{z_1}} < t_{i_{z_1+1}} = \ldots = t_{i_{z_1+z_2}} < \ldots < t_{i_{z_1+..+z_{m-1}+1}} = \ldots = t_{i_{z_1+..+z_m}} (= t_{i_r}).
$$

We begin with the probability

$$
\begin{aligned}
&p^*_{k;1,\ldots,r}(t_{i_1}, \ldots, t_{i_r}, t_i, s_1, \ldots, s_r) \\
&\quad = \mathrm{P}\left(X_{1:n} = t_{i_1}, \ldots, X_{r:n} = t_{i_r}, X_{r+1:n} > t_i, S_1 = s_1, \ldots, S_r = s_r\right).
\end{aligned}
$$

Notice that

$$
p^*_{k;1,\ldots,r}(t_{i_1}, \ldots, t_{i_r}, t_i, s_1, \ldots, s_r) = \prod_{w=1}^{l} \mathrm{P}\left(A^{t_{i_{z_1}}, \ldots, t_{i_{z_1+..+z_m}}, t_i}_{s^{(w)}_{z_1}, \ldots, s^{(w)}_{z_1+\ldots+z_m}}\right), \tag{13}
$$

where

$$A^{t_{i_{z_1}},\dots,t_{i_{z_1+..+z_m}},t_i}_{s^{(w)}_{z_1},\dots,s^{(w)}_{z_1+\dots+z_m}} = \Big\{\text{exactly } s^{(w)}_{z_1} \text{ of } X_i \sim F_w \text{ are } = t_{i_{z_1}},$$
$$\text{exactly } s^{(w)}_{z_1+z_2} - s^{(w)}_{z_1} \text{ of } X_i \sim F_w \text{ are } = t_{i_{z_1+z_2}},$$
$$\vdots$$
$$\text{exactly } s^{(w)}_{z_1+..+z_m} - s^{(w)}_{z_1+..+z_{m-1}} \text{ of } X_i \sim F_w \text{ are} = t_{i_r},$$
$$\text{and the rest } n_w - s^{(w)}_{z_1+\dots+z_m} \text{ of } X_i \sim F_w \text{ are } > t_i\Big\}.$$

Thus

$$\mathrm{P}\left(A^{t_{i_{z_1}},\dots,t_{i_{z_1+..+z_m}},t_i}_{s^{(w)}_{z_1},\dots,s^{(w)}_{z_1+\dots+z_m}}\right) = \frac{n_w!}{\left(n_w - s^{(w)}_{z_1+..+z_m}\right)! \prod_{h=1}^{m}\left(s^{(w)}_{z_1+..+z_h} - s^{(w)}_{z_1+..+z_{h-1}}\right)!} \cdot \left(\prod_{h=1}^{m}\left(p_w(t_{i_{z_1+\dots+z_h}})\right)^{s^{(w)}_{z_1+\dots+z_h} - s^{(w)}_{z_1+\dots+z_{h-1}}}\right)\left(\overline{F}_w(t_i)\right)^{n_w - s^{(w)}_{z_1+\dots+z_m}}, \tag{14}$$

with $s^{(w)}_{z_0} = 0$.

Now we will obtain the probability $p^{**}_{k;1,\dots,r}(t_{i_1},\dots,t_{i_r},t_i,t_i+x,s_1,\dots,s_r)$ of the event $\{X_{1:n} = t_{i_1},\dots,X_{r:n} = t_{i_r}, X_{r+1:n} > t_i, X_{n-k+1:n} > t_i + x, S_1 = s_1,\dots,S_r = s_r\}$. Since $z_1 + \dots + z_m = r$, with the notation (6), we get

$$\begin{aligned} &p^{**}_{k;1,\dots,r}(t_{i_1},\dots,t_{i_r},t_i,t_i+x,s_1,\dots,s_r) \\ &= \sum_{v=0}^{n-r-k}\left(\prod_{w=1}^{l-1}\sum_{v_w=0}^{\delta_w(v)}\mathrm{P}\left(B^{t_{i_{z_1}},\dots,t_{i_{z_1+..+z_m}},t_i,t_j}_{s^{(w)}_{z_1},\dots,s^{(w)}_{z_1+\dots+z_m},v_w}\right)\right) \\ &\quad\cdot\left(\mathrm{P}\left(B^{t_{i_{z_1}},\dots,t_{i_{z_1+..+z_m}},t_i,t_j}_{s^{(l)}_{z_1},\dots,s^{(l)}_{z_1+\dots+z_m},v-\tilde{v}_{l-1}}\right)\mathrm{I}(v-\tilde{v}_{l-1} \le n_l - s^{(l)}_r)\right), \end{aligned} \tag{15}$$

where t_j is defined in (5) and

$$B^{t_{i_{z_1}},\dots,t_{i_{z_1+..+z_m}},t_i,t_j}_{s^{(w)}_{z_1},\dots,s^{(w)}_{z_1+\dots+z_m},v_w} = \Big\{\text{exactly } s^{(w)}_{z_1} \text{ of } X_i \sim F_w \text{ are } = t_{i_{z_1}},$$
$$\text{exactly } s^{(w)}_{z_1+z_2} - s^{(w)}_{z_1} \text{ of } X_i \sim F_w \text{ are } = t_{i_{z_1+z_2}},$$
$$\vdots$$
$$\text{exactly } s^{(w)}_{z_1+..+z_m} - s^{(w)}_{z_1+..+z_{m-1}} \text{ of } X_i \sim F_w \text{ are} = t_{i_r},$$
$$\text{exactly } v_w \text{ of } X_i \sim F_w \text{ are } \in \big(t_i, t_j\big],$$
$$\text{and the rest } n_w - s^{(w)}_{z_1+..+z_m} - v_w \text{ of } X_i \sim F_w \text{ are} > t_j\Big\}. \tag{16}$$

Observe that $B^{t_{i_{z_1}},\dots,t_{i_{z_1+\dots+z_m}},t_i,t_j}_{s^{(l)}_{z_1},\dots,s^{(l)}_{z_1+\dots+z_m},v-\tilde{v}_{l-1}}$ is given by (16) with $w = l$ and v_w replaced by $v - \tilde{v}_{l-1}$. Using (9) we have, for $w = 1, \dots, l$,

$$\mathrm{P}\left(B^{t_{i_{z_1}},\dots,t_{i_{z_1+\dots+z_m}},t_i,t_j}_{s^{(w)}_{z_1},\dots,s^{(w)}_{z_1+\dots+z_m},v_w}\right) = \frac{n_w!}{\left(n_w - s^{(w)}_{z_1+\dots+z_m}\right)! \prod_{h=1}^{m}\left(s^{(w)}_{z_1+\dots+z_h} - s^{(w)}_{z_1+\dots+z_{h-1}}\right)!} \cdot \left(\prod_{h=1}^{m}\left(p_w(t_{i_{z_1+\dots+z_h}})\right)^{s^{(w)}_{z_1+\dots+z_h}-s^{(w)}_{z_1+\dots+z_{h-1}}}\right) f_w(t_i, t_j, v_w), \tag{17}$$

where $v_l = v - \tilde{v}_{l-1}$. Now combining (13) with (14) and (15) with (17), we are able to determine the conditional probability

$$p_{k;r}(t_i + x|t_i, s_1, \dots, s_n) = \frac{p^{**}_{k;1,\dots,r}(t_{i_1}, \dots, t_{i_r}, t_i, t_i + x, s_1, \dots, s_r)}{p^{*}_{k;1,\dots,r}(t_{i_1}, \dots, t_{i_r}, t_i, s_1, \dots, s_r)} = \frac{\sum_{v=0}^{n-r-k}\left(\prod_{w=1}^{l-1}\sum_{v_w=0}^{\delta_w(v)} f_w(t_i, t_j, v_w)\right)\left(f_l(t_i, t_j, v - \tilde{v}_{l-1})\,\mathrm{I}(v - \tilde{v}_{l-1} \le n_l - s^{(l)}_r)\right)}{\prod_{w=1}^{l}\left[\overline{F}_w(t_i)\right]^{n_w - s^{(w)}_{z_1+\dots+z_m}}}. \tag{18}$$

Applying (18), we obtain the conditional probability that the system break down at t_j because

$$\begin{aligned}&\mathrm{P}\big(T_{k,n} = t_j | X_{1:n} = t_{i_1}, \dots, X_{r:n} = t_{i_r}, X_{r+1:n} > t_i, S_1 = s_1, \dots, S_r = s_r\big)\\ &\quad = p_{k;r}(t_{j-1}|t_i, s_1, \dots, s_n) - p_{k;r}(t_j|t_i, s_1, \dots, s_n).\end{aligned}$$

Notice that the formula in (18) is the same as in (11). Then we have the following result.

Theorem 2 *Under the above assumptions and notation, for $r < n - k + 1$ and $t_{i_1} \le t_{i_2} \le \dots \le t_{i_r} \le t_i$, the conditional distribution of $T_{k,n}$ given $X_{1:n} = t_{i_1}, X_{2:n} = t_{i_2}, \dots, X_{r:n} = t_{i_r}, X_{r+1:n} > t_i, S_1 = s_1, \dots, S_r = s_r$,*

(i) does not depend on $t_{i_1}, t_{i_2}, \dots, t_{i_r}$ nor on $s_1, s_2, \dots, s_{r-1}$,
(ii) is the same as the unconditional distribution of $T^Y_{k,n-r}$ the lifetime of k-out-of-$(n-r)$ system consisting of components with independent lifetimes $Y_1, \dots, Y_{n-r}$, where there are exactly $n_w - s^{(w)}_r$ of Y_i's of type w having cdf F^Y_w, $w = 1, \dots, l$, defined in (12).

Remark 1 Theorems (1) and (2) corresponds to Theorem III.4 of Dembińska (2018).

3 Summary and conclusions

A k-out-of-n system is a technical device which plays an important role in the reliability theory. It has various applications in engineering. For example, it is used as the design of servers in internet service or the design of engines in the automotive industries. In this paper, we have considered k-out-of-n systems that consist of multiple types of components. Although such systems are more common in real life situations, their reliability analysis is more difficult and complicated. The operators of the systems are interested in getting inference about the reliability or other specifications of the system but they usually have only some partial information about the lifetime of the system e.g. they registered the times of failures of the components up to and including the fixed time. The presented results would allow the operators for greater planning and more efficient use of resources to reduce unexpected costs of utilization.

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Declarations

Conflict of interest The author states that there is no conflict of interest.

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