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Random discretization of stationary continuous time processes

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Abstract

This paper investigates second order properties of a stationary continuous time process after random sampling. While a short memory process always gives rise to a short memory one, we prove that long-memory can disappear when the sampling law has very heavy tails. Despite the fact that the normality of the process is not maintained by random sampling, the normalized partial sum process converges to the fractional Brownian motion, at least when the long memory parameter is preserved.

Keywords Gaussian process · Long memory · Partial sum · Random sampling · Regularly varying covariance

1 Introduction

Most of the papers on time series analysis assume that observations are equally spaced in time. However, irregularly spaced time series data appear in many applications, for instance in astrophysics, climatology, high frequency finance, signal processing. Elorrieta et al. (2019) and Eyheramendy et al. (2018) propose generalisations of autoregressive models for irregular time series motivated by an application in astronomy. The spectral analysis of these data is also studied in many papers with applications into astrophysics, climatology, physics [see for instance Scargle (1982), Broersen (2007), Mayo (1978), Masry and Lui (1975)].

A possible way to address the problem of non-equally spaced data is to transform the data into equally spaced observations using some methods of interpolation [see for instance Adorf (1995), Friedman (1962), Nieto-Barajas and Sinha (2014)].

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An alternative one consists in assuming that time series can be embedded into a continuous time process. The data are then interpreted as a realization of a continuous temporal process observed at random times [see for instance Jones (1981), Jones and Tryon (1987), Brockwell et al. (2007)]. This approach requires the study of the effects of random sampling on the properties of continuous time process, as well as the development of inference methods for these models. Mykland (2003) studies the effects of the random sampling on the parameter estimation of time-homogeneous diffusion [see also Duffie and Glynn (2004)]. Masry (1994) establishes the properties of spectral density function estimation.

In this paper we focus particularly on time series with long-range dependence [see Beran et al. (2013), Giraitis et al. (2012) for a review of available results for long-memory processes].

Some models and estimation methods have been proposed for continuous-time processes [see Tsai and Chan (2005a), Viano et al. (1994), Comte and Renault (1996), Comte (1996)]. Tsai and Chan (2005a) introduced the continuous-time autoregressive fractionally integrated moving average (CARFIMA(p,d,q)) model. Under the long-range dependence condition $d \in (0, 1/2)$, they calculate the autocovariance function of the stationary CARFIMA process and its spectral density function [see Tsai and Chan (2005b)]. These properties are extended to the case $d \in (-1/2, 1/2)$ in Tsai (2009). In Viano et al. (1994), continuous-time fractional ARMA processes are constructed. They establish the L^2 properties (spectral density and autocovariance function) and the dependence structure. Comte and Renault (1996) study the continuous time moving average fractional process, a family of long memory model. The statistical inference for continuous-time processes is generally constructed from the sampled process [see Tsai and Chan (2005a, b), Chambers (1996), Comte (1996)]. Different schemes of sampling can be considered. In Tsai and Chan (2005a), the estimation method is based on the maximum likelihood estimation for irregularly spaced deterministic time series data. Under the assumption of identifiability, Chambers (1996) considers the estimation of the long memory parameter of a continuous time fractional ARMA process with discrete time data using the low-frequency behaviour of the spectrum. Comte (1996) studied two methods for the estimation with regularly spaced data: Whittle likelihood method and the semiparametric approach of Geweke and Porter-Hudak. In this article we are interested in irregularly spaced data when the sampling intervals are independent and identically distributed positive random variables. In the light of previous results in discrete time, there was an effect of the random sampling on the dependence structure of the process. Indeed, Philippe and Viano (2010) show that the intensity of the long memory is preserved when the law of sampling intervals has finite first moment, but they also pointed out situations where a reduction of the long memory is observed.

We adopt the most usual definition of second order long memory process. Namely, a stationary process $\mathbf{U}$ has the long memory property if its autocovariance function σ_U satisfies the condition

$$\int_{\mathbb{R}^+} |\sigma_U(x)| \, \mathrm{d}x = \infty \quad \text{in the continuous-time case,}$$

$$\sum_{h \geq 0} |\sigma_U(h)| = \infty \quad \text{in the discrete-time case.}$$

We study the effect of random sampling on the properties of a stationary continuous time process. More precisely, we start with $\mathbf{X} = (X_t)_{t \in \mathbb{R}^+}$, a second-order stationary continuous time process. We assume that it is observed at random times $(T_n)_{n \geq 0}$ where $(T_n)_{n \geq 0}$ is a non-decreasing positive random walk independent of $\mathbf{X}$. We study the discrete-time process $\mathbf{Y}$ defined by

$$Y_n = X_{T_n}, \quad n \in \mathbb{N}. \tag{1.1}$$

The process $\mathbf{Y}$ obtained by random sampling is called the sampled process.

In this paper, we study the properties of this. In particular, we show that the results obtained by Philippe and Viano (2010) on the auto-covariance function are preserved for continuous time process $\mathbf{X}$. The large-sample statistical inference relies often on limit theorems of probability theory for partial sums. We show that Gaussianity is lost by random sampling. However, we prove that the asymptotic normality of the partial sum is preserved with the same standard normalization [see Giraitis et al. (2012), Chapter 4 for a review].

In Sect. 2, we study the behavior of the sampled process (1.1) for the general case. We establish that Gaussianity of $\mathbf{X}$ is not transmitted to $\mathbf{Y}$. Under rather weak conditions on the covariance σ_X, the weak dependence is preserved. A stronger assumption on the first moment of T_1 is necessary to preserve the long memory property. Under the condition $\mathbb{E}[T_1] < \infty$, if X is a long-memory process then Y also in the sense of definition above. In Sect. 3, we present the more specific situation of a regularly varying covariance where preservation or non-preservation of the memory can be quantified. In particular, we prove that for heavy tailed sampling distribution, a long memory process $\mathbf{X}$ can give raise to a short memory process $\mathbf{Y}$. In Sect. 4, we establish a Donsker's invariance principle when the initial process $\mathbf{X}$ is Gaussian and the long memory parameter is preserved.

2 General properties

Throughout this document we assume that the following properties hold on the initial process $\mathbf{X}$ and the random sampling scheme:

Assumption $\mathcal{H}$:

- $\mathcal{H}1$: $\mathbf{X} = (X_t)_{t \in \mathbb{R}^+}$ is a second-order stationary continuous time process with zero mean and autocovariance function σ_X.
- $\mathcal{H}2$: The random walk $(T_n)_{n \geq 0}$ is independent of $\mathbf{X}$.
- $\mathcal{H}3$: $T_0 = 0$.
- $\mathcal{H}4$: The increments $\Delta_j = T_{j+1} - T_j$ ($j \in \mathbb{N}$) are independent and identically distributed. The common distribution admits a probability density function s (with respect to the Lebesgue measure) supported by $\mathbb{R}^+$.

Remark 1 In Assumption $\mathcal{H}3$, we impose the specific initialization $T_0 = 0$ only to simplify our notations since it implies that $\Delta_j = T_{j+1} - T_j$ for all $j \in \mathbb{N}$. However, all the results remain true if we take $T_0 = \Delta_0$ and $\Delta_j = T_j - T_{j-1}$, for $j \geq 1$.

The following proposition gives the L^2-properties of the sampled process $\mathbf{Y}$.

Proposition 2.1 *Under Assumption $\mathcal{H}$, the discrete-time process $\mathbf{Y}$ defined in (1.1) is also second-order stationary with zero mean and its autocovariance sequence is*

$$\begin{cases} \sigma_Y(0) = \sigma_X(0), \\ \sigma_Y(h) = \mathbb{E}\left[\sigma_X(T_h)\right], \quad h \geq 1. \end{cases} \tag{2.1}$$

Proof These properties are obtained by applying Fubini's theorem, and using the independence between $\mathbf{X}$ and $(T_n)_{n\geq 0}$. Indeed, for all $h \in \mathbb{N}$, we have

$$\begin{aligned} \mathbb{E}\left[Y_h\right] = \mathbb{E}\left[X_{T_h}\right] &= \int\int x_t \,\mathrm{d}P_{\mathbf{X},T_h}(x,t) \\ &= \int\left\{\int x_t \,\mathrm{d}P_{\mathbf{X}}(x)\right\}\mathrm{d}P_{T_h}(t) = \int \mathbb{E}\left[X_t\right]\mathrm{d}P_{T_h}(t) = 0 \end{aligned}$$

and

$$\begin{aligned} \mathrm{Cov}(Y_n, Y_{n+h}) = \mathbb{E}\left[X_{T_n}X_{T_{n+h}}\right] &= \int\int x_t x_s \,\mathrm{d}P_{\mathbf{X},T_n,T_{n+h}}(x,t,s) \\ &= \int\left\{\int x_t x_s \,\mathrm{d}P_{\mathbf{X}}(x)\right\}\mathrm{d}P_{T_n,T_{n+h}}(t,s) \\ &= \int \mathrm{Cov}(X_t, X_s)\,\mathrm{d}P_{T_n,T_{n+h}}(t,s) \\ &= \int \sigma_X(t-s)\,\mathrm{d}P_{T_0,T_h}(t,s) \\ &= E\left[\sigma_X(T_h)\right]. \end{aligned}$$

□

2.1 Distribution of the sampled process

This part is devoted to the properties of the finite-dimensional distributions of the process $\mathbf{Y}$.

Proposition 2.2 *If $\mathbf{X}$ is a strictly stationary process satisfying $\mathcal{H}2 - \mathcal{H}4$, then the sampled process $\mathbf{Y}$ is a strictly stationary discrete-time process.*

Proof We arbitrarily fix $n \geq 1$, $p \in \mathbb{N}^*$ and $k_1, \ldots k_n \in \mathbb{N}$ such that $0 \leq k_1 < \cdots < k_n$. We show that the joint distribution of $(Y_{k_1+p}, \ldots, Y_{k_n+p})$ does not depend on $p \in \mathbb{N}$.

For $(y_1, \ldots, y_n) \in \mathbb{R}^n$, we have

$$P(Y_{k_1+p} \leq y_1, \dots, Y_{k_n+p} \leq y_n) = P\left(X_{T_{k_1+p}} \leq y_1, \dots, X_{T_{k_n+p}} \leq y_n\right)$$
$$= \mathbb{E}\left[P\left(X_{\Delta_0+\dots+\Delta_{k_1+p-1}} \leq y_1, \dots, X_{\Delta_0+\dots+\Delta_{k_n+p-1}} \leq y_n | \Delta_0, \dots \Delta_{k_n+p-1}\right)\right],$$

where $(\Delta_j)_{j\in\mathbb{N}}$ are the increments defined in Assumption $\mathcal{H}$. By the strict stationarity of $\mathbf{X}$ the right-hand-side of the last equation is equal to

$$\mathbb{E}\left[P\left(X_{\Delta_p+\dots+\Delta_{k_1+p-1}} \leq y_1, \dots, X_{\Delta_p+\dots+\Delta_{k_n+p-1}} \leq y_n | \Delta_0, \dots \Delta_{k_n+p-1}\right)\right]$$
$$= P(X_{U_0+\dots+U_{k_1-1}} \leq y_1, \dots, X_{U_0+\dots+U_{k_n-1}} \leq y_n) = P(Y_{k_1} \leq y_1, \dots, Y_{k_n} \leq y_n),$$

where $U_i = \Delta_{i+p}$ are i.i.d with density s. This concludes the proof. □

The following proposition is devoted to the particular case of a Gaussian process. We establish that the Gaussianity is not preserved by random sampling.

Proposition 2.3 *Under Assumption $\mathcal{H}$, if $\mathbf{X}$ is a Gaussian process then the marginals of the sampled process $\mathbf{Y}$ are Gaussian. Furthermore, if σ_X is not almost everywhere constant on the set $\{x : s(x) > 0\}$, then $\mathbf{Y}$ is not a Gaussian process.*

Proof We first prove the normality of marginal distributions.

Let U be a random variable, we denote Φ_U its characteristic function. We have, for all $t \in \mathbb{R}$

$$\Phi_{Y_k}(t) = \mathbb{E}\left[\mathbb{E}[e^{itX_{T_k}}|T_k]\right].$$

Conditionally on T_k, the probability distribution of X_{T_k} is the Gaussian distribution with zero mean and variance $\sigma_X(0)$. We get

$$\Phi_{Y_k}(t) = e^{-\sigma_X(0)t^2/2},$$

and thus Y_k is a Gaussian variable with zero mean and variance $\sigma_X(0)$.

We are now proving that the process is not Gaussian by contraposition, i.e. if $\mathbf{Y}$ is a Gaussian process then σ_X is almost everywhere constant on the set $\{x : s(x) > 0\}$. If $\mathbf{Y}$ is a Gaussian process then the random variable $Y_1 + Y_2$ has a Gaussian distribution (since it is a linear combination of two components of $\mathbf{Y}$),

$$\Phi_{Y_1+Y_2}(t) = e^{-\mathrm{Var}(Y_1+Y_2)t^2/2} = e^{-\sigma_X(0)t^2} e^{-t^2\mathbb{E}[\sigma_X(T_2-T_1)]},$$

and

$$\Phi_{Y_1+Y_2}(t) = \Phi_{X_{T_1}+X_{T_2}}(t)$$
$$= \mathbb{E}\left[\exp\left\{-\frac{t^2}{2}\begin{pmatrix}1\\1\end{pmatrix}^T \begin{pmatrix}\sigma_X(0) & \sigma_X(T_2-T_1)\\ \sigma_X(T_2-T_1) & \sigma_X(0)\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\right\}\right]$$

$$= e^{-\sigma_X(0)t^2}\mathbb{E}\left[e^{-t^2\sigma_X(T_2-T_1)}\right].$$

Then, for all $t \in \mathbb{R}$,

$$e^{-t^2\mathbb{E}[\sigma_X(T_2-T_1)]} = \mathbb{E}\left[e^{-t^2\sigma_X(T_2-T_1)}\right].$$

According to Jensen's inequality, this equality is achieved if and only if $\sigma_X(T_2 - T_1)$ is constant almost everywhere. □

Example 1 In Fig. 1, we illustrate the non-Gaussianity of the sampled process from the distribution of (Y_1, Y_2). To simulate a realization from the distribution of (Y_1, Y_2), we proceed as follows:

1. Generate the time interval $T_2 - T_1$ according to an exponential distribution with mean 1.
2. Generate (Y_1, Y_2) as a Gaussian vector with zero mean and covariance

$$\begin{pmatrix} \sigma_X(0) & \sigma_X(T_2 - T_1) \\ \sigma_X(T_2 - T_1) & \sigma_X(0) \end{pmatrix}.$$

We take $\mathbf{X}$ a long memory Gaussian process with the standard form of autocovariance function,

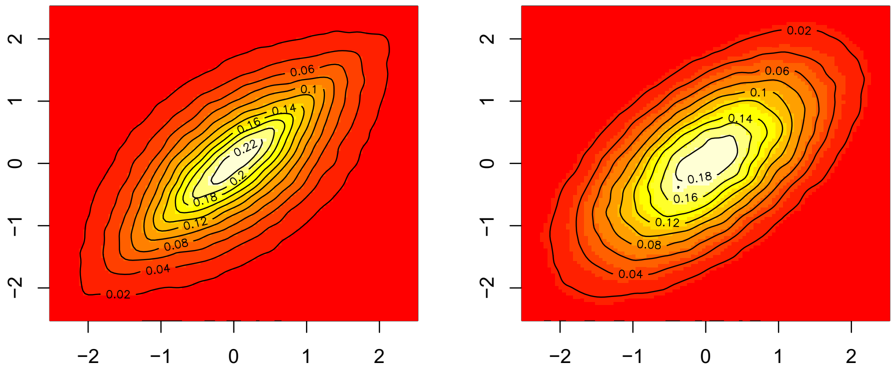
$$\sigma_X(t) = (1 + t^{1-2d})^{-1} \tag{2.2}$$

and $(T_i)_{i\in\mathbb{N}}$ is a homogeneous Poisson counting process with rate 1. The parameter $d \in (0, 1/2)$ in 2.2 measures the intensity of long-range dependence. In Sect 3 we study this specific form of autocovariance function, and specify the effect of subsampling on the value of d.

We simulate from the distribution of (Y_1, Y_2) a sample of size p. In Fig. 1a we represent the kernel estimate of the joint probability density function of (Y_1, Y_2). In order to compare the probability distribution of the sampled process with the corresponding Gaussian one, we simulate a sample of centered Gaussian vector (W_1, W_2) having the same variance matrix as (Y_1, Y_2) i.e.

$$\Sigma_{Y_1,Y_2} = \begin{pmatrix} \sigma_X(0) & \mathbb{E}[\sigma_X(T_1)] \\ \mathbb{E}[\sigma_X(T_1)] & \sigma_X(0) \end{pmatrix} = \begin{pmatrix} 1 & \Sigma_{1,2} \\ \Sigma_{1,2} & 1 \end{pmatrix},$$

where $\Sigma_{1,2} = \int_0^\infty \sigma_X(t)e^{-t}\,\mathrm{d}t = \int_0^\infty e^{-t}(1 + t^{1-2d})^{-1}\,\mathrm{d}t$ can be calculated numerically. In Fig. 1b, we represent the kernel estimate of the density of (W_1, W_2). The simulations are done with $d = .05$. We see that the form of the distribution of sampled process differs widely from Gaussian distribution. Note that for stronger long memory, the difference is visually more difficult to detect.



(a) (b)

Fig. 1 **a** The estimated density of the centered couple (Y_1, Y_2) is represented for intervals Δ_j having an exponential distribution with mean 1 and Gaussian initial process with autocovariance function $\sigma_X(t) = (1 + t^{0.9})^{-1}$. **b** Represents the estimated density of the centered Gaussian vector (W_1, W_2) with the same covariance matrix Σ_{Y_1,Y_2} as (Y_1, Y_2). Estimations are calculated on sample of size $p = 50{,}000$

2.2 Dependence of the sampled process

We are interested in the dependence structure of the **Y** process. In the following propositions, we provide sufficient conditions to preserve the weak (respectively long) memory after sampling.

Proposition 2.4 *Assume Assumption $\mathcal{H}$ holds. Let p be a real greater than 1 ($p \geq 1$). If there is a positive bounded function $\sigma_*(.)$, non-increasing on $\mathbb{R}^+$, such that*

1. $|\sigma_X(t)| \leq \sigma_*(t), \quad \forall t \in \mathbb{R}^+$
2. $\displaystyle\int_{\mathbb{R}^+} \sigma_*^p(t)dt < \infty$

then, the sampled process Y has an autocovariance function (2.1) in ℓ^p, i.e $\displaystyle\sum_{h\geq 0} |\sigma_Y(h)|^p < \infty$.

Remark 2 The proposition confirms an intuitive claim: random sampling cannot produce long memory from short memory. The particular case $p = 1$ implies that if **X** has short memory then, the sampled process **Y** has short memory too.

Proof It is clearly enough to prove that

$$\sum_{h\geq 1} \mathbb{E}\left[\sigma_*^p(T_h)\right] < \infty. \tag{2.3}$$

Since σ_* is a decreasing function, we have

$$\Delta_h \sigma_*^p(T_h + \Delta_h) = (T_{h+1} - T_h)\sigma_*^p(T_{h+1}) \leq \int_{T_h}^{T_{h+1}} \sigma_*^p(t)\,\mathrm{d}t, \quad \forall h \geq 0. \tag{2.4}$$

Taking the expectation of the left-hand-side and noting that Δ_h and T_h are independent, we obtain, for every $a > 0$,

$$\begin{aligned}
\mathbb{E}\left[\Delta_h \sigma_*^p(T_h+\Delta_h)\right] &= \int_{\mathbb{R}^+} u\mathbb{E}\left[\sigma_*^p(T_h+u)\right]\,\mathrm{d}P_{\Delta_0}(u)\\
&= \int_0^a u\mathbb{E}\left[\sigma_*^p(T_h+u)\right]\,\mathrm{d}P_{\Delta_0}(u) + \int_a^{+\infty} u\mathbb{E}\left[\sigma_*^p(T_h+u)\right]\,\mathrm{d}P_{\Delta_0}(u)\\
&\geq \int_0^a u\mathbb{E}\left[\sigma_*^p(T_h+u)\right]\,\mathrm{d}P_{\Delta_0}(u) + a\int_a^{+\infty} \mathbb{E}\left[\sigma_*^p(T_h+u)\right]\,\mathrm{d}P_{\Delta_0}(u)\\
&= \int_0^a u\mathbb{E}\left[\sigma_*^p(T_h+u)\right]\,\mathrm{d}P_{\Delta_0}(u) + a\left(\int_{\mathbb{R}^+} \mathbb{E}\left[\sigma_*^p(T_h+u)\right]\,\mathrm{d}P_{\Delta_0}(u)\right.\\
&\quad\left. - \int_0^a \mathbb{E}\left[\sigma_*^p(T_h+u)\right]\,\mathrm{d}P_{\Delta_0}(u)\right)\\
&= \int_0^a (u-a)\mathbb{E}\left[\sigma_*^p(T_h+u)\right]\,\mathrm{d}P_{\Delta_0}(u) + a\mathbb{E}\left[\sigma_*^p(T_{h+1})\right].
\end{aligned}$$

Since $\sigma_*^p(T_h+u) \leq \sigma_*^p(T_h)$ and $u-a \leq 0$, we get

$$\mathbb{E}\left[\Delta_h \sigma_*^p(T_h+\Delta_h)\right] \geq \left(\int_{[0,a[} (u-a)\,\mathrm{d}P_{\Delta_0}(u)\right)\mathbb{E}\left[\sigma_*^p(T_h)\right] + a\mathbb{E}\left[\sigma_*^p(T_{h+1})\right]. \tag{2.5}$$

It is possible to choose a such that $P(\Delta_0 \in [0,a]) < 1$. For such a choice we obtain

$$0 \leq -\int_{[0,a[} (u-a)\,\mathrm{d}P_{\Delta_0}(u) =: \ell(a) \leq aP(\Delta_0 \in [0,a]) < a.$$

After summation, the inequalities (2.5) give, for every $K \geq 0$

$$\begin{aligned}
\mathbb{E}\left[\sum_{h=1}^{\infty} \Delta_h \sigma_*^p(T_{h+1})\right] &\geq \sum_{h=1}^{K}\left[-\ell(a)\mathbb{E}[\sigma_*^p(T_h)] + a\mathbb{E}[\sigma_*^p(T_{h+1})]\right]\\
&= a\left(\mathbb{E}[\sigma_*^p(T_{K+1})] - \mathbb{E}[\sigma_*^p(T_1)]\right) + (a-\ell(a))\sum_{h=1}^{K}\mathbb{E}\left[\sigma_*^p(T_h)\right]\\
&\geq -a\sigma_*^p(0) + (a-\ell(a))\sum_{h=1}^{K}\mathbb{E}\left[\sigma_*^p(T_h)\right],
\end{aligned}$$

which implies

$$\mathbb{E}\left[\sum_{h=1}^{\infty} \Delta_h \sigma_*^p(T_{h+1})\right] \geq -a\sigma_*^p(0) + (a-\ell(a))\sum_{h\geq 1}\mathbb{E}\left[\sigma_*^p(T_h)\right].$$

Then, using (2.4)

$$\mathbb{E}\left[\sum_{h\geq 1}\Delta_h\sigma_*^p(T_{h+1})\right] \leq \mathbb{E}\left[\sum_{h\geq 1}\int_{T_h}^{T_{h+1}}\sigma_*^p(t)\,\mathrm{d}t\right] \leq \int_{\mathbb{R}^+}\sigma_*^p(t)\,\mathrm{d}t < \infty,$$

and consequently, as $a - \ell(a) > 0$

$$\sum_{h=1}^{\infty}\mathbb{E}\left[\sigma_*^p(T_h)\right] < \infty. \tag{2.6}$$

□

We now consider the case of long memory processes. We give conditions on T_1 that ensure the preservation of the long memory property.

Proposition 2.5 *Assume Assumption $\mathcal{H}$ holds. We suppose that $\sigma_X(.)$ is ultimately positive and non-increasing on $\mathbb{R}^+$, i.e there exists $t_0 \geq 0$ such that $\sigma_X(.)$ is positive and non-increasing on the interval $[t_0, \infty)$. If $\mathbb{E}[T_1] < \infty$, then the long memory is preserved after the subsampling, i.e. $\int_{\mathbb{R}^+}|\sigma_X(x)|\,\mathrm{d}x = \infty$ implies $\sum_{h\geq 0}|\sigma_Y(h)| = \infty$.*

Remark 3 In this proposition, we only show that the long memory is preserved in the sense of the non-summability of autocovariance function. Additional assumptions are required to compare the convergence rates of σ_X and σ_Y. This question is addressed in Sect. 3 where we impose semi parametric form on σ_X.

Remark 4 The assumptions on positivity and the decrease of the auto-covariance function are not too restrictive. They are satisfied in most of studied models. The condition of integrability of intervals Δ_j is the most difficult to verify since the underlying process is generally not observed.

Proof Let h_0 be the (random) first index such that $T_{h_0} \geq t_0$. For every $h \geq h_0$,

$$\int_{T_h}^{T_{h+1}}\sigma_X(t)\,\mathrm{d}t \leq (T_{h+1} - T_h)\sigma_X(T_h). \tag{2.7}$$

Summing up gives

$$\sum_{h\geq 1}\mathbb{I}_{h\geq h_0}\int_{T_h}^{T_{h+1}}\sigma_X(t)\,\mathrm{d}t \leq \sum_{h\geq 1}\mathbb{I}_{h\geq h_0}\Delta_h\sigma_X(T_h).$$

Now, taking expectations, and noting that, since $\mathbb{E}[T_1] = \mathbb{E}[\Delta_1] > 0$, the law of large numbers implies that $T_h \xrightarrow{a.s.} \infty$, and in particular $h_0 < \infty$ a.s., whence

$$\mathbb{E}\left[\int_{T_{h_0}}^{\infty}\sigma_X(t)\,\mathrm{d}t\right] \leq \mathbb{E}\left[\sum_{h=1}^{\infty}\Delta_h\sigma_X(T_h)\,\mathbb{I}_{h_0\leq h}\right].$$

The left hand side is infinite. Since Δ_h is independent of $\sigma_X(T_h)\,\mathbb{I}_{h_0\le h}$, the right hand side is $\mathbb{E}[T_1]\sum_{h\ge 1}\mathbb{E}[\sigma_X(T_h)\,\mathbb{I}_{h_0\le h}]$. Consequently, since $\mathbb{E}[T_1]<\infty$, we have

$$\sum_{h\ge 1}\mathbb{E}[\sigma_X(T_h)\,\mathbb{I}_{h_0\le h}]=\infty. \tag{2.8}$$

It remains to be noted that $\mathbb{E}[h_0]<\infty$ [see for example Feller (1966) p. 185], which implies

$$\sum_{h\ge 1}\mathbb{E}[|\sigma_X(T_h)|\,\mathbb{I}_{h_0>h}]\le \sigma_X(0)\sum_{h\ge 1}P(h_0\ge h)\le \sigma_X(0)\mathbb{E}[h_0]<\infty,$$

leading, via (2.8) to $\sum_{h\ge 1}|\mathbb{E}[\sigma_X(T_h)]|=\infty$. □

3 Long memory processes

We consider a long memory process **X** and we impose a semi parametric form to autocovariance function. We assume that the autocovariance σ_X is regularly varying function at infinity of the form

$$\sigma_X(t)=t^{-1+2d}L(t),\quad \forall t\ge 1 \tag{3.1}$$

where $0<d<1/2$ and L is ultimately non-increasing and slowly varying at infinity, in the sense that L is positive on $[t_0,\infty)$ for some $t_0>0$ and

$$\lim_{x\to+\infty}\frac{L(ax)}{L(x)}=1,\quad \forall a>0.$$

This class of models contains for instance CARFIMA models.

The parameter d characterizes the intensity of the memory of **X**. In the following propositions, we evaluate the long memory parameter of the sampled process **Y** as a function of d and the probability distribution of T_1.

3.1 Preservation of the memory when $\mathbb{E}[T_1]<\infty$

Theorem 3.1 *Under Assumption $\mathcal{H}$ and (3.1), if $0<\mathbb{E}[T_1]<\infty$, the discrete time process* **Y** *has a long memory and its covariance function behaves as*

$$\sigma_Y(h)\sim (h\mathbb{E}[T_1])^{-1+2d}L(h),\quad h\to\infty.$$

Remark 5 We can rewrite

$$\sigma_Y(h)=h^{-1+2d}\tilde{L}(h)$$

where $\tilde{L}$ is slowly varying at infinity and $\tilde{L}(h) \sim (\mathbb{E}[T_1])^{-1+2d} L(h)$ as $h \to \infty$. In particular, **X** and **Y** have the same memory parameter d.

Proof • We show first that

$$\liminf_{h\to\infty} \frac{\sigma_Y(h)}{(h\mathbb{E}[T_1])^{-1+2d} L(h)} \geq 1.$$

Let $0 < c < \mathbb{E}[T_1]$, and $h \in \mathbb{N}$ such that $ch \geq 1$,

$$\sigma_Y(h) \geq \mathbb{E}\left[\sigma_X(T_h)\, \mathbb{I}_{T_h>ch}\right] \geq \inf_{t>ch}\{L(t)t^{2d}\}\mathbb{E}\left[\frac{\mathbb{I}_{T_h>ch}}{T_h}\right].$$

Thanks to Hölder inequality,

$$(P(T_h > ch))^2 \leq \mathbb{E}[T_h]\mathbb{E}\left[\frac{\mathbb{I}_{T_h>ch}}{T_h}\right],$$

that is

$$\mathbb{E}\left[\frac{\mathbb{I}_{T_h>ch}}{T_h}\right] \geq \frac{(P(T_h > ch))^2}{h\mathbb{E}[T_1]}.$$

Summarizing,

$$\sigma_Y(h) \geq \inf_{t>ch}\{L(t)t^{2d}\}\frac{(P(T_h > ch))^2}{h\mathbb{E}[T_1]},$$

$$\frac{\sigma_Y(h)}{(h\mathbb{E}[T_1])^{-1+2d} L(h)} \geq \inf_{t>ch}\{L(t)t^{2d}\}\frac{(P(T_h > ch))^2}{(h\mathbb{E}[T_1])^{2d} L(h)}. \tag{3.2}$$

Using Bingham et al. (1989) (Th 1.5.3, p23), we obtain, since $d > 0$

$$\inf_{t\geq ch}\{L(t)t^{2d}\} \sim L(ch)(ch)^{2d}, \text{ as } h \to \infty. \tag{3.3}$$

The law of large numbers implies that $T_h/h \xrightarrow{a.s.} \mathbb{E}[T_1]$. As $c < \mathbb{E}[T_1]$, we have $P(T_h > ch) \to 1$ and the r.h.s. of (3.2) tends to $(c/\mathbb{E}[T_1])^{2d}$ as $h \to \infty$. Finally, for all $c < \mathbb{E}[T_1]$,

$$\liminf_{h\to\infty} \frac{\sigma_Y(h)}{(h\mathbb{E}[T_1])^{-1+2d} L(h)} \geq \left(\frac{c}{\mathbb{E}[T_1]}\right)^{2d}.$$

Taking the limit as $c \to \mathbb{E}[T_1]$, we get the lower bound.

- Let us now prove

$$\limsup_{h\to\infty} \frac{\sigma_Y(h)}{(h\mathbb{E}[T_1])^{-1+2d} L(h)} \leq 1.$$

We use a proof similar to that presented in Shi et al. (2010) (Theorem 1). We denote for $h \geq 1$ and $0 < s < 1$,

$$\begin{aligned}\mu_h &= \mathbb{E}[T_h] = h\mathbb{E}[T_1],\\ T_{h,s} &= \sum_{j=0}^{h-1} \Delta_j \, \mathbb{I}_{\Delta_j \leq \mu_h^s/\sqrt{h}},\\ \mu_{h,s} &= \mathbb{E}\left[T_{h,s}\right] = h\mathbb{E}\left[\Delta_0 \, \mathbb{I}_{\Delta_0 \leq \mu_h^s/\sqrt{h}}\right].\end{aligned}$$

Since $\mathbb{E}[T_1] < \infty$, we have for $\frac{1}{2} < s < 1$, $\mu_{h,s} \sim \mu_h$ as $h \to \infty$.
Let $\frac{1}{2} < s < \tau < 1$, t_0 such that $L(.)$ is non-increasing on $[t_0, \infty)$ and h such that $\mu_{h,s} - \mu_{h,s}^\tau \geq t_0$,

$$\begin{aligned}\sigma_Y(h) &= \mathbb{E}\left[T_h^{-1+2d} L(T_h) \, \mathbb{I}_{T_{h,s} \geq \mu_{h,s} - \mu_{h,s}^\tau}\right] + \mathbb{E}\left[T_h^{-1+2d} L(T_h) \, \mathbb{I}_{T_{h,s} < \mu_{h,s} - \mu_{h,s}^\tau}\right]\\ &=: M_1 + M_2.\end{aligned}$$

We will now establish upper bounds for the terms M1 and M2. We have

$$\begin{aligned}M_1 &\leq \mathbb{E}\left[T_{h,s}^{-1+2d} L(T_{h,s}) \, \mathbb{I}_{T_{h,s} \geq \mu_{h,s} - \mu_{h,s}^\tau}\right] \leq \left(\mu_{h,s} - \mu_{h,s}^\tau\right)^{-1+2d} L(\mu_{h,s} - \mu_{h,s}^\tau)\\ &= (h\mathbb{E}[T_1])^{-1+2d} L(h) \left(\frac{\mu_{h,s} - \mu_{h,s}^\tau}{h\mathbb{E}[T_1]}\right)^{-1+2d} \frac{L(\mu_{h,s} - \mu_{h,s}^\tau)}{L(h)}. \qquad (3.4)\end{aligned}$$

As $\tau < 1$ and $1/2 < s < 1$, $\left(\frac{\mu_{h,s} - \mu_{h,s}^\tau}{h\mathbb{E}[T_1]}\right)^{-1+2d} \to 1$ as $h \to \infty$. Then,

$$\frac{L(\mu_{h,s} - \mu_{h,s}^\tau)}{L(h)} = \frac{L\left(h\mathbb{E}[T_1]\frac{\mu_{h,s} - \mu_{h,s}^\tau}{h\mathbb{E}[T_1]}\right)}{L(h\mathbb{E}[T_1])} \frac{L(h\mathbb{E}[T_1])}{L(h)}.$$

As we have uniform convergence of $\lambda \mapsto \frac{L(h\mathbb{E}[T_1]\lambda)}{L(h\mathbb{E}[T_1])}$ to 1 (as $h \to \infty$) in each interval $[a, b]$ and as $\frac{\mu_{h,s} - \mu_{h,s}^\tau}{h\mathbb{E}[T_1]} \to 1$, we get

$$\frac{L(\mu_{h,s} - \mu_{h,s}^\tau)}{L(h)} \to 1,$$

as $h \to \infty$. We obtain

$$M_1 \leq \left(\mu_{h,s} - \mu_{h,s}^\tau\right)^{-1+2d} L(\mu_{h,s} - \mu_{h,s}^\tau) \sim (h\mathbb{E}[T_1])^{-1+2d} L(h). \qquad (3.5)$$

Since $\sup_{t \in \mathbb{R}^+} |\sigma_X(t)| = \sigma_X(0) < \infty$, we have

$$M_2 \leq \sigma_X(0) P\left(T_{h,s} < \mu_{h,s} - \mu_{h,s}^\tau\right) = \sigma_X(0) P\left(-T_{h,s} + \mathbb{E}[T_{h,s}] > \mu_{h,s}^\tau\right).$$

We apply Hoeffding inequality to variables $Z_j = -\Delta_j \mathbb{I}_{\Delta_j \leq \mu_h^s/\sqrt{h}}$ which are a.s in $[-\frac{\mu_h^s}{\sqrt{h}}, 0]$ to get,

$$M_2 \leq \sigma_X(0) \exp\left(-2\left(\frac{\mu_{h,s}^\tau}{\mu_h^s}\right)^2\right)$$

and $\left(\frac{\mu_{h,s}^\tau}{\mu_h^s}\right)^2 \sim (h\mathbb{E}[T_1])^{2(\tau-s)}$. Finally

$$M_2 = o((h\mathbb{E}[T_1])^{-1+2d} L(h)). \tag{3.6}$$

With (3.5) and (3.6), we get the upper bound.

□

3.2 Decrease of memory

The phenomenon is the same as in the discrete case [see Philippe and Viano (2010)]: starting from a long memory process, a heavy tailed sampling distribution can lead to a short memory process.

Proposition 3.2 *Assume that the covariance of* $\mathbf{X}$ *satisfies*

$$|\sigma_X(t)| \leq c \min(1, t^{-1+2d}) \qquad \forall t \in \mathbb{R}^+, \tag{3.7}$$

where $0 < d < 1/2$. *If there exists* $\beta \in (0, 1)$ *such that*

$$\liminf_{x\to\infty} \left(x^\beta P(T_1 > x)\right) > 0 \tag{3.8}$$

then, there exists $C > 0$ *such that*

$$|\sigma_Y(h)| \leq Ch^{\frac{-1+2d}{\beta}}. \tag{3.9}$$

Remark 6 The condition (3.8) implies that $\mathbb{E}[T_1^\beta] = \infty$. As $\beta \in (0, 1)$, the first moment of increments is infinite contrary to assumption of Sect. 3.1.

Proof From assumption (3.7),

$$|\sigma_Y(h)| \leq \mathbb{E}[|\sigma_X(T_h)|] \leq c\mathbb{E}[\min\{1, T_h^{-1+2d}\}].$$

We have

$$\begin{aligned}\mathbb{E}[\min\{1, T_h^{-1+2d}\}] &= \mathbb{E}[\mathbb{I}_{T_h \leq 1} + T_h^{-1+2d}\mathbb{I}_{T_h>1}] \\ &= P(T_h \leq 1) + \int_1^\infty x^{-1+2d}\, \mathrm{d}P_{T_h}(x).\end{aligned}$$

As $\mathbb{E}[\min\{1, T_h^{-1+2d}\}] \leq 1$, the integral in the right hand side is finite

$$\int_1^\infty x^{-1+2d} \,\mathrm{d}P_{T_h}(x)\,\mathrm{d}x < \infty.$$

Since $-2+2d \in [-2, -1[$, we also have

$$\int_1^\infty x^{-2+2d} P(T_h \leq x)\,\mathrm{d}x < \infty.$$

Thus, the integration by parts can be applied, and we get

$$\int_1^\infty x^{-1+2d} \,\mathrm{d}P_{T_h}(x) = (1-2d)\int_1^\infty x^{-2+2d} P(T_h \leq x)\,\mathrm{d}x - P(T_h \leq 1)$$

and

$$\mathbb{E}[\min\{1, T_h^{-1+2d}\}] = (1-2d)\int_1^\infty x^{-2+2d} P(T_h \leq x)\,\mathrm{d}x. \tag{3.10}$$

From assumption (3.8) on the tail of the sampling law, it follows that, there exists $C > 0$ and $x_0 \geq 1$ such that

$$\forall x \geq x_0, \quad P(T_1 > x) \geq Cx^{-\beta}.$$

Furthermore for $x \in [1, x_0]$,

$$x^\beta P(T_1 > x) \geq P(T_1 > x_0) \geq Cx_0^{-\beta}.$$

We obtain: $\forall x \geq 1,\ P(T_1 > x) \geq \tilde{C}x^{-\beta}$ with $\tilde{C} = Cx_0^{-\beta}$, and then

$$P(T_h \leq x) \leq P\left(\max_{0\leq l\leq h-1} \Delta_l \leq x\right) = P\left(T_1 \leq x\right)^h \leq \left(1-\tilde{C}x^{-\beta}\right)^h \leq e^{-\frac{\tilde{C}h}{x^\beta}}. \tag{3.11}$$

Gathering (3.10) and (3.11) then gives

$$\begin{aligned}\mathbb{E}[\min\{1, T_h^{-1+2d}\}] &\leq (1-2d)\int_1^\infty x^{-2+2d} e^{-\frac{\tilde{C}h}{x^\beta}}\,\mathrm{d}x \\ &= \frac{1-2d}{\beta} h^{-(1-2d)/\beta}\int_0^h u^{(1-2d)/\beta-1} e^{-\tilde{C}u}\,\mathrm{d}u.\end{aligned}$$

Since

$$\int_0^h u^{(1-2d)/\beta-1} e^{-\tilde{C}u}\,\mathrm{d}u \xrightarrow{h\to\infty} D := \int_0^\infty u^{(1-2d)/\beta-1} e^{-\tilde{C}u}\,\mathrm{d}u < \infty$$

we obtain

$$|\sigma_Y(h)|\, h^{\frac{1-2d}{\beta}} \leq c\mathbb{E}[\min\{1, T_h^{-1+2d}\}]\, h^{\frac{1-2d}{\beta}} \leq cD\frac{1-2d}{\beta}.$$

Thus (3.9) is proven with $C = cD\frac{1-2d}{\beta}$. □

Under some additional assumptions, we show that the bound obtained in Proposition 3.2 is equal to the convergence rate (up to a multiplicative constant).

Proposition 3.3 *Assume that*

$$\sigma_X(t) = t^{-1+2d} L(t)$$

where $0 < d < 1/2$ *and where L is slowly varying at infinity and ultimately monotone. We denote*

$$\beta := \sup\{\gamma \in \mathbb{R}^+ : \mathbb{E}[T_1^\gamma] < \infty\}. \tag{3.12}$$

If $\beta \in (0, 1)$ *then, for every* $\varepsilon > 0$*, there exists* $C_\varepsilon > 0$ *such that*

$$\sigma_Y(h) \geq C_\varepsilon h^{-\frac{1-2d}{\beta}-\varepsilon}, \qquad \forall h \geq 1. \tag{3.13}$$

Proof Let $\varepsilon > 0$. We have

$$\frac{\sigma_X(T_h)}{h^{-\frac{1-2d}{\beta}-\varepsilon}} = \frac{T_h^{-1+2d}}{h^{-\frac{1-2d}{\beta}-\varepsilon}} L(T_h) = \frac{T_h^{-1+2d-\frac{\beta\varepsilon}{2}}}{h^{-\frac{1-2d}{\beta}-\varepsilon}} T_h^{\frac{\beta\varepsilon}{2}} L(T_h) = \left(\frac{T_h}{h^\delta}\right)^{-1+2d-\frac{\beta\varepsilon}{2}} T_h^{\frac{\beta\varepsilon}{2}} L(T_h)$$

where

$$\delta = \frac{(1-2d)/\beta + \varepsilon}{1-2d+\frac{\beta\varepsilon}{2}} = \frac{1}{\beta}\left(\frac{1-2d+\beta\varepsilon}{1-2d+\beta\varepsilon/2}\right).$$

Using Proposition 1.3.6 in Bingham et al. (1989),

$$T_h^{\frac{\beta\varepsilon}{2}} L(T_h) \xrightarrow{a.s.} \infty \quad \text{as} \quad h \to \infty. \tag{3.14}$$

Moreover $\delta > \frac{1}{\beta}$. From (3.12), this implies $\mathbb{E}[T_1^{1/\delta}] < \infty$. Then, the law of large numbers of Marcinkiewicz-Zygmund [see Stout (1974) Theorem 3.2.3] yields

$$\frac{T_h}{h^\delta} \xrightarrow{a.s.} 0 \quad \text{as} \quad h \to \infty. \tag{3.15}$$

From (3.14) and (3.15) we obtain

$$\frac{\sigma_X(T_h)}{h^{-\frac{1-2d}{\beta}-\varepsilon}} \xrightarrow{a.s.} \infty \quad \text{as} \quad h \to \infty.$$

Therefore by Fatou's Lemma, we get

$$\liminf_{h\to\infty} \frac{\sigma_Y(h)}{h^{-\frac{1-2d}{\beta}-\varepsilon}} = \liminf_{h\to\infty} \mathbb{E}\left[\frac{\sigma_X(T_h)}{h^{-\frac{1-2d}{\beta}-\varepsilon}}\right] \geq \mathbb{E}\left[\liminf_{h\to\infty} \frac{\sigma_X(T_h)}{h^{-\frac{1-2d}{\beta}-\varepsilon}}\right] \xrightarrow{h\to\infty} \infty.$$

This convergence implies the inequality (3.13). □

Remark 7 In this context the long memory parameter d of the initial process $\mathbf{X}$ is not identifiable using the sampled process. Information on probability distribution of Δ_1 is required.

4 Limit theorems in semiparametric case

We consider the process of partial sums

$$S_n(\tau) = \sum_{j=1}^{[n\tau]} Y_j, \quad 0 \leq \tau \leq 1. \tag{4.1}$$

In Theorem 4.2, we show that if $\mathbf{X}$ is a Gaussian process and $\mathbf{X}$ and $\mathbf{Y}$ have the same long memory parameter, the normalized partial sum process converges to a fractional Brownian motion. According to Proposition 2.3, Gaussianity is lost after sampling, however we get the classical behavior obtained by Taqqu (1975) and Davydov (1970).

4.1 Convergence of the partial sum process

To prove the convergence of the normalized partial sum process, we first need a result on the convergence in probability of conditional variance of S_n.

Lemma 4.1 *Let* $\mathbf{X}$ *be a Gaussian process with regularly varying covariance function*

$$\sigma_X(t) = L(t)t^{-1+2d},$$

where $0 < d < 1/2$ *and where* L *is slowly varying at infinity and ultimately non-increasing.*
If $\mathbb{E}[T_1] < \infty$*, then we have*

$$L(n)^{-1} n^{-1-2d} \mathrm{Var}(X_{T_1} + \cdots + X_{T_n} | T_1, \ldots, T_n) \xrightarrow[n\to\infty]{p} \gamma_d, \tag{4.2}$$

where $\gamma_d := \frac{\mathbb{E}[T_1]^{-1+2d}}{d(1+2d)}$.

Proof See "Appendix". □

Theorem 4.2 *Assume Assumption* $\mathcal{H}$ *holds. If* $\mathbf{X}$ *is a Gaussian process with regularly varying covariance function* $\sigma_X(t) = L(t)t^{-1+2d}$*, with* $0 < d < 1/2$ *and* L *slowly* *varying at infinity and ultimately non-increasing (Hypothesis 3.1). Then, if* $\mathbb{E}[T_1] < \infty$*, we get*

$$\gamma_d^{-1/2} L(n)^{-1/2} n^{-1/2-d} S_n(.) \Rightarrow B_{\frac{1}{2}+d}(.), \quad \text{in } \mathcal{D}[0,1] \text{ with the uniform metric,} \tag{4.3}$$

where $B_{\frac{1}{2}+d}$ *is the fractional Brownian motion with parameter* $\frac{1}{2}+d$*, and where* γ_d *is defined in Lemma 4.1.*

Proof We first prove the weak convergence in finite-dimensional distributions of

$$\gamma_d^{-1/2} L(n)^{-1/2} n^{-1/2-d} S_n(.)$$

to the corresponding finite-dimensional distributions of $B_{\frac{1}{2}+d}(.)$.
It suffices to show that for every $k \geq 1$, $(b_1, \ \dots \ , b_k) \in \mathbb{R}^k$, $0 \leq t_1, \dots, t_k \leq 1$,

$$A_n := \gamma_d^{-1/2} L(n)^{-1/2} n^{-1/2-d} \sum_{i=1}^{k} b_i S_n(t_i)$$

satisfies

$$A_n \xrightarrow{d} \sum_{i=1}^{k} b_i B_{\frac{1}{2}+d}(t_i).$$

If $t_1 = \cdots = t_k = 0$, then $\gamma_d^{-1/2} L(n)^{-1/2} n^{-1/2-d} \sum_{i=1}^k b_i S_n(t_i) = \sum_{i=1}^k b_i B_{\frac{1}{2}+d}(t_i) = 0$. So we fix n large enough to have $[n \max_i(t_i)] \geq 1$ and denote $T^{(n)} = (T_1, \dots, T_{[n \max_i(t_i)]})$. The characteristic function of A_n is

$$\Phi_{A_n}(t) = \mathbb{E}[e^{itA_n}] = \mathbb{E}[e^{-\frac{t^2}{2}\operatorname{Var}(A_n|T^{(n)})}].$$

Moreover, we have

$$\begin{aligned}
&\operatorname{Var}(A_n|T^{(n)}) \\
&= \sum_{i,j=1}^{k} b_i b_j \gamma_d^{-1} L(n)^{-1} n^{-1-2d} \mathbb{E}[S_n(t_i) S_n(t_j)|T^{(n)}] \\
&= \sum_{i,j=1}^{k} \frac{b_i b_j \gamma_d^{-1} L(n)^{-1} n^{-1-2d}}{2} \Big[\operatorname{Var}(S_n(t_i)|T^{(n)}) \\
&\quad + \operatorname{Var}(S_n(t_j)|T^{(n)}) - \operatorname{Var}(S_n(t_i) - S_n(t_j)|T^{(n)})\Big].
\end{aligned}$$

By Lemma 4.1,

$$L(n)^{-1} n^{-1-2d} \operatorname{Var}(Y_1 + \cdots + Y_n | T_1, \dots, T_n) \xrightarrow[n\to\infty]{p} \gamma_d.$$

Therefore

$$\gamma_d^{-1} L(n)^{-1} n^{-1-2d} \mathrm{Var}(S_n(t_i)|T^{(n)}) \xrightarrow[n\to\infty]{p} t_i^{1+2d},$$

and for $t_i > t_j$

$$\gamma_d^{-1} L(n)^{-1} n^{-1-2d} \mathrm{Var}(S_n(t_i) - S_n(t_j)|T^{(n)}) = \gamma_d^{-1} L(n)^{-1} n^{-1-2d} \mathrm{Var}(Y_{[nt_i]+1} + \cdots + Y_{[nt_j]}|T^{(n)}) \xrightarrow[n\to\infty]{p} (t_i - t_j)^{1+2d}.$$

Finally, we have

$$\mathrm{Var}(A_n|T^{(n)}) \xrightarrow[n\to\infty]{p} \sum_{i,j=1}^{k} b_i b_j r_{\frac{1}{2}+d}(t_i, t_j),$$

where $r_{\frac{1}{2}+d}$ is the covariance function of a fractional Brownian motion, and hence

$$\exp\left(-\frac{t^2}{2}\mathrm{Var}(A_n|T^{(n)})\right) \xrightarrow[n\to\infty]{p} \exp\left(-\frac{t^2}{2}\sum_{i,j=1}^{k} b_i b_j r_{\frac{1}{2}+d}(t_i, t_j)\right).$$

Therefore, applying dominated convergence theorem, we get

$$\Phi_{A_n}(t) \xrightarrow[n\to\infty]{} \exp\left(-\frac{t^2}{2}\sum_{i,j=1}^{k} b_i b_j r_{\frac{1}{2}+d}(t_i, t_j)\right) = \Phi_{\sum_{i=1}^{k} b_i B_{\frac{1}{2}+d}(t_i)}(t).$$

The sequence of partial-sum processes $L(n)^{-1/2} n^{-1/2-d} S_n(.)$ is tight with respect to the uniform norm [see Giraitis et al. (2012) Prop 4.4.2 p78, for the proof of the tightness] and then we get the convergence in $\mathcal{D}[0, 1]$ with the uniform metric. □

4.2 Estimation of the long memory parameter

An immediate consequence of this limit theorem is to provide a nonparametric estimation of the long memory parameter d using the well-known R/S statistics. This is a heuristic method for estimating the long memory parameter. To validate the convergence of more efficient estimates (e.g. Whittle's estimate or estimators based on the spectral approach), a non trivial study of the asymptotic properties of periodogram is required. Indeed the sampled process do not satisfied the classical assumptions (Gaussian process, Linear process with independent and identically distributed innovations) under which the properties are established [see Giraitis et al. (2012), Beran et al. (2013)].

The R/S statistic is defined as the quotient between R_n and S_n where

$$R_n := \max_{1\le k\le n} \sum_{j=1}^{k} (Y_j - \overline{Y_n}) - \min_{1\le k\le n} \sum_{j=1}^{k} (Y_j - \overline{Y_n}) \tag{4.4}$$

and

$$S_n := \left(\frac{1}{n} \sum_{j=1}^{n} (Y_j - \overline{Y_n})^2 \right)^{1/2}. \tag{4.5}$$

Proposition 4.3 *Under the same assumptions as Theorem 4.2, we have*

$$\frac{1}{L(n)^{1/2} n^{1/2+d}} \frac{R_n}{S_n} \xrightarrow[n\to\infty]{d} \mathcal{R} := \sqrt{\frac{\gamma_d}{\sigma_X(0)}} \left(\max_{0\le t\le 1} B^0_{\frac{1}{2}+d}(t) - \min_{0\le t\le 1} B^0_{\frac{1}{2}+d}(t) \right) \tag{4.6}$$

where $B^0_{\frac{1}{2}+d}(t) = B_{\frac{1}{2}+d}(t) - tB_{\frac{1}{2}+d}(1)$ *is a fractional Brownian bridge and* γ_d *is a constant defined in Lemma 4.1.*

Proof Using the equality

$$\sum_{j=1}^{k} (Y_j - \overline{Y_n}) = \sum_{j=1}^{k} Y_j - \frac{k}{n} \sum_{j=1}^{n} Y_j = S_n\left(\frac{k}{n}\right) - \frac{k}{n} S_n(1)$$

and the convergence of the partial-sum process given in Theorem 4.2, we get

$$\frac{R_n}{L(n)^{1/2} n^{1/2+d}} \xrightarrow[n\to\infty]{d} \sqrt{\gamma_d} \left(\max_{0\le t\le 1} B^0_{\frac{1}{2}+d}(t) - \min_{0\le t\le 1} B^0_{\frac{1}{2}+d}(t) \right).$$

Then, we establish the convergence in probability of S_n^2 defined in (4.5). As

$$\text{Var}\left(\sum_{j=1}^{n} Y_j \right) \sim C n^{1+2d} \quad \text{as } n \to \infty,$$

we have for $\varepsilon > 0$

$$P\left(\left| \frac{1}{n} \sum_{j=1}^{n} Y_j \right| > \varepsilon \right) \le \frac{1}{n^2 \varepsilon^2} \text{Var}\left(\sum_{j=1}^{n} Y_j \right) \xrightarrow[n\to\infty]{} 0$$

and

$$P\left(\left| \frac{1}{n} \sum_{j=1}^{n} Y_j^2 - \sigma_X(0) \right| > \varepsilon \right) \le \frac{1}{n^2 \varepsilon^2} \text{Var}\left(\sum_{j=1}^{n} Y_j^2 \right)$$

$$= \frac{1}{n^2\varepsilon^2} \sum_{j=1}^{n} \sum_{k=1}^{n} \mathrm{Cov}\left(Y_j^2, Y_k^2\right)$$

$$= \frac{1}{n^2\varepsilon^2} \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\mathbb{E}\left[\mathbb{E}[X_{T_j}^2 X_{T_k}^2 | T_j, T_k]\right] - \sigma_X(0)^2\right).$$

For $(s, t) \in (\mathbb{R}^+)^2$, we decompose X_s^2 and X_t^2 in the complete orthogonal system of Hermite polynomials $(H_k)_{k\geq 0}$:

$$\left(\frac{X_s}{\sqrt{\sigma_X(0)}}\right)^2 = H_0\left(\frac{X_s}{\sqrt{\sigma_X(0)}}\right) + H_2\left(\frac{X_s}{\sqrt{\sigma_X(0)}}\right),$$

thus, we get

$$\frac{\mathbb{E}[X_s^2 X_t^2]}{\sigma_X(0)^2} = \mathbb{E}\left[H_0\left(\frac{X_s}{\sqrt{\sigma_X(0)}}\right) H_0\left(\frac{X_t}{\sqrt{\sigma_X(0)}}\right)\right] + \mathbb{E}\left[H_2\left(\frac{X_s}{\sqrt{\sigma_X(0)}}\right) H_0\left(\frac{X_t}{\sqrt{\sigma_X(0)}}\right)\right]$$
$$+ \mathbb{E}\left[H_0\left(\frac{X_s}{\sqrt{\sigma_X(0)}}\right) H_2\left(\frac{X_t}{\sqrt{\sigma_X(0)}}\right)\right] + \mathbb{E}\left[H_2\left(\frac{X_s}{\sqrt{\sigma_X(0)}}\right) H_2\left(\frac{X_t}{\sqrt{\sigma_X(0)}}\right)\right].$$

Using the orthogonality property of Hermite polynomials for a bivariate normal density with unit variances [see Giraitis et al. (2012), Prop 2.4.1], we obtain

$$\mathbb{E}[X_s^2 X_t^2] = \sigma_X^2(0)\left[1 + 2\mathrm{Cov}^2\left(\frac{X_s}{\sqrt{\sigma_X(0)}}, \frac{X_t}{\sqrt{\sigma_X(0)}}\right)\right]$$
$$= \sigma_X^2(0) + 2\sigma_X^2(t-s).$$

Finally,

$$P\left(\left|\frac{1}{n}\sum_{j=1}^{n} Y_j^2 - \sigma_X(0)\right| > \varepsilon\right) \leq \frac{2}{n^2\varepsilon^2} \sum_{j=1}^{n} \sum_{k=1}^{n} \mathbb{E}\left[\sigma_X^2(T_j - T_k)\right]$$
$$= \frac{4}{n^2\varepsilon^2} \sum_{j=0}^{n-1} (n-j)\mathbb{E}\left[\sigma_X^2(T_j)\right].$$

If $0 \leq d \leq 1/4$, we apply Proposition 2.4 with $p = 1$ and the function σ_X^2 to obtain

$$\frac{1}{n^2} \sum_{j=0}^{n-1} (n-j)\mathbb{E}\left[\sigma_X^2(T_j)\right] \leq \frac{1}{n} \sum_{j=0}^{\infty} \mathbb{E}\left[\sigma_X^2(T_j)\right] \xrightarrow[n\to\infty]{} 0.$$

If $1/4 < d < 1/2$, Theorem 3.1 can be applied to σ_X^2, and we get

$$\mathbb{E}\left[\sigma_X^2(T_h)\right] \sim L^2(h)\mathbb{E}[T_1]^{-2+4d} h^{-2+4d} \quad \text{as } h \to \infty.$$

Since L is positive, ultimately non-increasing, it admits a limit $L(x) \xrightarrow{x\to\infty} L_\infty$, and so

$$\mathbb{E}\left[\sigma_X^2(T_h)\right] \sim L_\infty^2 \mathbb{E}[T_1]^{-2+4d} h^{-2+4d} \quad \text{as } h \to \infty.$$

According to Giraitis et al. (2012) (Proposition 3.3.1 page 43), we get

$$\frac{1}{n^2}\sum_{j=0}^{n-1}(n-j)\mathbb{E}\left[\sigma_X^2(T_j)\right] \sim L_\infty^2 \mathbb{E}[T_1]^{-2+4d} \frac{2}{4d(4d-1)} n^{-2+4d} \quad \text{as } n \to \infty.$$

Therefore, we get in both cases

$$P\left(\left|\frac{1}{n}\sum_{j=1}^{n} Y_j^2 - \sigma_X(0)\right| > \varepsilon\right) \xrightarrow[n\to\infty]{} 0.$$

We conclude that $S_n \xrightarrow[n\to\infty]{p} \sqrt{\sigma_X(0)}$ and

$$\frac{1}{L(n)^{1/2} n^{1/2+d}} \frac{R_n}{S_n} \xrightarrow[n\to\infty]{d} \mathcal{R} := \sqrt{\frac{\gamma_d}{\sigma_X(0)}} \left(\max_{0\le t\le 1} B^0_{\frac{1}{2}+d}(t) - \min_{0\le t\le 1} B^0_{\frac{1}{2}+d}(t)\right).$$

□

An application of this result is the estimation of long-memory parameter by the R/S estimate [see e.g. Mandelbrot and Wallis (1969), Li et al. (2019)]. We assume that the autocovariance function satisfies the following condition $\sigma_X(t) \sim ct^{2d-1}$ as $t \to \infty$ (or equivalently $L(t) \to c > 0$ as $t \to \infty$). Taking the logarithm of both sides of (4.6) we get the heuristic identity

$$\log\left(\frac{R_n}{S_n}\right) = (1/2 + d)\log(n) + \log(\sqrt{c}\mathcal{R}) + \varepsilon_n,$$

where $(\varepsilon_n)_n$ is a sequence of random variables which converges to zero in probability. Then we estimate the slope of the regression line of $(\log(n), \log(R_n/S_n))$, which gives the R/S estimate of d. Because of the asymptotic linear relation, we do not fit the straight line on all points, but only on values of n large enough.

Example 2 Let us consider the same model as in Example 1, i.e. the intervals Δ_j have an exponential distribution and $\mathbf{X}$ is a Gaussian process with autocovariance function (2.2).

To evaluate the effects of random sampling on the R/S estimator, we also estimate the long memory parameter of Gaussian FARIMA$(0, d, 0)$ processes. A FARIMA$(0, d, 0)$ process is a stationary discrete-time process whose autocovariance function behaves as

$$\sigma(k) \sim \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} k^{2d-1}, \qquad k \in \mathbb{N}, k \to \infty.$$

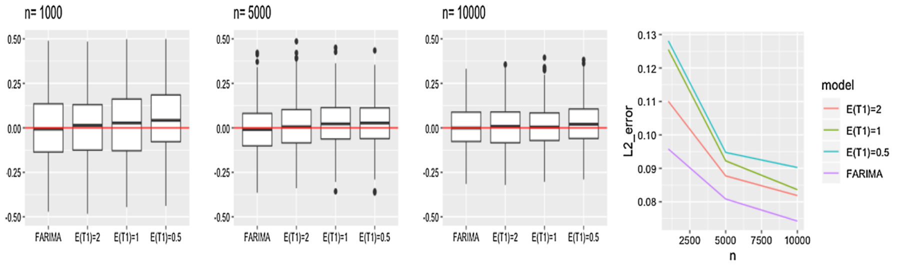


Fig. 2 Boxplots of the estimation error $\hat{d}_n - d$ and representation of the mean squared error (MSE) $\mathbb{E}\left[(\hat{d}_n - d)^2\right]$ for different values of n and $\mathbb{E}[T_1]$. The samples are simulated from the models defined in Example 2. The true value of the long memory parameter is $d = 0.25$. Estimations are done on $p = 500$ independent copies

See Giraitis et al. (2012) and Beran et al. (2013) for detailed presentations of this model.

We compare the performance of R/S estimate $\hat{d}_n$ for different values of $\mathbb{E}[T_1] \in \{1/2,\ 1,\ 2\}$ and different sample sizes $n \in \{1000,\ 5000,\ 100{,}000\}$. We fix $d = 0.25$, the long memory parameter is the same for all simulated processes. We regress $\log(R_k/S_k)$ against $\log(k)$ with $k > m$. The value of m is fixed according to the bias-variance tradeoff on the FARIMA model. In Fig. 2 we represent the boxplots of estimation errors. For all the models, the bias and the variance decrease as function of the sample size n. The boxplots show that the precision is of the same order of magnitude for sampled processes and for the FARIMA process. The R/S estimate is less efficient in terms of mean squared error for the sampled processes in particular for the small values of $\mathbb{E}[T_1]$. This effect can be explained by the fact that continuous time process is observed inside a random time interval $[0, T_n]$ with $T_n \sim \mathbb{E}[T_1]n$ as $n \to \infty$.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

5 Appendix

To prove Lemma 4.1, we need the following intermediate result:

Lemma 5.1 *If* $\mathbb{E}[T_1] < \infty$ *and* $\mathbf{X}$ *has a regularly varying covariance function*

$$\sigma_X(t) = L(t)t^{-1+2d}$$

with $0 < d < 1/2$ *and* L *slowly varying at infinity and ultimately non-increasing. Then,*

$$\mathrm{Var}(\sigma_X(T_h)) = \circ(L(h)^2 h^{-2+4d}), \qquad \text{as } h \to \infty. \tag{5.1}$$

Proof By Theorem 3.1, we have $\mathbb{E}[\sigma_X(T_h)] \underset{h\to\infty}{\sim} L(h)(h\mathbb{E}[T_1])^{-1+2d}$. To get the result, it is enough to prove that

$$\mathbb{E}[\sigma_X(T_h)^2] \underset{h\to\infty}{\sim} L(h)^2(h\mathbb{E}[T_1])^{-2+4d}.$$

To prove the asymptotic behavior of $\mathbb{E}[\sigma_X(T_h)^2]$, we will follow a similar proof as theorem 3.1:

- Let $0 < c < \mathbb{E}[T_1]$, and $h \in \mathbb{N}$ such that $ch \geq 1$,

$$\mathbb{E}[\sigma_X(T_h)^2] \geq \mathbb{E}\left[\sigma_X(T_h)^2\,\mathbb{I}_{T_h>ch}\right] \geq \mathbb{E}\left[L(T_h)^2 T_h^{-2+4d}\,\mathbb{I}_{T_h>ch}\right]$$
$$\geq \inf_{t>ch}\{L(t)^2 t^{4d}\}\mathbb{E}\left[\frac{\mathbb{I}_{T_h>ch}}{T_h^2}\right].$$

  Thanks to Jensen and Hölder inequalities,

$$\mathbb{E}\left[\frac{\mathbb{I}_{T_h>ch}}{T_h^2}\right] \geq \mathbb{E}\left[\frac{\mathbb{I}_{T_h>ch}}{T_h}\right]^2 \text{ and } P(T_h > ch)^2 \leq \mathbb{E}[T_h]\mathbb{E}\left[\frac{\mathbb{I}_{T_h>ch}}{T_h}\right],$$

  that is

$$\mathbb{E}\left[\frac{\mathbb{I}_{T_h>ch}}{T_h^2}\right] \geq \frac{P(T_h > ch)^4}{\mathbb{E}[T_h]^2}.$$

  Summarizing,

$$\frac{\mathbb{E}[\sigma_X(T_h)^2]}{L(h)^2(h\mathbb{E}[T_1])^{-2+4d}} \geq \frac{\inf_{t>ch}\{L(t)^2 t^{4d}\}}{L(h)^2 h^{4d}\mathbb{E}[T_1]^{4d}} P(T_h > ch)^4. \tag{5.2}$$

  Then, for $c < \mathbb{E}[T_1]$, we have $P(T_h > ch) \to 1$ and $\inf_{t>ch}\{L(t)^2 t^{4d}\} \sim L(ch)^2(ch)^{4d}$. Finally, for all $c < \mathbb{E}[T_1]$,

$$\liminf_{h\to\infty} \frac{\mathbb{E}[\sigma_X(T_h)^2]}{L(h)^2(h\mathbb{E}[T_1])^{-2+4d}} \geq \left(\frac{c}{\mathbb{E}[T_1]}\right)^{4d}.$$

  Taking the limit as $c \to \mathbb{E}[T_1]$, we get

$$\liminf_{h\to\infty} \frac{\mathbb{E}[\sigma_X(T_h)^2]}{L(h)^2(h\mathbb{E}[T_1])^{-2+4d}} \geq 1.$$

- Let $\frac{1}{2} < s < \tau < 1$, t_0 such that $L(.)$ is non-increasing and positive on $[t_0, \infty)$ and h such that $\mu_{h,s} - \mu_{h,s}^{\tau} \geq t_0$, with the same notation as Theorem 3.1,

$$\begin{aligned}\mathbb{E}[\sigma_X(T_h)^2] &= \mathbb{E}\left[L(T_h)^2 T_h^{-2+4d}\, \mathbb{I}_{T_{h,s} \geq \mu_{h,s} - \mu_{h,s}^{\tau}}\right] + \mathbb{E}\left[\sigma(T_h)^2\, \mathbb{I}_{T_{h,s} < \mu_{h,s} - \mu_{h,s}^{\tau}}\right] \\ &\leq L(\mu_{h,s} - \mu_{h,s}^{\tau})^2 \left(\mu_{h,s} - \mu_{h,s}^{\tau}\right)^{-2+4d} + \sigma_X(0)^2 P\left(T_{h,s} < \mu_{h,s} - \mu_{h,s}^{\tau}\right).\end{aligned}$$

We get

$$\begin{aligned}\frac{\mathbb{E}[\sigma_X(T_h)^2]}{L(h)^2(h\mathbb{E}[T_1])^{-2+4d}} \leq &\left(\frac{L(\mu_{h,s} - \mu_{h,s}^{\tau})}{L(h)}\right)^2 \left(\frac{\mu_{h,s} - \mu_{h,s}^{\tau}}{h\mathbb{E}[T_1]}\right)^{-2+4d} \\ &+ \sigma_X(0)^2 \frac{P\left(T_{h,s} < \mu_{h,s} - \mu_{h,s}^{\tau}\right)}{L(h)^2(h\mathbb{E}[T_1])^{-2+4d}},\end{aligned}$$

and finally

$$\limsup_{h\to\infty} \frac{\mathbb{E}[\sigma_X(T_h)^2]}{L(h)^2(h\mathbb{E}[T_1])^{-2+4d}} \leq 1.$$

□

Proof of Lemma 4.1:
Denote

$$\begin{aligned}W_n &= L(n)^{-1} n^{-1-2d} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_X(T_j - T_i) = L(n)^{-1} n^{-1-2d} \mathrm{Var}(X_{T_1} \\ &\quad + \cdots + X_{T_n} | T_1, \ldots, T_n).\end{aligned}$$

We want to prove that W_n converges in probability to γ_d. To do this, we will show that $\mathbb{E}[W_n] \xrightarrow[n\to\infty]{} \gamma_d$ and $\mathrm{Var}(W_n) \xrightarrow[n\to\infty]{} 0$.

- As $\mathbf{X}$ is a centered process $E[W_n] = L(n)^{-1} n^{-1-2d} \mathrm{Var}(Y_1 + \cdots + Y_n)$. By Theorem 3.1, we have

$$\sigma_Y(h) \sim L(h)(h\mathbb{E}[T_1])^{-1+2d} \qquad h \to \infty,$$

then

$$L(n)^{-1} n^{-1-2d} \mathrm{Var}(Y_1 + \cdots + Y_n) \xrightarrow[n\to\infty]{} \gamma_d, \tag{5.3}$$

[see Giraitis et al. (2012) Proposition 3.3.1, page 43].
Therefore we obtain

$$E[W_n] \xrightarrow[n\to\infty]{} \gamma_d.$$

- Furthermore,

$$\begin{aligned}\mathrm{Var}(W_n) &= L(n)^{-2}n^{-2-4d}\mathrm{Var}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_X(T_j-T_i)\right)\\ &\le L(n)^{-2}n^{-2-4d}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\sqrt{\mathrm{Var}(\sigma_X(T_j-T_i))}\right)^2\\ &= \left(2n^{-1-2d}L(n)^{-1}\sum_{h=1}^{n}(n-h)\sqrt{\mathrm{Var}(\sigma_X(T_h))}\right)^2.\end{aligned}$$

Then, by Lemma 5.1, $\sqrt{\mathrm{Var}(\sigma_X(T_h))} = \circ(L(h)h^{-1+2d})$ and $2\sum_{h=1}^{n}(n-h)L(h)h^{-1+2d} \sim \frac{L(n)n^{1+2d}}{d(1+2d)}$.
We get

$$2\sum_{h=1}^{n}(n-h)\sqrt{\mathrm{Var}(\sigma_X(T_h))} = \circ(L(n)n^{1+2d}).$$

Finally, $\mathrm{Var}(W_n) = \circ(1)$ which means that $\mathrm{Var}(W_n) \xrightarrow[n\to\infty]{} 0$. We obtain

$$W_n \xrightarrow[n\to\infty]{L^2,\ p} \gamma_d.$$

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