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Quadrupling: construction of uniform designs with large run sizes

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Abstract

Fractional factorial designs are widely used because of their various merits. Foldover or level permutation are usually used to construct optimal fractional factorial designs. In this paper, a novel method via foldover and level permutation, called quadrupling, is proposed to construct uniform four-level designs with large run sizes. The relationship of uniformity between the initial design and the design obtained by quadrupling is investigated, and new lower bounds of wrap-around L_2-discrepancy for such designs are obtained. These results provide a theoretical basis for constructing uniform four-level designs with large run sizes by quadrupling successively. Furthermore, the analytic connection between the initial design and the design obtained by quadrupling is presented under generalized minimum aberration criterion.

Keywords Level permutation · Foldover · Uniform design · Quadruple design · Generalized minimum aberration · Wrap-around L_2-discrepancy · Lower bound

Mathematics Subject Classification 62K15 · 62K10 · 62K99

1 Introduction

Fractional factorial designs are widely used in areas such as science, engineering and industry. Hence the construction of fractional factorial designs is an important issue. Designs obtained from foldover have excellent geometrical symmetric structure and good statistical properties, so the technique of foldover has been widely applied in the construction of optimal designs. Based on two specific types of foldover, Chen

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and Cheng (2006) proposed the method of doubling to construct two-level double designs with high resolution. A general complementary two-level design theory for doubling was discussed in Xu and Cheng (2008), and the analytical connection of the wordlength patterns of each pair of complementary projection designs by repeated doubling was built. Lei and Qin (2014) studied two-level designs by doubling from viewpoint of uniformity, and obtained some lower bounds of centered L_2-discrepancy for these kinds of designs. For further results on doubling we refer to Ou and Qin (2010, 2017). The foldover of two-level designs is simplest, which only changes the symbols of two levels ± 1. However, the method is out of place in high-level designs. Note that the foldover of two-level designs is a level permutation of factors in essence.

There is a lot of research on level permutations of factors. Cheng and Ye (2004) indicated level permutations of factors could alter their geometrical structures and statistical properties. Tang et al. (2012) constructed uniform minimum aberration designs by level permutations of factors and showed that the uniformity of this kind of designs is better than of those constructed by existing methods. Tang and Xu (2013) provided the construction method of multi-level uniform designs under centered L_2-discrepancies by level permutations of factors of a generalized minimum aberration design. Tang and Xu (2014) provided a justification of the minimum aberration criterion for quantitative factors and studied level permutations of factors for regular fractional factorial designs in order to improve their efficiency for screening quantitative factors. Based on level permutations of factors, Xu et al. (2014) built the relationship between average wrap-around L_2-discrepancy and generalized wordlength pattern and proposed a general method for identifying designs with smaller wrap-around L_2-discrepancy.

According to all level permutations of a three-level design $\mathcal{A}$ and the technique of foldover, Ou et al. (2019) extended the method of doubling to the method of tripling and proposed the Triple design $\mathcal{T}(\mathcal{A})$, which has been used to construct three-level fractional factorial designs. In Ou et al. (2019), the connection between the wrap-around L_2-discrepancy of the Triple design and the wordlength pattern of its initial design is built. A tight lower bound of the wrap-around L_2-discrepancy of Triple design is obtained, and an efficient method for constructing uniform minimum aberration designs is proposed based on the projection of Triple design.

The discussions on the construction of two-level or three-level uniform designs are rich. However, in the construction of high-level designs various difficulties exist due to their complex structure. The present paper aims to extend the results in Ou et al. (2019) further. The method of quadrupling via level permutations of a four-level design and foldover is proposed, which constructs uniform four-level designs with large sizes.

The paper is organized as follows. In Sect. 2, some notations, the definition of quadrupling and the Quadruple design are provided. Section 3 presents an important lemma, which is a basis to study latter problems. In Sect. 4, the relationship of uniformity between the Quadruple design and the initial design is considered, and new lower bounds of wrap-around L_2-discrepancy for such designs are also obtained, which can be used as a benchmark to construct and measure the uniformity of a four-level design and the Quadruple design. In Sect. 5, the analytic connection between the Quadruple design and the initial design under generalized minimum aberration criterion is built. Some illustrative examples are shown to support our theoretical results in Sect. 6. Concluding remarks are given in the final Sect. 7.

2 Preliminaries

Let $\mathcal{U}(n; 4^s)$ denote a class of U-type designs with n runs and s factors each at four levels. A design $\mathcal{F}$ in $\mathcal{U}(n; 4^s)$ can be presented as an $n \times s$ matrix with entries 0, 1, 2, 3, each entry appears equally often in each column, where each row corresponds to a run, and each column corresponds to a factor. An orthogonal array of strength t and size n with s constrains, denoted by $OA(n, 4^s, t)$, is a factorial design of n runs and s four-level factors such that all the level-combinations for any t factors appear equally often. The design $\mathcal{F}$ in $\mathcal{U}(n; 4^s)$, of course, is an orthogonal array of strength 1.

Consider a four-level design $\mathcal{F} \in \mathcal{U}(n; 4^s)$, the four kinds of level permutation of $\mathcal{F}$ and the corresponding designs obtained from these level permutations are listed in Table 1.

Inspired by the construction of two-level Double designs and three-level Triple designs, and based on the level permutations of $\mathcal{F}$ given in Table 1, the method of quadrupling for constructing uniform four-level designs is proposed in following definition.

Definition 1 Suppose $\mathcal{F} \in \mathcal{U}(n; 4^s)$, the $4n \times 4s$ matrix

$$\mathcal{Q}(\mathcal{F}) = \begin{pmatrix} \mathcal{F} & \mathcal{F} & \mathcal{F} & \mathcal{F} \\ \mathcal{F} & \mathcal{F}_{(1)} & \mathcal{F}_{(2)} & \mathcal{F}_{(3)} \\ \mathcal{F} & \mathcal{F}_{(2)} & \mathcal{F}_{(3)} & \mathcal{F}_{(1)} \\ \mathcal{F} & \mathcal{F}_{(3)} & \mathcal{F}_{(1)} & \mathcal{F}_{(2)} \end{pmatrix},$$

is called the quadrupling of $\mathcal{F}$, where $\mathcal{F}_{(i)}$ is shown in Table 1, $i = 1, 2, 3$. $\mathcal{Q}(\mathcal{F})$ is called the Quadruple design of $\mathcal{F}$, $\mathcal{F}$ is called the initial design of $\mathcal{Q}(\mathcal{F})$.

For $\mathcal{F} \in \mathcal{U}(n; 4^s)$, it is obvious that $\mathcal{Q}(\mathcal{F})$ is also a four-level design in $\mathcal{U}(4n; 4^{4s})$ which quadruples both the number of runs and factors of $\mathcal{F}$. The following example shows that the Quadruple design $\mathcal{Q}(\mathcal{F})$ is just the orthogonal combination of the four kinds of level permutations of $\mathcal{F}$ given in Table 1. From this viewpoint, the Quadruple design $\mathcal{Q}(\mathcal{F})$ is expected to have nice properties.

Example 1 Consider the simplest initial four-level design with four runs and one factor, i.e., $\mathcal{F} \in \mathcal{U}(4; 4^1)$ given as $\mathcal{F} = (0 \quad 1 \quad 2 \quad 3)'$. According to Definition 1, the

Table 1 Four kinds of level permutation of $\mathcal{F}$ and the corresponding images

Permutation no.	Initial design	Permutation method	Image
1	$\mathcal{F}$	$(0, 1, 2, 3) \mapsto (0, 1, 2, 3)$	$\mathcal{F}$
2	$\mathcal{F}$	$(0, 1, 2, 3) \mapsto (1, 0, 3, 2)$	$\mathcal{F}_{(1)}$
3	$\mathcal{F}$	$(0, 1, 2, 3) \mapsto (2, 3, 0, 1)$	$\mathcal{F}_{(2)}$
4	$\mathcal{F}$	$(0, 1, 2, 3) \mapsto (3, 2, 1, 0)$	$\mathcal{F}_{(3)}$

Quadruple design $\mathcal{Q}(\mathcal{F}) \in \mathcal{U}(16; 4^4)$ of $\mathcal{F}$ is easily obtained as follows

$$\mathcal{Q}(\mathcal{F}) = \begin{pmatrix} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 1 & 0 & 3 & 2 & 2 & 3 & 0 & 1 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 2 & 3 & 0 & 1 & 3 & 2 & 1 & 0 & 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 3 & 3 & 2 & 1 & 0 & 1 & 0 & 3 & 2 & 2 & 3 & 0 & 1 \end{pmatrix}'.$$

One can find that $\mathcal{Q}(\mathcal{F})$ is an orthogonal array of strength two.

For any design $\mathcal{F} \in \mathcal{U}(n; 4^s)$, the distance distribution $(E_0(\mathcal{F}), \ldots, E_s(\mathcal{F}))$ of $\mathcal{F}$ is defined as

$$E_i(\mathcal{F}) = \frac{1}{n}|\{(a, b) : d_H(a, b) = i, a \text{ and } b \text{ are two runs of } \mathcal{F}\}|, \quad 0 \le i \le s, \quad (1)$$

where $d_H(a, b)$ is the Hamming distance between two rows a and b, that is, the number of places where they differ, $|\Omega|$ is the cardinality of Ω. Based on the distance distribution $(E_0(\mathcal{F}), \ldots, E_s(\mathcal{F}))$ of design $\mathcal{F}$, the generalized wordlength pattern $(A_1(\mathcal{F}), \ldots, A_s(\mathcal{F}))$ of $\mathcal{F}$ is defined as

$$A_j(\mathcal{F}) = \frac{1}{n}\sum_{i=0}^{s} P_j(i; s, 4)E_i(\mathcal{F}), \quad j = 1, \ldots, s, \qquad (2)$$

where $P_j(i; s, 4) = \sum_{r=0}^{j}(-1)^r 3^{j-r}\binom{i}{r}\binom{s-i}{j-r}$ is Krawtchouk polynomial, $\binom{x}{y} = x(x-1)\cdots(x-y+1)/y!$, $\binom{x}{0} = 1$, for $x < y$, $\binom{x}{y} = 0$. The generalized minimum aberration (for simplicity, GMA) criterion is to sequentially minimize $A_j(\mathcal{F})$ for $j = 1, \ldots, s$. About GMA criterion, one can refer to Xu and Wu (2001) for more details.

In this paper, the wrap-around L_2-discrepancy (for simplicity, WD) is considered as the measurement of uniformity of a design. For any design $\mathcal{F} = (x_{ij})_{n\times s} \in \mathcal{U}(n; 4^s)$, its WD value, denoted as $WD(\mathcal{F})$, can be expressed as

$$[WD(\mathcal{F})]^2 = -\left(\frac{4}{3}\right)^s + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\prod_{l=1}^{s}\left[\frac{3}{2} - |u_{il} - u_{jl}|(1 - |u_{il} - u_{jl}|)\right], \quad (3)$$

where $u_{il} = \frac{2x_{il}+1}{8}, i = 1, \ldots, n, l = 1, \ldots, s$. The uniformity criterion favors designs with the smallest $WD(\mathcal{F})$.

For any $\mathcal{F} \in \mathcal{U}(n; 4^s)$, denote $\delta_{\mathcal{F}(i)\mathcal{F}(j)}(a, b)$ as the number of position where rows i and j of $\mathcal{F}$ take pair (a, b), where $a, b = 0, 1, 2, 3, i, j = 1, \ldots, n$. Define $\Omega_1 = \{(0, 0), (1, 1), (2, 2), (3, 3)\}$, $\Omega_2 = \{(0, 1), (1, 2), (2, 3), (0, 3), (1, 0), (2, 1), (3, 2), (3, 0)\}$, $\Omega_3 = \{(0, 2), (1, 3), (2, 0), (3, 1)\}$. Thus, the expression of $WD(\mathcal{F})$ in (3) can be rewritten as follows

$$[WD(\mathcal{F})]^2 = -\left(\frac{4}{3}\right)^s + \frac{1}{n}\left(\frac{3}{2}\right)^s + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\left(\frac{3}{2}\right)^{\lambda_{ij}(\mathcal{F},\mathcal{F})}\left(\frac{21}{16}\right)^{\tau_{ij}(\mathcal{F},\mathcal{F})}\left(\frac{5}{4}\right)^{\sigma_{ij}(\mathcal{F},\mathcal{F})}, \quad (4)$$

where $\lambda_{ij}(\mathcal{F},\mathcal{F}) = \sum_{(a,b)\in\Omega_1}\delta_{\mathcal{F}(i)\mathcal{F}(j)}(a,b)$, $\tau_{ij}(\mathcal{F},\mathcal{F}) = \sum_{(a,b)\in\Omega_2}\delta_{\mathcal{F}(i)\mathcal{F}(j)}(a,b)$, $\sigma_{ij}(\mathcal{F},\mathcal{F}) = \sum_{(a,b)\in\Omega_3}\delta_{\mathcal{F}(i)\mathcal{F}(j)}(a,b)$.

3 An important lemma

Consider a four-level design $\mathcal{F} \in \mathcal{U}(n; 4^s)$, $\mathcal{Q}(\mathcal{F})$ is the Quadruple design of $\mathcal{F}$, denote

$$\lambda^*_{ij}(\mathcal{Q}(\mathcal{F}), \mathcal{Q}(\mathcal{F})) = \sum_{(a,b)\in\Omega_1}\delta_{\mathcal{Q}(\mathcal{F})(i)\mathcal{Q}(\mathcal{F})(j)}(a,b),$$
$$\tau^*_{ij}(\mathcal{Q}(\mathcal{F}), \mathcal{Q}(\mathcal{F})) = \sum_{(a,b)\in\Omega_2}\delta_{\mathcal{Q}(\mathcal{F})(i)\mathcal{Q}(\mathcal{F})(j)}(a,b),$$
$$\sigma^*_{ij}(\mathcal{Q}(\mathcal{F}), \mathcal{Q}(\mathcal{F})) = \sum_{(a,b)\in\Omega_3}\delta_{\mathcal{Q}(\mathcal{F})(i)\mathcal{Q}(\mathcal{F})(j)}(a,b).$$

The following lemma respectively provides the analytic connections between $\lambda_{ij}(\mathcal{F},\mathcal{F})$ and $\lambda^*_{ij}(\mathcal{Q}(\mathcal{F}), \mathcal{Q}(\mathcal{F}))$, $\tau_{ij}(\mathcal{F},\mathcal{F})$ and $\tau^*_{ij}(\mathcal{Q}(\mathcal{F}), \mathcal{Q}(\mathcal{F}))$, $\sigma_{ij}(\mathcal{F},\mathcal{F})$ and $\sigma^*_{ij}(\mathcal{Q}(\mathcal{F}), \mathcal{Q}(\mathcal{F}))$ in $\mathcal{F}$ and $\mathcal{Q}(\mathcal{F})$, which is a basis for studying the rest problems in this paper.

Lemma 1 *Suppose $\mathcal{F} \in \mathcal{U}(n; 4^s)$, $\mathcal{Q}(\mathcal{F}) \in \mathcal{U}(4n; 4^{4s})$ is the Quadruple design of $\mathcal{F}$. Then the analytic connections between $\lambda_{ij}(\mathcal{F},\mathcal{F})$ and $\lambda^*_{ij}(\mathcal{Q}(\mathcal{F}), \mathcal{Q}(\mathcal{F}))$, $\tau_{ij}(\mathcal{F},\mathcal{F})$ and $\tau^*_{ij}(\mathcal{Q}(\mathcal{F}), \mathcal{Q}(\mathcal{F}))$, $\sigma_{ij}(\mathcal{F},\mathcal{F})$ and $\sigma^*_{ij}(\mathcal{Q}(\mathcal{F}), \mathcal{Q}(\mathcal{F}))$ in $\mathcal{F}$ and $\mathcal{Q}(\mathcal{F})$ are as follows,*
(i)

$$\lambda^*_{(i+kn)(j+ln)}(\mathcal{Q}(\mathcal{F}), \mathcal{Q}(\mathcal{F})) = \begin{cases} 4\lambda_{ij}(\mathcal{F},\mathcal{F}), & 1 \le i, j \le n, k = l; k, l = 0, 1, 2, 3, \\ s, & 1 \le i, j \le n, k \neq l; k, l = 0, 1, 2, 3. \end{cases}$$

(ii)

$$\tau^*_{(i+kn)(j+ln)}(\mathcal{Q}(\mathcal{F}), \mathcal{Q}(\mathcal{F})) = \begin{cases} 4\tau_{ij}(\mathcal{F},\mathcal{F}), & 1 \le i, j \le n, k = l; k, l = 0, 1, 2, 3, \\ 2s, & 1 \le i, j \le n, k \neq l; k, l = 0, 1, 2, 3. \end{cases}$$

(iii)

$$\sigma^*_{(i+kn)(j+ln)}(\mathcal{Q}(\mathcal{F}), \mathcal{Q}(\mathcal{F})) = \begin{cases} 4\sigma_{ij}(\mathcal{F},\mathcal{F}), & 1 \le i, j \le n, k = l; k, l = 0, 1, 2, 3, \\ s, & 1 \le i, j \le n, k \neq l; k, l = 0, 1, 2, 3. \end{cases}$$

Proof We firstly prove the (i). When $k = l = 0$,

$$\lambda^*_{ij}(\mathcal{Q}(\mathcal{F}), \mathcal{Q}(\mathcal{F})) = \lambda_{ij}(\mathcal{F}, \mathcal{F}) + \lambda_{ij}(\mathcal{F}, \mathcal{F}) + \lambda_{ij}(\mathcal{F}, \mathcal{F}) + \lambda_{ij}(\mathcal{F}, \mathcal{F}) = 4\lambda_{ij}(\mathcal{F}, \mathcal{F}).$$

When $k = l = 1, 2, 3$, the proofs of such cases are similar.

Next, we prove $\lambda^*_{(i+kn)(j+ln)}(\mathcal{Q}(\mathcal{F}), \mathcal{Q}(\mathcal{F})) = s$, when $k \neq l, k, l = 0, 1, 2$. Note that $\mathcal{F}_{(1)}$ is obtained by the level permutations $(0, 1, 2, 3) \to (1, 0, 3, 2)$ in the $\mathcal{F}$, so

$$\begin{aligned}\lambda_{ij}(\mathcal{F}, \mathcal{F}_{(1)}) &= \delta^{(0,0)}_{\mathcal{F}(i),\mathcal{F}_{(1)}(j)} + \delta^{(1,1)}_{\mathcal{F}(i),\mathcal{F}_{(1)}(j)} + \delta^{(2,2)}_{\mathcal{F}(i),\mathcal{F}_{(1)}(j)} + \delta^{(3,3)}_{\mathcal{F}(i),\mathcal{F}_{(1)}(j)}\\ &= \delta^{(0,1)}_{\mathcal{F}(i),\mathcal{F}(j)} + \delta^{(1,0)}_{\mathcal{F}(i),\mathcal{F}(j)} + \delta^{(2,3)}_{\mathcal{F}(i),\mathcal{F}(j)} + \delta^{(3,2)}_{\mathcal{F}(i),\mathcal{F}(j)}.\end{aligned}$$

Similarly, $\mathcal{F}_{(2)}$ is obtained by the level permutations $(0, 1, 2, 3) \to (2, 3, 0, 1)$ in the $\mathcal{F}$, $\mathcal{F}_{(3)}$ is obtained by the level permutation $(0, 1, 2, 3) \to (3, 2, 1, 0)$ in the $\mathcal{F}$, so

$$\begin{aligned}\lambda_{ij}(\mathcal{F}, \mathcal{F}_{(2)}) &= \delta^{(0,0)}_{\mathcal{F}(i),\mathcal{F}_{(2)}(j)} + \delta^{(1,1)}_{\mathcal{F}(i),\mathcal{F}_{(2)}(j)} + \delta^{(2,2)}_{\mathcal{F}(i),\mathcal{F}_{(2)}(j)} + \delta^{(3,3)}_{\mathcal{F}(i),\mathcal{F}_{(2)}(j)}\\ &= \delta^{(0,2)}_{\mathcal{F}(i),\mathcal{F}(j)} + \delta^{(1,3)}_{\mathcal{F}(i),\mathcal{F}(j)} + \delta^{(2,0)}_{\mathcal{F}(i),\mathcal{F}(j)} + \delta^{(3,1)}_{\mathcal{F}(i),\mathcal{F}(j)},\end{aligned}$$

and

$$\begin{aligned}\lambda_{ij}(\mathcal{F}, \mathcal{F}_{(3)}) &= \delta^{(0,0)}_{\mathcal{F}(i),\mathcal{F}_{(3)}(j)} + \delta^{(1,1)}_{\mathcal{F}(i),\mathcal{F}_{(3)}(j)} + \delta^{(2,2)}_{\mathcal{F}(i),\mathcal{F}_{(3)}(j)} + \delta^{(3,3)}_{\mathcal{F}(i),\mathcal{F}_{(3)}(j)}\\ &= \delta^{(0,3)}_{\mathcal{F}(i),\mathcal{F}(j)} + \delta^{(1,2)}_{\mathcal{F}(i),\mathcal{F}(j)} + \delta^{(2,1)}_{\mathcal{F}(i),\mathcal{F}(j)} + \delta^{(3,0)}_{\mathcal{F}(i),\mathcal{F}(j)}.\end{aligned}$$

When $k = 0$ and $l = 1$,

$$\begin{aligned}\lambda_{i,j+n}(\mathcal{Q}(\mathcal{F}), \mathcal{Q}(\mathcal{F})) &= \lambda_{ij}(\mathcal{F}, \mathcal{F}) + \lambda_{ij}(\mathcal{F}, \mathcal{F}_{(1)}) + \lambda_{ij}(\mathcal{F}, \mathcal{F}_{(2)}) + \lambda_{ij}(\mathcal{F}, \mathcal{F}_{(3)})\\ &= \lambda_{ij}(\mathcal{F}, \mathcal{F}) + \delta^{(0,1)}_{\mathcal{F}(i),\mathcal{F}(j)} + \delta^{(1,0)}_{\mathcal{F}(i),\mathcal{F}(j)} + \delta^{(2,3)}_{\mathcal{F}(i),\mathcal{F}(j)} + \delta^{(3,2)}_{\mathcal{F}(i),\mathcal{F}(j)}\\ &\quad + \delta^{(0,2)}_{\mathcal{F}(i),\mathcal{F}(j)} + \delta^{(1,3)}_{\mathcal{F}(i),\mathcal{F}(j)} + \delta^{(2,0)}_{\mathcal{F}(i),\mathcal{F}(j)} + \delta^{(3,1)}_{\mathcal{F}(i),\mathcal{F}(j)}\\ &\quad + \delta^{(0,3)}_{\mathcal{F}(i),\mathcal{F}(j)} + \delta^{(1,2)}_{\mathcal{F}(i),\mathcal{F}(j)} + \delta^{(2,1)}_{\mathcal{F}(i),\mathcal{F}(j)} + \delta^{(3,0)}_{\mathcal{F}(i),\mathcal{F}(j)}\\ &= \lambda_{ij}(\mathcal{F}, \mathcal{F}) + \tau_{ij}(\mathcal{F}, \mathcal{F}) + \sigma_{ij}(\mathcal{F}, \mathcal{F})\\ &= s.\end{aligned}$$

When $k \neq l$, the proofs of the other cases are similar. The proofs of the (ii) and (iii) are also similar to the (i). □

4 Relationship of uniformity between $\mathcal{Q}(\mathcal{F})$ and $\mathcal{F}$

The relationship between $\mathcal{Q}(\mathcal{F})$ and $\mathcal{F}$ is considered under uniformity criterion measured by WD, furthermore, new lower bounds of WD for the Quadruple design $\mathcal{Q}(\mathcal{F})$ and the initial design $\mathcal{F}$ are obtained in this section.

The following lemma shows that the WD value of $\mathcal{Q}(\mathcal{F})$ is completely decided by $\{\lambda_{ij}(\mathcal{F},\mathcal{F})\}$, $\{\tau_{ij}(\mathcal{F},\mathcal{F})\}$ and $\{\sigma_{ij}(\mathcal{F},\mathcal{F})\}$ of $\mathcal{F}$.

Lemma 2 *Let $\mathcal{F} \in \mathcal{U}(n; 4^s)$, $\mathcal{Q}(\mathcal{F}) \in \mathcal{U}(4n; 4^{4s})$ is the Quadruple design of $\mathcal{F}$. Then*

$$[WD(\mathcal{Q}(\mathcal{F}))]^2 = -\left(\frac{4}{3}\right)^{4s} + \frac{3}{4}\left(\frac{6615}{2048}\right)^{s} + \frac{1}{4n}\left(\frac{81}{16}\right)^{s} + \frac{1}{4n^2}\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\left(\frac{81}{16}\right)^{\lambda_{ij}(\mathcal{F},\mathcal{F})}\left(\frac{194481}{65536}\right)^{\tau_{ij}(\mathcal{F},\mathcal{F})}\left(\frac{625}{256}\right)^{\sigma_{ij}(\mathcal{F},\mathcal{F})} \tag{5}$$

where $\lambda_{ij}(\mathcal{F},\mathcal{F})$, $\tau_{ij}(\mathcal{F},\mathcal{F})$, $\sigma_{ij}(\mathcal{F},\mathcal{F})$ are shown in (4).

Proof According to Eq. (3) and Lemma 1,

$$\begin{aligned}
&[WD(\mathcal{Q}(\mathcal{F}))]^2 \\
&= -\left(\frac{4}{3}\right)^{4s} + \frac{1}{(4n)^2}\sum_{i=1}^{4n}\sum_{j=1}^{4n}\prod_{k=1}^{4s}\left[\frac{3}{2} - |u_{il}-u_{jl}|(1-|u_{il}-u_{jl}|)\right] \\
&= -\left(\frac{4}{3}\right)^{4s} + \frac{1}{(4n)^2}\left(\sum_{i=1}^{n} + \sum_{i=n+1}^{2n} + \sum_{i=2n+1}^{3n} + \sum_{i=3n+1}^{4n}\right) \\
&\quad \left(\sum_{j=1}^{n} + \sum_{j=n+1}^{2n} + \sum_{j=2n+1}^{3n} + \sum_{j=3n+1}^{4n}\right) \\
&\quad \times \prod_{k=1}^{s}\prod_{k=s+1}^{2s}\prod_{k=2s+1}^{3s}\prod_{k=3s+1}^{4s}\left[\frac{3}{2} - |u_{il}-u_{jl}|(1-|u_{il}-u_{jl}|)\right] \\
&= -\left(\frac{4}{3}\right)^{4s} + \frac{1}{(4n)^2}\left[4\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{3}{2}\right)^{4\lambda_{ij}(\mathcal{F},\mathcal{F})}\left(\frac{21}{16}\right)^{4\tau_{ij}(\mathcal{F},\mathcal{F})}\left(\frac{5}{4}\right)^{4\sigma_{ij}(\mathcal{F},\mathcal{F})}\right. \\
&\quad \left. + 12\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{3}{2}\right)^{s}\left(\frac{21}{16}\right)^{2s}\left(\frac{5}{4}\right)^{s}\right] \\
&= -\left(\frac{4}{3}\right)^{4s} + \frac{3}{4}\left(\frac{6615}{2048}\right)^{s} + \frac{1}{4n}\left(\frac{81}{16}\right)^{s} \\
&\quad + \frac{1}{4n^2}\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\left(\frac{81}{16}\right)^{\lambda_{ij}(\mathcal{F},\mathcal{F})}\left(\frac{194481}{65536}\right)^{\tau_{ij}(\mathcal{F},\mathcal{F})}\left(\frac{625}{256}\right)^{\sigma_{ij}(\mathcal{F},\mathcal{F})},
\end{aligned}$$

which completes the proof of Lemma 2. □

The lower bounds of discrepancy can be used as a benchmark for searching and constructing uniform designs. For $\mathcal{F} \in \mathcal{U}(n; 4^s)$, in order to obtain the lower bounds of the WD value of $\mathcal{F}$ and its Quadruple design $\mathcal{Q}(\mathcal{F}) \in \mathcal{U}(4n; 4^{4s})$, the following two lemmas are required.

Lemma 3 *Let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ be $2n$ nonnegative integers and $\sum_{i=1}^{n} x_i = c_1$, $\sum_{i=1}^{n} y_i = c_2$. Define $z_i = ax_i + by_i, i = 1, \ldots, n$, $c = ac_1 + bc_2$, where a and b are nonnegative number. Let $z_{(1)}, z_{(2)}, \ldots, z_{(l)}$ be the ordered arrangements of the distinct possible values of $z_1, z_2, \ldots, z_n$, and k be the largest integer such that $z_{(k)} \leq c/n < z_{(k+1)}$. Then for any positive integer t,*

$$\sum_{i=1}^{n} z_i^t \geq p z_{(k)}^t + q z_{(k+1)}^t,$$

where p and q are integers such that $p + q = n$ and $pz_{(k)} + qz_{(k+1)} = c$. The lower bound can be attained if only if all z_i are as equal as possible and equal to c/n.

Proof It is enough to prove that if we deviate slightly from the above choice, the value of $\sum_{i=1}^{n} z_i^t$ will not decrease. For $\epsilon_1 > 0, \epsilon_2 > 0$, let us consider the following simplest choice of $\sum_{i=1}^{n} z_i^t$

$$\epsilon_1 z_{(k-1)}^t + (p - \epsilon_1) z_{(k)}^t + (q - \epsilon_2) z_{(k+1)}^t + \epsilon_2 z_{(k+2)}^t,$$

where

$$\begin{aligned} &\epsilon_1 z_{(k-1)} + (p - \epsilon_1) z_{(k)} + (q - \epsilon_2) z_{(k+1)} + \epsilon_2 z_{(k+2)} = c \\ &\Rightarrow \epsilon_1 (z_{(k)} - z_{(k-1)}) = \epsilon_2 (z_{(k+2)} - z_{(k+1)}) \end{aligned}$$

It is easy to note that

$$\epsilon_1 + (p - \epsilon_1) + (q - \epsilon_2) + \epsilon_2 = p + q = n.$$

Let $\Delta_1 = pz_{(k)}^t + qz_{(k+1)}^t$ and $\Delta_2 = \epsilon_1 z_{(k-1)}^t + (p - \epsilon_1) z_{(k)}^t + (q - \epsilon_2) z_{(k+1)}^t + \epsilon_2 z_{(k+2)}^t$. Then

$$\begin{aligned} \Delta_2 - \Delta_1 &= \epsilon_1 z_{(k-1)}^t + (p - \epsilon_1) z_{(k)}^t + (q - \epsilon_2) z_{(k+1)}^t + \epsilon_2 z_{(k+2)}^t - p z_{(k)}^t - q z_{(k+1)}^t \\ &= -\epsilon_1 (z_{(k)}^t - z_{(k-1)}^t) + \epsilon_2 (z_{(k+2)}^t - z_{(k+1)}^t) \\ &= -\epsilon_1 (z_{(k)} - z_{(k-1)})(z_{(k)}^{t-1} + z_{(k)}^{t-2} z_{(k-1)} + \cdots + z_{(k)} z_{(k-1)}^{t-2} + z_{(k-1)}^{t-1}) \\ &\quad + \epsilon_2 (z_{(k+2)} - z_{(k+1)})(z_{(k+2)}^{t-1} + z_{(k+2)}^{t-2} z_{(k+1)} + \cdots \\ &\quad + z_{(k+2)} z_{(k+1)}^{t-2} + z_{(k+1)}^{t-1}) \\ &= -\epsilon_2 (z_{(k+2)} - z_{(k+1)})(z_{(k)}^{t-1} + z_{(k)}^{t-2} z_{(k-1)} + \cdots + z_{(k)} z_{(k-1)}^{t-2} + z_{(k-1)}^{t-1}) \\ &\quad + \epsilon_2 (z_{(k+2)} - z_{(k+1)})(z_{(k+2)}^{t-1} + z_{(k+2)}^{t-2} z_{(k+1)} + \cdots \\ &\quad + z_{(k+2)} z_{(k+1)}^{t-2} + z_{(k+1)}^{t-1}) \\ &= \epsilon_2 (z_{(k+2)} - z_{(k+1)}) \Big[(z_{(k+2)}^{t-1} - z_{(k)}^{t-1}) + (z_{(k+2)}^{t-2} z_{(k+1)} - z_{(k)}^{t-2} z_{(k-1)}) \\ &\quad + \cdots + (z_{(k+2)} z_{(k+1)}^{t-2} - z_{(k)} z_{(k-1)}^{t-2}) + (z_{(k+1)}^{t-1} - z_{(k-1)}^{t-1}) \Big] \\ &> 0. \end{aligned}$$

Lemma 4 *Let $\mathcal{F} \in \mathcal{U}(n; 4^s)$ be a four-level design. Then* □

(i) $\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\lambda_{ij}(\mathcal{F},\mathcal{F}) = ns\left(\frac{n}{4}-1\right)$,

(ii) $\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\tau_{ij}(\mathcal{F},\mathcal{F}) = 2\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\sigma_{ij}(\mathcal{F},\mathcal{F}) = \frac{sn^2}{2}$.

The proof of Lemma 4 is obvious by the definitions of $\lambda_{ij}(\mathcal{F},\mathcal{F})$, $\tau_{ij}(\mathcal{F},\mathcal{F})$ and $\sigma_{ij}(\mathcal{F},\mathcal{F})$.

Consider designs belonging to the class $\mathcal{F} \in \mathcal{U}(n; 4^s)$, let $\Phi = \{\phi_{ij}|\phi_{ij} = \ln\left(\frac{6}{5}\right)\lambda_{ij}(\mathcal{F},\mathcal{F}) + \ln\left(\frac{21}{20}\right)\tau_{ij}(\mathcal{F},\mathcal{F}), i, j = 1,\ldots,n, i \neq j\}$, denote $z_1, z_2, \ldots, z_{n(n-1)}$ as $n(n-1)$ elements in the set Φ, let $z_{(1)}, z_{(2)}, \ldots, z_{(l)}$ be the ordered arrangements of the distinct possible values of $z_1, z_2, \ldots, z_{n(n-1)}$. By Lemma 4, $c = \sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\phi_{ij} = \ln\left(\frac{6}{5}\right)ns\left(\frac{n}{4}-1\right) + \ln\left(\frac{21}{20}\right)\frac{sn^2}{2}$, let k be the largest integer such that $z_{(k)} \leq \frac{c}{n(n-1)} < z_{(k+1)}$.

On other hand, according to the definition of $\lambda_{ij}(\mathcal{F},\mathcal{F})$ and $\tau_{ij}(\mathcal{F},\mathcal{F})$, the possible choices of $\lambda_{ij}(\mathcal{F},\mathcal{F})$ and $\tau_{ij}(\mathcal{F},\mathcal{F})$ are both $0, 1, \ldots, s$. All possible ϕ_{ij} values corresponds to the following $(s+1)^2$ combinations $(\lambda_{ij}(\mathcal{F},\mathcal{F}), \tau_{ij}(\mathcal{F},\mathcal{F}))$:$\{(0,0), (0,1), \ldots, (0,s), (1,0), \ldots, (1,s), \ldots, (s,s)\}$, denote $z'_1, z'_2, \ldots, z'_{(s+1)^2}$ as all possible ϕ_{ij} values, let $z'_{(1)}, z'_{(2)}, \ldots, z'_{((s+1)^2)}$ be the ordered arrangements of $z'_1, z'_2, \ldots, z'_{(s+1)^2}$, k be the largest integer such that $z'_{(k)} \leq \frac{c}{n(n-1)} < z'_{(k+1)}$.

From above, it is obvious that

$$z_{(k)} \leq z'_{(k)} \leq \frac{c}{n(n-1)} < z'_{(k+1)} \leq z_{(k+1)}. \tag{6}$$

According to Lemma 3 and (6), we easily obtain the following relationship,

$$\sum_{i=1}^{n} z_i^t \geq p z'^t_{(k)} + q z'^t_{(k+1)}, \tag{7}$$

where p and q are integers such that $p + q = n(n-1)$ and $pz'_{(k)} + qz'_{(k+1)} = c$.

A new lower bound of WD for a four-level design $\mathcal{F} \in \mathcal{U}(n; 4^s)$ is presented as follows.

Theorem 1 *Let $\mathcal{F} \in \mathcal{U}(n; 4^s)$ be a four-level design. Then*

$$[WD(\mathcal{F})]^2 \geq LB[WD(\mathcal{F})],$$

where

$$LB[WD(\mathcal{F})] = \Delta + \frac{pe^{z'_{(k)}} + qe^{z'_{(k+1)}}}{n^2}\left(\frac{5}{4}\right)^s, \tag{8}$$

$\Delta = -\left(\frac{4}{3}\right)^s + \frac{1}{n}\left(\frac{3}{2}\right)^s$, *$p, q, z'_{(k)}$ and $z'_{(k+1)}$ are shown in (7). The lower bound can be attained if and only if among the $n(n-1)$ number of ϕ_{ij}, p of them take $z'_{(k)}$, q of them take $z'_{(k+1)}$.*

Proof According to (4),

$$[WD(\mathcal{F})]^2 = \Delta + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\left(\frac{3}{2}\right)^{\lambda_{ij}(\mathcal{F},\mathcal{F})}\left(\frac{21}{16}\right)^{\tau_{ij}(\mathcal{F},\mathcal{F})}$$
$$\left(\frac{5}{4}\right)^{s-[\lambda_{ij}(\mathcal{F},\mathcal{F})+\tau_{ij}(\mathcal{F},\mathcal{F})]}$$
$$= \Delta + \frac{1}{n^2}\left(\frac{5}{4}\right)^{s}\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\left(\frac{6}{5}\right)^{\lambda_{ij}(\mathcal{F},\mathcal{F})}\left(\frac{21}{20}\right)^{\tau_{ij}(\mathcal{F},\mathcal{F})}$$
$$= \Delta + \frac{1}{n^2}\left(\frac{5}{4}\right)^{s}\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n} \mathrm{e}^{\lambda_{ij}(\mathcal{F},\mathcal{F})\ln\left(\frac{6}{5}\right)+\tau_{ij}(\mathcal{F},\mathcal{F})\ln\left(\frac{21}{20}\right)}.$$

Denote $\phi_{ij} = \lambda_{ij}(\mathcal{F},\mathcal{F})\ln\left(\frac{6}{5}\right) + \tau_{ij}(\mathcal{F},\mathcal{F})\ln\left(\frac{21}{20}\right)$, then

$$[WD(\mathcal{F})]^2 = \Delta + \frac{1}{n^2}\left(\frac{5}{4}\right)^{s}\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}\sum_{t=0}^{\infty}\frac{(\phi_{ij})^t}{t!}$$
$$= \Delta + \frac{1}{n^2}\left(\frac{5}{4}\right)^{s}\sum_{t=0}^{\infty}\frac{1}{t!}\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}(\phi_{ij})^t.$$

By Lemma 4 and (7),

$$\sum_{i=1}^{n}\sum_{j(\neq i)=1}^{n}(\phi_{ij})^t \geq pz'^{t}_{(k)} + qz'^{t}_{(k+1)}.$$

Thus

$$[WD(\mathcal{F})]^2 \geq \Delta + \frac{p\mathrm{e}^{z'_{(k)}} + q\mathrm{e}^{z'_{(k+1)}}}{n^2}\left(\frac{5}{4}\right)^{s},$$

which completes the proof of Theorem 1. □

Similarly, a new lower bound of the Quadruple design $\mathcal{Q}(\mathcal{F})$ is obtained in the following corollary.

Corollary 1 *Let $\mathcal{F} \in \mathcal{U}(n; 4^s)$ and $\mathcal{Q}(\mathcal{F}) \in \mathcal{U}(4n; 4^{4s})$ be the Quadruple design of $\mathcal{F}$. Then*

$$[WD(\mathcal{Q}(\mathcal{F}))]^2 \geq LB[WD(\mathcal{Q}(\mathcal{F}))],$$

where

$$LB[WD(\mathcal{Q}(\mathcal{F}))] = \hat{\Delta} + \frac{pe^{4z'_{(k)}} + qe^{4z'_{(k+1)}}}{4n^2}\left(\frac{5}{4}\right)^{4s}, \tag{9}$$

$\hat{\Delta} = -\left(\frac{4}{3}\right)^{4s} + \frac{3}{4}\left(\frac{6615}{2048}\right)^{s} + \frac{1}{4n}\left(\frac{3}{2}\right)^{4s}$. *The necessary and sufficient conditions for attaining the lower bound are same to that ones in Theorem 1.*

From (9), the lower bound of WD value for the Quadruple design $\mathcal{Q}(\mathcal{F})$ is determined by the lower bound of WD value for the initial design $\mathcal{F}$, i.e., The WD value of $\mathcal{Q}(\mathcal{F})$ achieves the lower bound $LB[WD(\mathcal{Q}(\mathcal{F}))]$ in (9) if and only if the WD value of $\mathcal{F}$ achieves the lower bound $LB[WD(\mathcal{F})]$ in (8), so the relationship of uniformity measured by WD between $\mathcal{F}$ and $\mathcal{Q}(\mathcal{F})$ is built in the following theorem.

Theorem 2 *Let $\mathcal{F} \in \mathcal{U}(n; 4^s)$ and $\mathcal{Q}(\mathcal{F})$ be its Quadruple design. Then uniformity of $\mathcal{Q}(\mathcal{F})$ and $\mathcal{F}$ is almost equivalent.*

Proof The WD value of $\mathcal{Q}(\mathcal{F})$ achieves the lower bound $LB[WD(\mathcal{Q}(\mathcal{F}))]$ in (9), that is, $\mathcal{Q}(\mathcal{F})$ is a uniform design, if and only if the equality in Theorem 1 holds. Then $[WD(\mathcal{F})]^2 = LB[WD(\mathcal{F})]$, and $\mathcal{F}$ is also a uniform design. Vice versa. The proof is finished. □

Remark 1 According to Theorem 2, if $\mathcal{F}$ is a uniform four-level design, a series of uniform four-level designs with large run sizes can be obtained by successively quadrupling.

5 Analytic connection between $\mathcal{Q}(\mathcal{F})$ and $\mathcal{F}$ under GMA criterion

In this section, the analytic connection between $\mathcal{Q}(\mathcal{F})$ and $\mathcal{F}$ is invesgated in terms of GMA criterion. According to Lemma 1, we obtain another important lemma as follows.

Lemma 5 *Suppose $\mathcal{F} \in \mathcal{U}(n; 4^s)$, $\mathcal{Q}(\mathcal{F}) \in \mathcal{U}(4n; 4^{4s})$ is the Quadruple design of $\mathcal{F}$, $\{E_i(\mathcal{F})\}$, $\{d_{ij}(\mathcal{F})\}$ and $\{E_i^*(\mathcal{Q}(\mathcal{F}))\}$, $\{d_{ij}^*(\mathcal{Q}(\mathcal{F}))\}$ denote the distance distribution and Hamming distances of $\mathcal{F}$ and $\mathcal{Q}(\mathcal{F})$ respectively. Then $\{d_{ij}(\mathcal{F})\}$ and $\{d_{ij}^*(\mathcal{Q}(\mathcal{F}))\}$, $\{E_i(\mathcal{F})\}$ and $\{E_i^*(\mathcal{Q}(\mathcal{F}))\}$ have the following relationships:*

(i)

$$d^*_{(i+kn)(j+ln)}(\mathcal{Q}(\mathcal{F})) = \begin{cases} 4d_{ij}(\mathcal{F}), & 1 \le i, j \le n, k = l; k, l = 0, 1, 2, 3, \\ 3s, & 1 \le i, j \le n, k \ne l; k, l = 0, 1, 2, 3. \end{cases}$$

(ii) *If s mod $4 = 0$, then*

$$E^*_{4i}(\mathcal{Q}(\mathcal{F})) = \begin{cases} E_i(\mathcal{F}), & i = 0, 1, \ldots, s, \ \textit{and} \ \ i \ne \frac{3s}{4}, \\ E_i(\mathcal{F}) + 3n, & i = \frac{3s}{4}, \end{cases}$$

and

$$E^*_{4i+1}(\mathcal{Q}(\mathcal{F})) = E^*_{4i+2}(\mathcal{Q}(\mathcal{F})) = E^*_{4i+3}(\mathcal{Q}(\mathcal{F})) = 0, \ \ i = 0, 1, \ldots, s-1.$$

If s mod $4 = 1$*, then*

$$\begin{aligned} E^*_{4i}(\mathcal{Q}(\mathcal{F})) &= E_i(\mathcal{F}), \quad i = 0, 1, \ldots, s, \\ E^*_{4i+1}(\mathcal{Q}(\mathcal{F})) &= E^*_{4i+2}(\mathcal{Q}(\mathcal{F})) = 0, \quad i = 1, \ldots, s-1, \end{aligned}$$

and

$$E^*_{4i+3}(\mathcal{Q}(\mathcal{F})) = \begin{cases} 0, & i = 0, 1, \ldots, s-1 \;\; \textit{and} \;\; i \neq \frac{3s-3}{4}, \\ 3n, & i = \frac{3s-3}{4}. \end{cases}$$

If s mod $4 = 2$*, then*

$$\begin{aligned} E^*_{4i}(\mathcal{Q}(\mathcal{F})) &= E_i(\mathcal{F}), \quad i = 0, 1, \ldots, s, \\ E^*_{4i+1}(\mathcal{Q}(\mathcal{F})) &= E^*_{4i+3}(\mathcal{Q}(\mathcal{F})) = 0, \quad i = 1, \ldots, s-1, \end{aligned}$$

and

$$E^*_{4i+2}(\mathcal{Q}(\mathcal{A})) = \begin{cases} 0, & i = 0, 1, \ldots, s-1 \;\; \textit{and} \;\; i \neq \frac{3s-2}{4}, \\ 3n, & i = \frac{3s-2}{4}. \end{cases}$$

If s mod $4 = 3$*, then*

$$\begin{aligned} E^*_{4i}(\mathcal{Q}(\mathcal{F})) &= E_i(\mathcal{F}), \quad i = 0, 1, \ldots, s, \\ E^*_{4i+2}(\mathcal{Q}(\mathcal{F})) &= E^*_{4i+3}(\mathcal{Q}(\mathcal{F})) = 0, \quad i = 1, \ldots, s-1, \end{aligned}$$

and

$$E^*_{4i+1}(\mathcal{Q}(\mathcal{F})) = \begin{cases} 0, & i = 0, 1, \ldots, s-1 \;\; \textit{and} \;\; i \neq \frac{3s-1}{4}, \\ 3n, & i = \frac{3s-1}{4}. \end{cases}$$

Proof The result (i) is obvious by Lemma 1 and we only prove result (ii). If s mod 4=0, by Eq. (6) and (i), we have $E^*_{4i+1}(\mathcal{Q}(\mathcal{F})) = E^*_{4i+2}(\mathcal{Q}(\mathcal{F})) = E^*_{4i+3}(\mathcal{Q}(\mathcal{F})) = 0$, for $i = 0, 1, \ldots, s-1$. When $i = 0, 1, \ldots, s$ and $i \neq \frac{3s}{4}$,

$$\begin{aligned} E^*_{4i}(\mathcal{Q}(\mathcal{F})) &= \frac{1}{4n} |\{(x^*_i, x^*_j) : d^*_H(x^*_i, x^*_j) = 4i\}| \\ &= \frac{1}{4n} \times 4|\{(x_i, x_j) : d_H(x_i, x_j) = i\}| \\ &= \frac{1}{4n} \times 4n E_i(\mathcal{F}) = E_i(\mathcal{F}), \end{aligned}$$

and for $i = \frac{3s}{4}$,

$$\begin{aligned} E^*_{3s}(\mathcal{Q}(\mathcal{F})) &= \frac{1}{4n}|\{(x^*_i, x^*_j) : d^*_H(x^*_i, x^*_j) = 3s\}| \\ &= \frac{1}{4n}(4nE_{3s/4}(\mathcal{F}) + 12n^2) = 3n + E_{3s/4}(\mathcal{F}). \end{aligned}$$

The proofs of other cases are similar to the case one, which completes the proof of Lemma 5. □

The following theorem provides the analytical connection between $\mathcal{F}$ and $\mathcal{Q}(\mathcal{F})$ under GMA criterion.

Theorem 3 *Let $\mathcal{F} \in \mathcal{U}(n; 4^s)$ and $\mathcal{Q}(\mathcal{F})$ be the Quadruple design of $\mathcal{F}$. $\{A^*_j(\mathcal{Q}(\mathcal{F}))\}$ and $\{A_j(\mathcal{F})\}$ is the generalized wordlength pattern of $\mathcal{Q}(\mathcal{F})$ and $\mathcal{F}$, respectively. For $1 \le j \le 4s$, we have*

$$A^*_j(\mathcal{Q}(\mathcal{F})) = \frac{1}{4^{s+1}} \sum_{i=0}^{s} \sum_{v=0}^{s} P_j(4i; 4s, 4) P_i(v; s, 4) A_v(\mathcal{F}) + \frac{3}{4} P_j(3s; 4s, 4). \quad (10)$$

Proof If $s \bmod 4 = 0$, from (8) and the (ii) of Lemma 5, for $1 \le j \le 4s$ we have

$$\begin{aligned} A^*_j(\mathcal{Q}(\mathcal{F})) &= \frac{1}{4n} \sum_{i=0}^{4s} P_j(i; 4s, 4) E^*_i(\mathcal{Q}(\mathcal{F})) \\ &= \frac{1}{4n} \Bigg[\sum_{i=0}^{s} P_j(4i; 4s, 4) E^*_{4i}(\mathcal{Q}(\mathcal{F})) \\ &\quad + \sum_{i=0}^{s-1} P_j(4i+1; 4s, 4) E^*_{4i+1}(\mathcal{Q}(\mathcal{F})) \\ &\quad + \sum_{i=0}^{s-1} P_j(4i+2; 4s, 4) E^*_{4i+2}(\mathcal{Q}(\mathcal{F})) \\ &\quad + \sum_{i=0}^{s-1} P_j(4i+3; 4s, 4) E^*_{4i+3}(\mathcal{Q}(\mathcal{F})) \Bigg] \\ &= \frac{1}{4n} \left[\sum_{i=0}^{s} P_j(4i; 4s, 4) E_i(\mathcal{F}) + 3n P_j(3s; 4s, 4) \right]. \end{aligned}$$

Since $E_i(\mathcal{F}) = \frac{n}{4^s}\sum_{v=0}^{s} P_i(v; s, 4)A_v(\mathcal{F})$ (Ma and Fang 2001),

$$A_j^*(\mathcal{Q}(\mathcal{F})) = \frac{1}{4n}\left[\sum_{i=0}^{s} P_j(4i; 4s, 4)\frac{n}{4^s}\sum_{v=0}^{s} P_i(v; s, 4)A_v(\mathcal{F}) + 3nP_j(3s; 4s, 4)\right]$$
$$= \frac{1}{4^{s+1}}\sum_{i=0}^{s}\sum_{v=0}^{s} P_j(4i; 4s, 4)P_i(v; s, 4)A_v(\mathcal{F}) + \frac{3}{4}P_j(3s; 4s, 4).$$

For the other three cases, (10) also holds, which completes the proof of Theorem 3. □

6 Numerical examples

Next the nice properties of the Quadruple design and its projection designs are illustrated by some examples in this section.

Example 2 Consider the original four-level design $\mathcal{F} \in \mathcal{U}(8; 4^7)$ given as follows.

$$\mathcal{F} = \begin{pmatrix} 1 & 1 & 2 & 3 & 1 & 2 & 3 \\ 3 & 2 & 0 & 0 & 1 & 1 & 2 \\ 3 & 3 & 1 & 2 & 2 & 3 & 3 \\ 2 & 2 & 2 & 2 & 0 & 0 & 1 \\ 2 & 3 & 3 & 0 & 3 & 2 & 0 \\ 1 & 0 & 3 & 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 & 3 & 1 & 1 \end{pmatrix}.$$

$\mathcal{Q}(\mathcal{F}) \in \mathcal{U}(32; 4^{28})$ is the Quadruple design of $\mathcal{F}$. Successively quadrupling $\mathcal{F}$ twice yields the design $\mathcal{Q}^*(\mathcal{F}) \in \mathcal{U}(128; 4^{112})$. From Eqs. (4) and (8), we have $[WD(\mathcal{F})]^2 = LB[WD(\mathcal{F})] = 0.73$, namely, $\mathcal{A}$ is a uniform design measured by WD. From Eqs. (5) and (9), we have $[WD(\mathcal{Q}(\mathcal{F}))]^2 = LB[WD(\mathcal{Q}(\mathcal{F}))] = 2776.1$, $[WD(\mathcal{Q}^*(\mathcal{F}))]^2 = LB[WD(\mathcal{Q}^*(\mathcal{F}))] = 4.12\times 10^{17}$. Therefore, both $\mathcal{Q}(\mathcal{F})$ and $\mathcal{Q}^*(\mathcal{F})$ from quadrupling are also uniform designs measured by WD.

The uniformity of projection designs of the Quadruple design $\mathcal{Q}(\mathcal{F})$ is presented in Table 2, the WD values corresponding to the last column in Table 2 are from the homepage of uniform designs (UD). From Table 2, we easily find that the WD values of projection designs of the Quadruple design are not larger than WD on the homepage of uniform designs, some are even better.

Example 3 Consider the initial four-level design $\mathcal{F}$ given below, which is an orthogonal array $OA(16, 4^5, 2)$, $\mathcal{Q}(\mathcal{F}) \in \mathcal{U}(64; 4^{20})$ is the Quadruple design of $\mathcal{F}$.

$$\mathcal{F} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 & 1 & 0 & 3 & 2 & 2 & 3 & 0 & 1 & 3 & 2 & 1 & 0 \\ 0 & 2 & 3 & 1 & 1 & 3 & 2 & 0 & 2 & 0 & 1 & 3 & 3 & 1 & 0 & 2 \\ 0 & 3 & 1 & 2 & 1 & 2 & 0 & 3 & 2 & 1 & 3 & 0 & 3 & 0 & 2 & 1 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \end{pmatrix}'.$$

Table 2 Comparison of uniformity between projection designs and existing uniform designs

Runs (n)	Levels (q)	Projection factors (s)	WD of projection design	WD on UD homepage
32	4	2	0.027886	0.027886
32	4	3	0.056021	0.056021
32	4	4	0.10045	0.10045
32	4	5	0.169469	0.169469
32	4	6	0.275788*	0.276559
32	4	7	0.438446*	0.443409
32	4	8	0.692559*	0.69806

$s = 2$ is $\mathcal{Q}(\mathcal{F})$ mapping to columns 1, 8
$s = 3$ is $\mathcal{Q}(\mathcal{F})$ mapping to columns 1, 10, 12
$s = 4$ is $\mathcal{Q}(\mathcal{F})$ mapping to columns 1, 8, 10, 23
$s = 5$ is $\mathcal{Q}(\mathcal{F})$ mapping to columns 1, 8, 10, 24, 25
$s = 6$ is $\mathcal{Q}(\mathcal{F})$ mapping to columns 3, 8, 10, 13, 21, 23
$s = 7$ is $\mathcal{Q}(\mathcal{F})$ mapping to columns 3, 8, 10, 13, 21, 23, 25
$s = 8$ is $\mathcal{Q}(\mathcal{F})$ mapping to columns 1, 4, 8, 10, 13, 21, 23, 25
*The WD of the projection designs is smaller than WD on UD homepage

The comparison of uniformity between the projection designs of the Quadruple design $\mathcal{Q}(\mathcal{F})$ and existing uniform designs are presented in Table 3, the WD values corresponding to the last but one column in Table 3 are from Xu et al. (2014), which is the WD values of designs constructed by level permutations of factors of orthogonal array. The WD values corresponding to the last column in Table 3 are WD values of uniform designs by stochastic optimization method (SOM). From Table 3, we easily find that the uniformity of the projection designs of the Quadruple design are better than existing results, where the WD values are marked with symbols * and $\circ$.

According to the analytical connection between $\mathcal{F}$ and $\mathcal{Q}(\mathcal{F})$ under GMA criterion in Theorem 3, theoretically, using the minimum aberration design as the initial design is the best, but until now the available minimum aberration designs are two-level or three-level designs. So in the implementation of the proposed method, we have to go for second best, that is to say, the initial design has less aberration. As an obvious alternative, orthogonal array (OA) with strength two is appropriate, because it has relatively small wordlength pattern ($A_1 = 0, A_2 = 0$), which ensures that there is no aberration among main effects of factors. We choose the $OA(16, 4^5, 2)$ in Example 3 as the initial design, the projection designs of the Quadruple design of $OA(16, 4^5, 2)$ and the corresponding generalized wordlength pattern are given in Table 4, respectively. The numerical results in Table 4 show that if the initial design is an orthogonal design of strength two, then both the Quadruple design and its projection designs are also orthogonal designs of strength two, i.e., $A_1 = 0$, $A_2 = 0$. Therefore, the first two components of the generalized wordlength pattern of each projection design are omitted, and the other components except for A_3 and A_4 also are omitted for simplicity in Table 4. From above, we choose an orthogonal design with small number of factors and runs as initial design, lots of orthogonal designs with large number of factors and runs can be constructed by successively quadrupling.

Table 3 Comparison of uniformity between projection designs and existing uniform designs

Runs (n)	Level (q)	Projection factors (s)	WD of projection design	WD of permuted orthogonal designs	WD of UD constructed by SOM
64	4	3	0.055991	0.055991	0.055991
64	4	4	0.099948	0.099948	0.099948
64	4	5	0.167346°	0.167346°	0.167945
64	4	6	0.269157°	0.269157°	0.271437
64	4	7	0.42436°*	0.425343°	0.427147
64	4	8	0.654555°	0.654555°	0.662169
64	4	9	0.993731°*	0.995336°	1.011113
64	4	10	1.497617°*	1.502147°	1.526563
64	4	11	2.23429°*	2.243185°	2.28761
64	4	12	3.31139°*	3.339569°	3.412004
64	4	13	4.90255°*	4.907737°	5.055312
64	4	14	7.217505°*	7.303771°	7.478702

$s = 3$ is $\mathcal{Q}(\mathcal{A})$ mapping to columns 1, 2, 6
$s = 4$ is $\mathcal{Q}(\mathcal{A})$ mapping to columns 1, 2, 6, 15
$s = 5$ is $\mathcal{Q}(\mathcal{A})$ mapping to columns 1, 2, 6, 7, 15
$s = 6$ is $\mathcal{Q}(\mathcal{A})$ mapping to columns 1, 2, 6, 7, 15, 20
$s = 7$ is $\mathcal{Q}(\mathcal{A})$ mapping to columns 1, 2, 3, 6, 7, 11, 13
$s = 8$ is $\mathcal{Q}(\mathcal{A})$ mapping to columns 1, 2, 3, 6, 7, 8, 11, 12
$s = 9$ is $\mathcal{Q}(\mathcal{A})$ mapping to columns 1, 2, 3, 6, 7, 8, 11, 12, 13
$s = 10$ is $\mathcal{Q}(\mathcal{A})$ mapping to columns 1, 2, 3, 4, 6, 7, 8, 11, 12, 13
$s = 11$ is $\mathcal{Q}(\mathcal{A})$ mapping to columns 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13
$s = 12$ is $\mathcal{Q}(\mathcal{A})$ mapping to columns 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14
$s = 13$ is $\mathcal{Q}(\mathcal{A})$ mapping to columns 1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 17
$s = 14$ is $\mathcal{Q}(\mathcal{A})$ mapping to columns 1, 2, 3, 4, 7, 8, 9, 11, 12, 13, 14, 17, 18,19
*The WD of the projection designs is smaller than WD in Xu et al. (2014)
°The WD of the projection designs is smaller than WD of UD constructed by SOM

7 Concluding remarks

In this paper, a novel method, called quadrupling, is proposed for constructing uniform four-level designs with large run sizes. Quadrupling method is regarded as a generalization of tripling method proposed by Ou et al. (2019). There are some similarities between the two methods, for example, both use level permutations to construct larger designs from small designs. However, there are some differences between the two methods. Firstly, the objects of study are different, one is a three-level design, the other is a four-level design. Secondly, tripling method uses all level permutations of a three-level design while quadrupling method only uses partial level permutations of a four-level design.

Table 4 Projection designs of Quadruple design of $OA(16, 4^5, 2)$ and the wordlength patterns

Columns of projection	Wordlength pattern $(A_3, A_4, \ldots)$
1 2 6	(0)
1 2 6 7	(0, 3)
1 2 6 7 15	(0, 15, . . .)
1 2 6 7 15 20	(0, 45, . . .)
1 2 3 6 7 11 13	(9, 69, . . .)
1 2 3 6 7 8 10 15	(18, 120, . . .)
1 2 3 6 7 8 10 15 20	(27, 216, . . .)
1 2 3 4 6 7 8 10 15 20	(45, 327, . . .)
1 2 3 4 6 7 8 9 10 15 20	(63, 510, . . .)
1 2 3 4 6 7 8 9 11 12 13 14	(84, 765, . . .)
1 2 3 4 6 7 8 9 10 11 12 13 14	(120, 1029, . . .)
1 2 3 4 6 7 8 9 10 11 12 13 14 15	(156, 1419, . . .)
1 2 3 4 6 7 8 9 10 11 12 13 14 15 20	(195, 1935, . . .)
1 2 3 4 6 7 8 9 11 12 13 14 16 17 18 19	(240, 2580, . . .)
1 2 3 4 5 6 7 8 9 11 12 13 14 16 17 18 19	(312, 3204, . . .)
1 2 3 4 5 6 7 8 9 10 11 12 13 14 16 17 18 19	(384, 4044, . . .)
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19	(459, 5100, . . .)
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20	(540, 6375, . . .)

On the other hand, the construction of Q($\mathcal{F}$) in Definition 1 is not unique, because there are many kinds of level permutations of $\mathcal{F}$. When a simplest four-level design $\mathcal{F} = (0\ 1\ 2\ 3)'$ is chosen as the initial design, one can obtain the Quadruple design $\mathcal{Q}(\mathcal{F})$ by combining level permutations of $\mathcal{F}$ such that $\mathcal{Q}(\mathcal{F})$ is an $OA(16, 4^4, 2)$. This way other types of the Quadruple design possessed with excellent properties are found.

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