Metrika (2019) 82:589–605
https://doi.org/10.1007/s00184-018-0691-y

On the reliability modeling of weighted *k*-out-of-*n* systems with randomly chosen components

M. Salehi[1] · Z. Shishebor[1] · M. Asadi[2,3]

Received: 15 May 2018 / Published online: 11 October 2018

Abstract
The weighted k-out-of-n (briefly denoted as weighted k/n) systems are among the most important kind of redundancy structures. We consider a weighted k/n system with dependent components where the system is built up from two classes $\mathfrak{C}_X$ and $\mathfrak{C}_Y$ of components that are categorized according to their weights and reliability functions. It is assumed that a random number M, $M = 0, 1, \ldots, m$, of the components are chosen from set $\mathfrak{C}_X$ whose components are distributed as F_X and the remaining $n - M$ components selected from the set $\mathfrak{C}_Y$ whose components have distribution function F_Y. We further assume that the structure of dependency of the components can be modeled by a copula function. The reliability of the system, at any time t, is expressed as a mixture of reliability of weighted k/n systems with fixed number of the components of types $\mathfrak{C}_X$ and $\mathfrak{C}_Y$ in terms of the probability mass function M. Some stochastic orderings are made between two different weighted k/n systems. It is shown that when the random mechanism of the chosen components for two systems are ordered in usual stochastic (*st*) order then, under some conditions, the lifetimes of the two systems are also ordered in *st* order. We also compare the lifetimes of two different systems in the sense of stochastic precedence concept. The results are examined by several illustrative examples under different conditions.

Keywords Reliability · Weighted k-out-of-n system · Copulas · Stochastic order · Stochastic precedence

✉ Z. Shishebor
shisheb@shirazu.ac.ir

M. Salehi
salehi_marziye@shirazu.ac.ir

M. Asadi
m.asadi@sci.ui.ac.ir

1 Department of Statistics, Faculty of science, Shiraz University, Shiraz 7146713565, Iran

2 Department of Statistics, University of Isfahan, Isfahan 81744, Iran

3 School of Mathematics, Institute of Research in Fundamental Sciences (IPM), Tehran, Iran

Mathematics Subject Classification 90B25 · 62H20

1 Introduction

The k-out-of-n (briefly denoted as k/n) systems are important redundancy structures in reliability engineering. A n-component system is said to be a k/n system if it operates as long as at least k components out of the n components operate. In these kind of systems all components have an equal portion to the performance of the entire system. Hence the number of working components specifies the operation of the system. The stochastic and aging properties of k/n systems have been extensively investigated in the reliability literature (see, for example, Kuo and Zuo 2003; Li and Zhao 2008; Eryilmaz 2011). Recently k/n systems are extended to weighted k/n systems where the weight associated with each component can be considered as load/capacity of that component. If the performance of the k/n system is characterized by the total weights of operating components, then the system is said to have a weighted k/n structure. In other words, a weighted k/n system has an operational level of at least k if the total weight of working components is k or more. For a specific example of a weighted k/n system in real life applications, we refer to Samaniego and Shaked (2008). The reliability assessment and stochastic properties of weighted k/n systems under the assumption of independence between the components lifetimes, are studied by several authors (see, among others, Ball et al. 1995; Chen and Yang 2005; Higashiyama 2001; Rushdi 1990; Wu and Chen 1994a, b). Eryilmaz (2015) studied the capacity loss and residual capacity in binary weighted-k-out-of-n:G systems. The mean instantaneous performance of the systems with weighted components was presented by Eryilmaz (2013). For an extensive survey on applications of the systems with weighted components see Samaniego and Shaked (2008). Recently, attempts have been made to study the properties of the weighted k/n systems consisting of dependent components. Copula functions are one of the popular applied methods for modeling dependence among lifetimes of the components. For a comprehensive study on the theory of copulas, we refer the reader to Nelsen (2006). Analysis of the systems reliability using copula functions can be found in Jia and Cui (2012), Navarro et al. (2010, 2017), Tang et al. (2013a, b) and Wang and Pham (2012).

According to the definition of a weighted k/n system, the system lies among the structures that are made of leastwise two different kinds of components (for more description and reliability properties of such systems, we refer the reader to Kochar and Xu 2010; Cui and Xie 2005; Navarro et al. 2015). Recently Eryilmaz and Sarikaya (2013) and Eryilmaz (2014), have considered a weighted k/n system including two kinds of components. Such a system is assumed to be composed of n components in which a fixed number m out of n is from a class $\mathfrak{C}_X$ of identically distributed components and the rest of $n-m$ components are from a class $\mathfrak{C}_Y$ of identically distributed components. The cited authors have obtained some results on the stochastic and aging properties of the lifetimes of weighted k/n systems based on various copulas. Li et al. (2016) have studied on weighted k/n system with statistically dependent component lifetimes. Further resources on these subjects can also be found in Coolen and Coolen-Maturi (2012), Eryilmaz et al. (2018) and Samaniego and Navarro (2016).

Crescenzo (2007) presented the random strategy in the field of reliability theory. He showed that, in some situations, a random strategy might be a better option when our choice must be between two kinds of units such that one is more reliable than the other. Some useful applications related to the random strategy are in Navarro and Spizzichino (2010), Crescenzo and Pellerey (2011), Hazra and Nanda (2014), Navarro et al. (2015) and Hazra et al. (2017). Some applications of the above strategy may arise in reality. First, assume that a manufacturer uses a mix of two types of units for the production of its products. The reliability of manufactured units depends directly on the materials of the unit selected from each type and human factors, conditions of production, etc. The second application of the randomness is in physical systems e.g. stochastic resonance, Berdichevsky and Gitterman (1998) and Gammaitoni et al. (1998), Brownian ratchets, Bier (1997). The systems under the influence of the shock processes can be considered as the third example. In justifying the application of random theory Navarro et al. (2015) mentioned "Of course, the best systems are those which include only the best components. However, we shall assume that this option is not possible, maybe simply because we do not know which are the best components, and that we want to use both kinds of components. In these situations, random strategies may lead to the best systems."

The goal of the present paper is to investigate the reliability and stochastic properties of weighted k/n systems which consist of a random number of components when the components are from two different types. In other words, we consider a weighted k/n system and assume that among the n number of the components of the system, a random number M, $M = 0, 1, \ldots, m$, of components are chosen from class $\mathfrak{C}_X$ and the rest of $n - M$ components are from class $\mathfrak{C}_Y$ where the components in $\mathfrak{C}_X$ (and $\mathfrak{C}_Y$) are independent and identically distributed.

The paper is organized as follows: in Sect. 2, we recall some definitions and preliminary results that are useful in other sections. These include the definitions of some partial stochastic orders, the notion of "stochastic precedence" and the notion of copula. In Sect. 3, we first give a precise description of our proposed weighted k/n system. Then, we obtain the reliability of the system lifetime as a mixture representation. Several examples are also examined computationally and graphically in this section. Section 4 is devoted to comparisons between the lifetimes of two weighted k/n systems under different conditions. We first prove that if $M_i, i = 1, 2$, components are chosen from $\mathfrak{C}_X$ such that M_1 and M_2 are stochastically ordered, then under some conditions the lifetimes of the systems consisting of M_1 and M_2, respectively, are also stochastically ordered. The rest of Sect. 4 is centered on comparison between two systems in the sense of stochastic precedence. In other words, we study conditions under which for two weighted k/n systems with lifetimes T_1 and T_2, $P(T_1 \leq T_2) \geq 1/2$. Finally, some concluding remarks are presented in Sect. 5.

2 Preliminaries

In this section, we present some notions which are useful in deriving the main results. Throughout the paper, for a continuous random variable (r.v) X, let $\bar{F}$, f and $\lambda_F =$

$f/\bar{F}$ be survival function, density function and failure rate function, respectively. For a continuous random variable Y, functions $\bar{G}$, g and $\lambda_G = g/\bar{G}$ are defined similarly.

Definition 1 The random variable X is said to be smaller than the random variable Y in the

(i) usual stochastic order (denoted by $X \leq_{st} Y$) if for all t, $\bar{F}(t) \leq \bar{G}(t)$;
(ii) hazard rate order (denoted by $X \leq_{hr} Y$) if for all t, $\lambda_F(t) \geq \lambda_G(t)$;
(iii) likelihood ratio order (denoted by $X \leq_{lr} Y$) if $g(t)/f(t)$ is an increasing function of t.

It is well known that the following implications hold between these orderings:

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y.$$

The concepts of partial orderings are extended to the multivariate case. For n-dimensional random vector $\mathbf{X}$, let $\bar{F}$ and f be survival function (i.e., $\bar{F}(\mathbf{t}) = P(\mathbf{X} > \mathbf{t})$) and density function, respectively. For n-dimensional random vector $\mathbf{Y}$, functions $\bar{G}$ and g are defined similarly. We denote $\min\{\mathbf{t}_1, \mathbf{t}_2\} = (\min\{t_{1,1}, t_{1,2}\}, \ldots, \min\{t_{n,1}, t_{n,2}\})$ and $\max\{\mathbf{t}_1, \mathbf{t}_2\} = (\max\{t_{1,1}, t_{1,2}\}, \ldots, \max\{t_{n,1}, t_{n,2}\})$.

(i) The random vector $\mathbf{X}$ is smaller than the random vector $\mathbf{Y}$ in the multivariate stochastic order if
$$P(\mathbf{X} \in \mathfrak{U}) \leq P(\mathbf{Y} \in \mathfrak{U}),$$
for all sets $\mathfrak{U} \subseteq R^n$ (where $\mathfrak{U}$ is an upper set).
(ii) The random vector $\mathbf{X}$ is said to be smaller than the random vector $\mathbf{Y}$ in the multivariate hazard rate order if
$$\bar{F}(\mathbf{t}_1)\bar{G}(\mathbf{t}_2) \leq \bar{F}(\min\{\mathbf{t}_1, \mathbf{t}_2\})\bar{G}(\max\{\mathbf{t}_1, \mathbf{t}_2\}) \quad \forall\, \mathbf{t}_1, \mathbf{t}_2 \in \mathbb{R}^n,$$
(iii) The random vector $\mathbf{X}$ is said to be smaller than the random vector $\mathbf{Y}$ in the multivariate likelihood ratio order if
$$f(\mathbf{t}_1)g(\mathbf{t}_2) \leq f(\min\{\mathbf{t}_1, \mathbf{t}_2\})g(\max\{\mathbf{t}_1, \mathbf{t}_2\}) \quad \forall\, \mathbf{t}_1, \mathbf{t}_2 \in \mathbb{R}^n,$$

Standard references for more details on basic features and applications of these orders include Barlow and Proschan (1981), Li and Li (2013) and Shaked and Shanthikumar (2007).

Dealing with vectors of dependent random variables, a common approach to describe the dependence between the random variables is using the *copula*. Any joint distribution function H of a random vector $(T_1, \ldots, T_n)$ with the marginal distribution functions $F_1, \ldots, F_n$, can be written (by the Sklar theorem) as

$$H(t_1, \ldots, t_n) = C_\alpha(F_1(t_1), \ldots, F_n(t_n)), \tag{1}$$

for all $t_i \in \mathbb{R}$, $i = 1, \ldots, n$, where the n-dimensional function $C_\alpha : [0, 1]^n \to [0, 1]$ is called a copula function. The quantity α is the parameter of the copula which describes the dependency between $T_1, \ldots, T_n$. Also, the copula C_α is unique if marginals $F_1, \ldots, F_n$ are continuous and is given by

$$C_\alpha(u_1, \ldots, u_n) = H(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)),$$

where $F_i^{-1}(u) = inf\{t : F_i(t) \geq u\}$, $i = 1, \ldots, n$. Conversely, the joint distribution of random vector $(T_1, \ldots, T_n)$ can be determined by (1) if the marginal distributions of $T_1, \ldots, T_n$ and the copula function are known. For more details on copulas and their properties see, Nelsen (2006).

There are many parametric copula families in which these parameters control the strength of dependence between the marginals. Indeed, the copula parameter α determines the properties of $C_\alpha(u_1, \ldots, u_n)$. In some cases, there is a one to one relationship between copula parameter and Kendall's tau that allow to reparameterize parameter space into an easier interpretable one. For an n-dimensional copula $C_\alpha(\mathbf{u})$, $\mathbf{u} = (u_1, \ldots, u_n)$, the Kendall's τ correlation is defined as follows

$$\tau_n(\alpha) = \frac{1}{2^{n-1} - 1}\left[2^n \int_0^1 \cdots \int_0^1 C_\alpha(\mathbf{u}) dC_\alpha(\mathbf{u}) - 1\right],$$

Genest et al. (2011). In the independence case, $C_\alpha(\mathbf{u}) = u_1 \ldots u_n$ and

$$\int_0^1 \cdots \int_0^1 C_\alpha(\mathbf{u}) dC_\alpha(\mathbf{u}) = 1/2^n,$$

which implies that $\tau_n(\alpha) = 0$. When the Kendall coefficient is given, the copula parameter can be achieved.

Among the class of copulas, an interesting and important one is the class of Archimedean copulas, which can be written as

$$C(u_1, \ldots, u_n) = \varphi(\varphi^{-1}(u_1) + \cdots + \varphi^{-1}(u_n)),$$

where $\varphi : [0, \infty] \to [0, 1]$ is a continuous and strictly decreasing function with $\varphi(0) = 1, \varphi(\infty) = 0$ and the inverse function is denoted as φ^{-1}. The function φ is named Archimedean generator of C. Many classical copulas such as Ali-Mikhail-Haq, Clayton, Frank and Gumbel families belong to this class. An important member of Archimedean copulas family is the product copula i.e. a copula for the case of independence.

Now, we present the definition of symmetric copula.

Definition 2 For a given copula $C(u_1, \ldots, u_n)$, if

$$C(u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_{k-1}, u_k, u_{k+1}, \ldots, u_n)$$
$$= C(u_1, \ldots, u_{i-1}, u_k, u_{i+1}, \ldots, u_{k-1}, u_i, u_{k+1}, \ldots, u_n),$$

then u_i and u_k are said to be exchangeable. For any pair $u_i, u_k \in [0, 1]$, if u_i and u_k are exchangeable, then the copula $C(u_1, \ldots, u_n)$ is a symmetric copula.

In reliability, the dependence structure among the component lifetimes are commonly positive, so this positive dependence should be considered whenever we choose an appropriate copula. In this context we consider two families called Clayton and FGM, the first is an Archimedean copula but the latter is not, while both of them are symmetric copulas. For ease of reference, we recall the exact formula of the Clayton and FGM copulas.

(i) The n-variate Clayton copula is defined by

$$C_\alpha(u_1, \ldots, u_n) = (u_1^{-\alpha} + \cdots + u_n^{-\alpha} - n + 1)^{-1/\alpha} \quad \alpha \in [-1, \infty)\backslash\{0\}.$$

For this family of copulas the value of Kendall's tau is given by

$$\tau_n(\alpha) = \frac{1}{2^{n-1} - 1}\left[2^n \prod_{i=0}^{n-1}\left(\frac{1 + i\alpha}{2 + i\alpha}\right) - 1\right].$$

(ii) The n-variate form of FGM family of copulas is defined by

$$C_\theta(u_1, \ldots, u_n) = \left(\prod_{i=1}^{n} u_i\right)\left\{1 + \theta \prod_{i=1}^{n}(1 - u_i)\right\} \quad \theta \in (-1, 1).$$

In this case, the value of Kendall's tau is given by

$$\tau_n(\theta) = \frac{\theta}{3^n(2^{n-1} - 1)}\{1 + (-1)^n\}.$$

It is necessary to mention that the FGM copula describes positive (negative) dependence for $\theta > 0$ ($\theta < 0$).

3 System description

In the following, we consider a weighted k/n system consists of n dependent components. The components belong to two distinct classes $\mathfrak{C}_X = \{X_1, \ldots, X_m\}$ and $\mathfrak{C}_Y = \{Y_{m+1}, \ldots, Y_n\}$ with sizes m and $n - m$, respectively. Let F_X and F_Y denote the distribution functions of the components lifetime in $\mathfrak{C}_X$ and $\mathfrak{C}_Y$, respectively. We assume that the components have two states at any time, either working or failed. When a selected component from $\mathfrak{C}_X(\mathfrak{C}_Y)$ is in a working state, it has the weight

w (w^*). So, the system is in performance level k if and only if the total weights of operating components is k or above. In other words, components of the system are chosen from two distinct classes of components such that m of them have the weight w with lifetime distribution F_X, and the remaining $n-m$ components have the weight w^* with lifetime distribution F_Y. If $X_1, \ldots, X_m, Y_{m+1}, \ldots, Y_n$ denote the lifetimes of the system components in the two classes, then the total weight of the system is expressed by the stochastic process $W_n(t)$ at time $t \geq 0$,

$$W_n(t) = \sum_{i=1,\ldots,m} wI(X_i > t) + \sum_{i=m+1,\ldots,n} w^* I(Y_i > t),$$

where the symbol I(.) denotes the indicator random variable. The lifetime of the weighted k/n system, T, is

$$T = \inf\{t : W_n(t) < k\}.$$

Hence the system reliability is

$$R(t) = P(T > t) = P(W_n(t) \geq k), \quad \forall t \geq 0.$$

Suppose that the dependence between $X_1, \ldots, X_m, Y_{m+1}, \ldots, Y_n$ is modeled by the n-dimensional copula function C, i.e. the joint distribution function of $X_1, \ldots, X_m$, $Y_{m+1}, \ldots, Y_n$ is displayed as

$$H(t_1, \ldots, t_n) = C(F_X(t_1), \ldots, F_X(t_m), F_Y(t_{m+1}), \ldots, F_Y(t_n)).$$

The reliability properties of a system with two different kind of dependent components, when dependency is modeled by copulas, was studied by Eryilmaz (2014).

In what follows, we consider the case when n-copula is symmetric and the size of two classes $\mathfrak{C}_X$ and $\mathfrak{C}_Y$ are random. Let M be a random variable with support contained in {0,1,…,n}. If m (the number of components from $\mathfrak{C}_X$) is selected randomly according to the random variable M, then the system reliability function can be represented as

$$\begin{aligned} R_M(t) &= P(T_M > t) \\ &= \sum_{m=0}^{n} P\left(w \sum_{i=1,\ldots,m} I(X_i > t) + w^* \sum_{i=m+1,\ldots,n} I(Y_i > t) \geq k \right) P(M = m) \\ &= \sum_{m=0}^{n} R_m(t) P(M = m) \\ &= \sum_{m=0}^{n} \sum\sum_{\substack{wy+w^*z\geq k \\ 0\leq y\leq m \\ 0\leq z\leq n-m}} \binom{m}{y}\binom{n-m}{z} \sum_{l=0}^{y}\sum_{s=0}^{z} (-1)^{(l+s)} \binom{y}{l}\binom{z}{s} \end{aligned}$$

$$\times C(\underbrace{F_X(t), \ldots, F_X(t)}_{(m-y+l)\text{times}}, \underbrace{F_Y(t), \ldots, F_Y(t)}_{(n-m-z+s)\text{times}})$$
$$\times P(M = m).$$

The last equality follows from relation (4) in Eryilmaz (2014).
Let us look at the following examples which rely on the notations introduced just now.

Example 1 Consider two weighted 3/4 systems with $w = 2$, $w^* = 1$, $F_X(t) = 1 - \exp\{-0.1t\}$, $F_Y(t) = 1 - \exp\{-0.4t\}$. Let the dependence structure among components be generated by FGM copula. Assume that in the first (second) system the number of components from C_X is selected randomly according to random variables M_1 (M_2). If the random variables M_1 and M_2 follow Binomial distribution $B(4, 0.1)$ and $B(4, 0.9)$, respectively, then for $j = 1, 2$,

$$\begin{aligned}
R_{M_i}(t) &= \sum_{m=0}^{4} P\left(2 \sum_{i=1,\ldots,m} I(X_i > t) + \sum_{i=m+1,\ldots,n} I(Y_i > t) \geq 3\right) P(M_j = m) \\
&= \sum_{m=0}^{4} R_m(t) P(M_j = m) \\
&= \sum_{m=0}^{4} \sum_{\substack{2y+z\geq 3\\ 0\leq y\leq m\\ 0\leq z\leq 4-m}} \binom{m}{y}\binom{4-m}{z} \sum_{l=0}^{y}\sum_{s=0}^{z}(-1)^{(l+s)}\binom{y}{l}\binom{z}{s} \\
&\quad \times (1-\exp\{-0.1t\})^{m-y+l}(1-\exp\{-0.4t\})^{4-m-z+s} \\
&\quad \times (1+\theta(\exp\{-0.1t\})^{m-y+l}(\exp\{-0.4t\})^{4-m-z+s}) P(M_j = m)
\end{aligned}$$

Figure 1a, shows reliability results of these systems when $\tau = 0.001$.

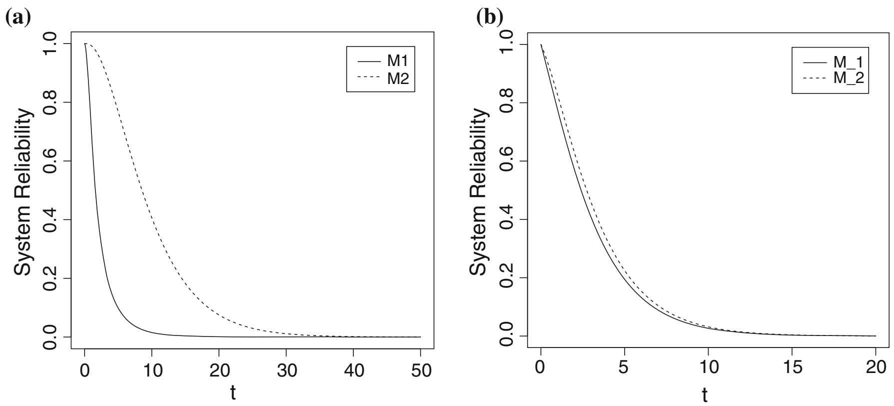

Fig. 1 The systems reliability **a** related to Example 1, **b** related to Example 2

The following example considers the case where the components are independent (i.e. the product copula is used).

Example 2 Consider two weighted 3/3 systems and assume that $w = 3$, $w^* = 2$, $F_X(t) = 1 - \exp\{-0.2\sqrt{t}\}$, $F_Y(t) = 1 - \exp\{-0.3\sqrt{t}\}$ where the components are assumed to be independent. Let the number of components of the first (second) system be selected randomly from C_X according to the random variable M_1 (M_2). If M_1 and M_2 are chosen randomly from {0,1,2,3} with probabilities {5/26, 6/26, 7/26, 8/26} and {1/10, 2/10, 3/10, 4/10}, respectively, then

$$R_{M_1}(t) = \sum_{m=0}^{3} \sum_{\substack{3a+2b\geq 3 \\ 0\leq a\leq m \\ 0\leq b\leq 3-m}}\sum \binom{m}{a}\binom{3-m}{b}(\exp\{-0.2\sqrt{t}\})^a(1-\exp\{-0.2\sqrt{t}\})^{m-a}$$
$$\times(\exp\{-0.3\sqrt{t}\})^b(1-\exp\{-0.3\sqrt{t}\})^{3-m-b}(m+5)/26$$

and

$$R_{M_2}(t) = \sum_{m=0}^{3} \sum_{\substack{3a+2b\geq 3 \\ 0\leq a\leq m \\ 0\leq b\leq 3-m}}\sum \binom{m}{a}\binom{3-m}{b}(\exp\{-0.2\sqrt{t}\})^a(1-\exp\{-0.2\sqrt{t}\})^{m-a}$$
$$\times(\exp\{-0.3\sqrt{t}\})^b(1-\exp\{-0.3\sqrt{t}\})^{3-m-b}(m+1)/10$$

In Fig. 1b, the reliability plots of these systems are depicted.

4 Comparison between two systems based on usual stochastic ordering

In this section we consider two independent weighted k/n systems with lifetimes T_1 and T_2. We aim to investigate conditions under which the reliability of the systems are ordered in two cases. In the first case, let T_i be the lifetime of the system built up in the case when M_i components are chosen randomly from the class $\mathfrak{C}_{X_i}$, $i = 1, 2$. The reliability of the two systems are compared in Theorem 1. In the second case, let T_i be the lifetime of the system built up in the case when the distribution functions of the components in $\mathfrak{C}_{X_i}$ and $\mathfrak{C}_{Y_i}$ are stochastically ordered, $i = 1, 2$. A comparison of systems reliability is shown in Theorem 2. To organize the proof of the theorems, we present the following lemmas.

Lemma 1 *Let $X_1, \ldots, X_n$ be continuous random variables with symmetric copula $C_{X_1,\ldots,X_n}$, marginal distribution functions $F_{X_1}, \ldots, F_{X_n}$, and joint distribution function $H_{X_1,\ldots,X_n}$. Let $Z_i = I(X_i > t)$, $i = 1, \ldots, n$. Then there exists a symmetric selection for the copula $C_{Z_1,\ldots,Z_n}$.*

Proof We prove this for the case of 2-copula. The proof for the case of n-copula is similar. First notice that

$$H_{Z_1,Z_2}(s_1,s_2) = P(I(X_1>t)\le s_1, I(X_2>t)\le s_2)$$
$$= \begin{cases} 0 & \text{for } s_1<0 \text{ or } s_2<0,\\ C_{X_1,X_2}(F_{X_1}(t),F_{X_2}(t)) & \text{for } 0\le s_1<1, 0\le s_2<1,\\ C_{X_1,X_2}(F_{X_1}(t),1) & \text{for } 0\le s_1<1, s_2\ge 1,\\ C_{X_1,X_2}(1,F_{X_2}(t)) & \text{for } s_1\ge 1, 0\le s_2<1,\\ C_{X_1,X_2}(1,1) & \text{for } s_1\ge 1, s_2\ge 1. \end{cases}$$

Define subcopula C'_{Z_1,Z_2} as

$$C'_{Z_1,Z_2}(u_1,u_2) = \begin{cases} 0 & \text{for } u_1=0 \text{ or } u_2=0,\\ C_{X_1,X_2}(F_{X_1}(t),F_{X_2}(t)) & \text{for } u_1=F_{X_1}(t), u_2=F_{X_2}(t),\\ C_{X_1,X_2}(F_{X_1}(t),1) & \text{for } u_1=F_{X_1}(t), u_2=1,\\ C_{X_1,X_2}(1,F_{X_2}(t)) & \text{for } u_1=1, u_2=F_{X_2}(t),\\ C_{X_1,X_2}(1,1) & \text{for } u_1=1, u_2=1. \end{cases}$$

Let D be the set of all permutations of all points in $RanF_{Z_1}\times RanF_{Z_2}$. Define subcopula C''_{Z_1,Z_2} on D in the manner that the value of C''_{Z_1,Z_2} is the same for all permutations of a point. It is possible since the copula $C_{X,Y}$ is symmetric. That is,

$$C''_{Z_1,Z_2}(u_1,u_2) = \begin{cases} 0 & \text{for } u_1=0 \text{ or } u_2=0,\\ C_{X_1,X_2}(F_{X_1}(t),F_{X_2}(t)) & \text{for } u_1=F_{X_1}(t), u_2=F_{X_2}(t),\\ C_{X_1,X_2}(F_{X_1}(t),F_{X_2}(t)) & \text{for } u_1=F_{X_2}(t), u_2=F_{X_1}(t),\\ C_{X_1,X_2}(F_{X_1}(t),1) & \text{for } u_1=F_{X_1}(t), u_2=1,\\ C_{X_1,X_2}(F_{X_1}(t),1) & \text{for } u_1=1, u_2=F_{X_1}(t),\\ C_{X_1,X_2}(1,F_{X_2}(t)) & \text{for } u_1=1, u_2=F_{X_2}(t),\\ C_{X_1,X_2}(1,F_{X_2}(t)) & \text{for } u_1=F_{X_2}(t), u_2=1,\\ C_{X_1,X_2}(1,1) & \text{for } u_1=1, u_2=1. \end{cases}$$

Then the extension of C''_{Z_1,Z_2} via *Sklar theorem* is a symmetric copula named it C_{Z_1,Z_2}. The copula C_{Z_1,Z_2} is a function that links F_{Z_1} and F_{Z_2} to H_{Z_1,Z_2}. □

Lemma 2 *Assume that R_m represents the reliability of a weighted k/n system with two distinct classes $\mathfrak{C}_X$ and $\mathfrak{C}_Y$ where the size of the class $\mathfrak{C}_X$ is m. Suppose that the components of the system are dependent with a symmetric copula. Let also w and w^* be the weights of the components and $\bar{F}_X$ and $\bar{F}_Y$ denote the reliability functions of the components in $\mathfrak{C}_X$ and $\mathfrak{C}_Y$, respectively. If $w\le w^*$ and $\bar{F}_X(t)\le\bar{F}_Y(t)$, for all t, then $R_m\le R_{m-1}$, for all $m=0,\ldots,n$, with convention $R_{-1}=1$, where*

$$R_m(t) = P\left(w\sum_{i=1}^{m} I(X_i > t) + w^*\sum_{i=1}^{n-m} I(Y_i > t) \geq k\right).$$

Proof First note that under the assumptions $w \leq w^*$ and $\bar{F}_X(t) \leq \bar{F}_Y(t)$, for all t, we have for all values $s \in R^+$

$$P(wI(X > t) \geq s) \leq P(w^*I(Y > t) \geq s). \tag{2}$$

From (2), it follows that $V_m(t) \leq_{st} W_{n-m+1}(t)$ where $V_i(t) = wI(X_i > t)$, $i = 1, \ldots, m$ and $W_i(t) = w^*I(Y_i > t)$, $i = 1, \ldots, n-m+1$. It is well-known that for a symmetric n-copula C, $n \geq 3$, all its $(n-1)$-margins are symmetric and they are equal. From this fact and Lemma 1

$$C_{V_1,\ldots,V_{m-1},W_1,\ldots,W_{n-m},W_{n-m+1}} = C_{V_1,\ldots,V_{m-1},V_m,W_1,\ldots,W_{n-m}}.$$

Put

$$W_{n,m}(t) = \sum_{i=1}^{m} V_i(t) + \sum_{i=1}^{n-m} W_i(t),$$

and

$$W_{n,m-1}(t) = \sum_{i=1}^{m-1} V_i(t) + \sum_{i=1}^{n-m+1} W_i(t).$$

Then from Theorems 6.B.14 and 6.B.16.(a) of Shaked and Shanthikumar (2007), we have

$$W_{n,m}(t) \leq_{st} W_{n,m-1}(t).$$

That is, $R_m \leq R_{m-1}$, for all $m = 1, \ldots, n$. This completes the proof. □

Now, we are ready to prove the main theorems.

Theorem 1 *Suppose that T_i, $i = 1, 2$, represent the lifetime of a weighted k/n system where the system components are dependent with a symmetric copula and the components are chosen randomly from two distinct classes $\mathfrak{C}_X$ and $\mathfrak{C}_Y$. Let also w and w^* be the weights of the components and $\bar{F}_X$ and $\bar{F}_Y$ denote the reliability functions of the components in $\mathfrak{C}_X$ and $\mathfrak{C}_Y$. Assume that the random variables M_1 and M_2 are the number of selected components from the set C_X in the first and second system, respectively.*

(i) If $M_1 \leq_{st} M_2$, $w \leq w^*$ and for all t, $\bar{F}_X(t) \leq \bar{F}_Y(t)$, then $T_1 \geq_{st} T_2$.
(ii) If $M_1 \leq_{st} M_2$, $w \geq w^*$ and for all t, $\bar{F}_X(t) \geq \bar{F}_Y(t)$, then $T_1 \leq_{st} T_2$.

Proof We only prove part (i), the proof of part (ii) is similar. We have,

$$\begin{aligned} R_M(t) &= \sum_{m=0}^{n} R_m(t)P(M=m) \\ &= \sum_{m=0}^{n-1} R_m(t)[P(M \geq m) - P(M \geq m+1)] + R_n(t)P(M=n) \\ &= \sum_{m=0}^{n} (R_m(t) - R_{m-1}(t))P(M \geq m), \end{aligned}$$

with convention $R_{-1} = 1$. From, $M_1 \leq_{st} M_2$, we have for all m, $P(M_1 \geq m) \leq P(M_2 \geq m)$. Also from Lemma 2, $R_m \leq R_{m-1}$, $m = 1, \ldots, n$. Hence,

$$(R_m(t) - R_{m-1}(t))P(M_1 \geq m) \geq (R_m(t) - R_{m-1}(t))P(M_2 \geq m), m = 1, \ldots, n,$$

which, in turn, implies that, for all t, $R_{M_1}(t) > R_{M_2}(t)$. □

The following example gives an illustration of Theorem 1.

Example 3 Consider a weighted 2/2 system with $w = 1$, $w^* = 2$, $F_X(t) = 1 - \exp\{-0.2t\}$, $F_Y(t) = 1 - \exp\{-0.1t\}$ and assume that the dependence structure among components is generated by Clayton copula. We consider two different cases for random variable M. In the first one we assume that $M = M_1$ takes its values randomly from {0,1,2} with uniform probabilities and in the second case $M = M_2$ takes its values on the same set with probabilities {1/6, 2/6, 3/6}.

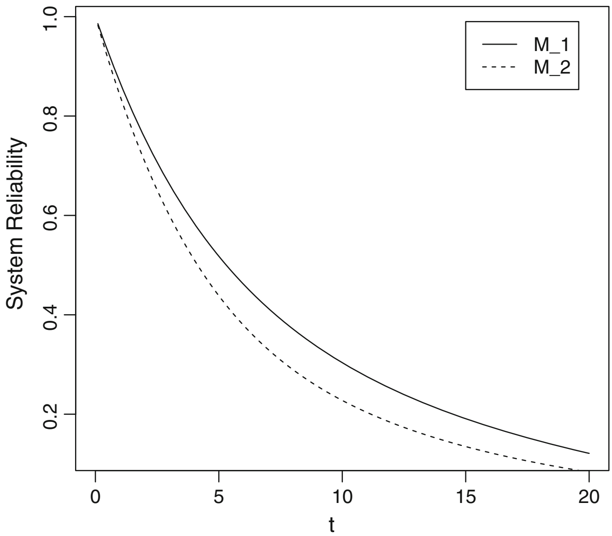

Fig. 2 The system reliability for system in Example 3

We leave it to the reader to verify that $M_1 \leq_{st} M_2$ and $\bar{F}_X(t) \leq \bar{F}_Y(t)$. So the conditions of Theorem 1 are satisfied. Hence $R_{M_1}(t) \geq R_{M_2}(t)$. The plots of the reliability functions of the systems are depicted in Fig. 2 when $\tau = 0.5$. In the case that the Clayton copula evinces negative dependency, the obtained result is similar to the positive dependency.

To complete the discussion, hazard rate (*hr*) and likelihood ratio (*lr*) orders are investigated as well. In this case, it is seen that the ordering for M is not transferred to the system lifetime T_M. This is shown in the next example.

Example 4 Consider a weighted 1/1 system with $w = 2$, $w^* = 1$, $F_X(t) = 1 - \exp\{-0.1t\}$, $F_Y(t) = 1 - \exp\{-0.2t\}$. If $M = M_1$ takes its values randomly from {0,1} with probabilities {0.1,0.9} and $M = M_2$ takes its values on the same set with probabilities {0.00901,0.990990}, it is plain to see $M_1 \leq_{hr} M_2$ and $X \geq_{hr} Y$. However, a situation can be found in which $T_{M_1} \leq_{hr} T_{M_2}$ doesn't hold, i.e., $P(T_{M_1} \geq t_2)P(T_{M_2} \geq t_1) \geq P(T_{M_1} \geq t_1)P(T_{M_2} \geq t_2), \forall t_1 \leq t_2$. It will be established if take $t_1 = 3.465$ and $t_2 = 4.581$. Since the *hr* order is the necessary condition for *lr* order and in this example, $M_1 \leq_{lr} M_2$ and $X \geq_{lr} Y$, hence $T_{M_1} \leq_{lr} T_{M_2}$ doesn't hold.

In the following, we explore the conditions under which the stochastic ordering between the components of two weighted k/n systems imply the stochastic ordering between the lifetimes of the systems.

Theorem 2 *Suppose that T_j represents the lifetime of a weighted k/n system with two classes $\mathfrak{C}_{X_j}$ and $\mathfrak{C}_{Y_j}$ where the size of the class $\mathfrak{C}_{X_j}$ is random, $j = 1, 2$. Let M be a random variable with support contained in $\{0, 1, \ldots, n\}$. Suppose the size of the class $\mathfrak{C}_{X_j}$ is selected randomly according to the random variable M. Moreover, Let F_{X_j} and F_{Y_j} indicate the distribution functions of the components in $\mathfrak{C}_{X_j}$ and $\mathfrak{C}_{Y_j}$. Assume that when a selected component from $\mathfrak{C}_{X_j}$ ($\mathfrak{C}_{Y_j}$) lies in a working situation, it has the weight w (w^*), $j = 1, 2$. Let the dependence structure among components be built with a common symmetric copula. If $X_1 \geq_{st} X_2$ and $Y_1 \geq_{st} Y_2$, then $T_1 \geq_{st} T_2$.*

Proof Let X_{ij} and Y_{lj} be the lifetimes of the components in the classes $\mathfrak{C}_{X_j}$ and $\mathfrak{C}_{Y_j}$, $j = 1, 2$, $i = 1, \ldots, m$ and $l = 1, \ldots, n - m$. It suffices to show that

$$\left(w \sum_{\substack{i=1 \\ i \in \mathfrak{C}_{X_1}}}^{m} I(X_{i1} > t) + w^* \sum_{\substack{i=1 \\ i \in \mathfrak{C}_{Y_1}}}^{n-m} I(Y_{i1} > t) \right) \geq_{st} \left(w \sum_{\substack{i=1 \\ i \in \mathfrak{C}_{X_2}}}^{m} I(X_{i2} > t) + w^* \sum_{\substack{i=1 \\ i \in \mathfrak{C}_{Y_2}}}^{n-m} I(Y_{i2} > t) \right),$$

for each fixed m. According to the assumption

$$X_{i1} \geq_{st} X_{i2}, \quad Y_{l1} \geq_{st} Y_{l2}, \quad i = 1, \ldots, m, l = 1, \ldots, n - m. \tag{3}$$

Then from what we have argued in Theorem 1, $wI(X_{i1} > t) \geq_{st} wI(X_{i2} > t)$ and $w^*I(Y_{l1} > t) \geq_{st} w^*I(Y_{l2} > t)$. From Lemma 1 the components in random vectors

$$(I(X_{11} > t), \ldots, I(X_{m1} > t), I(Y_{11} > t), \ldots, I(Y_{(n-m)1} > t))$$

and

$$(I(X_{12} > t), \ldots, I(X_{m2} > t), I(Y_{12} > t), \ldots, I(Y_{(n-m)2} > t))$$

are dependent with a symmetric common copula. Then, from the inequality (3), the result follows using Theorems 6.B.14 and 6.B.16.(a) of Shaked and Shanthikumar (2007). □

Remark 1 Note that the random variables T_1 and T_2 in part (i) of Theorem 1 and Theorem 2 are independent and since

$$P(T_1 \geq T_2) = \int_0^\infty P(T_1 \geq t)dF_{T_2}(t),$$

then it is easy to see that the usual stochastic order implies the *sp* order (see Arcones et al. 2002; Santis et al. 2015). Recall that for two random variables X_1 and X_2, the random variable X_1 is said to be stochastically precede X_2 (written $X_1 \leq_{sp} X_2$) if and only if $P(X_1 \leq X_2) \geq 1/2$. The random variables X_1 and X_2 are *sp*-equivalent if and only if they satisfy $P(X_1 \leq X_2) = 1/2$.

In the following, we give two examples to measure the difference between $P(T_1 \geq T_2)$ and $1/2$ in two cases. The first case is when the dependence structure is Clayton and the second one is when the component lifetimes are independent.

Example 5 Consider a weighted $k/10$ system for $k = 1, \ldots, 11$ with $w = 1$, $w^* = 2$, $F_{X_1}(t) = 1 - \exp\{-0.1t\}$, $F_{Y_1}(t) = 1 - \exp\{-0.2t\}$, $F_{X_2}(t) = 1 - \exp\{-0.5t\}$, $F_{Y_2}(t) = 1 - \exp\{-0.6t\}$. So the conditions of Theorem 2 are satisfied. Thus by Remark 1, we have $P(T_1 \geq T_2) \geq 1/2$. Now, assume that the dependence structure among components is generated by Clayton copula when $\tau = 0.2$ and $\tau = 0.5$. In the case that the random variable M follows Binomial distribution $B(10, 0.7)$, the values of $P(T_1 \geq T_2)$ for $k = 1, \ldots, 11$ are calculated. The corresponding results are presented in Table 1.

Example 6 Consider a weighted $k/10$ system for $k = 1, \ldots, 11$ with $w = 1$, $w^* = 2$, $F_{X_1}(t) = 1 - \exp\{-0.1t\}$, $F_{Y_1}(t) = 1 - \exp\{-0.2t\}$, $F_{X_2}(t) = 1 - \exp\{-0.5t\}$, $F_{Y_2}(t) = 1 - \exp\{-0.6t\}$. Again the conditions of Theorem 2 are satisfied. Then Remark 1 gives $P(T_1 \geq T_2) \geq 1/2$. Here, assume that the components are independent and the random variable M follows Binomial distribution $B(10, 0.7)$. The third column in Table 1 shows $P(T_1 \geq T_2)$ for $k = 1, \ldots, 11$.

Table 1 The values of $P(T_1 \geq T_2)$: the first column ($\tau = 0.2$), the second column ($\tau = 0.5$), for system in Example 5, the third column (independence) for system in Example 6

k	Clayton $\tau = 0.2$	Clayton $\tau = 0.5$	Independence $\tau = 0$
1	0.9150	0.8327	0.8455
2	0.8939	0.8260	0.8425
3	0.8823	0.8269	0.8266
4	0.8716	0.8265	0.8017
5	0.8626	0.8259	0.7745
6	0.8542	0.8243	0.7470
7	0.8463	0.8216	0.7196
8	0.8384	0.8172	0.6923
9	0.8298	0.8109	0.6646
10	0.8195	0.8027	0.6353
11	0.7633	0.7502	0.5748

5 Conclusion

In this paper, we studied some reliability and stochastic properties of weighted k-out-of-n systems. We assumed that the system is built up from two different classes of components that are classified with respect to the weights and reliability functions of the components. It was assumed that a random number of components, say M, are chosen from the first class and the rest of components are selected from the second class. Different copula functions were considered for modeling the structure of dependency of the system component lifetimes. The reliability of the system, was expressed as a mixture of the reliability of weighted k-out-of-n systems, with fixed number of components of the two types, in terms of the probability mass function of the random variable M. We showed that when the random mechanism of the chosen components for different two weighted k-out-of-n systems are ordered in usual stochastic (*st*) order then, under some conditions on the weights and the distributions of components lifetime, the lifetimes of the two systems are also ordered in *st* order. In addition, the lifetimes of two different systems were compared in the sense of stochastic precedence concept. We presented some illustrative examples for the results under different conditions.

Acknowledgements The authors express their sincere thanks to the Editor, and associate editor, and two referees for providing constructive comments which led to improvement of the paper. Asadi's research work was carried out in IPM Isfahan branch and was in part supported by a Grant from IPM (No. 96620411).

References

Arcones MA, Kvam PH, Samaniego FJ (2002) Nonparametric estimation of a distribution subject to a stochastic precedence constraint. J Am Stat Assoc 97:170–182

Ball MO, Hagstrom JN, Provan JS (1995) Threshold reliability of networks with small failure sets. Networks 25:101–115

Barlow R.E, Proschan F (1981) Statistical theory of reliability and life testing: Probability models. Silver Spring, MD

Berdichevsky V, Gitterman M (1998) Stochastic resonance and ratchets new manifestations. Physica A 249:88–95

Bier M (1997) Brownian ratchets in physics and biology. Contemp Phys 38:371–379

Chen Y, Yang Q (2005) Reliability of two-stage weighted k-out-of-n systems with components in common. IEEE Trans Reliab 54:431–440

Coolen FPA, Coolen-Maturi T (2012) Generalizing the signature to systems with multiple types of components. Complex Syst Dependability 115–130

Cui L, Xie M (2005) On a generalized k-out-of-n system and its reliability. Int J Syst Sci 36:267–274

De Santis E, Fantozzi F, Spizzichino F (2015) Relations between stochastic orderings and generalized stochastic precedence. Probab Eng Inf Sci 29:329–343

Di Crescenzo A (2007) A Parrondo paradox in reliability theory. Math Sci 32:17–22

Di Crescenzo A, Pellerey F (2011) Stochastic comparisons of series and parallel systems with randomized independent components. Oper Res Lett 39:380–384

Eryilmaz S (2011) Dynamic behavior of k-out-of-n: G systems. Oper Res Lett 39:155–159

Eryilmaz S (2013) Mean instantaneous performance of a system with weighted components that have arbitrarily distributed lifetimes. Reliab Eng Syst Saf 119:290–293

Eryilmaz S (2014) Multivariate copula based dynamic reliability modeling with application to weighted-k-out-of-n systems of dependent components. Struct Saf 51:23–28

Eryilmaz S (2015) Capacity loss and residual capacity in weighted k-out-of-n: G systems. Reliab Eng Syst Saf 136:140–144

Eryilmaz S, Coolen FPA, Coolen-Maturi T (2018) Marginal and joint reliability importance based on survival signature. Reliab Eng Syst Saf 172:118–128

Eryilmaz S, Sarikaya K (2013) Modeling and analysis of weighted-k-out-of-n: G system consisting of two different types of components. Proc Inst Mech Eng Part O J Risk Reliab 201:1–7

Gammaitoni L, Hanggi P, Jung P, Marchesoni F (1998) Stochastic resonance. Rev Mod Phys 70:223–287

Genest C, Neslehova J, Ghorbal NB (2011) Estimators based on Kendall's tau in multivariate copula models. Australian New Zealand J Stat 53:157–177

Hazra NK, Finkelstein M, Cha JH (2017) On optimal grouping and stochastic comparisons for heterogenous items. J Multivar Anal 160:146–156

Hazra NK, Nanda AK (2014) Some results on series and parallel systems of randomized components. Oper Res Lett 42:132–136

Higashiyama Y (2001) A factored reliability formula for weighted-k-out-of-n system. Asia-Pacific J Oper Res 18:61–66

Jia X, Cui L (2012) Reliability research of k-out-of-n: G supply chain system based on copula. Commun Stat Theory Methods 41:4023–4033

Kochar S, Xu M (2010) On residual lifetimes of k-out-of-n systems with nonidentical components. Probab Eng Inf Sci 24:109–127

Kuo W, Zuo MJ (2003) Optimal reliability modeling: principles and applications. Wiley, New York

Li H, Li X (2013) Stochastic orders in reliability and risk. Springer, New York

Li X, You Y, Fang R (2016) On weighted k-out-of-n systems with statistically dependent component lifetimes. Probab Eng Inf Sci 30:533–546

Li X, Zhao P (2008) Stochastic comparison on general inactivity time and general residual life of k-out-of-n systems. Commun Stat Simul Comput 37:1005–1019

Navarro J, Del Aguila Y (2017) Stochastic comparisons of distorted distributions, coherent systems and mixtures with ordered components. Metrika 80:627–648

Navarro J, Pellerey F, Di Crescenzo A (2015) Orderings of coherent systems with randomized dependent components. Eur J Oper Res 240:127–139

Navarro J, Spizzichino F (2010) Comparisons of series and parallel systems with components sharing the same copula. Appl Stoch Models Bus Ind 26:775–791

Nelsen RB (2006) An introduction to copulas. Springer, New York

Rushdi AM (1990) Threshold systems and their reliability. Microelectron Reliab 30:299–312

Samaniego FJ, Navarro J (2016) On comparing coherent systems with heterogeneous components. Adv Appl Probab 48(1):88–111

Samaniego FJ, Shaked M (2008) Systems with weighted components. Stat Probab Lett 78:815–823

Shaked M, Shanthikumar JG (2007) Stochastic orders. Springer series in statistics. Springer, New York

Tang XS, Li DQ, Zhou CB, Phoon KK, Zhang LM (2013a) Impact of copulas for modeling bivariate distributions on system reliability. Struct Saf 44:80–90

Tang XS, Li DQ, Zhou CB, Zhang LM (2013b) Bivariate distribution models using copulas for reliability analysis. Proc Inst Mech Eng Part O J Risk Reliab 227:499–512

Wang Y, Pham H (2012) Modeling the dependent competing risks with multiple degradation processes and random shock using time-varying copulas. IEEE Trans Reliab 61:13–22

Wu JS, Chen RJ (1994a) An algorithm for computing the reliability of a weighted-k-out-of-n system. IEEE Trans Reliab 43:327–328

Wu JS, Chen RJ (1994b) Efficient algorithms for k-out-of-n and consecutive weighted k-out-of-n : F system. IEEE Trans Reliab 43:650–655