Metrika (2018) 81:797–820
https://doi.org/10.1007/s00184-018-0659-y

On consistency of the weighted least squares estimators in a semiparametric regression model

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Received: 17 July 2017 / Published online: 21 April 2018

Abstract This paper is concerned with the semiparametric regression model $y_i = x_i\beta + g(t_i) + \sigma_i e_i,\ \ i = 1, 2, \ldots, n$, where $\sigma_i^2 = f(u_i)$, (x_i, t_i, u_i) are known fixed design points, β is an unknown parameter to be estimated, $g(\cdot)$ and $f(\cdot)$ are unknown functions, random errors e_i are widely orthant dependent random variables. The p-th $(p > 0)$ mean consistency and strong consistency for least squares estimators and weighted least squares estimators of β and g under some more mild conditions are investigated. A simulation study is also undertaken to assess the finite sample performance of the results that we established. The results obtained in the paper generalize and improve some corresponding ones of negatively associated random variables.

Keywords Semiparametric regression model · Widely orthant dependent random error · Least squares estimator · Consistency

Mathematics Subject Classification 62F12 · 62G20

1 Introduction

As we known that semiparametric regression models (or partially linear models) rely on a dimension reduction assumption, while being still flexible enough due to the presence

Supported by the National Natural Science Foundation of China (11671012, 11501004, 11501005), the Natural Science Foundation of Anhui Province (1508085J06) and the Key Projects for Academic Talent of Anhui Province (gxbjZD2016005).

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of a nonparametric term. In recent years, semiparametric regression models have attracted a growing number of statisticians to study it. Based on the effect of weather on electricity demand, Engle et al. (1986) studied the following semiparametric regression model,

$$Y_i = X_i^{'}\beta + g(T_i) + e_i, \quad i = 1, 2, \ldots, n. \tag{1.1}$$

Hong (1991) studied the model (1.1), and gave the estimators $\hat{\beta}_n$ and g_n^* of β and g respectively by the methods of least squares and the nearest neighbor weight functions. In addition, he also obtained the asymptotic normality for $\hat{\beta}_n$ and the strong consistency for g_n^*. Based on the independent and identically distributed (i.i.d.) samples, Gao (1992) proposed the kernel estimator $\hat{g}_n(\cdot)$ of $g(\cdot)$ and the least squares estimator $\hat{\beta}_n$ of β in the semiparametric regression model as follows,

$$Y_i = X_i\beta + g(t_i) + e_i, \quad i = 1, 2, \ldots, n, \tag{1.2}$$

and obtained some strong and weak consistencies and convergence rates for estimators of β and $g(\cdot)$. Hu (1999) defined the least squares estimators $\tilde{\beta}_\tau$ of β and the estimator $\tilde{g}_\tau(t)$ of g, respectively. In the case of the independent random errors, he established some asymptotic properties for the estimators $\tilde{\beta}_\tau$ and $\tilde{g}_\tau(t)$, including the strong consistency, uniform strong consistency, r-th $(r > 2)$ mean consistency and r-th $(r > 2)$ mean uniform consistency. Pan et al. (2003) discussed the semiparametric model (1.2) with L^q mixingale errors, and obtained r-th $(r > 2)$ mean consistency and complete consistency for estimators of β and g. Hu (2006) studied the model (1.2) with linear time series errors and obtained the r-th $(r > 2)$ mean consistency and complete consistency for the estimators $\hat{\beta}_n$ and $\hat{g}_n(t)$ of β and g, respectively. Based on model (1.2), Gao et al. (1994) proposed a more general semiparametric regression model,

$$y_i = x_i\beta + g(t_i) + \sigma_i e_i, \quad i = 1, 2, \ldots, n, \tag{1.3}$$

where $\sigma_i^2 = f(u_i)$, (x_i, t_i, u_i) are known fixed design points, β is an unknown parameter to be estimated, $g(\cdot)$ and $f(\cdot)$ are unknown functions defined on compact set $A \subset \mathbb{R}$, e_i are random errors. Additionally, Gao et al. (1994) gave the least squares estimators (LSE) and the weighted least squares estimators (WLSE) of β and g and the estimator of f, and further proved the asymptotic normality for the two estimators of β under i.i.d. random errors. Chen et al. (1998) investigated the strong consistency for the two estimators of β if $\{e_i, i \geq 1\}$ is i.i.d.. Based on negatively associated random errors, Baek and Liang (2006) studied the strong consistency of estimators of β, g and f, and the asymptotic normality of β; Zhou and Hu (2010) obtained p-th $(p > 2)$ mean consistency for LSE and WLSE of β and g. For more details about the asymptotic properties of the estimators in semiparametric regression models, one can refer to Chen (1988), Speckman (1988), Hamilton and Truong (1997), Mammen and Van de Geer (1997), Aneiros and Quintela (2001), Zhou and Lin (2013) among others.

Inspired by the above literatures, we will study the p-th $(p > 0)$ mean consistency and strong consistency for LSE and WLSE of β and g in model (1.3) under the random errors $\{e_i, i \geq 1\}$ being zero mean widely orthant dependent random variables. We will give the details in Sect. 2. Now let us recall the definition of widely orthant dependence structure.

Definition 1.1 For the random variables $\{X_n, n \geq 1\}$, if there exists a finite real sequence $\{h_U(n), n \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, +\infty)$, $1 \leq i \leq n$,

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leq h_U(n) \prod_{i=1}^{n} P(X_i > x_i),$$

then we say that the $\{X_n, n \geq 1\}$ are widely upper orthant dependent (WUOD, in short); if there exists a finite real sequence $\{h_L(n), n \geq 1\}$ satisfying for each $n \geq 1$ and for all $x_i \in (-\infty, +\infty)$, $1 \leq i \leq n$,

$$P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n) \leq h_L(n) \prod_{i=1}^{n} P(X_i \leq x_i),$$

then we say that the $\{X_n, n \geq 1\}$ are widely lower orthant dependent (WLOD, in short); if they are both WUOD and WLOD, then we say that the $\{X_n, n \geq 1\}$ are widely orthant dependent, and $h_U(n), h_L(n), n \geq 1$ are called dominating coefficients.

The concept of WOD random variables was firstly introduced by Wang et al. (2013). And they therein gave some examples to show that the class of WOD random variables includes some common negatively dependent random variables, some positively dependent random variables and some others. Subsequently, various properties and applications were obtained. For instance, Liu et al. (2012) gave the asymptotically equivalent formula for the finite-time ruin probability under a dependent risk model with constant interest rate, Shen (2013a) established the Bernstein type inequality for WOD random variables and gave some applications, He et al. (2013) provided the asymptotic lower bounds of precise large deviations with nonnegative and dependent random variables, Wang et al. (2014) further established the complete convergence for arrays of row-wise WOD random variables and gave its applications in nonparametric regression models, Wang and Hu (2015a) studied the consistency of the nearest neighbor estimator of the density function based on WOD samples, Shen et al. (2016) provided some exponential probability inequalities for WOD sequence and gave applications in complete convergence and complete moment convergence, Chen et al. (2016) established a more accurate inequality of WOD random variables, and obtained some limit theorems including the strong law of large numbers, the complete convergence, the almost sure elementary renewal theorem and the weighted elementary renewal theorem, and so on.

Obviously, $h_U(n) \geq 1, h_L(n) \geq 1, n \geq 1$. If $h_U(n) = h_L(n) = M$ for any $n \geq 1$, it is easily seen that the random variables $\{X_n, n \geq 1\}$ are extended negatively dependent (END, in short), where M is a positive constant. More particularly, if $M = 1$, then the random variables $\{X_n, n \geq 1\}$ are called negatively orthant dependent (NOD, in short). In other words, NOD is a special case of END. For details about NOD and END sequence, one can refer to Volodin (2002), Asadian et al. (2006), Liu (2009), Wang and Wang (2013), Shen (2013b), Shen et al. (2015), Wang et al. (2015b), and so forth. Furthermore, Joag-Dev and Proschan (1983) pointed out that negatively associated (NA, in short) random variables are NOD. Meanwhile, Hu (2000) introduced the concept of negatively superadditive dependence (NSD, in short) and pointed out that NSD implies NOD [see Property 2 of Hu (2000)]. By the above description, the class of WOD random variables contains END random variables, NOD random variables, NSD random variables, NA random variables and independent random variables as special cases. Thus, it is of practical significance to study the mean consistency and the strong consistency of estimators in the semiparametric model (1.3) with WOD random errors.

The organization of the paper is as follows. In Sect. 2, we first present the LSE and the WLSE of β and $g(\cdot)$, and some basic assumptions; and then we will establish the main results, including the mean consistency and strong consistency for the LSE and the WLSE of β and $g(\cdot)$; a numerical simulation to study the consistency of LSE for β and $g(\cdot)$ is also carried out; finally, some important lemmas to prove the main results are provided. In Sect. 3, we mainly give the proofs of the main results. In "Appendix", we present the proofs of Lemmas 2.4 and 2.5.

Throughout the paper, denote $h(n) = \max\{h_U(n), h_L(n)\}$. $a_n = O(b_n)$ denotes that there exists a positive constant C such that $a_n \le Cb_n$. Let $c, c_1, c_2, C, C_1, C_2, \ldots$ denote the positive constants whose values may vary at each occurrence.

2 Main results and lemmas

2.1 Estimators and basic assumptions

The LSE and the WLSE of β and $g(\cdot)$ given in Gao et al. (1994) are as follows:

$$\hat{\beta}_n = S_n^{-2}\sum_{i=1}^{n}\tilde{x}_i\tilde{y}_i,\quad \hat{g}_n(t) = \sum_{i=1}^{n}W_{ni}(t)(y_i - x_i\hat{\beta}_n), \tag{2.1}$$

$$\tilde{\beta}_n = T_n^{-2}\sum_{i=1}^{n}a_i\tilde{x}_i\tilde{y}_i,\quad \tilde{g}_n(t) = \sum_{i=1}^{n}W_{ni}(t)(y_i - x_i\tilde{\beta}_n), \tag{2.2}$$

where $W_{ni}(\cdot)$ are weight functions only depending on the designed points t_i $(i = 1, 2, \ldots, n)$, $\tilde{x}_i = x_i - \sum_{j=1}^{n}W_{nj}(t_i)x_j$, $\tilde{y}_i = y_i - \sum_{j=1}^{n}W_{nj}(t_i)y_j$, $S_n^2 = \sum_{i=1}^{n}\tilde{x}_i^2$, $a_i = \frac{1}{f(u_i)}$, $T_n^2 = \sum_{i=1}^{n}a_i\tilde{x}_i^2$.

In this paper, we will consider the following assumptions:

H₁ (i) $\lim\limits_{n\to\infty}\frac{1}{n(h(n))^{2r}}\sum_{i=1}^{n}\tilde{x}_i^2 = \Gamma\ (0 < \Gamma < \infty), \exists\, r \ge 1$;

$(i)'$ $\lim\limits_{n\to\infty}\frac{1}{n(h(n))^{r}}\sum_{i=1}^{n}\tilde{x}_i^2 = \Gamma\ (0 < \Gamma < \infty), \exists\, r > 0$;

(ii) $0 < m_0 \le \min\limits_{1\le i\le n} f(u_i) \le \max\limits_{1\le i\le n} f(u_i) \le M_0 < \infty$;

(iii) $g(\cdot)$ and $f(\cdot)$ are continuous on compact set A.

H₂ $\max\limits_{1\le j\le n}\left|\sum_{i=1}^{n}W_{ni}(t_j) - 1\right| = o(1)$;

H₃ $\max\limits_{1\le j\le n}\sum_{i=1}^{n}|W_{ni}(t_j)|I(|t_i - t_j| > a) = o(1),\ \forall\, a > 0$;

H₄ $\max\limits_{1\le j\le n}\sum_{i=1}^{n}|W_{ni}(t_j)| = O(1)$;

H5 $\max_{1\le i,j\le n} |W_{ni}(t_j)| = O(n^{-s}(h(n))^{-r}),\ \exists\, s > 0,\ r \ge 1;$

H5′ $\max_{1\le i,j\le n} |W_{ni}(t_j)| = O(n^{-s}(h(n))^{-r}),\ \exists\, s > 0,\ r > 0.$

Remark 2.1 $(H_1)(i)$ (with $h(n) = 1$) (ii) are some regular conditions, which are assumed in Gao et al. (1994), Chen et al. (1998), Baek and Liang (2006) and so on. Moreover, it can be deduced from $(H_1)(i)$ (or $(i)'$) (ii) here that

$$S_n^{-2}\sum_{i=1}^n |\tilde{x}_i| \le C,\quad T_n^{-2}\sum_{i=1}^n |a_i\tilde{x}_i| \le C. \tag{2.3}$$

Remark 2.2 Remark 2.3 in Baek and Liang (2006) mentioned that the following two weight functions satisfy assumptions (H_2)–(H_5) with $h(n) = \log n$, $s = \frac{1}{2}$ and $r = 1$:

$$W_{ni}^{(1)}(t) = \frac{1}{h_n}\int_{s_{i-1}}^{s_i} K\left(\frac{t-s}{h_n}\right) ds,$$

$$W_{ni}^{(2)}(t) = K\left(\frac{t-t_i}{h_n}\right)\left[\sum_{j=1}^n K\left(\frac{t-t_j}{h_n}\right)\right]^{-1},$$

where $s_i = (t_i + t_{i+1})/2,\ i = 1, 2, \ldots, n-1,\ s_0 = 0,\ s_n = 1,\ 0 \le t_1 \le t_2 \le \cdots \le t_n \le 1$, $K(\cdot)$ is the Parzen-Rosenblatt kernel function, and h_n is a bandwidth parameter.

2.2 Consistency

Let $\{e_i, i \ge 1\}$ be a sequence of mean zero WOD random errors with dominating coefficient $h(n)$, which is stochastically dominated by a random variable e, that is

$$P(|e_i| > x) \le CP(|e| > x)$$

for all $x \ge 0$, $n \ge 1$ and some $C > 0$.

Theorem 2.1 (mean consistency) *Let* $p > 0$. *Suppose that conditions* (H_1)*(i, ii, iii) and* (H_2)–(H_5) *hold. If* $Ee^2 < \infty$ *for* $0 < p \le 2$ *or* $E|e|^p < \infty$ *for* $p > 2$, *then*

$$\lim_{n\to\infty} E|\hat{\beta}_n - \beta|^p = 0, \tag{2.4}$$

$$\lim_{n\to\infty} E|\tilde{\beta}_n - \beta|^p = 0. \tag{2.5}$$

In addition, if $\max_{1\le j\le n}\left|\sum_{i=1}^n W_{ni}(t_j)x_i\right| = O(1)$, *then*

$$\lim_{n\to\infty}\max_{1\le i\le n} E|\hat{g}_n(t_i) - g(t_i)|^p = 0, \tag{2.6}$$

$$\lim_{n\to\infty}\max_{1\le i\le n} E|\tilde{g}_n(t_i) - g(t_i)|^p = 0. \tag{2.7}$$

In particular, if $h(n)$ is a constant function, we can get the following corollary by Theorem 2.1.

Corollary 2.1 *Let $p > 0$. Assume that $\{e_i, i \geq 1\}$ be a sequence of END random errors with mean zero. Suppose that conditions (H_1)–(H_5) hold with $h(n) = 1$. If $\sup_i Ee_i^2 < \infty$ for $0 < p \leq 2$ or $\sup_i E|e_i|^p < \infty$ for $p > 2$, then (2.4) and (2.5) hold. In addition, if $\max_{1\leq j\leq n} \left|\sum_{i=1}^n W_{ni}(t_j)x_i\right| = O(1)$, then (2.6) and (2.7) hold.*

Remark 2.3 Under the NA sequence and conditions (H_1)–(H_5) with $h(n) = 1$ and $s = 1/2$, Zhou and Hu (2010) obtained results (2.4)–(2.7) under $\sup_i E|e_i|^p < \infty$ for some $p > 2$, while Corollary 2.1 in our paper gives these results for some $p > 0$. Since NA sequence is END, we extend Theorem 2.1 of Zhou and Hu (2010) to the case of END sequence. Furthermore, $s > 0$ in condition (H_5) of Corollary 2.1 is more general than $s = 1/2$ in Theorem 2.1 of Zhou and Hu (2010).

The next theorem gives the strong consistency of estimators under some analogous conditions.

Theorem 2.2 (strong consistency) *Suppose that conditions (H_1) $(i^{'}$, ii, iii), (H_2)–(H_4) and $(H_5^{'})$ hold. If $Ee^2 < \infty$ and $\sum_{i=1}^n i^{-s}(h(i))^{-r} = O(n^s)$ for some $s > 0$ and $r > 0$, then*

$$\hat{\beta}_n \to \beta \ \ a.s., \ \ n \to \infty, \tag{2.8}$$

$$\tilde{\beta}_n \to \beta \ \ a.s., \ \ n \to \infty. \tag{2.9}$$

In addition, if $\max_{1\leq j\leq n} \left|\sum_{i=1}^n W_{ni}(t_j)x_i\right| = O(1)$, then

$$\max_{1\leq i\leq n} |\hat{g}_n(t_i) - g(t_i)| \to 0 \ \ a.s., \ \ n \to \infty, \tag{2.10}$$

$$\max_{1\leq i\leq n} |\tilde{g}_n(t_i) - g(t_i)| \to 0 \ \ a.s., \ \ n \to \infty. \tag{2.11}$$

Remark 2.4 Since $h(i) \geq 1$, $\sum_{i=1}^n i^{-s}(h(i))^{-r} = O(n^s)$ in Theorem 2.2 always holds as long as $s \geq 1/2$ and $r > 0$.

Particularly, if $h(n)$ is a constant function and $s = 1/2$ in Theorem 2.2, we can get the following corollary.

Corollary 2.2 *Assume that $\{e_i, i \geq 1\}$ be a sequence of mean zero END random errors, which is stochastically dominated by a random variable e. Suppose that conditions (H_1)–(H_5) hold with $s = 1/2$, $h(n) = 1$. If $Ee^2 < \infty$, then (2.8) and (2.9) hold. In addition, if $\max_{1\leq j\leq n} \left|\sum_{i=1}^n W_{ni}(t_j)x_i\right| = O(1)$, then (2.10) and (2.11) hold.*

Remark 2.5 Under mean zero NA random errors, Theorem 2.1 of Baek and Liang (2006) gave the results (2.8)–(2.11) under the conditions (H_1)–(H_5) with $s = 1/2$, $h(n) = 1$ and $\sup_i E|e_i|^p < \infty$ for some $p > 2$. Compared with it, Corollary 2.2 (i) extends the case of NA random variables to END random variables; (ii) lowers the order of the moment from $p > 2$ to 2.

2.3 Simulation

In this section, we will carry out a numerical simulation to study the consistency of LSE for β and $g(\cdot)$. The data is generated from the model (1.3). Choose $\sigma_i = 1$, $x_i = (-1)^i \frac{i}{n}$, $i = 1, 2, \ldots, n$. We take random error vector $(e_1, e_2, \ldots, e_n)' \sim N(\mathbf{0}, \Sigma)$, where $\mathbf{0}$ is a zero column vector, and

$$\Sigma=\begin{pmatrix} \frac{1}{2}+\nu^2 & -\nu & 0 & \ldots & 0 & 0 & 0 \\ -\nu & \frac{1}{2}+\nu^2 & -\nu & \ldots & 0 & 0 & 0 \\ 0 & -\nu & \frac{1}{2}+\nu^2 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \frac{1}{2}+\nu^2 & -\nu & 0 \\ 0 & 0 & 0 & \ldots & -\nu & \frac{1}{2}+\nu^2 & -\nu \\ 0 & 0 & 0 & \ldots & 0 & -\nu & \frac{1}{2}+\nu^2 \end{pmatrix}_{n\times n}, \quad \nu = 0.1.$$

It is obvious that $e_1, e_2, \ldots, e_n$ generated as the above method are NA by Joag-Dev and Proschan (1983), which is a special case of WOD ($h(n) = 1$). Especially, we choose the nearest neighbor weights to be weight functions $W_{ni}(\cdot)$. Without loss of generality, let $A = [0, 1]$ and $t_i = \frac{i}{n}$, $i = 1, 2, \ldots, n$. For any $t \in A$, we rewrite $|t_1 - t|, |t_2 - t|, \ldots, |t_n - t|$ as follows

$$|t_{R_1(t)} - t| \le |t_{R_2(t)} - t| \le \cdots \le |t_{R_n(t)} - t|,$$

if $|t_i - t| = |t_j - t|$, $|t_i - t|$ is permuted before $|t_j - t|$ if $i < j$. Let $k_n = \lfloor n^{0.6} \rfloor$ and define the nearest neighbor weight functions as follows

$$W_{ni}(t) = \begin{cases} \frac{1}{k_n}, & \text{if } |t_i - t| \le |t_{R_{k_n}(t)} - t|, \\ 0, & \text{otherwise.} \end{cases}$$

For any $t = t_i$, $i = 1, 2, \ldots, n$, it is easily checked that

$$\begin{aligned} \sum_{i=1}^{n} W_{ni}(t) &= \sum_{i=1}^{n} W_{nR_i(t)}(t) = \sum_{i=1}^{k_n} \frac{1}{k_n} = 1, \\ \max_{1\le i\le n} W_{ni}(t) &= \frac{1}{k_n} \le Cn^{-0.6}, \\ \sum_{i=1}^{n} W_{ni}(t) I(|t_i - t| > a) &\le \sum_{i=1}^{n} W_{ni}(t) \frac{(t_i - t)^2}{a^2} \\ &\le \sum_{i=1}^{k_n} \frac{(t_{R_i(t)} - t)^2}{k_n a^2} \le \sum_{i=1}^{k_n} \frac{(i/n)^2}{k_n a^2} \end{aligned}$$

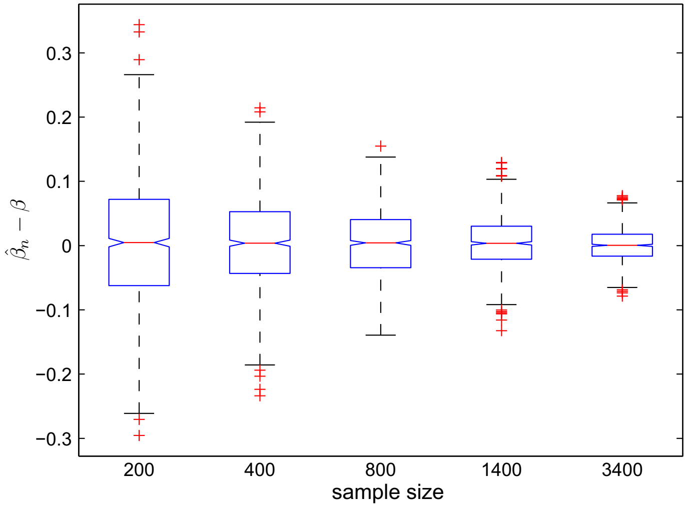

Fig. 1 Boxplots of $\hat{\beta}_n - \beta$ with $\beta = 2$, t = 1/n, g(t) = t^2

$$\leq \left(\frac{k_n}{na}\right)^2 \leq \frac{C}{n^{0.8}},$$

$$\left|\sum_{i=1}^{n} W_{ni}(t)x_i\right| \leq \sum_{i=1}^{n} W_{ni}(t) = 1,$$

which imply that the assumptions in our results are satisfied. Next, we compute $\hat{\beta}_n - \beta$ and $\hat{g}_n(t) - g(t)$ for 1000 times and obtain the corresponding boxplots by taking $t = \frac{1}{n}, \frac{100}{n}$ and the sample sizes n as 200, 400, 800, 1400, 3400 respectively when β and $g(t)$ are in two different forms.

Case 1 $\beta = 2$, $g(t) = t^2$.
Case 2 $\beta = 3$, $g(t) = \sin t$.

In Figs. 1, 2, 3, 4, 5, 6, 7 and 8, $\hat{\beta}_n - \beta$ and $\hat{g}_n(t) - g(t)$, regardless of the values of t, fluctuate to zero and the variation ranges decrease markedly as the sample size n increases. These verify the validity of our results.

3 Proof of main results

It is easy to see that

$$\hat{\beta}_n - \beta = S_n^{-2}\left[\sum_{i=1}^{n} \sigma_i \tilde{x}_i e_i - \sum_{i=1}^{n} \tilde{x}_i \left(\sum_{j=1}^{n} W_{nj}(t_i)\sigma_j e_j\right) + \sum_{i=1}^{n} \tilde{x}_i \tilde{g}(t_i)\right], \tag{3.1}$$

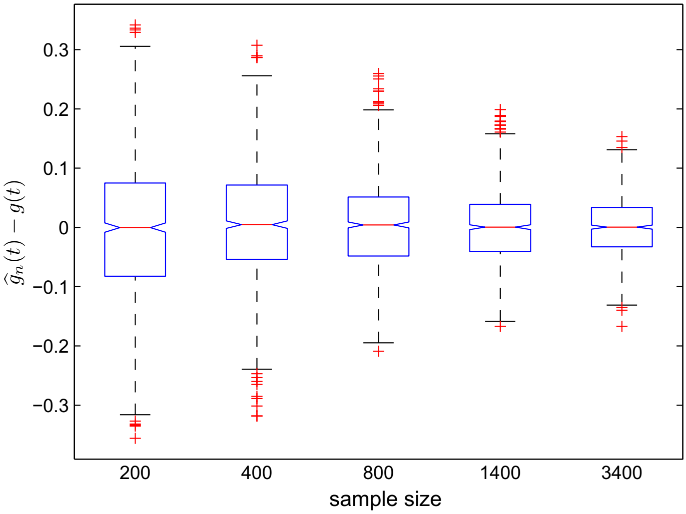

Fig. 2 Boxplots of $\hat{g}_n(t) - g(t)$ with $\beta = 2$, t = 1/n, g(t) = t^2

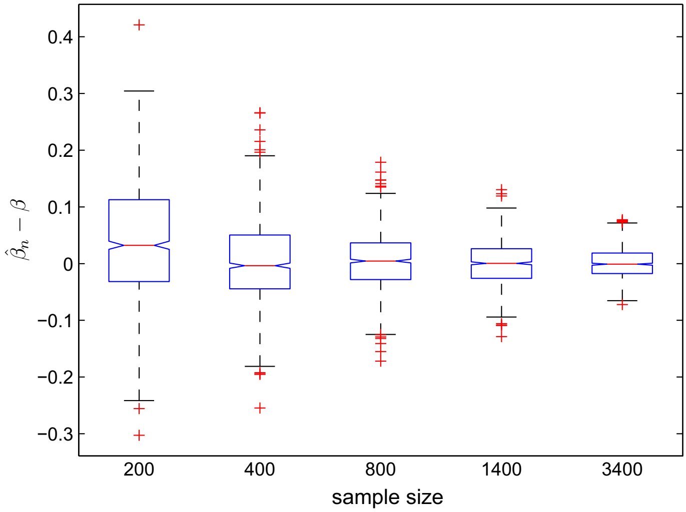

Fig. 3 Boxplots of $\hat{\beta}_n - \beta$ with $\beta = 2$, t = 100/n, g(t) = t^2

$$\tilde{\beta}_n - \beta = T_n^{-2}\left[\sum_{i=1}^{n} a_i \sigma_i \tilde{x}_i e_i - \sum_{i=1}^{n} a_i \tilde{x}_i \left(\sum_{j=1}^{n} W_{nj}(t_i)\sigma_j e_j\right) + \sum_{i=1}^{n} a_i \tilde{x}_i \tilde{g}(t_i)\right], \tag{3.2}$$

$$\hat{g}_n(t_i) - g(t_i) = \sum_{j=1}^{n} W_{nj}(t_i)\sigma_j e_j - (\hat{\beta}_n - \beta)\sum_{j=1}^{n} W_{nj}(t_i) x_j - \tilde{g}(t_i), \tag{3.3}$$

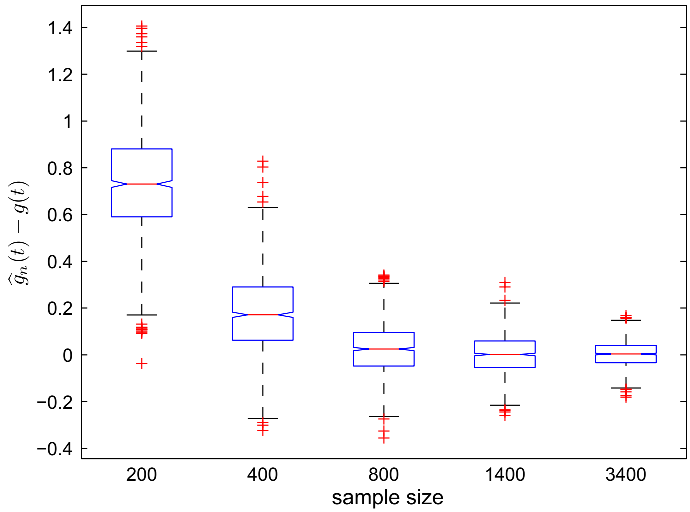

Fig. 4 Boxplots of $\hat{g}_n(t) - g(t)$ with $\beta = 2$, t = 100/n, g(t) = t^2

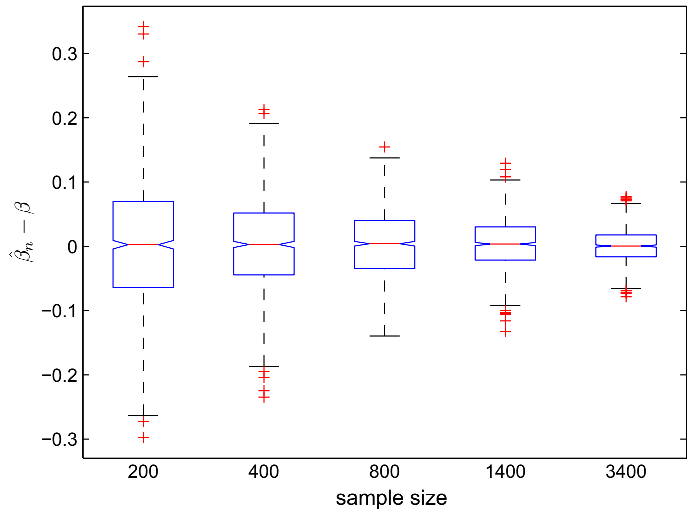

Fig. 5 Boxplots of $\hat{\beta}_n - \beta$ with $\beta = 3$, t = 1/n, g(t) = sint

$$\tilde{g}_n(t_i) - g(t_i) = \sum_{j=1}^{n} W_{nj}(t_i)\sigma_j e_j - (\tilde{\beta}_n - \beta)\sum_{j=1}^{n} W_{nj}(t_i)x_j - \tilde{g}(t_i), \tag{3.4}$$

where $\tilde{g}(t_i) = g(t_i) - \sum_{j=1}^{n} W_{nj}(t_i)g(t_j)$.

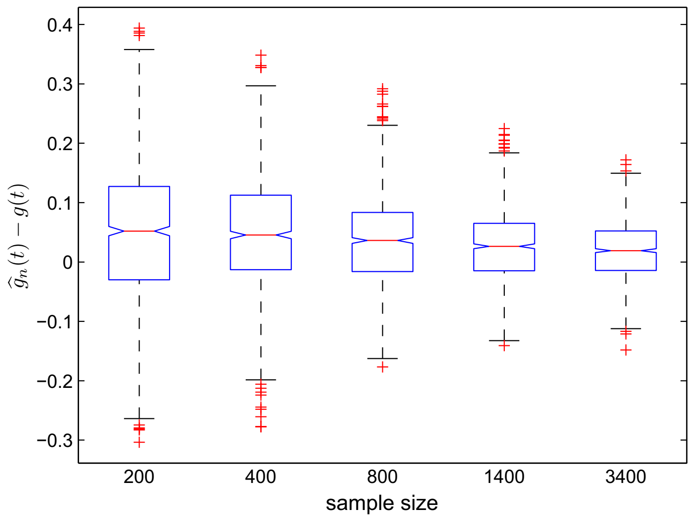

Fig. 6 Boxplots of $\hat{g}_n(t) - g(t)$ with $\beta = 3$, t = 1/n , g(t) = sint

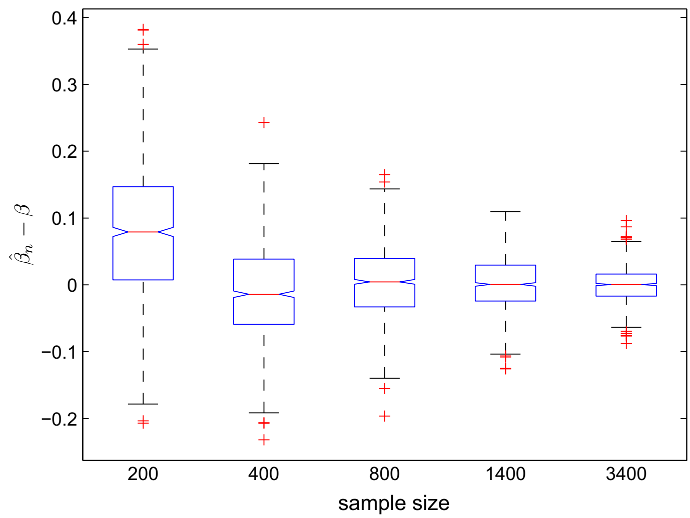

Fig. 7 Boxplots of $\hat{\beta}_n - \beta$ with $\beta = 3$, t = 100/n, g(t) = sint

Proof of Theorem 2.1 We only prove (2.5) and (2.7), as the proofs of (2.4) and (2.6) are respectively analogous. Denote

$$H_{1n} = T_n^{-2} \sum_{i=1}^{n} a_i \sigma_i \tilde{x}_i e_i, \ H_{2n} = T_n^{-2} \sum_{i=1}^{n} a_i \tilde{x}_i \left(\sum_{j=1}^{n} W_{nj}(t_i) \sigma_j e_j \right),$$

$$H_{3n} = T_n^{-2} \sum_{i=1}^{n} a_i \tilde{x}_i \tilde{g}(t_i).$$

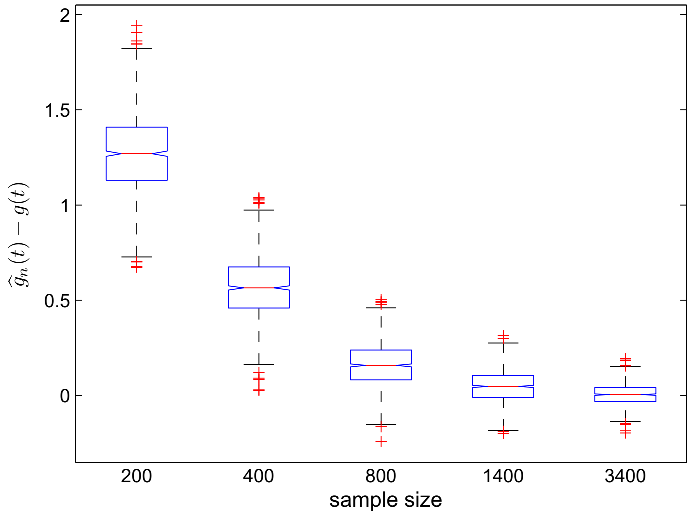

Fig. 8 Boxplots of $\hat{g}_n(t) - g(t)$ with $\beta = 3$, t = 100/n , g(t) = sint

From (3.2) and C_r inequality, we have

$$E|\tilde{\beta}_n - \beta|^p \le C(E|H_{1n}|^p + E|H_{2n}|^p + E|H_{3n}|^p). \tag{3.5}$$

Note that $H_{1n} = \sum_{i=1}^n (T_n^{-2} a_i \sigma_i \tilde{x}_i) e_i \doteq \sum_{i=1}^n b_{ni} e_i$, and

$$\max_{1\le i\le n} |b_{ni}| \le \max_{1\le i\le n} \frac{|a_i \tilde{x}_i|}{T_n} \cdot \max_{1\le i\le n} \sigma_i \cdot \frac{1}{T_n} = O(n^{-1/2}(h(n))^{-r}), \tag{3.6}$$

$$\sum_{i=1}^n |b_{ni}| \le \sum_{i=1}^n \frac{|a_i \tilde{x}_i|}{T_n^2} \cdot \max_{1\le i\le n} \sigma_i = O(1), \tag{3.7}$$

by $H_1(i, ii)$ and (2.3). Hence, we obtain by Lemma A.1 that

$$\lim_{n\to\infty} E|H_{1n}|^p = 0. \tag{3.8}$$

Observe that $H_{2n} = \sum_{j=1}^n \left(\sum_{i=1}^n T_n^{-2} a_i \tilde{x}_i W_{nj}(t_i) \sigma_j\right) e_j \doteq \sum_{j=1}^n d_{nj} e_j$, and

$$\max_{1\le j\le n} |d_{nj}| \le \max_{1\le j\le n} \sigma_j \cdot \max_{1\le i,j\le n} |W_{nj}(t_i)| \cdot \sum_{i=1}^n \frac{|a_i \tilde{x}_i|}{T_n^2} = O(n^{-s}(h(n)^{-r})), \tag{3.9}$$

$$\begin{aligned} \sum_{j=1}^n |d_{nj}| &\le \sum_{j=1}^n \left| \sum_{i=1}^n T_n^{-2} a_i \tilde{x}_i W_{nj}(t_i) \sigma_j \right| \le \sum_{i=1}^n \sum_{j=1}^n \sigma_j |W_{nj}(t_i)| \frac{|a_i \tilde{x}_i|}{T_n^2} \\ &\le \max_{1\le j\le n} \sigma_j \cdot \max_{1\le i\le n} \sum_{j=1}^n |W_{nj}(t_i)| \cdot \sum_{i=1}^n \frac{|a_i \tilde{x}_i|}{T_n^2} = O(1), \end{aligned} \tag{3.10}$$

by $H_1(ii)$, H_5 and (2.3). Therefore, we have by Lemma A.1 that

$$\lim_{n\to\infty} E|H_{2n}|^p = 0. \tag{3.11}$$

We now discuss H_{3n}. It follows from $H_1(iii)$, H_2, H_3 and H_4 that

$$H_{3n} \le \left(\max_{1\le i\le n} |\tilde{g}(t_i)|\right)\left(T_n^{-2}\sum_{i=1}^{n}|a_i\tilde{x}_i|\right)$$

and

$$\begin{aligned}\max_{1\le i\le n}|\tilde{g}(t_i)| \le& \max_{1\le i\le n}|g(t_i)|\left|\sum_{j=1}^{n}W_{nj}(t_i)-1\right| \\ &+\max_{1\le i\le n}\sum_{j=1}^{n}|W_{nj}(t_i)||g(t_i)-g(t_j)|I(|t_i-t_j|>a) \\ &+\max_{1\le i\le n}\sum_{j=1}^{n}|W_{nj}(t_i)||g(t_i)-g(t_j)|I(|t_i-t_j|\le a) \\ =& \, o(1).\end{aligned} \tag{3.12}$$

So, we can obtain by (3.12) and (2.3) that

$$\lim_{n\to\infty} E|H_{3n}|^p = 0,$$

which, together with (3.5), (3.8) and (3.11), yields (2.5).

Now we turn to prove (2.7). It can be seen by (3.4) that

$$\begin{aligned}&\max_{1\le i\le n} E|\tilde{g}_n(t_i)-g(t_i)|^p \\ &\le C\max_{1\le i\le n} E\left|\sum_{j=1}^{n}W_{nj}(t_i)\sigma_j e_j\right|^p + C\max_{1\le i\le n}\left|\sum_{j=1}^{n}W_{nj}(t_i)x_j\right|^p E|\tilde{\beta}_n-\beta|^p \\ &\quad + C\max_{1\le i\le n}|\tilde{g}(t_i)|^p \\ &\doteq Q_{1n}+Q_{2n}+Q_{3n}.\end{aligned} \tag{3.13}$$

We can obtain from $H_1(ii)$, H_4 and H_5 that $Q_{1n} \to 0$, $n \to \infty$ by applying Lemma A.1. From (2.5) and the assumption $\max_{1\le j\le n}\left|\sum_{i=1}^{n} W_{ni}(t_j)x_i\right| = O(1)$, we can get $Q_{2n} \to 0$, $n \to \infty$. $Q_{3n} \to 0$, $n \to \infty$ follows from (3.12). Therefore, the desired result (2.7) follows from (3.13) and $Q_{1n} \to 0$, $Q_{2n} \to 0$, $Q_{3n} \to 0$, $n \to \infty$. This completes the proof of the theorem. □

Proof of Theorem 2.2 Using the notations in the proof of Theorem 2.1, we know that

$$\tilde{\beta}_n - \beta = H_{1n} + H_{2n} + H_{3n}.$$

Applying Lemma A.2, we have by (3.6) and (3.7) that $H_{1n} \to 0$ $a.s.$, $n \to \infty$. Likewise, by (3.9) and (3.10), $H_{2n} \to 0$ $a.s.$, $n \to \infty$. From (2.3) and (3.12), we can easily obtain that

$$H_{3n} \le \left(\max_{1\le i\le n} |\tilde{g}(t_i)|\right)\left(T_n^{-2}\sum_{i=1}^{n} a_i \tilde{x}_i\right) \to 0, \quad n \to \infty.$$

So (2.9) is proved. It follows from (3.4) that

$$\begin{aligned}
&\max_{1\le i\le n} |\tilde{g}_n(t_i) - g(t_i)| \\
&\le C \max_{1\le i\le n}\left|\sum_{j=1}^{n} W_{nj}(t_i)\sigma_j e_j\right| + C \max_{1\le i\le n}\left|\sum_{j=1}^{n} W_{nj}(t_i)x_j\right| |\tilde{\beta}_n - \beta| \\
&\quad + C \max_{1\le i\le n} |\tilde{g}(t_i)| \\
&\doteq R_{1n} + R_{2n} + R_{3n}.
\end{aligned} \tag{3.14}$$

From $H_1(ii)$, H_4 and H_5, we obtain that $R_{1n} \to 0$ $a.s.$, $n \to \infty$ by applying Lemma A.2. According to (2.9) and the assumption $\max_{1\le j\le n}\left|\sum_{i=1}^{n} W_{ni}(t_j)x_i\right| = O(1)$, we can get $R_{2n} \to 0$ $a.s.$, $n \to \infty$. $R_{3n} \to 0$, $n \to \infty$ follows from (3.12). Therefore, the desired result (2.11) follows from (3.14) and $R_{1n} \to 0$ $a.s.$, $R_{2n} \to 0$ $a.s.$, $R_{3n} \to 0$, $n \to \infty$. The proof is completed. □

Acknowledgements The authors are grateful to the Referee for carefully reading the manuscript and for providing helpful comments and constructive criticism which enabled them to improve the paper.

Appendix

Lemma A.1 *Let $p > 0$ and $\{X_n, n \ge 1\}$ be a sequence of zero mean WOD random variables with dominating coefficient $h(n)$, which is stochastically dominated by a random variable X. Assume that $\{a_{ni}(\cdot), 1 \le i \le n, n \ge 1\}$ is a function array defined on compact set A satisfying*

$$\max_{1\le j\le n} \sum_{i=1}^{n} |a_{ni}(z_j)| = O(1) \tag{3.15}$$

and

$$\max_{1\le i,j\le n} |a_{ni}(z_j)| = O(n^{-\alpha}(h(n))^{-\beta}),\ \exists\, \alpha > 0,\ \beta \ge 1. \tag{3.16}$$

If $EX^2 < \infty$ *for* $0 < p \le 2$, *then*

$$\lim_{n\to\infty} \max_{1\le j\le n} E\left|\sum_{i=1}^{n} a_{ni}(z_j)X_i\right|^p = 0. \tag{3.17}$$

If $E|X|^p < \infty$ *for* $p > 2$, *then (3.17) still holds.*

Remark A.1 Lemma A.1 also holds when the moment condition $EX^2 < \infty$ is changed to $\sup_i EX_i^2 < \infty$, $E|X|^p < \infty$ is changed to $\sup_i E|X_i|^p < \infty$ and the condition of stochastic domination is deleted. Under the similar modification, Theorem 2.1 also holds true.

Proof of Lemma A.1 Without loss of generality, we can assume that $a_{ni}(z_j) > 0$.

If $0 < p \le 2$, by Jensen's inequality, Marcinkiewicz-Zygmund-type inequality (one can refer to Wang et al. (2014) for instance), (3.15), (3.16) and $EX^2 < \infty$, we have

$$\begin{aligned}
&\max_{1\le j\le n} E\left|\sum_{i=1}^{n} a_{ni}(z_j)X_i\right|^p \\
&\le C\left(EX^2\right)^{p/2}\left(h(n)\max_{1\le i,j\le n} a_{ni}(z_j)\right)^{p/2}\left(\max_{1\le j\le n}\sum_{i=1}^{n} a_{ni}(z_j)\right)^{p/2} \\
&\le Cn^{-\alpha p/2}(h(n))^{(1-\beta)p/2} \to 0,\quad n\to\infty.
\end{aligned}$$

If $p > 2$, we denote

$$X_{ni}^j = n^{1/p}(h(n))^{\beta/p} a_{ni}(z_j)X_i,$$

thus, we only need to prove

$$\frac{1}{n(h(n))^\beta}\max_{1\le j\le n} E\left|\sum_{i=1}^{n} X_{ni}^j\right|^p \to 0,\quad n\to\infty.$$

For any $t > 0$, denote

$$\begin{aligned}
Y_{ni}^j &= -t^{1/p} I(X_{ni}^j < -t^{1/p}) + X_{ni}^j I(|X_{ni}^j| \le t^{1/p}) + t^{1/p} I(X_{ni}^j > t^{1/p}), \\
Z_{ni}^j &= (X_{ni}^j + t^{1/p}) I(X_{ni}^j < -t^{1/p}) + (X_{ni}^j - t^{1/p}) I(X_{ni}^j > t^{1/p}).
\end{aligned}$$

For fixed $t > 0$ and $1 \le j \le n$, we can see that $\{Y_{ni}^j, 1 \le i \le n, n \ge 1\}$ and $\{Z_{ni}^j, 1 \le i \le n, n \ge 1\}$ are both arrays of rowwise WOD random variables. Noting that $X_{ni}^j = Y_{ni}^j - EY_{ni}^j + Z_{ni}^j - EZ_{ni}^j$, we have

$$
\begin{aligned}
&\frac{1}{n(h(n))^{\beta}} \max_{1\le j\le n} E\left|\sum_{i=1}^{n} X_{ni}^{j}\right|^{p} \\
&= \frac{1}{n(h(n))^{\beta}} \max_{1\le j\le n} \left[\int_{0}^{n\varepsilon} P\left(\left|\sum_{i=1}^{n} X_{ni}^{j}\right|^{p} > t\right) dt \right. \\
&\quad \left. + \int_{n\varepsilon}^{\infty} P\left(\left|\sum_{i=1}^{n} X_{ni}^{j}\right|^{p} > t\right) dt\right] \\
&\le \varepsilon + \frac{1}{n(h(n))^{\beta}} \max_{1\le j\le n} \int_{n\varepsilon}^{\infty} P\left(\left|\sum_{i=1}^{n}\left(Y_{ni}^{j} - EY_{ni}^{j}\right)\right| > t^{1/p}/2\right) dt \\
&\quad + \frac{1}{n(h(n))^{\beta}} \max_{1\le j\le n} \int_{n\varepsilon}^{\infty} P\left(\left|\sum_{i=1}^{n}\left(Z_{ni}^{j} - EZ_{ni}^{j}\right)\right| > t^{1/p}/2\right) dt \\
&\doteq \varepsilon + I_1 + I_2. \qquad (3.18)
\end{aligned}
$$

First, we prove $I_2 \to 0,\ n \to \infty$. Note that

$$
\begin{aligned}
\max_{1\le j\le n} \max_{t>n\varepsilon} \left|t^{-1/p} \sum_{i=1}^{n} EZ_{ni}^{j}\right| &\le Cn^{-1} \max_{1\le j\le n} \sum_{i=1}^{n} E|X_{ni}^{j}|^{p} I\left(|X_{ni}^{j}| > (n\varepsilon)^{1/p}\right) \\
&\le C(h(n))^{\beta} \max_{1\le j\le n} \sum_{i=1}^{n} a_{ni}^{p}(z_j) E|X_i|^{p} \\
&\le CE|X|^{p}(h(n))^{\beta} \max_{1\le i,j\le n} a_{ni}^{p-1}(z_j) \max_{1\le j\le n} \sum_{i=1}^{n} a_{ni}(z_j) \\
&\le Cn^{-\alpha(p-1)}(h(n))^{-\beta(p-2)} E|X|^{p} \to 0, \quad n \to \infty.
\end{aligned}
$$

Hence, for any $t > n\varepsilon$ and all n large enough, we have $\max_{1\le j\le n}\left|\sum_{i=1}^{n} EZ_{ni}^{j}\right| \le t^{1/p}/4$, which implies that for all n large enough,

$$
\begin{aligned}
I_2 &\le \frac{1}{n(h(n))^{\beta}} \max_{1\le j\le n} \int_{n\varepsilon}^{\infty} P\left(\left|\sum_{i=1}^{n} Z_{ni}^{j}\right| > t^{1/p}/4\right) dt \\
&\le \frac{1}{n(h(n))^{\beta}} \max_{1\le j\le n} \sum_{i=1}^{n} \int_{n\varepsilon}^{\infty} P\left(|X_{ni}^{j}| > t^{1/p}\right) dt \\
&\le \frac{1}{n(h(n))^{\beta}} \max_{1\le j\le n} \sum_{i=1}^{n} E|X_{ni}^{j}|^{p} I(|X_{ni}^{j}|^{p} > n\varepsilon) \\
&\le \max_{1\le j\le n} \sum_{i=1}^{n} a_{ni}^{p}(z_j) E|X_i|^{p} \le CE|X|^{p} \max_{1\le i,j\le n} a_{ni}^{p-1}(z_j) \max_{1\le j\le n} \sum_{i=1}^{n} a_{ni}(z_j) \\
&\le CE|X|^{p} n^{-\alpha(p-1)}(h(n))^{-\beta(p-1)} \to 0, \quad n \to \infty. \qquad (3.19)
\end{aligned}
$$

Next, we will show that $I_1 \to 0,\ n \to \infty$. Taking $q > p$, we have by Markov's inequality and Rosenthal-type inequality (one can refer to Wang et al. (2014) for instance) that

$$\begin{aligned} I_1 &\le \frac{C}{n(h(n))^\beta} \max_{1\le j\le n} \int_{n\varepsilon}^{\infty} t^{-q/p} E\left|\sum_{i=1}^{n}\left(Y_{ni}^{j} - EY_{ni}^{j}\right)\right|^q dt \\ &\le \frac{C}{n(h(n))^\beta} \max_{1\le j\le n} \int_{n\varepsilon}^{\infty} t^{-q/p} \sum_{i=1}^{n} E|Y_{ni}^{j}|^q dt \\ &\quad + \frac{Ch(n)}{n(h(n))^\beta} \max_{1\le j\le n} \int_{n\varepsilon}^{\infty} t^{-q/p} \left(\sum_{i=1}^{n} E\left(Y_{ni}^{j}\right)^2\right)^{q/2} dt \\ &\doteq I_{11} + I_{12}. \end{aligned} \tag{3.20}$$

According to the definition of Y_{ni}^{j}, we have

$$\begin{aligned} I_{11} &\le \frac{C}{n(h(n))^\beta} \max_{1\le j\le n} \int_{n\varepsilon}^{\infty} t^{-q/p} \sum_{i=1}^{n} E|X_{ni}^{j}|^q I\left(|X_{ni}^{j}| \le t^{1/p}\right) dt \\ &\quad + \frac{C}{n(h(n))^\beta} \max_{1\le j\le n} \int_{n\varepsilon}^{\infty} \sum_{i=1}^{n} P\left(|X_{ni}^{j}| > t^{1/p}\right) dt \\ &\doteq I_{111} + I_{112}. \end{aligned} \tag{3.21}$$

In view of the proof of I_2, we can get that $I_{112} \to 0,\ n \to \infty$. Next, we estimate the limit of I_{111} as $n \to \infty$. It is easy to check that

$$\begin{aligned} I_{111} &\le \frac{C}{n(h(n))^\beta} \max_{1\le j\le n} \int_{n\varepsilon}^{\infty} t^{-q/p} \sum_{i=1}^{n} E|X_{ni}^{j}|^q I\left(|X_{ni}^{j}|^p \le (n+1)\varepsilon\right) dt \\ &\quad + \frac{C}{n(h(n))^\beta} \max_{1\le j\le n} \int_{n\varepsilon}^{\infty} t^{-q/p} \sum_{i=1}^{n} E|X_{ni}^{j}|^q I\left((n+1)\varepsilon < |X_{ni}^{j}|^p \le t\right) dt \\ &= \frac{C}{n(h(n))^\beta} \max_{1\le j\le n} \int_{n\varepsilon}^{\infty} t^{-q/p} \sum_{i=1}^{n} E|X_{ni}^{j}|^q I\left(|X_{ni}^{j}|^p \le (n+1)\varepsilon\right) dt \\ &\quad + \frac{C}{n(h(n))^\beta} \max_{1\le j\le n} \int_{(n+1)\varepsilon}^{\infty} t^{-q/p} \sum_{i=1}^{n} E|X_{ni}^{j}|^q I\left((n+1)\varepsilon < |X_{ni}^{j}|^p \le t\right) dt \\ &\doteq I_{111}^{'} + I_{111}^{''}. \end{aligned} \tag{3.22}$$

Similar to the proof of (3.19), we have

$$I'_{111} \leq \frac{C}{n(h(n))^{\beta}} \max_{1\leq j\leq n} \sum_{i=1}^{n} E|X_{ni}^{j}|^{p} I\left(|X_{ni}^{j}|^{p} \leq (n+1)\varepsilon\right)$$
$$\leq C \max_{1\leq j\leq n} \sum_{i=1}^{n} a_{ni}^{p}(z_j) E|X_i|^{p} \to 0, \quad n\to\infty, \tag{3.23}$$

and

$$I''_{111} = \frac{C}{n(h(n))^{\beta}} \max_{1\leq j\leq n} \sum_{m=n+1}^{\infty} \int_{m\varepsilon}^{(m+1)\varepsilon} t^{-q/p} \sum_{i=1}^{n} E|X_{ni}^{j}|^{q} I((n+1)\varepsilon < |X_{ni}^{j}|^{p}$$
$$\leq t)dt$$
$$\leq \frac{C}{n(h(n))^{\beta}} \max_{1\leq j\leq n} \sum_{m=n+1}^{\infty} m^{-q/p} \sum_{i=1}^{n} E|X_{ni}^{j}|^{q} I((n+1)\varepsilon < |X_{ni}^{j}|^{p} \leq (m+1)\varepsilon)$$
$$= \frac{C}{n(h(n))^{\beta}} \max_{1\leq j\leq n} \sum_{m=n+1}^{\infty} m^{-q/p} \sum_{i=1}^{n} \sum_{k=n+1}^{m} E|X_{ni}^{j}|^{q} I(k\varepsilon < |X_{ni}^{j}|^{p} \leq (k+1)\varepsilon)$$
$$\leq \frac{C}{n(h(n))^{\beta}} \max_{1\leq j\leq n} \sum_{i=1}^{n} \sum_{k=n+1}^{\infty} k^{1-q/p} E|X_{ni}^{j}|^{q} I(k\varepsilon < |X_{ni}^{j}|^{p} \leq (k+1)\varepsilon)$$
$$\leq \frac{C}{n(h(n))^{\beta}} \max_{1\leq j\leq n} \sum_{i=1}^{n} E|X_{ni}^{j}|^{p} \to 0, \quad n\to\infty, \tag{3.24}$$

which imply that $I_{111} \to 0,\ n\to\infty$. Noting that $p > 2$, $\beta \geq 1$ and $EX^2 < \infty$, we have

$$I_{12} \leq \frac{Ch(n)}{n(h(n))^{\beta}} \max_{1\leq j\leq n} \int_{n\varepsilon}^{\infty} t^{-q/p} \left(\sum_{i=1}^{n} E\left(X_{ni}^{j}\right)^{2} I\left(|X_{ni}^{j}| \leq t^{1/p}\right)\right.$$
$$\left. + \sum_{i=1}^{n} t^{2/p} P\left(|X_{ni}^{j}| > t^{1/p}\right)\right)^{q/2} dt$$
$$\leq \frac{C}{n} \max_{1\leq j\leq n} \int_{n\varepsilon}^{\infty} t^{-q/p} \left(\sum_{i=1}^{n} E\left(X_{ni}^{j}\right)^{2}\right)^{q/2} dt$$
$$\leq \frac{C}{n} n^{1-q/p} \max_{1\leq j\leq n} \left(\sum_{i=1}^{n} E\left(n^{1/p}(h(n))^{\beta/p} a_{ni}(z_j) X_i\right)^{2}\right)^{q/2}$$
$$\leq C(EX^2)^{q/2} n^{-\alpha q/2} (h(n))^{(1/p-1/2)\beta q} \to 0, \quad n\to\infty. \tag{3.25}$$

The proof is completed. □

Lemma A.2 *Let* $\{X_n, n \geq 1\}$ *be a sequence of zero mean WOD random variables with dominating coefficient* $h(n)$*, which is stochastically dominated by a random variable* X*. Assume that* $\{a_{ni}(\cdot), 1 \leq i \leq n, n \geq 1\}$ *is a function array defined on compact set* A *satisfying*

$$\max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ni}(z_j)| = O(1) \tag{3.26}$$

and

$$\max_{1\leq i,j\leq n} |a_{ni}(z_j)| = O(n^{-\alpha}(h(n))^{-\beta}),\ \exists\, \alpha > 0,\ \beta > 0. \tag{3.27}$$

If $EX^2 < \infty$ *and* $\sum_{i=1}^{n} i^{-\alpha}(h(i))^{-\beta} = O(n^{\alpha})$ *for some* $\alpha > 0$ *and* $\beta > 0$*, then*

$$\max_{1\leq j\leq n} \left|\sum_{i=1}^{n} a_{ni}(z_j)X_i\right| \to 0\ \ a.s.,\ n \to \infty. \tag{3.28}$$

Proof Without loss of generality, we can assume that $a_{ni}(z_j) > 0$.

For any $\varepsilon > 0$, choose $0 < \delta < \alpha/2$ and large $N \geq 1$, which will be specialized later. Denote $X_{ni}(j) = a_{ni}(z_j)X_i$, and

$$\begin{aligned}
Y_{ni}^{(1)}(j) &= -n^{-\delta}(h(n))^{-\beta/4} I\left(X_{ni}(j) < -n^{-\delta}(h(n))^{-\beta/4}\right)\\
&\quad + X_{ni}(j) I\left(|X_{ni}(j)| \leq n^{-\delta}(h(n))^{-\beta/4}\right)\\
&\quad + n^{-\delta}(h(n))^{-\beta/4} I\left(X_{ni}(j) > n^{-\delta}(h(n))^{-\beta/4}\right),\\
Y_{ni}^{(2)}(j) &= \left(X_{ni}(j) + n^{-\delta}(h(n))^{-\beta/4}\right) I\left(X_{ni}(j) \leq -\frac{\varepsilon}{N}(h(n))^{-\beta/4}\right)\\
&\quad + \left(X_{ni}(j) - n^{-\delta}(h(n))^{-\beta/4}\right) I\left(X_{ni}(j) \geq \frac{\varepsilon}{N}(h(n))^{-\beta/4}\right),\\
Y_{ni}^{(3)}(j) &= \left(X_{ni}(j) - n^{-\delta}(h(n))^{-\beta/4}\right) I\left(n^{-\delta}(h(n))^{-\beta/4} \leq X_{ni}(j) < \frac{\varepsilon}{N}(h(n))^{-\beta/4}\right),\\
Y_{ni}^{(4)}(j) &= \left(X_{ni}(j) + n^{-\delta}(h(n))^{-\beta/4}\right) I\left(-\frac{\varepsilon}{N}(h(n))^{-\beta/4}\right) < X_{ni}(j)\\
&\leq -n^{-\delta}\left(h(n))^{-\beta/4}\right).
\end{aligned}$$

Then

$$\begin{aligned}
\max_{1\leq j\leq n}\left|\sum_{i=1}^{n} a_{ni}(z_j)X_i\right| &\leq \max_{1\leq j\leq n}\left|\sum_{i=1}^{n} Y_{ni}^{(1)}(j)\right| + \max_{1\leq j\leq n}\left|\sum_{i=1}^{n} Y_{ni}^{(2)}(j)\right|\\
&\quad + \max_{1\leq j\leq n}\left|\sum_{i=1}^{n} Y_{ni}^{(3)}(j)\right| + \max_{1\leq j\leq n}\left|\sum_{i=1}^{n} Y_{ni}^{(4)}(j)\right|\\
&\doteq J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{3.29}$$

To prove (3.28), it suffices to show $J_i \to 0$ $a.s.$, $n \to \infty$, $i = 1, 2, 3, 4$. We first prove $J_1 \to 0$ $a.s.$, $n \to \infty$. For each j, we know that $\{Y_{ni}^{(1)}(j), 1 \le i \le n, n \ge 1\}$ is still an array of rowwise WOD random variables. In view of $EX_i = 0$, (3.26), (3.27) and $EX^2 < \infty$, we get

$$
\begin{aligned}
&\max_{1\le j\le n}\left|\sum_{i=1}^{n} EY_{ni}^{(1)}(j)\right| \\
&\le \max_{1\le j\le n}\sum_{i=1}^{n}\Big[E|X_{ni}(j)|I\left(|X_{ni}(j)|>n^{-\delta}(h(n))^{-\beta/4}\right) \\
&\quad + n^{-\delta}(h(n))^{-\beta/4}P\left(|X_{ni}(j)|>n^{-\delta}(h(n))^{-\beta/4}\right)\Big] \\
&\le 2\max_{1\le j\le n}\sum_{i=1}^{n}E|X_{ni}(j)|I\left(|X_{ni}(j)|>n^{-\delta}(h(n))^{-\beta/4}\right) \\
&\le C\max_{1\le j\le n} n^{\delta}(h(n))^{\beta/4}\sum_{i=1}^{n}E|X_{ni}(j)|^2 I\left(|X_{ni}(j)|>n^{-\delta}(h(n))^{\beta}\right) \\
&\le Cn^{\delta}(h(n))^{\beta/4}\cdot\max_{1\le j\le n}\sum_{i=1}^{n}a_{ni}(z_j)\cdot\max_{1\le i,j\le n}a_{ni}(z_j)\cdot EX^2 \\
&\le Cn^{\delta-\alpha}(h(n))^{-3\beta/4}EX^2 \to 0,\ n\to\infty.
\end{aligned}
$$

Hence, for all n large enough, $\max_{1\le j\le n}\left|\sum_{i=1}^{n} EY_{ni}^{(1)}(j)\right| < \frac{\varepsilon}{2}$. Applying Markov's inequality and Rosenthal-type inequality, and taking

$$
q > \max\left\{\frac{2(\delta+1)-\alpha}{\delta}, \frac{4}{\alpha}, \frac{2}{\beta}, 2\right\},
$$

we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty}P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{n}Y_{ni}^{(1)}(j)\right|>\varepsilon\right) \\
&\le C\sum_{n=1}^{\infty}P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{n}\left(Y_{ni}^{(1)}(j)-EY_{ni}^{(1)}(j)\right)\right|>\frac{\varepsilon}{2}\right) \\
&\le C\sum_{n=1}^{\infty}\sum_{j=1}^{n}P\left(\left|\sum_{i=1}^{n}\left(Y_{ni}^{(1)}(j)-EY_{ni}^{(1)}(j)\right)\right|>\frac{\varepsilon}{2}\right) \\
&\le C\sum_{n=1}^{\infty}\sum_{j=1}^{n}\left[\sum_{i=1}^{n}E\left|Y_{ni}^{(1)}(j)\right|^{q}+h(n)\sum_{i=1}^{n}\left(E\left|Y_{ni}^{(1)}(j)\right|^{2}\right)^{q/2}\right] \\
&\doteq J_{11}+J_{12}.
\end{aligned}
\tag{3.30}
$$

Note that

$$J_{11} \le \sum_{n=1}^{\infty}\sum_{j=1}^{n}\sum_{i=1}^{n}\Big[n^{-\delta q}(h(n))^{-\frac{\beta q}{4}} P\left(|X_{ni}(j)| > n^{-\delta}(h(n))^{-\frac{\beta}{4}}\right)$$
$$+ \; E|X_{ni}(j)|^q I\left(|X_{ni}(j)| \le n^{-\delta}(h(n))^{-\frac{\beta}{4}}\right)\Big]$$
$$\le C\sum_{n=1}^{\infty}\sum_{j=1}^{n}\sum_{i=1}^{n} n^{-\delta(q-2)}(h(n))^{-\beta(q-2)/4}E|X_{ni}(j)|^2$$
$$\le CEX^2\sum_{n=1}^{\infty} n^{1-\alpha-\delta(q-2)}(h(n))^{-\beta(q+2)/4} < \infty, \tag{3.31}$$

and

$$J_{12} \le C\sum_{n=1}^{\infty}\sum_{j=1}^{n} h(n)\left(\sum_{i=1}^{n} E|X_{ni}(j)|^2\right)^{q/2}$$
$$\le C(EX^2)^{q/2}\sum_{n=1}^{\infty} n^{1-\alpha q/2}(h(n))^{1-\beta q/2} < \infty. \tag{3.32}$$

We can see that $J_1 \to 0\, a.s., n \to \infty$ by (3.30)–(3.32) and the Borel–Cantelli Lemma.

Next we turn to estimate J_2. It follows from (3.27) that

$$\max_{1\le j\le n}\left|\sum_{i=1}^{n} Y_{ni}^{(2)}(j)\right| \le C\max_{1\le j\le n}\sum_{i=1}^{n}|X_{ni}(j)|\, I\left(|X_{ni}(j)| \ge \frac{\varepsilon}{N}(h(n))^{-\beta/4}\right)$$
$$\le Cn^{-\alpha}(h(n))^{-\beta}\sum_{i=1}^{n}|X_i| I\left(|X_i| \ge Cn^{\alpha}(h(n))^{\beta}(h(n))^{-\beta/4}\right)$$
$$\le Cn^{-\alpha}(h(n))^{-\beta}\sum_{i=1}^{n}|X_i| I(|X_i| \ge Ci^{\alpha}). \tag{3.33}$$

Hence, to prove $J_2 \to 0\; a.s.,\; n \to \infty$, we only need to show

$$\sum_{i=1}^{\infty} i^{-\alpha}(h(i))^{-\beta}|X_i| I(|X_i| \ge Ci^{\alpha}) < \infty\; a.s.. \tag{3.34}$$

It can be checked by $\sum_{i=1}^{n} i^{-\alpha}(h(i))^{-\beta} = O(n^{\alpha})$ and $EX^2 < \infty$ that

$$\sum_{i=1}^{\infty} i^{-\alpha}(h(i))^{-\beta}E|X_i| I(|X_i| \ge Ci^{\alpha})$$
$$\le C\sum_{i=1}^{\infty} i^{-\alpha}(h(i))^{-\beta}\sum_{n=i}^{\infty} E|X| I(Cn^{\alpha} \le |X| < C(n+1)^{\alpha})$$

$$\leq C\sum_{n=1}^{\infty} n^{\alpha} E|X|I(Cn^{\alpha} \leq |X| < C(n+1)^{\alpha}),$$
$$\leq CEX^2 < \infty, \quad (3.35)$$

which implies that (3.34) holds. Consequently, according to (3.33), (3.34) and Kronecker's lemma, $J_2 \to 0$ $a.s.$, $n \to \infty$.

From the definition of $Y_{ni}^{(3)}(j)$, we know that

$$0 \leq Y_{ni}^{(3)}(j) < \frac{\varepsilon}{N}(h(n))^{-\beta/4} - n^{-\delta}(h(n))^{-\beta/4} < \frac{\varepsilon}{N}.$$

Therefore, by taking $N > \max\left\{\frac{2}{\alpha-2\delta}, \frac{2}{\beta}\right\}$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{n} Y_{ni}^{(3)}(j)\right| > \varepsilon\right) \\
&\leq \sum_{n=1}^{\infty}\sum_{j=1}^{n} P\left(\text{there are at least N's nonzero } Y_{ni}^{(3)}(j)\right) \\
&\leq \sum_{n=1}^{\infty}\sum_{j=1}^{n}\sum_{1\leq k_1<\cdots<k_N\leq n} \\
&\quad P\left(X_{n,k_1}(j) \geq n^{-\delta}(h(n))^{-\beta/4}, \ldots, X_{n,k_N}(j) \geq n^{-\delta}(h(n))^{-\beta/4}\right) \\
&\leq \sum_{n=1}^{\infty}\sum_{j=1}^{n}\sum_{1\leq k_1<\cdots<k_N\leq n} h(n)\prod_{i=1}^{N} P\left(X_{n,k_i}(j) \geq n^{-\delta}(h(n))^{-\beta/4}\right) \\
&\leq \sum_{n=1}^{\infty}\sum_{j=1}^{n} h(n)\left(\sum_{i=1}^{n} P\left(|X_{ni}(j)| \geq n^{-\delta}(h(n))^{-\beta/4}\right)\right)^N \\
&\leq \sum_{n=1}^{\infty}\sum_{j=1}^{n} h(n)\left(\sum_{i=1}^{n} n^{2\delta}(h(n))^{\beta/2} E|X_{ni}(j)|^2\right)^N \\
&\leq C(EX^2)^N \sum_{n=1}^{\infty} n^{1-(\alpha-2\delta)N}(h(n))^{1-\beta N/2} < \infty.
\end{aligned}$$

Hence, from the Borel–Cantelli lemma, we can obtain $J_3 \to 0$ $a.s.$ $n \to \infty$. Note that

$$-\frac{\varepsilon}{N} < -\frac{\varepsilon}{N}(h(n))^{-\beta/4} + n^{-\delta/4}(h(n))^{-\beta} < Y_{ni}^{(4)}(j) \leq 0.$$

Similar to the proof of J_3, we have $J_4 \to 0$ $a.s.$ $n \to \infty$. This completes the proof of lemma. □

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