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On the residual lifetime of coherent systems with heterogeneous components

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Abstract The residual lifetime is of significant interest in reliability and survival analysis. In this article, we obtain a mixture representation for the reliability function of the residual lifetime of a coherent system with heterogeneous components in terms of the reliability functions of residual lifetimes of order statistics. Some stochastic comparisons are made on the residual lifetimes of the systems. Some examples are also given to illustrate the main results.

Keywords Coherent system · Heterogeneous variables · Order statistics · Signature · Stochastic ordering

1 Introduction

Consider a coherent system whose n components have independent and identically distributed (i.i.d.) lifetimes $X_1, \dots, X_n$ with the common cumulative distribution function (cdf) F. The distribution of F is assumed to be continuous with support set $(0, \infty)$. Let T denote the lifetime of the system. Samaniego (1985) introduced an n-dimensional probability vector $\mathbf{s} = (s_1, \dots, s_n)$ as a system signature, where

$$s_i = \Pr(T = X_{i:n}), \ i = 1, \dots, n, \tag{1}$$

is equal to the probability that the ith component failure causes the system to fail. Samaniego (1985) also showed that under the i.i.d. assumption on the component lifetimes, the reliability function of the system may be represented as

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$$\bar{F}_T(t) = \sum_{i=1}^{n} s_i \bar{F}_{i:n}(t), \tag{2}$$

where $\bar{F}_{i:n}(t) = \Pr(X_{i:n} > t)$ and $X_{i:n}$ is the ith order statistic among $X_1, \ldots, X_n$. Setting $p = \bar{F}(t)$, the reliability function of the system given in (2) may be rewritten through the reliability polynomial $h(p)$ as:

$$h(p) = \sum_{i=1}^{n} s_i \sum_{j=n-i+1}^{n} \binom{n}{j} p^j (1-p)^{n-j}. \tag{3}$$

The reliability polynomial $h(p)$ is strictly increasing for $p \in (0, 1)$, with $h(0) = 0$ and $h(1) = 1$, see Barlow and Proschan (1975).

The mixture representation as given in (2) also holds for coherent systems in the exchangeable case; see Navarro and Rychlik (2007) and Navarro et al. (2008). Navarro et al. (2007) proved that the reliability function $\bar{F}_T$ of coherent systems with exchangeable components can also be written as

$$\bar{F}_T(t) = \sum_{i=1}^{n} a_i \bar{F}_{1:i}(t) = \sum_{i=1}^{n} b_i \bar{F}_{i:i}(t), \tag{4}$$

where $\bar{F}_{1:i}(t) = \Pr(X_{1:i} > t)$ and $\bar{F}_{i:i}(t) = \Pr(X_{i:i} > t)$ for $i = 1, 2, \ldots, n$. The vectors of coefficients $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ only depend on the structure function of the system and were called minimal signature and maximal signature, respectively. Navarro and Rubio (2010) showed that there exist two triangular (non-singular) matrices, A_n and B_n, such that $\mathbf{s} = \mathbf{a}A_n = \mathbf{b}B_n$. Therefore, if we know $\mathbf{s}$, $\mathbf{a}$, or $\mathbf{b}$, then we can compute the other two vectors.

If a coherent system with lifetime T has independent components then its reliability function can be written as

$$\bar{F}_T(t) = H(\bar{F}_1(t), \ldots, \bar{F}_n(t)), \tag{5}$$

where $\bar{F}_i(t)$, $i = 1, \ldots, n$, is the reliability function of ith component, and H is called the *structure reliability function* in Esary and Proschan (1963). Setting $p_i(t) = \bar{F}_i(t)$ for $i = 1, \ldots, n$, the reliability function of the system may be written as $H(p_1, \ldots, p_n)$. This polynomial is strictly increasing in $(0, 1)^n$ and is such that $H(0, \ldots, 0) = 0$ and $H(1, \ldots, 1) = 1$. Clearly the reliability polynomial in (3) can be expressed as $h(p) = H(p, \ldots, p)$. Using representation (5), Navarro et al. (2011) have established a mixture representations similar to (2) and (4) for the reliability function of a coherent system with heterogeneous components.

Mixture representations of the system reliability function are very useful tools in the comparison of the performance of competing systems. Some examples include Kochar et al. (1999), Li and Zhang (2008), Zhang (2010) and Navarro et al. (2008). Many authors have presented a mixture representation of the reliability function of the residual lifetime and inactivity time of a coherent system. For example, Navarro et al.

(2008) obtained several representations of the reliability functions of residual lifetimes of used coherent systems in terms of the reliability functions of residual lifetimes of order statistics. Zhang (2010) and Goliforushani and Asadi (2011) presented a mixture representation of the inactivity time of a system with i.i.d. components and obtained some ordering results among the inactivity times of the systems. Feng et al. (2013) extended the results of Navarro et al. (2008), and found a new mixture representation of the conditional residual lifetime of a coherent system in terms of conditional residual lifetimes of order statistics. Zhang and Balakrishnan (2015) presented several useful mixture representations for the reliability function of the inactivity time of systems with heterogeneous components based on order statistics, signatures and mean reliability functions. In this paper, We extend the results of Zhang and Balakrishnan (2015) for the residual lifetime of a system with heterogeneous components. We also obtained some stochastic ordering properties for the conditional residual lifetime of a coherent system with heterogeneous components based on stochastically ordered coefficient vectors.
The rest of paper is organized as follows. Section 2 contains some pertinent basic definitions and several lemmas which are useful in our derivations. In Sect. 3 we find a mixture representation for the residual lifetime of a system with heterogeneous component lifetimes. Some aging properties and stochastic comparisons of the residual lifetimes of two systems are finally discussed in Sect. 4.

2 Preliminaries and lemmas

First, we recall the basic definitions of some stochastic orders and aging properties that will be used in our subsequent discussions.

Definition 1 Let X and Y be the lifetimes of units, with distribution functions $F(x)$ and $G(x)$, reliability functions $\bar{F}(x)$ and $\bar{G}(x)$, and probability density functions $f(x)$ and $g(x)$, respectively. Then:

1. X is said to be less than Y in stochastic order (denoted by $X \leq_{st} Y$) when $\bar{F}(x) \leq \bar{G}(x)$ for all x;
2. X is said to be less than Y in hazard rate order (denoted by $X \leq_{hr} Y$) when $\bar{F}(x)/\bar{G}(x)$ is decreasing in x;
3. X is said to be less than Y in reversed hazard rate order (denoted by $X \leq_{rh} Y$) when $F(x)/G(x)$ is decreasing in x;
4. X is said to be less than Y in likelihood ratio order (denoted by $X \leq_{lr} Y$) when $f(x)/g(x)$ is decreasing in supports of $f(x)$ and $g(x)$.

and for two discrete probability distributions $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ we have

1. $\mathbf{p}$ is said to be less than $\mathbf{q}$ in stochastic order (written $\mathbf{p} \leq_{st} \mathbf{q}$) when $\sum_{j=i}^{n} p_j \leq \sum_{j=i}^{n} q_j$ for all $i = 1, \ldots, n$;
2. $\mathbf{p}$ is said to be less than $\mathbf{q}$ in likelihood ratio order (written $\mathbf{p} \leq_{lr} \mathbf{q}$) when p_i/q_i decrease in i.

We refer the readers to Shaked and Shanthikumar (2007) for more details on stochastic orderings and their applications.

Some basic aging properties of a system is recalled in the next definition, more details can be found in Barlow and Proschan (1975).

Definition 2 Let X be a non-negative random variable with cdf F, where $F(0) = 0$. Then, X is said to be

1. Increasing (decreasing) failure rate, written as IFR (DFR), if for fixed $y > 0$, $\bar{F}(x+y)/\bar{F}(x)$ is decreasing (increasing) in $x \geq 0$;
2. Increasing (deceasing) failure rate average, written as IFRA (DFRA), if $-\frac{1}{t}\ln \bar{F}(t)$ is increasing (decreasing) for all $t \geq 0$;
3. New better (worse) than used, written as NBU (NWU) if $\bar{F}(x+y) \leq (\geq) \bar{F}(x)\bar{F}(y)$ for all $x, y \geq 0$.

The following lemmas will be useful to the proofs of the main results in this paper.

Lemma 1 (Navarro et al. 2011) *Let T be the lifetime of a coherent system with independent component lifetimes $X_1, \ldots, X_n$ distributed according to reliability functions $\bar{F}_1, \ldots, \bar{F}_n$. Assume that h and H are the system's reliability polynomial and reliability structure function, respectively, and let $\bar{G}(t) = h^{-1}(H(\bar{F}_1(t), \ldots, \bar{F}_n(t)))$ be the reliability function of i.i.d. variables $Y_1, \ldots, Y_n$. Then*

$$\Pr(T > t) = \sum_{i=1}^{n} s_i \Pr(Y_{i:n} > t) = \sum_{i=1}^{n} s_i \bar{G}_{i:n}(t), \tag{6}$$

$$\Pr(T > t) = \sum_{i=1}^{n} a_i \Pr(Y_{1:i} > t) = \sum_{i=1}^{n} a_i \bar{G}_{1:i}(t),$$

$$\Pr(T > t) = \sum_{i=1}^{n} b_i \Pr(Y_{i:i} > t) = \sum_{i=1}^{n} b_i \bar{G}_{i:i}(t),$$

where $Y_{1:n}, \ldots, Y_{n:n}$ are the order statistics of i.i.d. component lifetimes $Y_1, \ldots, Y_n$, and $\mathbf{s} = (s_1, \ldots, s_n)$, $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ are the corresponding signature, minimal signature and maximal signature vectors.

The results of Lemma 1 are useful tool in comparing of two coherent systems with heterogeneous components.

Lemma 2 *Let $X_1, \ldots, X_n$ be n independent random variables with common distribution function F, and let $Y_1, \ldots, Y_n$ be other n independent random variables with common distribution function G. If $X \leq_{st} Y$, then $\frac{\bar{G}_{i:n}(t)}{\bar{F}_{i:n}(t)}$ is decreasing in i.*

Proof It is enough to show that for $t \geq 0$,

$$\frac{\bar{G}_{k:n}(t)}{\bar{F}_{k:n}(t)} \geq \frac{\bar{G}_{k+1:n}(t)}{\bar{F}_{k+1:n}(t)}.$$

The above equation is equivalent to

$$\frac{\sum_{i=n-k}^{n} \binom{n}{i} \bar{F}^i(t) F^{n-i}(t)}{\sum_{m=n-k+1}^{n} \binom{n}{m} \bar{F}^m(t) F^{n-m}(t)} \geq \frac{\sum_{i=n-k}^{n} \binom{n}{i} \bar{G}^i(t) G^{n-i}(t)}{\sum_{m=n-k+1}^{n} \binom{n}{m} \bar{G}^m(t) G^{n-m}(t)}.$$

If we define $\phi_1(t) = \bar{F}(t)/F(t)$ and $\phi_2(t) = \bar{G}(t)/G(t)$, then it is enough to show that

$$\frac{\sum_{i=n-k}^{n} \binom{n}{i} \phi_1^i(t)}{\sum_{m=n-k+1}^{n} \binom{n}{m} \phi_1^m(t)} \geq \frac{\sum_{i=n-k}^{n} \binom{n}{i} \phi_2^i(t)}{\sum_{m=n-k+1}^{n} \binom{n}{m} \phi_2^m(t)},$$

or equivalently

$$\sum_{i=n-k}^{n-k} \sum_{m=n-k+1}^{n} \binom{n}{i}\binom{n}{m} \phi_1^i(t)\phi_2^i(t)(\phi_2^{m-i}(t) - \phi_1^{m-i}(t)) \geq 0.$$

The condition $X \leq_{st} Y$ implies that $\phi_2^{m-i}(t) - \phi_1^{m-i}(t) \geq 0$, for $t \geq 0$ and $i < m$, and the proof is immediate. □

3 Mixture representation of residual lifetime

Residual lifetime is an important characteristic in reliability. The corresponding residual lifetime of a system with lifetime T is defined as the remaining life of a system that has survived until a certain time t, i.e., $\{T - t | T > t\}$. Many authors have studied various types of residual lifetimes of coherent systems; see, for example Li and Lu (2003), Asadi and Bairamov (2006), Khaledi and Shaked (2007), Navarro et al. (2011), Zhang and Li (2010) and Eryilmaz (2013). A mixture representation for the residual lifetime of a coherent system with heterogeneous independent components is presented in the next theorem.

Theorem 1 *Let T be the lifetime of a coherent system with heterogeneous independent component lifetimes $X_1, \ldots, X_n$, which X_i having the continuous distribution function F_i for $i = 1, \ldots, n$. Let $\mathbf{s} = (s_1, \ldots, s_n)$ be the signature vector of T and $\bar{G}(t) = h^{-1}(H(\bar{F}_1(t), \ldots, \bar{F}_n(t)))$ be the reliability function of i.i.d. component lifetimes $Y_1, \ldots, Y_n$, then for all $x \geq 0$ and $t \geq 0$*

$$\Pr(T - t > x | T > t) = \sum_{i=1}^{n} p_i(t) \Pr(Y_{i:n} > t + x | Y_{i:n} > t), \tag{7}$$

where coefficient $p_i(t) = s_i \Pr(Y_{i:n} > t)/\bar{F}_T(t)$ for $i = 1, \ldots, n$, such that $\sum_{i=1}^{n} p_i(t) = 1$, and $Y_{i:n}$ is the ith order statistic among $Y_1, \ldots, Y_n$.

Proof By using Lemma 1 we have

$$\begin{aligned} \Pr(T - t > x | T > t) &= \frac{\bar{F}_T(t + x)}{\bar{F}_T(t)} \\ &= \frac{\sum_{i=1}^{n} s_i \Pr(Y_{i:n} > t + x)}{\bar{F}_T(t)}. \end{aligned} \tag{8}$$

Since

$$\Pr(Y_{i:n} > t + x) = \Pr(Y_{i:n} > t + x | Y_{i:n} > t) \Pr(Y_{i:n} > t),$$

Equation (8) can be rewritten as

$$
\begin{aligned}
\Pr(T-t>x|T>t) &= \frac{\sum_{i=1}^{n} s_i \Pr(Y_{i:n}>t+x|Y_{i:n}>t)\Pr(Y_{i:n}>t)}{\bar{F}_T(t)} \\
&= \sum_{i=1}^{n} p_i(t)\Pr(Y_{i:n}>t+x|Y_{i:n}>t).
\end{aligned}
$$

This completes the proof. □

In the next example, we find the dynamic coefficient vectors of two coherent systems with 4 components by using the result of Theorem 1.

Example 1 Consider two coherent systems with lifetimes $T_1=\max(\min(X_{1,1}, X_{1,2}), \min(X_{1,3}, X_{1,4}))$ and $T_2=\max(X_{2,1}, X_{2,2}, X_{2,3}, X_{2,4})$. Let $X_{i,1}, X_{i,2}, X_{i,3}, X_{i,4}$ be distributed according to reliability functions $\bar{F}_{i,1}, \bar{F}_{i,2}, \bar{F}_{i,3}, \bar{F}_{i,4}$, for $i=1,2$, respectively. Then the reliability function of the first system is given by

$$
\begin{aligned}
\bar{F}_{T_1}(t) &= \Pr(\max(\min(X_{1,1}, X_{1,2}), \min(X_{1,3}, X_{1,4}))>t) \\
&= \Pr(\min(X_{1,1}, X_{1,2})>t)+\Pr(\min(X_{1,3}, X_{1,4})>t)-\Pr(X_{1,1:4}>t) \\
&= \bar{F}_{1,1}(t)\bar{F}_{1,2}(t)+\bar{F}_{1,3}(t)\bar{F}_{1,4}(t)-\bar{F}_{1,1}(t)\bar{F}_{1,2}(t)\bar{F}_{1,3}(t)\bar{F}_{1,4}(t) \\
&= H(\bar{F}_{1,1}(t), \bar{F}_{1,2}(t), \bar{F}_{1,3}(t), \bar{F}_{1,4}(t)).
\end{aligned}
$$

The system's reliability polynomial is $h_1(p)=2p^2-p^4$, and then $h_1^{-1}(x)=\sqrt{1-\sqrt{1-x}}$. Similarly the reliability function of the second system is given by

$$
\begin{aligned}
\bar{F}_{T_2}(t) &= \bar{F}_{2,1}(t)+\bar{F}_{2,2}(t)+\bar{F}_{2,3}(t)+\bar{F}_{2,4}(t)-\bar{F}_{2,1}(t)\bar{F}_{2,2}(t)-\bar{F}_{2,1}(t)\bar{F}_{2,3}(t) \\
&\quad -\bar{F}_{2,1}(t)\bar{F}_{2,4}(t)-\bar{F}_{2,2}(t)\bar{F}_{2,3}(t)-\bar{F}_{2,2}(t)\bar{F}_{2,4}(t)-\bar{F}_{2,3}(t)\bar{F}_{2,4}(t) \\
&\quad +\bar{F}_{2,1}(t)\bar{F}_{2,2}(t)\bar{F}_{2,3}(t)+\bar{F}_{2,1}(t)\bar{F}_{2,2}(t)\bar{F}_{2,4}(t)+\bar{F}_{2,2}(t)\bar{F}_{2,3}(t)\bar{F}_{2,4}(t) \\
&\quad +\bar{F}_{2,1}(t)\bar{F}_{2,3}(t)\bar{F}_{2,4}(t)-\bar{F}_{2,1}(t)\bar{F}_{2,2}(t)\bar{F}_{2,3}(t)\bar{F}_{2,4}(t) \\
&= H(\bar{F}_{2,1}(t), \bar{F}_{2,2}(t), \bar{F}_{2,3}(t), \bar{F}_{2,4}(t)).
\end{aligned}
$$

The system's reliability polynomial is $h_2(p)=4p-6p^2+4p^3-p^4$, and then $h_2^{-1}(x)=1-(1-x)^{1/4}$. Let $X_{i,1}, X_{i,2}, X_{i,3}, X_{i,4}$ follow from the exponential distributions with parameters 1, 2, 3 and 4, respectively. Then

$$
\begin{aligned}
\bar{G}_1(t) &= \sqrt{1-\sqrt{1-e^{-3t}-e^{-7t}+e^{-10t}}}, \\
\bar{G}_2(t) &= 1-\sqrt[4]{1-e^{-t}-e^{-2t}+2e^{-5t}-e^{-9t}-e^{-8t}+e^{-10t}}.
\end{aligned}
$$

After some computations, it can be shown that the coefficient vectors of the systems T_1 and T_2 are

$$\mathbf{p}_1(t) = \left(0, \frac{8\bar{G}_1(t) - 6\bar{G}_1^2(t)}{6 - 3\bar{G}_1^2(t)}, \frac{6 + 3\bar{G}_1^2(t) - 8\bar{G}_1(t)}{6 - 3\bar{G}_1^2(t)}, 0\right),$$
$$\mathbf{p}_2(t) = (0, 0, 0, 1).$$

Then we have

$$\begin{aligned}\Pr(T_1 - t > x | T_1 > t) &= \sum_{i=1}^{n} p_{1,i}(t) \Pr(Y_{1,i:n} - t > x | Y_{1,i:n} > t)\\ &= \frac{1}{6-3\bar{G}_1^2(t)} \left[\frac{(8\bar{G}_1(t) - 6\bar{G}_1^2(t))(4\bar{G}_1^3(t+x) - 3\bar{G}_1^3(t+x))}{4\bar{G}_1^3(t) - 3\bar{G}_1^3(t)} \right.\\ &\quad \left. + \frac{(6+3\bar{G}_1^2(t) - 8\bar{G}_1(t))(6\bar{G}_1^2(t+x) + 3\bar{G}_1^4(t+x) - 8\bar{G}_1^3(t+x))}{(6\bar{G}_1^2(t) + 3\bar{G}_1^2(t) - 8\bar{G}_1^3(t))} \right],\end{aligned}$$

and

$$\begin{aligned}\Pr(T_2 - t > x | T_2 > t) &= \sum_{i=1}^{n} p_{2,i}(t) \Pr(Y_{2,i:n} - t > x | Y_{2,i:n} > t)\\ &= \frac{4\bar{G}_2(t+x) - 6\bar{G}_2^2(t+x) + 4\bar{G}_2^3(t+x) - \bar{G}_2^4(t+x)}{4\bar{G}_2(t) - 6\bar{G}_2^2(t) + 4\bar{G}_2^3(t) - \bar{G}_2^4(t)}.\end{aligned}$$

In the following, we give two other mixture representations for the residual lifetime based on conditional minimal and maximal signatures. The proof of this theorem is similar to that of Theorem 1 and hence is omitted.

Theorem 2 . *Let* $\mathbf{a} = (a_1, \ldots, a_n)$ *and* $\mathbf{b} = (b_1, \ldots, b_n)$ *be the minimal and maximal signature vectors of a coherent system with lifetime* T. *Then, under the conditions of Theorem 1, we have*

$$\Pr(T - t > x | T > t) = \sum_{i=1}^{n} a_i(t) \Pr(Y_{1:i} > t + x | Y_{1:i} > t),$$

and

$$\Pr(T - t > x | T > t) = \sum_{i=1}^{n} b_i(t) \Pr(Y_{i:i} > t + x | Y_{i:i} > t),$$

where $a_i(t) = a_i \Pr(Y_{1:i} > t)/\bar{F}_T(t)$ *and* $b_i(t) = b_i \Pr(Y_{i:i} > t)/\bar{F}_T(t)$ *for* $i = 1, \ldots, n$, *such that* $\sum_{i=1}^{n} a_i(t) = 1$ *and* $\sum_{i=1}^{n} b_i(t) = 1$.

In the next theorem we find a mixture representation similar to Eq. (7) for the residual lifetime of a coherent system with heterogeneous dependent components.

Theorem 3 *Consider a coherent system with heterogeneous dependent component lifetimes $X_1, \ldots, X_n$, which X_i having the continuous distribution function F_i and reliability function $\bar{F}_i$ for $i = 1, \ldots, n$ and h and W be the system's reliability polynomial and structure-dependence function, respectively. Also assume that $\tilde{G}(t) = h^{-1}(W(\bar{F}_1(t), \ldots, \bar{F}_n(t)))$ is the reliability function of i.i.d. component lifetimes $Y_1, \ldots, Y_n$. Let $\mathbf{s} = (s_1, \ldots, s_n)$ be the signature vector of T, then*

$$\Pr(T - t > x | T > t) = \sum_{i=1}^{n} \tilde{p}_i(t) \Pr(Y_{i:n} > t + x | Y_{i:n} > t),$$

where coefficient $\tilde{p}_i(t) = s_i \Pr(Y_{i:n} > t)/\bar{F}_T(t)$, for $i = 1, \ldots, n$, such that $\sum_{i=1}^{n} \tilde{p}_i(t) = 1$, and $Y_{i:n}$ is the ith order statistic among $Y_1, \ldots, Y_n$.

Proof The proof can be found by considering Theorem 3.5 of Navarro et al. (2011), and following the same steps as used in Theorem 1. □

4 Ordering results

In this section, we obtain some ordering results for the residual lifetimes of two coherent systems with two sets of different heterogeneous components under some specified conditions. We first establish a result that compares the vector of coefficient as given in Theorem 1, in two different times t_1 and t_2, for $0 \leq t_1 \leq t_2$.

Lemma 3 *Let $\mathbf{p}(t)$ be the vector of coefficient as given in Theorem 1. Then $\mathbf{p}(t_1) \leq_{lr} \mathbf{p}(t_2)$ for all $0 \leq t_1 \leq t_2$.*

Proof It is enough to prove that the ratio $\frac{p_i(t_1)}{p_i(t_2)}$ is a decreasing function in i. We have

$$\frac{p_i(t_1)}{p_i(t_2)} = \frac{s_i \Pr(Y_{i:n} > t_1)\bar{F}_T(t_2)}{s_i \Pr(Y_{i:n} > t_2)\bar{F}_T(t_1)}.$$

The result holds if and only if

$$\frac{\Pr(Y_{i:n} > t_1)}{\Pr(Y_{i:n} > t_2)} \geq \frac{\Pr(Y_{j:n} > t_1)}{\Pr(Y_{j:n} > t_2)}, \quad for\ 1 \leq i < j \leq n.$$

The proof is now completed by noting the fact that the order statistics are hr-ordered in i.i.d. case. □

In the next theorem we find a condition that concludes the stochastic order between the dynamic coefficient vectors as given in Theorem 1.

Lemma 4 *Let T_i be the lifetime of a coherent system with i.i.d. component lifetimes $X_{i,1}, \ldots, X_{i,n}$ with respective reliability function $\bar{F}_i$ for $i = 1, 2$. If both systems have the same signature $\mathbf{s} = (s_1, \ldots, s_n)$, then we have*

$$If \quad X_{1,1} \leq_{st} X_{2,1}, then \quad \mathbf{p}_1(t) \geq_{st} \mathbf{p}_2(t),$$

where $\mathbf{p}_i(t)$ *is the coefficient vector of the system with lifetime* T_i, $i = 1, 2$.

Proof It should be showed that for any $t \geq 0$ and for all $i = 1, 2, \ldots, n$,

$$\sum_{j=i}^{n} \mathbf{p}_{1,j}(t) \geq \sum_{j=i}^{n} \mathbf{p}_{2,j}(t),$$

which is equivalent to

$$\sum_{j=i}^{n}\sum_{k=1}^{n} s_j s_k \Pr(X_{1,j:n} > t) \Pr(X_{2,k:n} > t) - \sum_{j=i}^{n}\sum_{k=1}^{n} s_j s_k \Pr(X_{2,j:n} > t) \Pr(X_{1,k:n} > t) \geq 0.$$

Note that

$$\sum_{j=i}^{n}\sum_{k=i}^{n} s_j s_k [\bar{F}_{1,j:n}(t)\bar{F}_{2,k:n}(t) - \bar{F}_{2,j:n}(t)\bar{F}_{1,k:n}(t)] = 0,$$

where $\bar{F}_{i,j:n}(t)$ is the distribution function of $X_{i,j:n}$, for $i = 1, 2$. Therefore, it is enough to show that for $k \leq j$,

$$\bar{F}_{1,j:n}(t)\bar{F}_{2,k:n}(t) - \bar{F}_{2,j:n}(t)\bar{F}_{1,k:n}(t) \geq 0.$$

The proof would be completed by using the result of Lemma 2. □

Now, we apply the result of Lemma 4 to compare the dynamic coefficient vectors of two coherent systems with heterogenous components.

Theorem 4 *Let* T_i *be the lifetime of a coherent system with heterogeneous component lifetimes* $X_{i,1}, \ldots, X_{i,n}$ *distributed according to reliability functions* $\bar{F}_{i,1}, \ldots, \bar{F}_{i,n}$, *for* $i = 1, 2$. *Suppose the systems have the same reliability polynomial and reliability structure function* h *and* H, *respectively. Let* $\bar{G}_i(t) = h^{-1}(H(\bar{F}_{i,1}(t), \ldots, \bar{F}_{i,n}(t)))$ *be the reliability function of i.i.d. component lifetimes* $Y_{i,1}, Y_{i,2}, \ldots, Y_{i,n}$; *for* $i = 1, 2$. *Then, we have*

$$If\ X_{1,i} \leq_{st} X_{2,i};\ i = 1, \ldots, n,\ then\ \mathbf{p}_1(t) \geq_{st} \mathbf{p}_2(t).$$

Proof The proof can be found from Eq. (6) and Lemma 4. □

In the next theorem we compare two coherent systems with the same structure and different components.

Theorem 5 *Under the conditions of Theorem 4, we have the following results.*

(i) If $Y_{1,1} \leq_{hr} Y_{2,1}$ *and* $yh'(y)/h(y)$ *is decreasing in* $(0, 1)$*, then* $(T_1 - t|T_1 > t) \leq_{hr} (T_2 - t|T_2 > t)$*;*
(ii) If $Y_{1,1} \leq_{rh} Y_{2,1}$ *and* $(1 - y)h'(y)/(1 - h(y))$ *is increasing in* $(0, 1)$*, then* $(T_1 - t|T_1 > t) \leq_{rh} (T_2 - t|T_2 > t)$*;*
(iii) If $Y_{1,1} \leq_{lr} Y_{2,1}$ *and* $yh''(y)/h'(y)$ *is non-negative and decreasing in* $(0, 1)$*, then* $(T_1 - t|T_1 > t) \leq_{lr} (T_2 - t|T_2 > t)$*.*

Proof (i) From Lemma 1, we can find that $\Pr(T_i > t) = h(\bar{G}_i(t))$ for $i = 1, 2$. Analogously, the hazard rate of $T_i - t|T_i > t; i = 1, 2$ can be written as

$$\begin{aligned} r_{T_i - t|T_i > t}(x) &= \frac{g_i(t + x)h'(\bar{G}_i(t + x))}{h(\bar{G}_i(t + x))} \\ &= \frac{g_i(t + x)}{\bar{G}_i(t + x)}\alpha(\bar{G}_i(t + x)) \\ &= r_i(t + x)\alpha(\bar{G}_i(t + x)); \quad i = 1, 2 \end{aligned}$$

where $g_i(t + x)$ and $r_i(t + x)$ are the density and hazard rate functions of T_i for $i = 1, 2$, respectively, and $\alpha(u) = uh'(u)/h(u)$. Now, the proof is an immediate consequence of Theorem 2.6 (ii) in Navarro et al. (2013).

(ii) The reversed hazard rate function of $T_i - t|T_i > t; i = 1, 2$ can be written as

$$\begin{aligned} \bar{r}_{T_i - t|T_i > t}(x) &= \frac{g_i(t + x)h'(\bar{G}_i(t + x))}{1 - h(\bar{G}_i(t + x))} \\ &= \frac{g_i(t + x)}{1 - \bar{G}_i(t + x)}\beta(\bar{G}_i(t + x)) \\ &= \bar{r}_i(t + x)\beta(\bar{G}_i(t + x)); \quad i = 1, 2, \end{aligned}$$

where $\beta(u) = (1 - u)h'(u)/(1 - h(u))$ and $\bar{r}_i(t)$ is the reversed hazard rate function of T_i; $i = 1, 2$. The rest of the proof is similar to Theorem 2.6 (iii) of Navarro et al. (2013).

(iii) The proof is an immediate consequence of Theorem 2.6 (iv) in Navarro et al. (2013).

□

In the following we compare two coherent systems with different structures and different components.

Theorem 6 *Let* T_i *be the lifetime of a coherent system with a coefficient vector* $\mathbf{p}_i(t) = (p_{i,1}(t), \ldots, p_{i,n}(t))$ *and having heterogeneous independent component lifetimes* $X_{i,1}, \ldots, X_{i,n}$ *distributed according to reliability functions* $\bar{F}_{i,1}, \ldots, \bar{F}_{i,n}$*, for* $i = 1, 2$*, respectively. Let* h_i *and* H_i *be the system's reliability polynomial and reliability structure function, respectively, and let* $\bar{G}_i(t) = h_i^{-1}(H_i(\bar{F}_{i,1}(t), \ldots, \bar{F}_{i,n}(t)))$ *be the reliability function of i.i.d. component lifetimes* $Y_{i,1}, \ldots, Y_{i,n}$*, for* $i = 1, 2$*. Then, we have If* $\mathbf{p}_1(t) \leq_{st} \mathbf{p}_2(t)$ *and* $Y_{1,1} \leq_{hr} Y_{2,1}$*, then* $(T_1 - t|T_1 > t) \leq_{st} (T_2 - t|T_2 > t)$*.*

Proof

$$\begin{aligned}\Pr(T_1 - t > x | T_1 > t) &= \sum_{i=1}^{n} p_{1,i}(t) \Pr(Y_{1,i:n} - t > x | Y_{1,i:n} > t) \\ &\le \sum_{i=1}^{n} p_{2,i}(t) \Pr(Y_{1,i:n} - t > x | Y_{1,i:n} > t) \\ &\le \sum_{i=1}^{n} p_{2,i}(t) \Pr(Y_{2,i:n} - t > x | Y_{2,i:n} > t) \\ &= \Pr(T_2 - t > x | T_2 > t),\end{aligned}$$

The first inequality follows from the facts that $p_1(t) \le_{st} p_2(t)$ and $\Pr(Y_{1,i:n} - t | Y_{1,i:n} > t)$ is increasing in i. The second inequality holds from the fact that the condition $\bar{G}_1(t) \le_{hr} \bar{G}_2(t)$ implies that $(Y_{1,i:n} - t | Y_{1,i:n} > t) \le_{hr} (Y_{2,i:n} - t | Y_{2,i:n} > t)$, for $i = 1, \ldots, n$, see Theorem 1.B.34 of Shaked and Shanthikumar (2007), and so $(Y_{1,i:n} - t | Y_{1,i:n} > t) \le_{st} (Y_{2,i:n} - t | Y_{2,i:n} > t)$. □

We must point out that in the case of two coherent systems with the same components but different structures the conditions of stochastic comparisons are equivalent to that of Theorem 6. The next example is given to explain the result of Theorem 6.

Example 2 Consider two coherent systems with lifetimes $T_1 = \min(X_{1,1}, \max(X_{1,2}, X_{1,3}))$ and $T_2 = \max(X_{2,1}, \min(X_{2,2}, X_{2,3}))$. Let $X_{i,1}$,$X_{i,2}$ and $X_{i,3}$ be distributed according to reliability functions $\bar{F}_{i,1}$, $\bar{F}_{i,2}$ and $\bar{F}_{i,3}$, for $i = 1, 2$. Assume that $\bar{G}_i(t)$ is the corresponding mean reliability function of i.i.d. component lifetimes $Y_{i,1}$, $Y_{i,2}$, $Y_{i,3}$ for $i = 1, 2$. The corresponding signatures of the systems are $\mathbf{s}_1 = (1/3, 2/3, 0)$ and $\mathbf{s}_2 = (0, 2/3, 1/3)$, respectively. Then we have

$$\begin{aligned}\mathbf{p}_1(t) &= \left(\frac{\bar{G}_1(t)}{6 - 3\bar{G}_1(t)}, \frac{6 - 4\bar{G}_1(t)}{6 - 3\bar{G}_1(t)}, 0\right), \\ \mathbf{p}_2(t) &= \left(0, \frac{6\bar{G}_2(t) - 4\bar{G}_2^2(t)}{3(1 + \bar{G}_2(t) - \bar{G}_2^2(t))}, \frac{3 - 3\bar{G}_2(t) + \bar{G}_2^2(t)}{3(1 + \bar{G}_2(t) - \bar{G}_2^2(t))}\right).\end{aligned}$$

Let $Y_{1,1}$ and $Y_{2,1}$ be distributed as exponential with parameters 2 and 1, respectively. Then $Y_{1,1} \le_{hr} Y_{2,1}$, and $\mathbf{p}_1(t) \le_{st} \mathbf{p}_2(t)$ for any $t \ge 0$, and we have

$$\begin{aligned}\Pr(T_1 - t > x | T_1 > t) &= \sum_{i=1}^{n} p_{1,i}(t) \Pr(Y_{1,i:n} - t > x | Y_{1,i:n} > t) \\ &= \frac{1}{6 - 3e^{-2t}} \left[e^{-2t-6x} + \frac{(6 - 4e^{-2t})(3e^{-4t-4x} - 2e^{-6t-6x})}{3e^{-4t} - 2e^{-6t}} \right], \\ \Pr(T_2 - t > x | T_2 > t) &= \sum_{i=1}^{n} p_{2,i}(t) \Pr(Y_{2,i:n} - t > x | Y_{2,i:n} > t)\end{aligned}$$

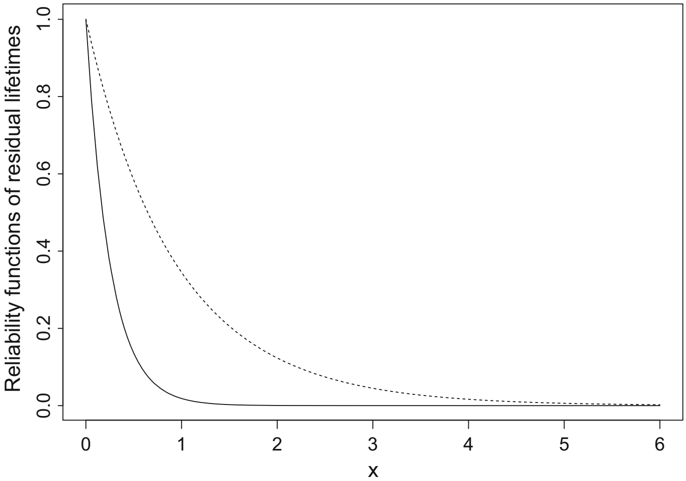

Fig. 1 Survival probability curves of $(T_1 - t|T_1 > t)$ (*thick line*) and $(T_2 - t|T_2 > t)$ (*dotted line*), presented in Example 2

$$= \frac{1}{3+3e^{-t}-3e^{-2t}}\left[\frac{(6e^{-t}-4e^{-2t})(3e^{-2(t+x)}-2e^{-3(t+x)})}{(3e^{-2t}-2e^{-3t})} + \frac{(3-3e^{-t}+e^{-2t})(3e^{-(t+x)}-3e^{-2(t+x)}+e^{-3(t+x)})}{(3e^{-t}-3e^{-2t}+e^{-3t})}\right].$$

By using the result of Theorem 6 we find that $(T_1 - t|T_1 > t) \leq_{st} (T_2 - t|T_2 > t)$. For further clarification, the reliability functions of the residual lifetimes of the systems are presented in Fig. 1, when t is equal to 2.

The representation in (7) is a useful tool to obtain the aging properties of the system lifetime when we have some information about the distribution of the order statistics $Y_{1:n}, \ldots, Y_{n:n}$. In the next theorem we prove that if $Y_{i:n}$ is NWU (DFR) for $i = 1, 2, \ldots, n$, then T is NWU (DFR) as well.

Theorem 7 *Under the assumptions of Theorem 1, we have*

(i) If $Y_{i:n}$ is NWU for $i = 1, 2, \ldots, n$, then T is NWU as well;
(ii) If $Y_{i:n}$ is DFR for $i = 1, 2, \ldots, n$, then T is DFR as well.

Proof (i) It is well known that the order statistics are likelihood ratio ordered in the i.i.d. case, i.e. $Y_{i:n} \leq_{lr} Y_{i+1:n}$ for $i = 1, 2, \ldots, n-1$; see (Shaked and Shanthikumar (2007), p. 54). It is also known that if two random variables are lr-ordered, then their residual lifetimes are lr-ordered as well, i.e., $(Y_{i:n} - t|Y_{i:n} > t) \leq_{lr} (Y_{i+1:n} - t|Y_{i+1:n} > t)$; see, for example, Theorem 1.C.6 of Shaked and Shanthikumar (2007). On the other hand, since $\mathbf{s} = \mathbf{p}(0)$, by using Lemma 3 we have $\mathbf{s} \leq_{st} \mathbf{p}(t)$. The proof can now be completed by using Theorem 1 and Theorem 1.A.6 of Shaked and Shanthikumar (2007).

(ii) Since $Y_{i:n}$ is DFR, we conclude that $(Y_{i:n}-t_1|Y_{i:n}>t_1) \leq_{st} (Y_{i:n}-t_2|Y_{i:n}>t_2)$, for $t_1 < t_2$. The proof then gets completed by Theorem 1.A.6 of Shaked and Shanthikumar (2007) and Theorem 1.

□

More aging properties similar to that given in Theorem 7 can be obtained from Lemma 1 and the results given in Navarro et al. (2014). Some of the immediate results are as follows.

(i) If $yh'(y)/h(y)$ is decreasing (increasing) in (0, 1) and Y_1 is IFR (DFR), then T is IFR (DFR).
(ii) If Y_1 is NBU(NWU) and $h(uv) \leq (\geq) h(u)h(v)$, for all $0 \leq u, v \leq 1$, then T is NBU(NWU).
(iii) If Y_1 is IFRA(DFRA) and $h(u^\alpha) \geq (\leq)(h(u))^\alpha$, for all $0 \leq u \leq 1$ and all $0 \leq \alpha \leq 1$, then T is IFRA(DFRA).

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