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# Bivariate distributions with conditionals satisfying the proportional generalized odds rate model

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**Abstract** New bivariate models are obtained with conditional distributions (in two different senses) satisfying the proportional generalized odds rate (PGOR) model. The PGOR semi-parametric model includes as particular cases the Cox proportional hazard rate (PHR) model and the proportional odds rate (POR) model. Thus the new bivariate models are very flexible and include, as particular cases, the bivariate extensions of PHR and POR models. Moreover, some well known parametric bivariate models are also included in these general models. The basic theoretical properties of the new models are obtained. An application to fit a real data set is also provided.

**Keywords** Conditionally specified distributions · Proportional hazard rate · Proportional odds rate · Bivariate Pareto distribution

## 1 Introduction

The extension of univariate models to the bivariate setup is a relevant topic in Probability Theory. Conditionally specified models are good options to get these extensions (see Arnold et al. 1993, 1999). The Cox proportional hazard rate (PHR) model is a very useful univariate semi-parametric model with applications in different areas such as Survival Analysis and Reliability Theory (see e.g. Guo and Zeng 2014; Meeker

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and Escobar 1998, and the references therein). Recently, Navarro and Sarabia (2013) extend the PHR model to the bivariate setup by using different conditional specification techniques. The basic properties of the new models are obtained. Similar studies of other bivariate models are given in Gupta (2001), Gupta and Gupta (2012), Gupta et al. (2013), Navarro et al. (2006).

Another popular univariate semi-parametric model is the proportional odds rate (POR) model (see e.g. Guo and Zeng 2014, and the references therein). The PHR and POR univariate models can be included in a more general model called proportional generalized odds rate (PGOR) model (see Bennett 1983; Dabrowska and Doksum 1988; Marshal and Olkin 2007; Zintzaras 2012, and the references therein).

In this paper, we extend the PGOR model to the bivariate set up by using different conditional specifications techniques. The basic properties of the new models are obtained. These extensions contain, as particular cases, the extensions of PHR and POR models. Moreover, we show that some well known parametric bivariate models are also included in the new bivariate models.

The rest of the paper is organized as follows. In Sect. 2 we introduce the univariate PGOR model and we obtain some preliminary properties. The PGOR model is extended to the bivariate setup in Sects. 3 and 4 by using different conditioning. The basic properties of the new models are included in these sections. An application to a real data set is presented in Sect. 5. Some conclusions are given in Sect. 6.

Throughout the paper we use the terms *increasing* and *decreasing* in a wide sense, that is, a function $g$ is increasing (resp. decreasing) if $g(x) \leq g(y)$ $(\geq)$ for all $x \leq y$.

## 2 The univariate PGOR model

Let $T$ be a nonnegative (lifetime) random variable with absolutely continuous survival (reliability) function $S(t) = \Pr(T > t)$. Then the *Generalized Odds (GO)* function for $T$ is defined by

$$\Lambda_T(t|c) = \begin{cases} \frac{1}{c}\frac{1-S^c(t)}{S^c(t)}, & \text{if} \quad c > 0 \\ -\ln S(t), & \text{if} \quad c = 0 \end{cases}$$

for $t \geq 0$ such that $S(t) > 0$. Note that $\Lambda_T(t|1) = (1 - S(t))/S(t)$ is the odds of occurring $T$ before time $t$, which is known in the literature as the survival odds of $T$ (i.e. the probability of failure at time $t$ divided by the probability of survival at time $t$). Moreover, $\lim_{c\to 0^+} \Lambda_T(t|c) = -\ln S(t)$ which is the cumulative hazard function of $T$. For other values of $c$, the quantity $c\Lambda_T(t|c)$ has an interpretation as a type of survival odds of the survival function $S^c(t)$. In particular (see Dabrowska and Doksum 1988), if $c$ is a positive integer, say $k$, and we consider a series system which consists in $k$ independent components whose lifetimes, $T_1, T_2, \ldots, T_k$, are distributed as $T$, then the lifetime of the series system $T_{1:k} = \min(T_1, T_2, \ldots, T_k)$ has survival function $\Pr(T_{1:k} > t) = S^k(t)$. Therefore, the survival odds of the system failure time $T_{1:k}$ occurring before time $t$ is

$$k\Lambda(t|k) = (1 - S^k(t))/S^k(t),$$

that is, $k\Lambda(t|k)$ is the odds rate for a series system with $k$ i.i.d. components with a common survival function $S(t)$. The function $\ln \Lambda_T(t|1)$ is called the log-odds function of $T$ (see Navarro et al. 2008; Sunoj et al. 2007).

The derivative of $\Lambda_T(t|c)$ with respect to $t$, denoted by $\lambda_T(t|c)$, is called the *generalized odds rate (GOR)* function and is given by

$$\lambda_T(t|c) = \frac{f(t)}{S^{c+1}(t)},$$

for $c \geq 0$, where $f(t) = -S'(t)$ is the probability density function (PDF) of $T$. In particular, $\lambda_T(t|0)$ is the hazard (or failure) rate function of $T$.

For $c > 0$, it is easy to prove that $\lambda_T(t|c) = \alpha > 0$ for $t \geq 0$ if, and only if, $T$ has a Pareto type-II distribution, denoted by $T \sim$ Pareto $(c\alpha, 1/c)$, with survival function

$$S(t) = (1 + c\alpha t)^{-1/c}, \quad t \geq 0, \ c, \alpha > 0. \tag{2.1}$$

For $c = 0$, it is well known that the hazard rate function satisfies $\lambda(t|0) = \alpha > 0$ for $t \geq 0$ if, and only, if $T$ has an exponential distribution with survival function $S(t) = \exp(-\alpha t)$ for $t \geq 0$. The exponential model can be obtained from the Pareto model (2.1) when $c \to 0^+$.

The proportional generalized odds rate model was proposed in the literature to analyse lifetime data (see among others, Bennett 1983; Dabrowska and Doksum 1988; Marshal and Olkin 2007). It is defined as follows: Two random variables $X$ and $Y$ satisfy the *proportional generalized odds rate* (PGOR) model if

$$\lambda_X(t|c) = \theta \lambda_Y(t|c),$$

for all $t \geq 0$, where $\lambda_X(t|c)$ and $\lambda_Y(t|c)$ are the respective GOR functions and where $\theta$ is a positive parameter. Obviously, this is equivalent to

$$\Lambda_X(t|c) = \theta \Lambda_Y(t|c),$$

for all $t \geq 0$, where $\Lambda_X(t|c)$ and $\Lambda_Y(t|c)$ are the respective GO functions. In the sequel, to simplify the notation, we drop the $c$ in $\Lambda_X$ and $\Lambda_Y$ and we take $\Lambda_X = \Lambda$. Then the univariate PGOR model is defined by

$$\Lambda(t;\theta) = \theta^{-1}\Lambda(t)$$

and its survival function is

$$S(t;\theta) = \left(1 + \frac{c}{\theta}\Lambda(t)\right)^{-1/c}, \tag{2.2}$$

for $t \geq 0$, where $c, \theta > 0$ and $\Lambda(t)$ is a given baseline GO function. We assume that $\Lambda(t)$ does not depend on parameters $\theta$ and $c$ but that it might depend on other parameters. Throughout the paper, we use the notation $T \sim \mathcal{PGOR}(\theta, c, \Lambda(t))$ for a random variable $T$ with survival function $S(t;\theta)$ satisfying (2.2). Notice that if $c \to 0^+$ and $\Lambda(t) = -\ln S(t)$, then (2.2) is equivalent to

$$S(t;\theta) = S^{1/\theta}(t),$$

that is, to the well known PHR model.

It is easy to prove that an arbitrary function $\Lambda(t)$ is a genuine GO function of an absolutely continuous survival function if, and only if, $\Lambda(t)$ is absolutely continuous, increasing, $\Lambda(0) = 0$ and $\Lambda(\infty) = \infty$. For $c > 0$, the PDF of $S(t;\theta)$ is given by

$$f(t;\theta) = \frac{\lambda(t)}{\theta}\left(1 + \frac{c}{\theta}\Lambda(t)\right)^{-1-1/c}, \tag{2.3}$$

where $\lambda(t) = \Lambda'(t)$ is the baseline generalized odds rate function.

It is easy to see that (2.2) defines a proper survival function for any baseline survival function $S$ (GO function $\Lambda$), any $c \geq 0$ and any $\theta > 0$ since it can be written as $S(t;\theta) = h(S(t))$, where $h(x) = (1 - 1/\theta + x^{-c}/\theta)^{-1/c}$ for $c > 0$ or $h(x) = x^{1/\theta}$ for $c = 0$. In both cases, $h$ is an strictly increasing continuous function in $[0, 1]$ such that $h(0) = 0$ and $h(1) = 1$ and so, $h(S(t))$ is a survival function. This is a particular case of the *distorted distributions* studied in Navarro et al. (2013, 2014, 2015) and in the references therein.

Some well known parametric models can be obtained from the PGOR model (2.2) as follows:

- $\Lambda(t) = t$ for $t \geq 0$, gives the Pareto distribution $P(c/\theta, 1/c)$ for $c > 0$ and the exponential distribution when $c \to 0^+$.
- $\Lambda(t) = t^\alpha$ for $t \geq 0$ and $\alpha > 0$, gives the log-logistic distribution for $c = 1$ and gives the Weibull distribution when $c \to 0^+$. If $c > 0$, the model can be called *generalized log-logistic distribution*. This model includes Pareto, exponential, log-logistic and Weibull models.

Therefore, the extensions to the bivariate case of the PGOR model obtained in the following sections will include as particular cases the extensions of these models.

## 3 Bivariate model with PGOR conditionals

In this section, we assume that $(X, Y)$ is a bivariate random vector with support $(0, \infty) \times (0, \infty)$. We are interested in specifying the bivariate distribution of $(X, Y)$, under the following conditions:

$$(X|Y = y) \sim \mathcal{PGOR}(\theta_1(y), c, \Lambda_1(x)) \tag{3.1}$$

and

$$(Y|X = x) \sim \mathcal{PGOR}(\theta_2(x), c, \Lambda_2(y)), \tag{3.2}$$

where $x, y, c \geq 0$, $\theta_1(y), \theta_2(x) \geq 0$ and $\Lambda_1(x)$ and $\Lambda_2(y)$ are two GO functions. The model is obtained in the following theorem.

**Theorem 3.1** *Let $\Lambda_1$ and $\Lambda_2$ be univariate generalized odds functions for $c > 0$ and let $(X, Y)$ be a random vector with an absolutely continuous distribution having support $(0, \infty) \times (0, \infty)$. If $(X, Y)$ satisfies (3.1) and (3.2), then its PDF is given by*

$$f(x, y) = \frac{K a_1 a_2 \lambda_1(x) \lambda_2(y)}{(a_0 + c a_1 \Lambda_1(x) + c a_2 \Lambda_2(y) + c\phi a_1 a_2 \Lambda_1(x) \Lambda_2(y))^{1+1/c}}, \tag{3.3}$$

*for $x, y \geq 0$, where $K, a_1, a_2 > 0$ and $a_0, \phi \geq 0$ are constants and $\lambda_i(t) = \Lambda_i'(t)$, $i = 1, 2$.*

The proof follows the lines of the proof of Theorem 2.1 in Navarro and Sarabia (2013) and can be obtained from the authors.

Clearly, if $a_0 > 0$, then, without loss of generality, we can take $a_0 = 1$. However, the value $a_0 = 0$ leads to a valid model whenever $c > 1$ (see case $(i)$ in Arnold et al. 1999, p.106). Additional conditions about the parameters are given in the next section. If $a_0 = 1$ and $c$ tends to zero in (3.3), then we obtain the model given in (2.3) of Navarro and Sarabia (2013). If $c > 0$ and $\Lambda_1(t) = \Lambda_2(t) = t$ for $t \geq 0$ in (3.3), then we obtain the bivariate Pareto model given in (5.7) (p. 105) of Arnold et al. (1999) (see also (2.2) in Gupta 2001).

### 3.1 Marginal and conditional distributions

From (3.3), the marginal PDF are given by

$$f_X(x) = K \frac{a_1 \lambda_1(x)}{1 + \phi a_1 \Lambda_1(x)} (a_0 + c a_1 \Lambda_1(x))^{-1/c}, \quad x > 0$$

and

$$f_Y(y) = K \frac{a_2 \lambda_2(y)}{1 + \phi a_2 \Lambda_2(y)} (a_0 + c a_2 \Lambda_2(y))^{-1/c}, \quad y > 0.$$

So, when $c \neq \phi$, the marginals of $X$ and $Y$ does not belong to the class of univariate PGOR models from $\Lambda_i$, $i = 1, 2$. The PDF of $Z = a_1 \Lambda_1(X)$ is

$$f_Z(z) = K \frac{1}{1 + \phi z} (a_0 + cz)^{-1/c}, \quad z > 0$$

and

$$K^{-1} = \int_0^\infty \frac{1}{1 + \phi x} (a_0 + cx)^{-1/c} dx.$$

Then the normalizing constant can be evaluated in the following way. If $a_0 > 0$, using formula 3.197-1 in Gradshteyn and Ryzhik (1994), we have

$$\int_0^\infty \frac{1}{(1+\phi x)(a_0+cx)^{1/c}}dx = a_0^{1-1/c} F\left(1, 1; 1+\frac{1}{c}; 1-\frac{a_0\phi}{c}\right),$$

where

$$F(a, b; c; z) = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt \tag{3.4}$$

is the hypergeometric function and $\beta(\cdot, \cdot)$ is the beta function. In consequence,

$$K^{-1} = a_0^{1-1/c} F\left(1, 1; 1+\frac{1}{c}; 1-\frac{a_0\phi}{c}\right).$$

If $a_0 = 0$, then

$$K^{-1} = \int_0^\infty \frac{c^{-1/c}}{x^{1/c}+\phi x^{1+1/c}}dx$$

and so we need $c > 1$ and $\phi > 0$. Analogously, if $\phi = 0$, then

$$K^{-1} = \int_0^\infty (a_0+cx)^{-1/c}dx$$

and so we need $c < 1$ and $a_0 > 0$. Then

$$K^{-1} = \frac{a_0^{1-1/c}}{1-c}.$$

The conditional PDF are given by

$$f_{X|Y=y}(x) = \frac{f(x, y)}{f_Y(y)} = \frac{a_1\lambda_1(x)(1+\phi a_2\Lambda_2(y))\,(a_0+ca_2\Lambda_2(y))^{1/c}}{(a_0+ca_1\Lambda_1(x)+ca_1\Lambda_2(y)+c\phi a_1a_2\Lambda_1(x)\Lambda_2(y))^{1+1/c}}$$

and

$$f_{Y|X=x}(y) = \frac{f(x, y)}{f_X(x)} = \frac{a_2\lambda_2(y)(1+\phi a_1\Lambda_1(x))\,(a_0+ca_1\Lambda_1(x))^{1/c}}{(a_0+ca_1\Lambda_1(x)+ca_2\Lambda_2(y)+c\phi a_1a_2\Lambda_1(x)\Lambda_2(y))^{1+1/c}},$$

for $x, y > 0$. Then the conditional survival functions are

$$\Pr(X > x|Y = y) = \left(1+c\frac{a_1+\phi a_1a_2\Lambda_2(y)}{a_0+ca_2\Lambda_2(y)}\Lambda_1(x)\right)^{-1/c}$$

and

$$\Pr(Y > y|X = x) = \left(1 + c\frac{a_2 + \phi a_1 a_2 \Lambda_1(x)}{a_0 + ca_1\Lambda_1(x)}\Lambda_2(y)\right)^{-1/c},$$

for $x, y > 0$. Of course, they are of the forms (3.1) and (3.2), respectively, with

$$\theta_1(y) = \frac{a_0 + ca_2\Lambda_2(y)}{a_1 + \phi a_1 a_2 \Lambda_2(y)}$$

and

$$\theta_2(x) = \frac{a_0 + ca_1\Lambda_1(x)}{a_2 + \phi a_1 a_2 \Lambda_1(x)}.$$

Note that $\theta_1(y)$ and $\theta_2(x)$ are strictly increasing (decreasing) functions when $c > \phi a_0$ ($<$). The special case $c = a_0\phi$ corresponds to the case of independence.

Then the conditional hazard rate functions of $(X|Y = y)$ and $(Y|X = x)$ are given by

$$h_{X|Y=y}(x) = \frac{\lambda_1(x)}{\theta_1(y) + c\Lambda_1(x)} \tag{3.5}$$

and

$$h_{Y|X=x}(y) = \frac{\lambda_2(y)}{\theta_2(x) + c\Lambda_2(y)},$$

respectively. Again, we see that if $c \to 0^+$, then the conditional distributions satisfy the PHR model (see Navarro and Sarabia 2013) with $h_{X|Y=y}(x) = \lambda_1(x)/\theta_1(y)$ and $h_{Y|X=x}(y) = \lambda_2(y)/\theta_2(x)$, where $\lambda_1(y)$ and $\lambda_2(x)$ are two given baseline hazard rate functions. Analogously, for the bivariate Pareto model, if $c > 0$ and $\Lambda_1(t) = \Lambda_2(t) = t$ for $t \geq 0$, then we obtain the expressions given in Gupta (2001).

Note that $h_{X|Y=y}(x)$ increases (decreases) in $y$ and $h_{Y|X=x}(y)$ increases (decreases) in $x$ whenever $c < \phi a_0$ ($>$). Analogously, $h_{X|Y=y}(x)$ increases (decreases) in $x$ when $\theta_1(y) + c\Lambda_1(x)$ is logconvex (logconcave) in $x$ and $h_{Y|X=x}(y)$ increases (decreases) in $y$ when $\theta_2(x) + c\Lambda_2(y)$ is logconvex (logconcave) in $y$.

## 3.2 Dependence

In this section we study the following dependence concept.

**Definition 3.2** Let $\mathcal{X}$ and $\mathcal{Y}$ be subsets of the real line. A function $f(x, y)$ is said to be totally positive of order 2 ($TP_2$) (reverse regular of order 2 ($RR_2$)) if

$$f(x_1, y_1)f(x_2, y_2) \geq f(x_1, y_2)f(x_2, y_1) \quad (\leq),$$

for all $x_1 \leq x_2$ in $\mathcal{X}$ and $y_1 \leq y_2$ in $\mathcal{Y}$.

Holland and Wang (1987) showed that this definition is closely related to the following local dependence function

$$\gamma(x, y) = \frac{\partial^2}{\partial x \partial y} \ln f(x, y).$$

In fact, they showed that under the assumption that $f(x, y)$ is a PDF with support $\mathcal{X} \times \mathcal{Y}$, $f(x, y)$ is $TP_2$ ($RR_2$) if and only if $\gamma(x, y) \geq 0$ ($\leq$) in $\mathcal{X} \times \mathcal{Y}$. Then, for the PDF given in (3.3), we obtain

$$\gamma(x, y) = \frac{a_1 a_2 (c+1) \lambda_1(x) \lambda_2(y)(c - \phi a_0)}{(a_0 + c a_1 \Lambda_1(x) + c a_2 \Lambda_2(y) + c\phi a_1 a_2 \Lambda_1(x) \Lambda_2(y))^2},$$

for $x, y > 0$. Hence, if $c > \phi a_0$, then $f(x, y)$ is $TP_2$ and $X$ an $Y$ are positively dependent. Also, if $c < \phi a_0$, then $f(x, y)$ is $RR_2$ and $X$ and $Y$ are negatively dependent. As we have already mentioned, the case $c = \phi a_0$ corresponds to the independence case.

### 3.3 Bivariate survival function and bivariate hazard rate

The bivariate survival function of the model (3.3) is obtained in the following proposition.

**Proposition 3.3** *The bivariate survival function of the PDF* (3.3) *can be obtained as*

$$S(x, y) = \begin{cases} \frac{cK\phi^{-1+1/c}}{(g(x,y))^{1/c}} F\left(\frac{1}{c}, \frac{1}{c}; \frac{1+c}{c}; \frac{c-a_0\phi}{g(x,y)}\right), & \frac{c-a_0\phi}{g(x,y)} > -1 \\ \frac{cK\phi^{-1+1/c}}{(g(x,y)+a_0\phi-c)^{1/c}} F\left(\frac{1}{c}, 1; \frac{1+c}{c}; \frac{a_0\phi-c}{g(x,y)-c+a_0\phi}\right), & \frac{c-a_0\phi}{g(x,y)} \leq -1 \end{cases} \tag{3.6}$$

*for* $x, y \geq 0$, *where* $g(x, y) = c(1 + \phi a_2 \Lambda_2(y))(1 + \phi a_1 \Lambda_1(x))$ *and* $F$ *is the hypergeometric function defined in* (3.4).

The proof follows the lines of Sect. 2.3 in Navarro and Sarabia (2013) and can be obtained from the authors. The marginal survival functions can be computed from (3.6) since $S_X(t) = \Pr(X > t) = S(t, 0)$ and $S_Y(t) = \Pr(Y > t) = S(0, t)$. Analogously, the bivariate hazard rate function defined by Basu (1971) as

$$r(x, y) = \frac{f(x, y)}{S(x, y)}$$

can be computed from (3.3) and (3.6).

### 3.4 Hazard gradient and marginal hazard functions

The hazard gradient corresponding to the vector $(X, Y)$ is defined as follows (see Johnson and Kotz 1975).

$$\mathbf{h}(x, y) = (h_1(x, y), h_2(x, y))' = -\nabla \ln S(x, y),$$

where $\nabla = (\partial/\partial x, \partial/\partial y)'$. Note that $h_1(x, y) = h_{X|Y>y}(x)$ and $h_2(x, y) = h_{Y|X>x}(y)$, where $h_{X|Y>y}(x)$ and $h_{Y|X>x}(y)$ are the hazard rate functions of $(X|Y > y)$ and $(Y|X > x)$, respectively, that is,

$$h_1(x, y) = -\frac{\partial}{\partial x} \ln S(x, y) = \frac{1}{S(x, y)} \int_y^{\infty} f(x, v) dv$$

and

$$h_2(x, y) = -\frac{\partial}{\partial y} \ln S(x, y) = \frac{1}{S(x, y)} \int_x^{\infty} f(u, y) du.$$

Hence, we get

$$h_1(x, y) = \frac{1}{S(x, y)} \frac{K a_1 \lambda_1(x)}{1 + \phi a_1 \Lambda_1(x)} m_1^{-1/c}(x, y) \tag{3.7}$$

and

$$h_2(x, y) = \frac{1}{S(x, y)} \frac{K a_2 \lambda_2(y)}{1 + \phi a_2 \Lambda_2(y)} m_1^{-1/c}(x, y), \tag{3.8}$$

where

$$m_1(x, y) = a_0 + c a_1 \Lambda_1(x) + c a_2 \Lambda_2(y) + c \phi a_1 a_2 \Lambda_1(x) \Lambda_2(y).$$

Notice that we have

$$\frac{h_1(x, y)}{h_2(x, y)} = \frac{a_1 \lambda_1(x)}{a_2 \lambda_2(y)} \frac{1 + \phi a_2 \Lambda_2(x)}{1 + \phi a_1 \Lambda_1(x)} = \frac{h_{X|Y=y}(x)}{h_{Y|X=x}(y)}. \tag{3.9}$$

In particular, if $c > 0$ and $\Lambda_1(t) = \Lambda_2(t) = t$ for $t \geq 0$, then we obtain the expressions for the hazard components of the bivariate Pareto model given in (3.3) and (3.4) of Gupta (2001).

The explicit expressions of the hazard components can be obtained from (3.6). The monotonicity of the hazard gradient components can be obtained by using the following result (Shaked 1977, Proposition 3.4 ).

**Lemma 3.4** *If $f(x, y)$ is $TP_2$ ($RR_2$), then $h_1(x, y)$ is decreasing (increasing) in $y$ and $h_2(x, y)$ is decreasing (increasing) in $x$.*

Then, by using the result given in Sect. 3.2, if $c \geq \phi a_0$ ($c \leq \phi a_0$), we conclude that $h_1(x, y)$ is decreasing (increasing) in $y$ and $h_2(x, y)$ is decreasing (increasing) in $x$. Moreover we have the following result.

**Proposition 3.5** *Let $c \leq a_0 \phi$ ($\geq$).*

*(i)* *If* $1 + \phi a_1 \Lambda_1(x)$ *is logconvex (logconcave), then* $h_1(x, y)$ *is increasing (decreasing) in* $x$.

*(ii)* *If* $1 + \phi a_2 \Lambda_2(y)$ *is logconvex (logconcave), then* $h_2(x, y)$ *is increasing (decreasing) in* $y$.

The proof follows the lines of the proof of Proposition 2.2 in Navarro and Sarabia (2013) and can be obtained from the authors.

In particular, for the bivariate Pareto model, if $c > 0$ and $\Lambda_1(t) = \Lambda_2(t) = t$ for $t \geq 0$, then $1 + \phi a_1 \Lambda_1(x) = 1 + \phi a_1 x$ and $1 + \phi a_2 \Lambda_2(y) = 1 + \phi a_2 y$ which are logconcave functions and hence $h_1(x, y)$ is decreasing in $x$ and $h_2(x, y)$ is decreasing in $y$. This result was obtained in Gupta (2001), p. 219. Analogously, if $c \to 0^+$, then we obtain Proposition 2.2 in Navarro and Sarabia (2013).

### 3.5 Clayton–Oakes measure

A well known measure of association corresponding to bivariate survival models is Clayton–Oakes measure (see e.g. Oakes 1989) defined by

$$\theta(x, y) = \frac{f(x, y) S(x, y)}{S_1(x, y) S_2(x, y)}. \tag{3.10}$$

where $S_1(x, y) = \frac{\partial}{\partial x} S(x, y)$ and $S_2(x, y) = \frac{\partial}{\partial y} S(x, y)$. It is well know (see e.g. Gupta 2001) that

$$\theta(x, y) = \frac{h_{X|Y=y}(x)}{h_1(x, y)}. \tag{3.11}$$

Then $\theta(x, y)$ can be computed from (3.5) and (3.7). Moreover, from Gupta (2001) (see p. 210), we have that $\theta(x, y) \geq 1$ $(\leq)$ if, and only if, $h_1(x, y)$ is increasing (decreasing) in $y$. Hence, from the results obtained in Sect. 3.4, $\theta(x, y) \geq 1$ if, and only, if $c \geq \phi a_0$ $(c \leq \phi a_0)$. Note in passing that there is a typo in the condition given in Gupta (2001), p. 210, line 13, for $\theta(x, y) > 1$ in the bivariate Pareto model (the second $\lambda_2$ should be replaced with $\lambda_0$).

### 3.6 Series and parallel systems

Let us consider series and parallel systems with two components having lifetimes $X$ and $Y$. Suppose that the PDF of $(X, Y)$ is of the form (3.3). Then, using (3.1) and (3.2), if it is known that a component has failed at time $t$, then the lifetime of the other component satisfies a PGOR model with parameter $\theta_i(t)$ and a baseline GO function $\Lambda_i$, $i = 1, 2$. Now if $T_{1:2} = \min(X, Y)$ is the lifetime of the series system, then its survival function is

$$S_{1:2}(t) = \Pr(\min(X, Y) > t) = S(t, t).$$

Hence, it can be computed from (3.6). Analogously, its hazard rate can be computed from (3.7) and (3.8), by using the fact that

$$h_{1:2}(t) = h_1(t,t) + h_2(t,t)$$

(see e.g. Gupta (2001)). Alternatively, from (3.9), it can be computed as

$$h_{1:2}(t) = h_2(t,t)\left(1 + \frac{h_1(t,t)}{h_2(t,t)}\right) = h_2(t,t)\left(1 + \frac{a_1\lambda_1(t)}{a_2\lambda_2(t)}\frac{1+\phi a_2\Lambda_2(t)}{1+\phi a_1\Lambda_1(t)}\right). \tag{3.12}$$

Hence, from Proposition 3.5, we have the following result.

**Proposition 3.6** *If $c \geq a_0\phi$ ($\leq$), $1+\phi a_2\Lambda_2(y)$ is logconcave (logconvex), $\lambda_1(t)/\lambda_2(t)$ is decreasing (increasing), $\lambda_1(t)/\lambda_2(t) \geq a_2/a_1$ ($\leq$) and $\lambda_1(t)/\lambda_2(t) \geq \Lambda_1(t)/\Lambda_2(t)$ ($\leq$), then $h_{1:2}(t)$ is decreasing (increasing).*

In particular, if $a_1 = a_2$ and $\Lambda_1(t) = \Lambda_2(t)$, then $h_{1:2}(t)$ is decreasing (increasing) whenever $c \geq a_0\phi$ ($\leq$) and $1+\phi a_2\Lambda_2(y)$ is logconcave (logconvex). For the bivariate Pareto model, we have $\Lambda_1(t) = \Lambda_2(t) = t$ and hence $1+\phi a_2\Lambda_2(y) = 1+\phi a_2 y$ which is logconcave. Therefore, $h_{1:2}(t)$ is decreasing whenever $c \geq a_0\phi$ and $a_1 \geq a_2$. Gupta (2001) proved, by using a different approach, that $h_{1:2}(t)$ is always decreasing.

Analogously, the lifetime of a parallel system with component lifetimes $X$ and $Y$ is $T_{2:2} = \max(X,Y)$ for which the survival function can be computed as

$$S_{2:2}(t) = S(t,0) + S(0,t) - S(t,t).$$

Hence $S_{2:2}(t)$ can be obtained from (3.6).

## 4 Bivariate models with PGOR conditional survival distributions

In this section we obtain an alternative bivariate model by assuming that its conditional survival distributions satisfy

$$(X|Y>y) \sim \mathcal{PGOR}(\theta_1(y), c_1(y), \Lambda_1(x)) \tag{4.1}$$

and

$$(Y|X>x) \sim \mathcal{PGOR}(\theta_2(x), c_2(x), \Lambda_2(y)), \tag{4.2}$$

for $x, y \geq 0$, where $\Lambda_1(x)$ and $\Lambda_2(y)$ are GO functions and $c_1(y), c_2(x) > 0$. In the next theorem, we obtain the unique models satisfying (4.1) and (4.2).

**Theorem 4.1** *If $\Lambda_1$ and $\Lambda_2$ are GO functions and $(X,Y)$ is a random vector with support $(0,\infty)\times(0,\infty)$ satisfying (4.1) and (4.2) for $x, y \geq 0$, then its survival function is*

$$S(x,y) = (1 + ca_1\Lambda_1(x) + ca_2\Lambda_2(y) + \theta c a_1 a_2 \Lambda_1(x)\Lambda_2(y))^{-1/c}, \tag{4.3}$$

*for* $x, y \geq 0$, *where* $a_1, a_2, c > 0$ *and* $0 \leq \theta \leq c+1$ *or*

$$S(x, y) = \exp(-\theta_1 \ln(1 + a_1\Lambda_1(x)) - \theta_2 \ln(1 + a_2\Lambda_2(y)) \\ - \theta_3 \ln(1 + a_1\Lambda_1(x)) \ln(1 + a_2\Lambda_2(y))) \tag{4.4}$$

*for* $x, y \geq 0$, *where* $a_1, a_2, \theta_1, \theta_2 > 0$ *and* $\theta_3 \geq 0$.

*Proof* As $X$ and $Y$ have support $(0, \infty)$, then $\Lambda_1$ and $\Lambda_2$ are strictly increasing functions in $(0, \infty)$. If $X^* = \Lambda_1(X)$ and $Y^* = \Lambda_2(Y)$, then $(X^*, Y^*)$ satisfies (4.1) and (4.2) for $\Lambda_1^*(x) = x$ and $\Lambda_2(y)^* = y$, that is, their conditional distributions satisfy

$$\Pr(X^* > x | Y^* > y) = \left(1 + \frac{x}{\theta_1^*(y)}\right)^{-1/c_1^*(y)},$$

$$\Pr(Y^* > y | X^* > x) = \left(1 + \frac{y}{\theta_2^*(x)}\right)^{-1/c_2^*(x)},$$

for $x, y \geq 0$. Hence, $(X^*, Y^*)$ satisfies the conditions (11.31) and (11.32) (p. 262) in Arnold et al. (1999) in the case (i). Therefore, the survival function $S^*$ of $(X^*, Y^*)$ satisfies (11.35) or (11.36) (p. 263) in Arnold et al. (1999).

In the first case, it can be written as

$$S^*(x, y) = (1 + ca_1x + ca_2y + c\theta a_1a_2xy)^{-1/c},$$

for $x, y > 0$, where $a_1, a_2, c > 0$ and $0 \leq \theta \leq c+1$. The last condition is needed in order to assure that its PDF is nonnegative. Note in passing that there is a mistake in p. 263 of Arnold et al. (1999) since we need $\theta \in [0, c+1]$ (instead of $\theta \in [0, 2]$) in order to assure that the model in (11.35) is a valid model. Hence, in this case, the survival function of $(X, Y)$ is given by $S(x, y) = S^*(\Lambda_1(x), \Lambda_2(y))$ and (4.3) holds.

In the second case, $S^*$ can be written as

$$S^*(x, y) = \exp(-\theta_1 \ln(1 + a_1x) - \theta_2 \ln(1 + a_2y) - \theta_3 \ln(1 + a_1x) \ln(1 + a_2y)),$$

for $x, y \geq 0$, where $a_1, a_2, \theta_1, \theta_2 > 0$ and $\theta_3 \geq 0$. Hence, in this case, the survival function of $(X, Y)$ is given by $S(x, y) = S^*(\Lambda_1(x), \Lambda_2(y))$ and (4.4) holds.

*Remark 4.2* If we restrict conditions (4.1) and (4.2) to the case $c_1(y) = c_2(x) = c > 0$ for $x, y \geq 0$, then the unique solution is (4.3) since, if (4.4) holds, then

$$\Pr(X > x | Y > y) = (1 + a_1\Lambda_1(x))^{-\theta_1 - \theta_3 \ln(1 + a_2\Lambda_2(y))}$$

for $x, y \geq 0$. Hence $\theta_3 = 0$ and $\theta_1 = 1/c$. In a similar way, it can be proved that $\theta_2 = 1/c$. Therefore, (4.4) reduces to

$$S(x, y) = (1 + a_1\Lambda_1(x))^{-1/c}(1 + a_2\Lambda_2(y))^{-1/c}$$

which is included in the model (4.3) (by taking $\theta = c$). So we will restrict our attention to the model (4.3) in the following sections.

As an immediate consequence we have that, if $S$ is given by (4.3), then

$$\lim_{c \to 0^+} S(x, y) = \exp\left(-a_1\Lambda_1(x) - a_2\Lambda_2(y) - \theta a_1 a_2 \Lambda_1(x)\Lambda_2(y)\right),$$

which is the survival function of the model obtained in (3.3) of Navarro and Sarabia (2013) under the condition that the conditional distributions of $(X|Y > y)$ and $(Y|X > x)$ satisfy the PHR model.

## 4.1 Marginal and conditional distributions

If (4.3) holds, then the marginal survival functions are

$$S_X(x) = (1 + ca_1\Lambda_1(x))^{-1/c},$$

for $x \geq 0$ and

$$S_Y(y) = (1 + ca_2\Lambda_2(y))^{-1/c},$$

for $y \geq 0$. Note that both satisfy the PGOR model. The marginal PDF are

$$f_X(x) = a_1\lambda_1(x)(1 + ca_1\Lambda_1(x))^{-1-1/c},$$

for $x \geq 0$ and

$$f_Y(y) = a_2\lambda_2(y)(1 + ca_2\Lambda_2(y))^{-1-1/c},$$

for $y \geq 0$. Hence the marginal hazard rate functions are

$$h_X(x) = \frac{a_1\lambda_1(x)}{1 + ca_1\Lambda_1(x)},$$

for $x \geq 0$ and

$$h_Y(y) = \frac{a_2\lambda_2(y)}{1 + ca_2\Lambda_2(y)},$$

for $y \geq 0$. Therefore, $h_X$ is increasing (decreasing) if and only if $1 + a_1\Lambda_1(x)$ is logconvex (logconcave). Analogously, $h_Y$ is increasing (decreasing) if and only if $1 + a_2\Lambda_2(y)$ is logconvex (logconcave). Then the conditional survival functions can be written as

$$\Pr(X > x|Y > y) = \left(1 + ca_1\frac{1 + \theta a_2\Lambda_2(y)}{1 + ca_2\Lambda_2(y)}\Lambda_1(x)\right)^{-1/c}$$

and

$$\Pr(Y > y|X > x) = \left(1 + ca_2 \frac{1 + \theta a_1 \Lambda_1(x)}{1 + ca_2 \Lambda_1(x)} \Lambda_2(y)\right)^{-1/c},$$

for $x, y \geq 0$. Therefore (4.1) and (4.2) hold for $c_1(y) = c_2(x) = c > 0$,

$$\theta_1(y) = \frac{1}{a_1} \frac{1 + ca_2 \Lambda_2(y)}{1 + \theta a_2 \Lambda_2(y)}$$

and

$$\theta_2(x) = \frac{1}{a_2} \frac{1 + ca_1 \Lambda_1(x)}{1 + \theta a_1 \Lambda_1(x)},$$

for $x, y \geq 0$. Hence $\theta_1(y)$ and $\theta_2(x)$ are increasing (decreasing) functions whenever $0 \leq \theta \leq c$ ($c \leq \theta \leq c + 1$). The case $c = \theta$ corresponds to the independence case with $\theta_1(y) = 1/a_1$ and $\theta_2(x) = 1/a_2$ for $x, y \geq 0$.

The conditional survival PDF are

$$f_{X|Y>y}(x) = a_1 \lambda_1(x) \frac{1 + \theta a_2 \Lambda_2(y)}{1 + ca_2 \Lambda_2(y)} \left(1 + ca_1 \frac{1 + \theta a_2 \Lambda_2(y)}{1 + ca_2 \Lambda_2(y)} \Lambda_1(x)\right)^{-1-1/c}$$

and

$$f_{Y|X>x}(y) = a_2 \lambda_2(y) \frac{1 + \theta a_1 \Lambda_1(x)}{1 + ca_2 \Lambda_1(x)} \left(1 + ca_2 \frac{1 + \theta a_1 \Lambda_1(x)}{1 + ca_2 \Lambda_1(x)} \Lambda_2(y)\right)^{-1-1/c},$$

for $x, y \geq 0$. Hence their hazard rate functions are

$$h_{X|Y>y}(x) = \frac{\lambda_1(x)}{c\Lambda_1(x) + \theta_1(y)}$$

and

$$h_{Y|X>x}(y) = \frac{\lambda_2(y)}{c\Lambda_2(y) + \theta_2(x)}.$$

Therefore, $h_{X|Y>y}(x)$ is decreasing (increasing) in $y$ and $h_{Y|X>x}(y)$ is decreasing (increasing) in $x$ whenever $0 \leq \theta \leq c$ ($c \leq \theta \leq c + 1$). Moreover, $h_{X|Y>y}(x)$ is increasing (decreasing) in $x$ if, and only, if $c\Lambda_1(x) + \theta_1(y)$ is logconvex (logconcave) in $x$. Analogously, $h_{Y|X>x}(y)$ is increasing (decreasing) in $y$ if, and only, if $c\Lambda_2(y) + \theta_2(x)$ is logconvex (logconvex) in $y$. Recall that these functions are the components of the hazard gradient $\mathbf{h} = (h_1, h_2)$ of $(X, Y)$, that is, $h_1(x, y) = h_{X|Y>y}(x)$ and $h_2(x, y) = h_{Y|X>x}(y)$.

### 4.2 Bivariate density and bivariate hazard rate

The bivariate PDF of the survival function (4.3) is

$$f(x,y)=a_1a_2\lambda_1(x)\lambda_2(y)\frac{c+1-\theta+\theta a_1\Lambda_1(x)+\theta a_2\Lambda_2(y)+\theta^2a_1a_2\Lambda_1(x)\Lambda_2(y)}{(m_2(x,y))^{2+1/c}},$$

for $x, y \geq 0$, where

$$m_2(x,y)=1+ca_1\Lambda_1(x)+ca_2\Lambda_2(y)+\theta ca_1a_2\Lambda_1(x)\Lambda_2(y).$$

Note that $f$ is nonnegative whenever $0 \leq \theta \leq c+1$. The bivariate hazard rate function $r(x, y) = f(x, y)/S(x, y)$ can be obtained as

$$r(x,y)=a_1a_2\lambda_1(x)\lambda_2(y)\frac{c-\theta+(1+\theta a_1\Lambda_1(x))(1+\theta a_2\Lambda_2(y))}{(m_2(x,y))^2}, \tag{4.5}$$

for $x, y \geq 0$. In the next theorem we show that this bivariate hazard rate function jointly with the marginal survival functions characterize the model (4.3) (under some assumptions). The proof is obtained from Theorem 1 in Navarro (2008).

**Theorem 4.3** *Let $\Lambda_1$ and $\Lambda_2$ be analytical GO functions in $(0,\infty)$ and let $S(x, y)$ be a bivariate survival function such that $\ln S(x, y)$ is analytical in $(0,\infty)\times(0,\infty)$. Then $S(x, y)$ satisfies (4.3) if and only if the following conditions hold:*

*(i)* $X \sim \mathcal{PGOR}(1/a_1, c, \Lambda_1(x))$,
*(ii)* $Y \sim \mathcal{PGOR}(1/a_2, c, \Lambda_2(y))$ *and*
*(iii)* $r(x, y)$ *is given by* (4.5).

### 4.3 Conditional densities and conditional hazard functions

Suppose that $(X, Y)$ has survival function of the form (4.3), then the (usual) conditional PDF are

$$f_{X|Y=y}(x)=a_1\lambda_1(x)\frac{c-\theta+(1+\theta a_1\Lambda_1(x))(1+\theta a_2\Lambda_2(y))}{(1+ca_2\Lambda_2(y))^{-1-1/c}(m_2(x,y))^{2+1/c}}$$

and

$$f_{Y|X=x}(y)=a_2\lambda_2(y)\frac{c-\theta+(1+\theta a_1\Lambda_1(x))(1+\theta a_2\Lambda_2(y))}{(1+ca_1\Lambda_1(x))^{-1-1/c}(m_2(x,y))^{2+1/c}},$$

for $x, y \geq 0$. The conditional hazard functions can be computed from the preceding expressions as

$$h_{X|Y=y}(x)=\frac{f_{X|Y=y}(x)}{\int_x^\infty f_{X|Y=y}(z)dz}$$

and

$$h_{Y|X=x}(y) = \frac{f_{Y|X=x}(y)}{\int_y^\infty f_{Y|X=x}(z)dz},$$

for $x, y \geq 0$. Alternative expressions are given in the following section.

### 4.4 Clayton–Oakes measure

If the joint survival function of $(X, Y)$ is given by (4.3), then the Clayton–Oakes measure of association $\theta(x, y)$ defined by (3.10) is given by

$$\theta(x, y) = 1 + \frac{c - \theta}{(1 + \theta a_1 \Lambda_1(x))(1 + \theta a_2 \Lambda_2(y))},$$

for $x, y \geq 0$. Note that $\theta(x, y)$ is decreasing (increasing) in both $x$ and $y$ whenever $0 \leq \theta \leq c$ ($c \leq \theta \leq c + 1$). Also note that $\theta(x, y) = 1$ if and only if $c = \theta$. Hence, from (3.11), the conditional hazard rate $h_{X|Y=y}(x)$ can be computed as

$$h_{X|Y=y}(x) = \theta(x, y) h_{X|Y>y}(x) = \theta(x, y) \frac{\lambda_1(x)}{c\Lambda_1(x) + \theta_1(y)}.$$

Analogously, $h_{Y|X=x}(y)$ can be computed as $h_{Y|X=x}(y) = \theta(x, y) h_{Y|X>x}(y)$.

### 4.5 Series and parallel systems

In this subsection we consider series and parallel systems with components lifetimes $X, Y$ where $(X, Y)$ has the joint survival function (4.3). This is equivalent from (4.1) and (4.2) to assume that if it is known that a component is working at age $t$, then the conditional distribution of the other component has a PGOR model with parameter $\theta_i(t)$ and GO functions $\Lambda_i$, $i = 1, 2$. Then the survival function of the lifetime of the series system $T_{1:2} = \min(X, Y)$ is

$$S_{1:2}(t) = S(t, t) = (m_2(t, t))^{-1/c}.$$

and its hazard rate function is

$$h_{1:2}(t) = h_1(t, t) + h_2(t, t) = \frac{\lambda_1(t)}{\theta_1(t) + c\Lambda_1(t)} + \frac{\lambda_2(t)}{\theta_2(t) + c\Lambda_2(t)}.$$

For the parallel system lifetime $T_{2:2} = \max(X, Y)$, the survival function is given by

$$S_{2:2}(t) = (1 + ca_1\Lambda_1(t))^{-1/c} + (1 + ca_2\Lambda_2(t))^{-1/c} - (m_2(t, t))^{-1/c}.$$

## 5 An application

In this section we fit our model (3.3) to the data set presented in Simiu and Filliben (1975) on annual maximal wind speeds (mph) at two locations (Eastport and North Head) in the United States of America for the period 1912–1948 (see also Arnold et al. 1999, Table 12.1, p. 287). In our model we assume that

$$\Lambda_1(x) = \frac{1-(1-\exp\{-\exp(-(x-5)/10)\})^{\alpha_1}}{\alpha_1(1-\exp\{-\exp(-(x-5)/10)\})^{\alpha_1}},$$
$$\Lambda_2(y) = \frac{1-(1-\exp\{-\exp(-(y-5)/10)\})^{\alpha_2}}{\alpha_2(1-\exp\{-\exp(-(y-5)/10)\})^{\alpha_2}},$$
$$a_0 = 1.$$

Arnold et al. (1999) fitted their models (model (12.21) on p. 279 and model (12.39) on p. 282) to this data set obtaining $\ln(L) = 110.992, 114.025$, respectively, where $L$ is the likelihood function (see also Arnold et al. 1998). The corresponding parameter estimations together with the estimated standard errors and the corresponding values of the log-likelihoods are displayed in Table 12.2, p. 288, of Arnold et al. (1999).

Under the above assumptions, the parameter estimations together with the estimated standard errors and the value of the log-likelihood function for our model (3.3) are displayed in Table 1. The parameters are estimated by using MLE with numerical procedures by using the interior-point method. The estimated standard errors are obtained by using numerical methods with computing the elements of Fisher information matrix. As the results show, our model fits better than the models considered by Arnold et al. (1999). Furthermore, according to Sect. 3.2, the values of $c$ and $\phi$ indicate a negative correlation as one can expect for this data set. The density contour plot for this model can be seen in Fig. 1.

Instead of the log-likelihood function we can use the AIC value to compare the models. For any statistical model, the AIC value is

$$AIC = 2k - 2\ln(L),$$

where $k$ is the number of parameters in the model, and $L$ is the maximized value of the likelihood function for the model. Given a set of candidate models for the

**Table 1** Parameter estimations together with the estimated standard errors and the value of the log-likelihood function for the model (3.3)

| Parameters | Estimations | Standard errors |
|---|---|---|
| $\alpha_1$ | 6.8118 | 0.0043 |
| $\alpha_2$ | 3.578 | 0.0074 |
| $a_1$ | 0.01 | 0.0056 |
| $a_2$ | 0.01 | 0.0038 |
| $\phi$ | 0.5526 | 0.0047 |
| c | 0.01 | 0.0048 |
| $\ln(L)$ | 170.17 | |

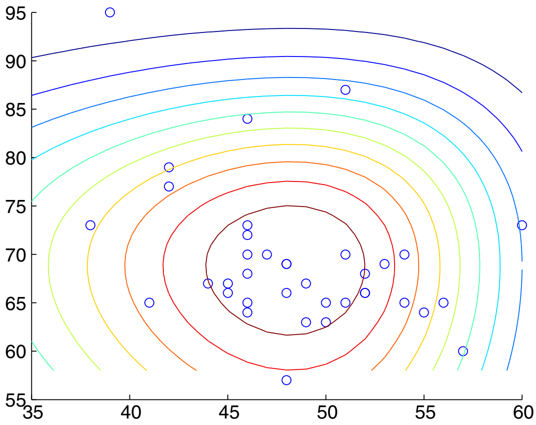


**Fig. 1** Density contour plot for model (3.3) for the maximum wind speeds data set

data, the model with the minimum AIC value is preferred. The AIC not only rewards goodness of fit, but also includes a penalty that is an increasing function of the number of estimated parameters. The AIC for the models in Arnold et al. (1999) are: $2 \cdot 5 - 2 \cdot 110.992 = -211.984$, and $2 \cdot 5 - 2 \cdot 114.025 = -218.5$ and the AIC of our model is: $2 \cdot 6 - 2 \cdot 170.17 = -318.34$. Hence, using the AIC criterion, our model is preferred to the models considered in Arnold et al. (1999).

## 6 Conclusions

In this paper two new bivariate models are obtained and studied from a theoretical point-of-view. The models are characterized by the property that their conditional distributions (in two different senses) belong to the proportional generalized odds rate (PGOR) model with parameter $c > 0$. The proposed models are very flexible and contain in the limit (when $c \to 0^+$) the models obtained in Navarro and Sarabia (2013) characterized by conditional distributions in the Cox PHR model. They also contain the bivariate Pareto models obtained and studied in Arnold et al. (1993, 1999) and Gupta (2001).

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