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Dependence properties of bivariate distributions with proportional (reversed) hazards marginals

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Abstract This paper considers two classes of bivariate distributions having proportional (reversed) hazard rates models as their marginals. Various dependence properties of the proposed models are studied through their copulas.

Keywords Copula · Dependence · Proportional hazard model

1 Introduction

For a given univariate cumulative distribution function (cdf) F, the function defined by

$$G(x) = F^{\alpha}(x), \quad x \geq 0, \tag{1}$$

is a distribution function for $\alpha > 0$. This model is called proportional reversed hazard rate (PRHR) model with the proportionality parameter α and proposed by Gupta et al. (1998) as a dual of the well–known proportional hazard rate (PHR) model

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$$H(x) = 1 - (1 - F(x))^{\alpha}, \quad x \geq 0. \tag{2}$$

The class of distributions of the form F^{α} is also known as the exponentiated class of distributions with baseline distribution function F. In recent years, several standard distributions have been generalized using the exponential function; for instance see (Gupta and Kundu 2007; Mudholkar and Huston 1996; Nadarajah 2006; Nassar and Eissa 2003).

In the bivariate case, the construction of distributions with given marginals has been a problem of interest to statisticians for many years. Today, in view of Sklar's theorem (Nelsen 2006), this problem can be reduced to the construction of a copula. Recently, various authors provided construction methods from the class of copulas to itself, or from a more general class of functions to another; see, e.g, (Durante 2009; Morillas 2005). One of the purposes for such constructions is to increase the availability for modeling purposes. In the bivariate case, it is of general interest to extend the models (1) and (2). Unlike the univariate set up, there is more than one definition for PHR and PRHR models in the bivariate case; for instance see (Clayton and Cuzick 1985; Finkelstein 2003; Kundu and Gupta 2010; Sankaran and Gleeja 2006). Our main aim in this paper is to formulate a suitable notion of bivariate proportional (reversed) hazard models. We do this by defining bivariate PHR and PRHR models in such a way that implies their marginals follow univariate PHR and PRHR distributions. Starting from a given bivariate cdf $F(x, y)$ with the univariate marginal distribution functions F_1 and F_2, a natural bivariate extension of the PHR model is a bivariate distribution of the form

$$H(x, y) = 1 - (1 - F(x, y))^{\alpha}, \tag{3}$$

and a generalization of the PRHR model is a bivariate distribution with the survival functions of the form

$$\bar{G}(x, y) = 1 - (1 - \bar{F}(x, y))^{\alpha}, \tag{4}$$

where $\bar{F}(x, y) = 1 - F_1(x) - F_2(y) + F(x, y)$ is the survival function associated with F. Note that the functions G and H have the univariate marginals of the form (1) and (2), respectively. Unlike the univariate case, the functions defined by (3) and (4) are not necessary bivariate cdfs. This paper, provides conditions that the proposed models (3) and (4) define bivariate cdfs and discusses the different dependence properties of these models through their associated copulas.

An interpretation of the proposed model in terms of bivariate (reversed) hazard rates is as follows. Let (X_1, X_2) be the lifetimes of a two component systems whose joint cdf is F. Then $1 - F(x_1, x_2) = P(X_1 > x_1 \text{ or } X_2 > x_2)$, gives the probability that at least of the components will survive beyond the time (x_1, x_2) and $1 - \bar{F}(x_1, x_2) = P(X_1 \leq x_1 \text{ or } X_2 \leq x_2)$, gives the probability that at least one of the components is not surviving up to time (x_1, x_2). Following Roy (2002), let $r_F(x_1, x_2) = (r_{1F}(x_1, x_2), r_{2F}(x_1, x_2))$, be the vector of reversed hazard rates define by

$$r_{iF}(x_1, x_2) = \frac{\partial}{\partial x_i} \log(1 - \bar{F}(x_1, x_2)).$$

Thus the model defined by (4) is PRHR in the sense that

$$r_{iG}(x_1, x_2) = \alpha r_{iF}(x_1, x_2), \quad i = 1, 2.$$

A similar argument holds for the model (3), in terms of the hazard rates defined by $r^*_{iF}(x_1, x_2) = -\frac{\partial}{\partial x_i} \log(1 - F(x_1, x_2))$. That is

$$r^*_{iH}(x_1, x_2) = \alpha r^*_{iF}(x_1, x_2), \quad i = 1, 2.$$

2 Two classes of bivariate distributions

In this section we determine the conditions under which the functions defined by (3) and (4) are bivariate distribution functions.

2.1 Genesis of the proposed models

Consider a sequence of independent Bernoulli trials in which the kth trial has probability of $\frac{\alpha}{k}$, $0 < \alpha < 1$, $k \in \{1, 2, 3, \ldots\}$. Let N be the trial number on which the first success occurs. Then N has Mittag-Leffler (Pillai and Jayakumar 1995) distribution with probability mass function

$$\begin{aligned} P(N = n) &= (1 - \alpha)\left(1 - \frac{\alpha}{2}\right) \ldots \left(1 - \frac{\alpha}{n-1}\right) \frac{\alpha}{n} \\ &= \frac{(-1)^{n-1} \alpha(\alpha - 1) \ldots (\alpha - n + 1)}{n!}. \end{aligned} \tag{5}$$

The probability generating function of N is then

$$g(t) = E\left(t^N\right) = 1 - (1 - t)^{\alpha}, \quad t \in [0, 1].$$

Let $(X_1, Y_1), (X_2, Y_2), \ldots$ be a sequence of independent and identically distributed (i.i.d.) random vectors from a continuous bivariate distribution function $F(x, y)$ with univariate marginal distributions F_1 and F_2. Let N be a discrete random variable independent of (X_i, Y_i) having probability mass function (5). Put

$$U_1 = \max(X_1, \ldots, X_N), \qquad U_2 = \max(Y_1, \ldots, Y_N),$$

and

$$V_1 = \min(X_1, \ldots, X_N), \qquad V_2 = \min(Y_1, \ldots, Y_N).$$

Then (U_1, U_2) has joint distribution function

$$H(x, y) = P\{U_1 \le x, U_2 \le y\} = \sum_{n=1}^{\infty} [P(X_i \le x, Y_i \le y)]^n P(N = n)$$
$$= g(F(x, y)) = 1 - (1 - F(x, y))^{\alpha},$$

whose univariate marginals are PHR models given by

$$H_i(x) = 1 - \bar{F}_i^{\alpha}(x), \quad i = 1, 2. \tag{6}$$

Similarly, the joint survival function of (V_1, V_2) is given by (4) with associated joint distribution function

$$G(x, y) = F_1^{\alpha}(x) + F_2^{\alpha}(y) - \{F_1(x) + F_2(y) - F(x, y)\}^{\alpha}, \tag{7}$$

whose marginal cdfs belong to the PRHR model given by

$$G_i(x) = F_i^{\alpha}(x), \quad i = 1, 2. \tag{8}$$

In short, we have proved the following result.

Proposition 1 *For any bivariate distribution function F and $0 < \alpha \le 1$, the functions defined by (3) and (7) are bivariate distribution functions with the PHR and PRHR marginals, respectively.*

Remark 1 Notice that for a given bivariate distribution function F, the models defined by (3) and (7) may fail to be bivariate distributions when $\alpha > 1$. For example, let $F(x, y) = \max(x + y - 1, 0)$, $x, y \in [0, 1]$ and let $x_1 = \frac{1}{2}$, $x_2 = 1$, $y_1 = \frac{1}{2}$ and $y_2 = 1$. Then for $H(x, y) = 1 - (1 - \max(x + y - 1, 0))^{\alpha}$, we have that

$$H(x_2, y_2) - H(x_1, y_2) - H(x_2, y_1) + H(x_1, y_1) = -1 + \frac{1}{2^{\alpha-1}} < 0,$$

for all $\alpha > 1$, and hence H is not a bivariate distribution function.

Example 1 Let $F(x, y) = (1 - e^{-\lambda_1 x})(1 - e^{-\lambda_2 y})$, $x, y \ge 0$. Then (3) defines a bivariate exponential distribution of the form

$$H(x, y) = 1 - \left\{1 - (1 - e^{-\lambda_1 x})(1 - e^{-\lambda_2 y})\right\}^{\alpha},$$

and (4) defines a bivariate distribution with survival function of the form

$$\overline{G}(x, y) = 1 - \left\{1 - e^{-(\lambda_1 x + \lambda_2 y)}\right\}^{\alpha},$$

whose univariate marginals belong to the family of generalized exponential distributions (Gupta and Kundu 2007) given by

$$G_i(x) = (1 - e^{-\lambda_i x})^{\alpha}, \quad i = 1, 2.$$

Remark 2 If (X_1, X_2) is a random vector with survival function

$$\overline{G}(x_1, x_2) = 1 - \left\{1 - \overline{F_1}(x_1)\overline{F_2}(x_2)\right\}^{\alpha},$$

with univariate marginals $G_i(x) = F_i^{\alpha}(x),\ i = 1, 2$, then the marginal random variables $Y_i = -\ln G_i(X_i),\ i = 1, 2$ are exponentially distributed with hazard rate α. The bivariate distribution function of (Y_1, Y_2) is then

$$P(Y_1 \leq y_1, Y_2 \leq y_2) = 1 - \left\{1 - (1 - e^{-y_1})(1 - e^{-y_2})\right\}^{\alpha},$$

which is a bivariate cdf of the form (3).

2.2 Mittag-Leffler stability

As shown in (Marshall and Olkin 1997), for a given bivariate distribution function F, the bivariate proportional odds models defined by

$$H(x, y) = \frac{\alpha F(x, y)}{1 - (1 - \alpha)F(x, y)} \quad \text{and} \quad \overline{G}(x, y) = \frac{\alpha \overline{F}(x, y)}{1 - (1 - \alpha)\overline{F}(x, y)},$$

satisfy the geometric-maximum stability (geometric-minimum stability) property. A similar property holds for the models (3) and (4) which we call *Mittag-Leffler stability*. To see this, let $(X_1, Y_1), (X_2, Y_2), \ldots$ be a sequence of independent and identically distributed random vectors with a common distribution in the family (3) and if N' is independent of (X_i, Y_i)'s has a Mittag-Leffler distribution with parameter β, then the random vector $(\max_{1 \leq i \leq N'}(X_i), \max_{1 \leq i \leq N'}(Y_i))$ has a distribution in the family of the form

$$H(x, y) = 1 - (1 - F(x, y))^{\alpha\beta}. \tag{9}$$

A similar stability property holds for the random vector of minima. As the extreme value distributions are limiting distributions for extrema, they are sometimes useful approximations. In practice, a random variable of interest may be the extreme of only a finite number N of random variables. When N has a Mittag-Leffler distribution, the random variable has this type of stability property.

3 Dependence properties

3.1 Underlying copulas

Let F be a given bivariate distribution function with associated copula D and univariate marginals F_1 and F_2. Let G_i and H_i, $i = 1, 2$, denote the univariate marginal distributions of G and H given by (6) and (8), respectively. In view of *Sklar's Theorem* (Nelsen 2006), solving the eqnarrays

$$C_\alpha[D]\{H_1(x), H_2(y)\} = H(x, y),$$

and

$$C^*_\alpha[D]\{G_1(x), G_2(y)\} = G(x, y),$$

for the functions $C_\alpha[D]$ and $C^*_\alpha[D]$, yields the copulas associated with the bivariate cdfs H and G, respectively, as

$$C_\alpha[D](u, v) = 1 - \left(1 - D\left(1 - (1-u)^{\frac{1}{\alpha}}, 1 - (1-v)^{\frac{1}{\alpha}}\right)\right)^\alpha, \tag{10}$$

and

$$C^*_\alpha[D](u, v) = u + v - \left(u^{\frac{1}{\alpha}} + v^{\frac{1}{\alpha}} - D\left(u^{\frac{1}{\alpha}}, v^{\frac{1}{\alpha}}\right)\right)^\alpha, \tag{11}$$

for all $u, v \in (0, 1)$ and $0 < \alpha \leq 1$. The representations (10) and (11) are "unique", in the sense that given a copula D, they generate unique copulas; that is if D_1 and D_2 are two copulas such that $C_\alpha[D_1] = C_\alpha[D_2]$ ($C^*_\alpha[D_1] = C^*_\alpha[D_2]$) for every $\alpha \in (0, 1]$, then $D_1 = D_2$.

Remark 3 Note that the copula $C_\alpha[D]$ is a special case of a transformation of the copula D by means of the function $\psi(t) = 1 - (1-t)^\alpha$; see, e.g, (Durante 2009; Klement et al. 2005; Morillas 2005).

A bivariate survival function $\overline{F}$ with marginal survival functions $\overline{F_1}$ and $\overline{F_2}$ can be usefully described by its survival copula $\widehat{D}$ through the relation $\widehat{D}(\overline{F_1}(x), \overline{F_2}(y)) = \overline{F}(x, y)$, where $\widehat{D}(u, v) = u + v - 1 + D(1-u, 1-v)$ (see, e.g., Nelsen 2006). The following result, whose proof is a straightforward calculation, provides a relationship between $C_\alpha[D]$ and $C^*_\alpha[D]$.

Proposition 2 *For a given copula D, let $C_\alpha[D]$ and $C^*_\alpha[D]$ be the copulas defined by (10) and (11), respectively. Then*

$$\widehat{C^*_\alpha}[D](u, v) = C_\alpha[\widehat{D}](u, v), \tag{12}$$

for all $u, v \in (0, 1)$ and $\alpha \in (0, 1]$, where $\widehat{C^*_\alpha}$ and $\widehat{D}$ are the survival copulas associated with C^*_α and D, respectively.

The above observation can also be interpreted stochastically: Let $(X_i, Y_i),\ 1 \leq i \leq N$, be independent and identically distributed random vectors from (X, Y) with copula D. Furthere let N be independent of (X_i, Y_i)'s and have a Mittag-Leffler distribution with parameter α.

Then the random vectors $(\max_{1\leq i\leq N}\ (X_i), \max_{1\leq i\leq N}\ (Y_i))$ and $(\min_{1\leq i\leq N}\ (X_i), \min_{1\leq i\leq N}\ (Y_i))$ have the copulas $C_\alpha[D]$ and $C^*_\alpha[D]$, respectively. Since the copula of $(-X_i, -Y_i)$ is $\widehat{D}$, then the copula structure of $(\max_{1\leq i\leq N}\ (-X_i),\ \max_{1\leq i\leq N}\ (-Y_i))$, i.e., $C_\alpha[\widehat{D}]$ is the same as the copula structure of $(-\min_{1\leq i\leq N}\ (X_i), -\min_{1\leq i\leq N}\ (Y_i))$, which is $\widehat{C}^*_\alpha[D]$.

In the following we provide several examples. First note that $C[M] = C^*[M] = M$, where $M(u, v) = \min(u, v)$ is the Fréchet–Hoeffding upper bound copula.

Example 2 Consider the Fréchet-Hoeffding lower bound copula $W(u, v) = \max(u + v - 1, 0)$. Then the copulas generated by (10) and (11) are given by

$$C_\alpha\,[W]\,(u, v) = \max\left\{1 - \left((1-u)^{\frac{1}{\alpha}} + (1-v)^{\frac{1}{\alpha}}\right)^\alpha, 0\right\},$$

which is an Archimedean copula (see, e.g., Nelsen 2006, for detail) with the non-strict generator $\phi(t) = (1-t)^{\frac{1}{\alpha}}$. Since $\widehat{W} = W$, it follows from (12) that

$$C^*_\alpha\,[W]\,(u, v) = \widehat{C}_\alpha\,[W]\,(u, v) = u + v - 1 - \max\{[u^{\frac{1}{\alpha}} + v^{\frac{1}{\alpha}})]^\alpha - 1, 0\}.$$

Example 3 Consider the product copula $\Pi(u, v) = uv$, which is the copula of independent random variables. Then we have

$$C_\alpha\,[\Pi]\,(u, v) = 1 - (1-u)(1-v)\{(1-u)^{-\frac{1}{\alpha}} + (1-v)^{-\frac{1}{\alpha}} - 1\}^\alpha, \qquad (13)$$

which is an Archimedean copula with strict generator $\phi(t) = -\ln(1 - (1-t)^{\frac{1}{\alpha}})$. A copula of the form (13) belongs to the known Joe's family of copulas (see Joe 1997, for more details). Since the copula Π satisfies $\widehat{\Pi} = \Pi$, we obtain

$$C^*_\alpha[\Pi](u, v) = \widehat{C}_\alpha[\Pi](u, v) = u + v - uv\left\{u^{-\frac{1}{\alpha}} + v^{-\frac{1}{\alpha}} - 1\right\}^\alpha.$$

The following result provides a representation for the stability property given in (9) in terms of copulas.

Proposition 3 *Given a copula D and $\alpha, \beta \in (0, 1]$, $C_\beta[C_\alpha[D]] = C_{\alpha\beta}[D]$ and $C^*_\beta[C^*_\alpha[D]] = C^*_{\alpha\beta}[D]$.*

3.2 Dependence orderings

If C_1 and C_2 are two copulas, we say that C_2 is *more concordant than* C_1 (written $C_1 \prec_c C_2$) if $C_1 \leq C_2$. A copula C is *positively quadrant dependent* (written PQD) if $\Pi \prec_c C$—reversing the sense of the inequality we have *negatively quadrant dependent*

(NQD). A totally ordered parametric family $\{C_\alpha\}$ of copulas is *positively ordered* if $C_{\alpha_1} \prec_c C_{\alpha_2}$ whenever $\alpha_1 \leq \alpha_2$ and *negatively ordered* if $C_{\alpha_2} \prec_c C_{\alpha_1}$ whenever $\alpha_1 \leq \alpha_2$ (Joe 1997; Nelsen 2006). In what follows, we provide several properties on ordering of the family of copulas given by (10) and (11).

Proposition 4 *Let D_1 and D_2 be two copulas such that $D_1 \prec_c D_2$. Then for every $\alpha \in (0,1)$, $C_\alpha[D_1] \prec_c C_\alpha[D_2]$ and $C^*_\alpha[D_1] \prec_c C^*_\alpha[D_2]$.*

Proof The proof can be obtained by routine calculations. □

Proposition 5 *If the copula D is PQD, then the copulas generated by* (10) *and* (11) *are PQD too, i.e. if $D \succ_c \Pi$ then $C_\alpha[D] \succ_c \Pi$ and $C^*_\alpha[D] \succ_c \Pi$.*

Proof If $D \succ_c \Pi$ then $\widehat{D} \succ_c \Pi$. By using Proposition 4 we have that $C_\alpha[D] \succ_c C_\alpha[\Pi]$ and from equality (12), $\widehat{C}^*_\alpha[\widehat{D}] \succ_c \widehat{C}^*_\alpha[\Pi] = C_\alpha[\Pi]$. Thus it is enough to show that $C_\alpha[\Pi] \succ_c \Pi$. Since $C_\alpha[\Pi]$ and Π are both Archimedean copulas with the generators $\phi_1(t) = -\ln(1-(1-t)^{\frac{1}{\alpha}})$ and $\phi_2(t) = -\ln(t)$, respectively, the required result follows from the fact that ϕ_2/ϕ_1 is nondecreasing on (0,1) (see Genest and MacKey 1986). □

3.3 Tail dependence coefficients

A reason for adding new parameters to a given copula is to produce families that exhibit some more flexible properties. In particular, copulas with different tail behaviour are often useful to build models for estimating the extreme and risky events (Joe 1997). In the following we show how the proposed model (10) and (11) may modify the tail behaviour of a given copula D, as measured by its tail dependence coefficients. For a given copula D, the lower (resp. upper) tail dependence coefficient, λ_L (resp. λ_U) is defined by (Joe 1997; Nelsen 2006)

$$\lambda_L(D) = \lim_{u\to 0^+} \frac{D(u,u)}{u}, \tag{14}$$

and the upper tail dependence coefficient

$$\lambda_U(D) = 2 - \lim_{u\to 1^-} \frac{1-D(u,u)}{1-u}. \tag{15}$$

Proposition 6 *For a given copula D let $C_\alpha[D]$ and $C^*_\alpha[D]$ be the copulas defined by* (10) *and* (11)*. Then*

$$\lambda_L(C_\alpha[D]) = \lambda_L(D), \quad \lambda_U(C_\alpha[D]) = 2-(2-\lambda_U(D))^\alpha,$$

and

$$\lambda_L(C^*_\alpha[D]) = 2-(2-\lambda_L(D))^\alpha, \quad \lambda_U(C^*_\alpha[D]) = \lambda_U(D).$$

Proof By taking into account (14), the lower tail dependence coefficient of $C_\alpha[D]$ can be expressed as

$$\begin{aligned}\lambda_L(C_\alpha[D]) &= \lim_{u\to 0^+} \frac{1-\{1-D(1-(1-u)^{\frac{1}{\alpha}}, 1-(1-u)^{\frac{1}{\alpha}})\}^\alpha}{u} \\ &= \lim_{u\to 0^+} \frac{1-(1-D(u,u))^\alpha}{1-(1-u)^\alpha} \\ &= \lim_{u\to 0^+} \frac{d}{du} D(u,u) = \lambda_L(D).\end{aligned}$$

By taking into account (15), the upper tail dependence coefficient of $C_\alpha[D]$ can be expressed as

$$\begin{aligned}\lambda_U(C_\alpha[D]) &= 2 - \lim_{u\to 1^-} \frac{\left\{1-D(1-(1-u)^{\frac{1}{\alpha}}, 1-(1-u)^{\frac{1}{\alpha}})\right\}^\alpha}{1-u} \\ &= 2 - \left(\lim_{u\to 1^-} \frac{1-D(u,u)}{1-u}\right)^\alpha \\ &= 2-(2-\lambda_U(D))^\alpha.\end{aligned}$$

Similar argument holds for the lower and upper tail dependence of $C^*_\alpha[D]$. □

3.4 Measures of association

The population version of three of the most common nonparametric measures of association between the components of a continuous random pair (X, Y) are *Kendall's tau* (τ), *Spearman's rho* (ρ) and *Blomqvist's medial correlation coefficient* (β) which depend only on the copula D of the pair (X, Y), and are given by

$$\tau(D) = 4\int_0^1\int_0^1 D(u,v)\,dD(u,v) - 1,$$

$$\rho(D) = 12\int_0^1\int_0^1 D(u,v)dudv - 3, \tag{16}$$

and

$$\beta(D) = 4D\left(\frac{1}{2},\frac{1}{2}\right) - 1. \tag{17}$$

See (Nelsen 2006) for details. The following result provides expressions for these measures associated with the copulas defined by (10) and (11).

Proposition 7 *For a given copula D, let $C_\alpha[D]$ be the copula defined by (10). Then for every $\alpha \in (0, 1]$,*

$$\tau(C_\alpha[D]) = \frac{1-4\alpha}{1-2\alpha} - 4\alpha^2 \int_0^1 (1-t)^{2\alpha-2} K_D(t)dt, \tag{18}$$

where $K_D(t) = P[D(U,V) \le t]$, $t \in (0,1)$, is the Kendall's distribution function of the copula D,

$$\rho(C_\alpha[D]) = 9 - 12\alpha^2 \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} \int_0^1 \int_0^1 [D(u,v)]^j (1-u)^{\alpha-1} (1-v)^{\alpha-1} du dv, \tag{19}$$

and

$$\beta(C_\alpha[D]) = 3 - 4\left(1 - D\left(1-2^{-\frac{1}{\alpha}}, 1-2^{-\frac{1}{\alpha}}\right)\right)^\alpha. \tag{20}$$

Proof For a given copula D, in view of Proposition 1 in (Genest and Rivest 2001) with the strictly increasing, differentiable bijection $\gamma(t) = 1-(1-t)^{\frac{1}{\alpha}}$ on the interval $(0, 1)$, the Kendall distribution function of $C_\alpha[D]$ could be obtained as

$$K_{C_\alpha[D]}(t) = t - \alpha(1-t)^{1-\frac{1}{\alpha}} \left\{1-(1-t)^{\frac{1}{\alpha}} - K_D(1-(1-t)^{\frac{1}{\alpha}})\right\}.$$

Now (18) follows from the fact that $K_{C_\alpha[D]}$ is related to the $\tau(C_\alpha[D])$ (see Nelsen 2006) via

$$\tau(C_\alpha[D]) = 3 - 4\int_0^1 K_{C_\alpha[D]}(t)dt.$$

The expression for $\rho(C_\alpha[D])$ may be easily deduced from formula (16) and using the binomial expansion

$$(1-x)^\alpha = \sum_{j=0}^{\infty} \binom{\alpha}{j} (-1)^j x^j \quad |x| < 1, \alpha \in \mathbb{R}.$$

The expression for $\beta(C_\alpha[D])$ follows from(10) and (17). □

The following result provides the lower bounds for Kendall's tau, Spearman's rho and Blomqvist's beta associated with the copula defined by (10).

Proposition 8 *For a given copula D let $C_\alpha[D]$ be the copula defined by (10). Then for each $\alpha \in (0, 1]$*

$$1 - 2\alpha \le \tau(C_\alpha[D]) \le 1, \quad 9 - 12\alpha^2 B(\alpha) \le \rho(C_\alpha[D]) \le 1 \quad \text{and}$$
$$3 - 2^{\alpha+1} \le \beta(C_\alpha[D]) \le 1,$$

where

$$B(\alpha) = \sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^k \binom{\alpha}{j}\binom{j}{k}\frac{B(\alpha+j+1,\alpha)}{\alpha+j-k},$$

where $B(a, b)$ denotes the Beta function $B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$.

Proof Since any copula D satisfies that $W \prec_c D \prec_c M$, by using Proposition 4 and that $C_\alpha[M] = M$, we have $C_\alpha[W] \prec_c C_\alpha[D] \prec_c M$ and then

$$\tau(C_\alpha[W]) \le \tau(C_\alpha[D]) \le 1, \quad \rho(C_\alpha[W]) \le \rho(C_\alpha[D]) \le 1 \text{ and}$$
$$\beta(C_\alpha[W]) \le \beta(C_\alpha[D]) \le 1.$$

By using (18) and the fact that $K_W(t) = 1$, $t \in [0, 1]$ we obtain that $\tau(C_\alpha[W]) = 1 - 2\alpha$. The lower bound for the Spearman's rho, $\rho(C_\alpha[D])$, can be calculated from (19) with $D = W$ and the fact that

$$\begin{aligned}
&\int_0^1\int_0^1 [W(u, v)]^j(1-u)^{\alpha-1}(1-v)^{\alpha-1}dudv \\
&= \int_0^1\int_{1-u}^1 (u+v-1)^j(1-u)^{\alpha-1}(1-v)^{\alpha-1}dvdu \\
&= \sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}\frac{1}{\alpha+j-k}\int_0^1 u^{\alpha+j}(1-u)^{\alpha-1}du \\
&= \sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}\frac{B(\alpha+j+1,\alpha)}{\alpha+j-k}.
\end{aligned}$$

The lower bound of Blomqvist's β follows from (20), which completes the proof. □

Remark 4 Note that for every copula-based measure of association κ, satisfying Scarsini's axioms (Scarsini 1984), $\kappa(C) = \kappa(\widehat{C})$. Therefore using (12) one can obtain the measures of association κ, related to the copula $C^*_\alpha[D]$ via

$$\kappa(C^*_\alpha[D]) = \kappa(C_\alpha[\widehat{D}]).$$

The following result provides the expressions for Kendall's tau, Spearman's rho and Blomqvist's beta associated with the copula $C_\alpha[\Pi]$.

Proposition 9 *Let $C_\alpha[D]$ be the copula defined by (10) with $D = \Pi$. Then for every $\alpha \in (0, 1]$,*

$$\rho(C_\alpha[\Pi]) = 9 - 12\alpha^2 \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} [B(j+1, a)]^2,$$
$$\tau(C_\alpha[\Pi]) = 1 + 4\alpha B(2, 2\alpha - 1)(\Psi(2) - \Psi(2\alpha + 1)),$$

and

$$\beta(C_\alpha[\Pi]) = 3 - \left(2\frac{1+\alpha}{\alpha} - 1\right)^\alpha,$$

where Ψ is the digamma function.

4 Concavity properties

A copula C is Schur-concave if and only if,

$$C(u, v) \leq C(\lambda u + (1-\lambda)v, (1-\lambda)u + \lambda v),$$

for all $u, v \in (0, 1)$ and $\lambda \in [0, 1]$ (see Nelsen 2006).

The following result shows that Schur-concavity of a given copula D is preserved under the constructions (10) and (11).

Proposition 10 *Let D be a Schur-concave copula. Then the generated copulas $C_\alpha[D]$ and $C_\alpha^*[D]$ defined by (10) and (11), respectively, are Schur-concave as well.*

Proof Let $h(t) = 1 - (1-t)^{\frac{1}{\alpha}}$. Then h is a concave function and

$$h(\lambda x + (1-\lambda)x) \geq \lambda h(x) + (1-\lambda)h(y)$$

and

$$h((1-\lambda)x + \lambda y) \geq (1-\lambda)h(x) + \lambda h(y).$$

Moreover, the Schur-concavity of D implies that

$$D(\lambda h(x) + (1-\lambda)h(x), (1-\lambda)h(x) + \lambda h(y)) \geq D(h(x), h(y)).$$

Since D is increasing in each variable, by definition of $C_\alpha[D]$, we have

$$C_\alpha[D](\lambda x + (1-\lambda)y, (1-\lambda)x + \lambda y) \geq C_\alpha[D](x, y).$$

The Schur-concavity of $C_\alpha^*[D]$ follows from the relation (12) and the fact that if D is Schur-concave then $\widehat{D}$ is Schur-concave as well. □

A copula C is said to be quasi-concave (Nelsen 2006) if for all $u, v, u', v' \in \mathbb{I}^2$ and all $\lambda \in [0, 1]$,

$$C(\lambda u + (1-\lambda)v, \lambda u' + (1-\lambda)v') \geq \min\{C(u, u'), C(v, v')\}.$$

The following result shows that quasi-concavity of a given copula D is preserved under the constructions (10) and (11).

Proposition 11 *If D is quasi-concave, then the generated copula $C_\alpha[D]$ given by* (10)*, is also quasi-concave.*

Proof Let u, u', v, v' and λ be in [0,1]. Since $h(t) = 1-(1-t)^{\frac{1}{\alpha}}$ is concave, we have that

$$h(\lambda u + (1-\lambda)v) \geq \lambda h(u) + (1-\lambda)h(v),$$

and

$$h(\lambda u' + (1-\lambda)v') \geq \lambda h(u') + (1-\lambda)h(v').$$

Moreover, since D is increasing in each variable and quasi-concave, we have

$$\begin{aligned} &D(h(\lambda u + (1-\lambda)v), h(\lambda u' + (1-\lambda)v')) \\ &\quad \geq D(\lambda h(u) + (1-\lambda)h(v), \lambda h(u') + (1-\lambda)h(v')) \\ &\quad \geq \min\left\{D(h(u), h(u')), D(h(v), h(v'))\right\}. \end{aligned}$$

But h is increasing so that by definition of $C_\alpha[D]$ we have that

$$C_\alpha[D](\lambda u + (1-\lambda)v, \lambda u' + (1-\lambda)v') \geq \min\{C_\alpha[D](u, u'), C_\alpha[D](v, v')\}.$$

□

Since each Archimedean copula is quasi-concave (see, e.g., Tibiletti 1995) as a consequence of Proposition (11) we have:

Corollary 1 *If D is an Archimedean copula then the copula $C_\alpha[D]$ generated by* (10) *is also quasi-concave.*

5 Discussion

We have introduced a method for adding a parameter to a given bivariate distribution to construct new families. The proposed families have proportional (reversed) hazard rate models as their univariate marginals. We study different dependence properties of the proposed models in terms of their associated copulas. Applications and advantages of these models to obtain certain bivariate distributions such as the bivariate exponential, general exponential, Pareto and Weibull distributions through data examples, is in progress. The reader will recognize that our method could be used to generalize any given d-dimensional distribution function. Let F be a given d-dimensional distribution function with univariate marginals $F_1, \ldots, F_d$. Then for $\alpha \in (0, 1]$, the function defined by

$$G(x_1, \ldots, x_d) = 1 - (1 - F(x_1, \ldots, x_d))^{\alpha}$$

is a new multivariate distribution function with univariate marginal's given by $G_i(x) = 1 - \overline{F_i}^{\alpha}(x),\ i = 1, \ldots, d$, and the function

$$\overline{H}(x_1, \ldots, x_d) = 1 - \left(1 - \overline{F}(x_1, \ldots, x_d)\right)^{\alpha}$$

defines a multivariate survival function with univariate marginal's $\overline{H}_i(x) = 1 - F_i^{\alpha}(x),\ i = 1, \ldots, d$.

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