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A new fluctuation test for constant variances with applications to finance

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Abstract We present a test to determine whether variances of time series are constant over time. The test statistic is a suitably standardized maximum of cumulative first and second moments. We apply the test to time series of various assets and find that the test performs well in applications. Moreover, we propose a portfolio strategy based on our test which hedges against potential financial crises and show that it works in practice.

Keywords Econometric modeling · Finance · Portfolio optimization · Structural breaks · Variance

1 Introduction

It is well known, in particular in empirical finance, that variances among many time series cannot be assumed to remain constant over longer stretches of time (Krishan et al. 2009). Especially, variances of stock indices seem to vary over time. A good example

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is the recent financial crisis, in which capital market volatilities and correlations raised quite dramatically. As a consequence, risk figures increased significantly as diversification effects were overestimated (Bissantz et al. 2011). In literature, this phenomenon is sometimes referred to as "Diversification Meltdown" (Campbell et al. 2008) and is well known also from other contexts.

A change in market parameters has serious consequences in practice, in particular for portfolio optimization which is based on diversification effects between several assets. If the relevant market parameters (e.g. volatilities) change, the optimization is no longer valid and the risk incorrectly calculated. Similar problems occur to applications in risk management and the valuation of financial instruments.

There are some methods to formally test for changes in volatilities, correlations or other dependence measures and/or procedures for estimation of change points; many of them work in a parametric environment (Chu 1995; Chen and Gupta 1997; Kokoszka and Leipus 2000; Dias and Embrechts 2004; Mikosch and Starica 2004; Andreou and Ghysels 2006; Galeano and Peña 2007), look at conditional parameters (Andreou and Ghysels 2002), assume that potential break points are known (Pearson and Wilks 1933; Jennrich 1970; Goetzmann et al. 2005), or simply estimate correlations from moving windows without giving a formal decision rule (Longin and Solnik 2002). See also Andreou and Ghysels (2010) who provide a review on some of these methods.

Only recently, Aue et al. (2009) and Wied et al. (2011) have proposed formal completely nonparametric tests for unconditional dependence measures in this context. They do not build upon prior knowledge as to the timing of potential shifts. Aue et al. (2009) propose a test to detect changes in the covariance structure, while Wied et al. (2011) present a method to test for changes in the correlation structure between assets. They are based on cumulated sums of second order empirical cross moments (in the style of Ploberger et al. 1989) and reject the null of constant covariance or correlation structure if these cumulated sums fluctuate too much.

This paper considers a non-parametric fluctuation test for constant variances over time. On the one hand, this test can be regarded as a special case of the Aue et al. (2009)-test for the one-dimensional case. On the other hand it goes beyond it by rigorously proving the asymptotic null distribution for the case that the expected values are estimated by arithmetic means basing on the first j observations (so that we compare successively estimated empirical variances). Moreover, we derive the asymptotic distribution of our test statistic under local alternatives.We use proving methods that were also used for the test for constant correlation described in Wied et al. (2011).

Our second contribution is the application to financial data and the derivation of an investment strategy. We analyze the volatility structure of four indices including stocks, bonds and commodities and see that the test performs well throughout the whole empirical application. The resulting dates of rejection seem to be reasonable. Besides, we suggest a simple investment strategy based on the test and evaluate it by an out-of-sample study.

The paper is organized as follows. First, we describe the test statistic and its asymptotic distribution in Sect. 2. Section 3 derives local power properties, Sect. 4 analyzes the finite sample performance of the test by a small simulation study, Sect. 5 applies the test to financial data and Sect. 6 concludes. Proofs are given below the summary in the Appendix.

2 Model and test statistic

Let $(X_t, t = 1, 2 \ldots)$ be a sequence of random variables with finite absolute $(4+\delta)$th moments. We want to test whether the variance of X_t,

$$\mathsf{Var}(X_t) = \mathsf{E}(X_t^2) - (\mathsf{E}(X_t))^2,$$

is constant over time, i.e. we test

$$H_0 : \mathsf{Var}(X_t) = \sigma^2 \ \forall t \in \{1, \ldots, T\} \text{ vs. } H_1 : \exists t \in \{1, \ldots, T-1\} : \mathsf{Var}(X_t) \neq \mathsf{Var}(X_{t+1})$$

for a constant σ^2. Our test statistic is

$$Q_T(X) = \max_{1 \le j \le T} \left| \hat{D} \frac{j}{\sqrt{T}} \left([\mathsf{Var}X]_j - [\mathsf{Var}X]_T \right) \right| \tag{1}$$

where

$$[\mathsf{Var}X]_l = \frac{1}{l} \sum_{t=1}^{l} X_t^2 - \left(\frac{1}{l} \sum_{t=1}^{l} X_t \right)^2 =: \overline{X_l^2} - \left(\overline{X_l} \right)^2$$

is the empirical variance calculated from the first l observations. Furthermore,

$$\hat{D} = \left(\left(1, -2\overline{X_T}\right) \hat{D}_1 \left(1, -2\overline{X_T}\right)' \right)^{-1/2}$$

with

$$\hat{D}_1 = \frac{1}{T} \sum_{t=1}^{T} \hat{U}_t \hat{U}_t' + 2 \sum_{j=1}^{T} k\left(\frac{j}{\gamma_T} \right) \frac{1}{T} \sum_{t=1}^{T-j} \hat{U}_t \hat{U}_{t+j}'$$

and

$$\hat{U}_l = \begin{pmatrix} X_l^2 - \overline{X_T^2} \\ X_l - \overline{X_T} \end{pmatrix},$$

$$k(x) = \begin{cases} 1 - |x|, & |x| \le 1 \\ 0, & otherwise \end{cases},$$

$$\gamma_T = \sqrt{T}.$$

The scalar $\hat{D}$ is needed for the asymptotic null distribution. It mainly captures the long-run-dependence and the fluctuations resulting from estimating the expected value. The test rejects the null hypothesis of constant variance if the empirical variances fluctuate too much, as measured by

$\max_{1\le j\le T}\left|[\mathsf{Var}X]_j - [\mathsf{Var}X]_T\right|$, with the weighting factor $\frac{j}{\sqrt{T}}$ scaling down deviations at the beginning of the sample where the $[\mathsf{Var}X]_j$ are more volatile.

The following technical assumptions are required for the limiting null distribution:

(A1) The sequence $(X_t, t = 1, 2\ldots)$ is weak-sense stationary.

(A2) For

$$U_t = \begin{pmatrix} X_t^2 - \mathsf{E}(X_1^2) \\ X_t - \mathsf{E}(X_1) \end{pmatrix}$$

and $S_j := \sum_{t=1}^{j} U_t$, we have

$$\lim_{T\to\infty} \mathsf{E}\left(\frac{1}{T} S_T S_T'\right) =: D_1 \text{ is finite and positive definite.}$$

(A3) The r-th absolute moments of the components of U_t are uniformly bounded for some $r > 2$.

(A4) The sequence $(X_t, t = 1, 2\ldots)$ is L_2-NED (near-epoch dependent) with size $-\frac{r-1}{r-2}$, with r from (A3), and constants (c_t), $t = 1, 2, \ldots$ on a sequence (V_t), $t = 1, 2, \ldots$, which is α-mixing of size $\phi^* := -\frac{r}{r-2}$, such that

$$c_t \le 2\left(\left\{\mathsf{E}|X_t^2 - \mathsf{E}(X_1^2)|^2 + \mathsf{E}|X_t - \mathsf{E}(X_1)|^2\right\}\right)^{\frac{1}{2}}.$$

Assumption (A4) guarantees that

$$U_t^* := \left(X_t^2,\ X_t\right)'$$

is L_2-NED with size $\frac{1}{2}$, see Davidson (1994). It could be modified to ϕ-mixing, requiring only finite 4-th moments, but this would admit less serial dependence than we allow here. In particular, assumption (A4) allows for GARCH-effects (see e.g. Hansen 1991 or Carrasco and Chen 2002), which are observed in financial data. Note that Assumption (A1) is already partly fulfilled because we assume constant variances under the null. The assumption of constant expected values is in line with Aue et al. (2009).

To investigate large sample properties, we make the transformation

$$Q_T(X) = \sup_{z\in[0,1]}\left|\hat{D}\frac{\tau(z)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(z)} - [\mathsf{Var}X]_T\right)\right|$$

with $\tau(z) = [1 + z(T-1)]$.

Theorem 2.1 *Under H_0 and Assumptions (A1)–(A4),*

$$Q_T(X) \to \sup_{z\in[0,1]} |B(z)|,$$

where $B(z)$ is a one-dimensional Brownian Bridge.

The limit distribution of $Q_T(X)$ is well known, see Billingsley (1968), and its quantiles provide an asymptotic test.

3 Local power

In this section, we analyze the local power properties of our test. Since the distribution of the time series now changes with T, we will deal with triangular arrays, i.e. the random variables $(X_t), t \in \mathbb{Z}$, and $(V_t), t \in \mathbb{Z}$, from assumption (A4) form a triangular array. However, we stick to the former notation for simplicity, i.e. $(X_t) := (X_{t,T}), (V_t) := (V_{t,T}), t \in \mathbb{Z}; T = 1, 2, \ldots$.

We replace assumption (A1) of weak-sense stationarity by

(A5) The sequence $(X_t, t = 1, 2, \ldots)$ fulfills all properties of weak-sense stationarity except for $\mathbb{E}(X_t^2) = m_x^2 + \frac{1}{\sqrt{T}} g\left(\frac{t}{T}\right)$ for a constant m_x^2 and a bounded function g which can be approximated by step functions and which is not identically 0 such that the function

$$\int_0^z g(u)du - z \int_0^1 g(u)du$$

is different from 0 for at least one $z \in [0, 1]$.

A typical example for the function g would be a step function with a jump from 0 to g_0 in a given point z_0 which implies that the variance jumps at time $[T \cdot z_0]$. A piecewise constant function g with multiple jumps would lead to multiple change points as in e.g. Inoue (2001) and using a continuous function g would lead to continuously changing variances. Such local alternatives are also considered in Ploberger and Krämer (1990) who analyze local power properties of the CUSUM and CUSUM of squares test.

Theorem 3.1 *Under Assumptions (A2)–(A5),*

$$Q_T(X) \to \sup_{z \in [0,1]} |B(z) + D(z)|,$$

where $D(z) = C\left(\int_0^z g(u)du - z \int_0^1 g(u)du\right)$ *and* C *is a positive constant which is given in the Appendix.*

$D(z)$ is a deterministic function which depends on the specific form of the local alternative under consideration, characterized by g.

In combination with Anderson's Lemma, Theorem 3.1 guarantees that the asymptotic power is always larger than or equal to α, see Andrews (1997), p. 1114.

The supremum is now taken over the absolute value of a Brownian Bridge plus a deterministic function $D(z)$. Its distribution is rather unwieldy, but it is possible to give a more simple result for the rejection probability for large g. To this purpose, rewrite assumption (A5) as $g(z) = Mh(z)$ for a function h and a factor M. The function h represents the structural form of the alternative, whereas M captures its amplitude.

Corollary 3.2 *Let Assumptions (A2)–(A5) be true with* $g(z) = Mh(z)$. *Let* $P_{H_1}(M)$ *be the rejection probability for given* M *under the alternative and let* $\epsilon > 0$. *Then there is a* M_0 *such that*

$$\lim_{T\to\infty} P_{H_1}(M) > 1 - \epsilon$$

for all $M > M_0$.

This means that local rejection probabilities become arbitrarily large as structural changes are increasing.

4 Finite sample behavior

In this section, we investigate the finite sample properties of our test. First, we analyze the size under fulfilled assumptions. Since Assumption (A3) is questionable in financial data due to heavy tails (the third or fourth moment might not exist), we also investigate the robustness against violations of this assumption. Next, we analyze the power properties of the test.

Finally, we analyze the size properties in an online study, i.e. if we want to do sequential testing by successively enlarging the data day-by-day. Theorem 2.1 shows that the test asymptotically keeps the size if it is applied once. The additional question here is how the size is affected in an online application if several tests are performed.

For the size analysis, we use an $AR(1)$-process with $\rho = 0.1$ and t_ν-distributed innovations with expectation 0, variance $\sigma^2 = 1$ and different values of ν. The assumptions require $\nu > 4$, but we also include smaller values of ν. Anyway, ν must be larger than 2 so that the variance exists. We vary the length of the time series T, always use 5,000 replications and a nominal level of $\alpha = 1$ and 5%, respectively. Tables 1 and 2

Table 1 Empirical size ($\alpha = 1\%$)

	$T = 200$	$T = 500$	$T = 800$	T =1,000
$\nu = 3$	<0.001	<0.001	0.001	0.001
$\nu = 4$	<0.001	0.001	0.003	0.001
$\nu = 5$	0.001	0.001	0.002	0.002
$\nu = 8$	0.001	0.002	0.002	0.003
$\nu = 20$	0.001	0.003	0.004	0.005

Table 2 Empirical size ($\alpha = 5\%$)

	$T = 200$	$T = 500$	$T = 800$	T =1,000
$\nu = 3$	0.009	0.011	0.018	0.014
$\nu = 4$	0.014	0.021	0.020	0.021
$\nu = 5$	0.016	0.019	0.023	0.027
$\nu = 8$	0.015	0.023	0.028	0.029
$\nu = 20$	0.019	0.025	0.031	0.040

Table 3 Empirical power: the variance is equal to $\sigma_1^2 = 1$ in the first half of the sample and equal to σ_2^2 in the second half ($\alpha = 1\%$)

	$T = 200$	$T = 500$	$T = 800$	$T = 1{,}000$
$\sigma_2^2 = 2$	0.023	0.335	0.672	0.796
$\sigma_2^2 = 4$	0.202	0.879	0.969	0.982
$\sigma_2^2 = 0.5$	0.013	0.304	0.650	0.788
$\sigma_2^2 = 0.25$	0.151	0.872	0.966	0.979

Table 4 Empirical power: the variance is equal to $\sigma_1^2 = 1$ in the first half of the sample and equal to σ_2^2 in the second half ($\alpha = 5\%$)

	$T = 200$	$T = 500$	$T = 800$	$T = 1{,}000$
$\sigma_2^2 = 2$	0.262	0.718	0.896	0.939
$\sigma_2^2 = 4$	0.718	0.972	0.991	0.993
$\sigma_2^2 = 0.5$	0.216	0.682	0.886	0.931
$\sigma_2^2 = 0.25$	0.675	0.968	0.987	0.991

Table 5 Results for the online detection

	$T = 200$	$T = 500$	$T = 800$	$T = 1{,}000$
$\alpha = 5\%$	0.140	0.228	0.270	0.294
$\alpha = 1\%$	0.008	0.016	0.022	0.026

give the results; we see that the test basically keeps the size, but is too conservative. Nevertheless, the size increases for increasing T and increasing ν, although it seems that there are convergence problems for $\nu \leq 4$.

The setup for the power analysis is the same as before with the only difference that the variance jumps from $\sigma_1^2 = 1$ to different values of σ_2^2 in the middle of the time series. The choices of σ_2^2 are quite realistic because volatilities vary a lot in practice (see, e.g., Bissantz et al. 2011). We consider different amounts of increasing as well as decreasing variances. We use t_5-distributed innovations. The results are written down in Tables 3 and 4, it is especially seen that the power increases with T.

For the online setup, we generate time series of length T with again $AR(1)$-process with $\rho = 0.1$, t_5-distributed innovations, variance 1 and the nominal levels $\alpha = 5\%$ and $\alpha = 1\%$. We perform tests in a sequential way, i.e. we first apply the test on the first 20-th data point (see the application section for a discussion of this choice), then on the 21-st data point and so on. The final test statistic is given by the maximum over all $T - 20 + 1$ test statistics. Table 5 gives the empirical size in this setup. We see that the actual size increases in T and is higher than the nominal levels (especially for $\alpha = 5\%$), but that it is still controlled, i.e. it does not reach 1. Nevertheless, the empirical size for $\alpha = 5\%$ seems to be too high for practical investigations. Hence, it would be worthwhile to implement an theoretical analysis about this issue to adjust the overall size to a given α using ideas of Chu et al. (1996), but this lies beyond the scope of the present paper. The overall size for $\alpha = 1\%$ is still acceptable for practical applications.

5 Applications

5.1 Historical rejection dates

In order to evaluate the quality of the test it is applied to several assets: two stock indices (S&P 500, DAX), a commodity index (CRB Spot Index) and a government bond index (REX), using daily data (final quote) for the time span 01.01.1988–01.04.2010. The procedure for the test is as follows. We start at the 20-th available data point and increase the period of time successively for 1 day. The choice of the starting point is due to the fact that approximately 20 data points are required for a reliable estimation of the volatility. For each of these time intervals the test is applied for $\alpha = 1\%$. This procedure is successively performed until the tests rejects the null hypothesis of constant volatility. If this is the case, the 20-th day after rejection is the new starting point and the procedure is repeated for the remaining time span. We have to wait these 20 days as the volatility cannot be assumed to be constant anymore, if the null hypothesis is rejected. A new reliable estimation requires another 20 data points after the point in time, where the volatility changed. Otherwise, the estimator would be biased as data of two different phases were mixed.

Table 6 includes the rejection dates of the null hypothesis for $\alpha = 1\%$. There are not too many break points detected, which coincides with our simulation study stating that there are no serious overrejection problems for $\alpha = 1\%$.

The results seem to be reasonable. For example, the rejection dates coincide with the Asian financial crisis in 1997, the LTCM collapse and the ruble crisis in 1998, the beginning of the war on Iraq in 2003, the bursting of the U.S. real estate bubble in 2007 or the Lehman bankruptcy in 2008.

Besides, large differences of the market parameters between the break points can be observed. Figure 1 and Table 7 illustrate this phenomenon for the DAX. Table 7 includes the annualized market parameters (returns and volatilites) for the respective period between two structural breaks. Figure 1 shows the average and the rolling 250-day volatility of the DAX. Besides, the rejection dates are given for $\alpha = 1\%$.

5.2 A trading strategy

The results above show that changes in market parameters can be detected reasonably for $\alpha = 1\%$. In order to derive a trading strategy, which is based on the proposed

Table 6 Rejection dates ($\alpha = 1\%$)

S&P	DAX	REX	CRB
02.12.1993	29.01.1988	10.10.1994	17.11.1998
27.03.1997	12.07.1989	18.03.2009	29.05.2009
15.08.2005	04.10.1994		26.06.2009
11.12.2007	21.10.1997		
01.12.2008	24.03.2003		
10.09.2009	23.12.2004		
	06.10.2008		

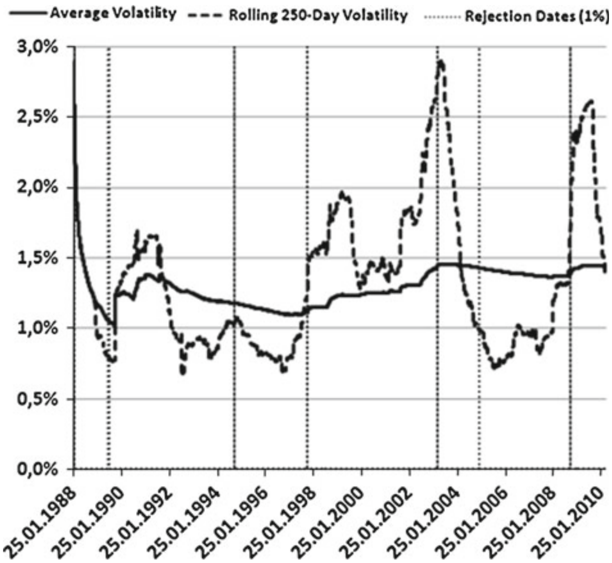

Fig. 1 Volatility and structural breaks of the DAX

Table 7 Rejection dates and annualized market parameters ($\alpha = 1\%$)

DAX	Returns (%)	Volatilities (%)
29.01.1988–12.07.1989	31.91	14.33
12.07.1989–04.10.1994	5.28	19.06
04.10.1994–21.10.1997	21.93	15.18
21.10.1997–24.03.2003	−7.03	29.92
24.03.2003–23.12.2004	24.35	21.21
23.12.2004–06.10.2008	7.91	16.89
06.10.2008–01.04.2010	4.70	34.28

test, we perform an out of sample study. In this study, we investigate a simple strategy which applies the proposed test.

The strategy is as follows. The available time span since the last detected change in volatility is used to calculate the historical return which is used as an estimator for the future. Moreover, an asset is allowed to be bought if at least 20 days have passed since the last structural break. Finally, the capital is equally distributed between all assets with positive expected future return.

Portfolio shiftings are done the day after the test rejected in order to design the study realistic. We choose $\alpha = 1\%$ for the test and neglect transaction costs. Besides, we assume daily rebalancing and neglect currency fluctuations.

The results can be found in Fig. 2 and Table 8.

The average return of the strategy is 1.06% higher than the average of the underlying assets. The volatility is lower, both compared to the arithmetic mean of all asset volatilities (30.27%) and compared to the volatility of the naive portfolio in which diversification effects are included (3.55%). Moreover, the portfolio development is

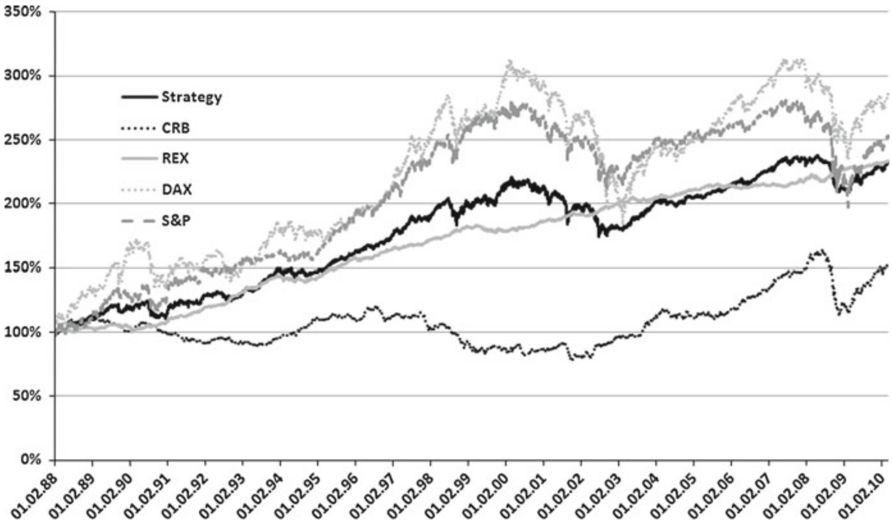

Fig. 2 Strategy and underlying assets

Table 8 Summary statistics for all indices and our strategy

	Strategy (%)	CRB (%)	REX (%)	DAX (%)	S&P (%)	Naive strategy (%)
Return p.a.	5.74	2.23	5.76	8.09	6.62	5.68
Volatility p.a.	8.77	6.51	3.30	22.65	17.84	9.09

relatively stable and only a little money is lost during financial crisis. This result is very remarkable as three risky assets are considered throughout the study.

6 Summary

In this paper, we introduced and proved a new test to determine whether variances of time series are constant over time. Thereby, the test statistic is a suitably standardized maximum of cumulative first and second moments. We applied the test to several time series of assets which are relevant for applications in finance and found that the test performs well in these applications. The market parameters fluctuate a lot comparing the different periods between structural breaks.

Moreover, we derived a simple trading strategy, which outperforms a strategy based on equal portfolio weights. More precisely, the return increased by 1.06% while at the same time the volatility decreased. This is remarkable because the trading strategy is very simple. We believe that refinements of the strategy will lead to further improvements. This topic will be in focus of our ongoing research.

Apart from such refinements, there are other aspects which might be worth investigating in the future.

The test statistic (1) is the supremum over the $[\mathsf{Var} X]_j$-series. Alternatively, other functionals are likewise possible, such as some standardized version of

$$\max_{1\le j\le T}\left([\mathsf{Var}X]_j-[\mathsf{Var}X]_T\right)-\min_{1\le j\le T}\left([\mathsf{Var}X]_j-[\mathsf{Var}X]_T\right),$$

or simply some suitable average (see Krämer and Schotman 1992, or Ploberger and Krämer 1992). Another interesting topic would be a detailed discussion of the change point locations. A CUSUM test for retrospective break detection always yields a natural estimator of the (dominating) change point in a given time series, if the test rejects the null hypothesis. In our case, it is the point where the weighted differences of the variances are maximal, i.e.

$$\text{argmax}_{1\le j\le T}\left|\hat{D}\frac{j}{\sqrt{T}}\left([\mathsf{Var}X]_j-[\mathsf{Var}X]_T\right)\right|. \tag{2}$$

It might be an interesting question for future work, if one obtains different change point locations with such a retrospective analysis.

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Appendix: Proofs

A.1 Main proofs

For the proof of Theorem 2.1 and 3.1, we need some lemmas and some notation: Let I be some index set, e.g. $I=[\epsilon,1]$ for some $\epsilon\in[0,1)$. For an integer $k\ge 1$, let $l_\infty(I,\mathbb{R}^k)$ be the set of all bounded functions $\theta: I\to\mathbb{R}^k$, equipped with supremum norm

$$||\theta||_\infty := \sup_{i\in I}||\theta(i)||,$$

where $||\cdot||$ denotes Euclidean norm.

At first, we consider the behavior under the null hypothesis, i.e. we prove Theorem 2.1.

Lemma A.1 *Under H_0 and Assumptions (A1)–(A4), in $l_\infty([0,1],\mathbb{R})$,*

$$\hat{D}\frac{\tau(\cdot)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(\cdot)}-\sigma^2\right)\to_d W_1(\cdot)$$

where $\sigma^2=\mathsf{E}(X_1^2)-(\mathsf{E}(X_1))^2$ and $W_1(z)$ is a one-dimensional Brownian Motion.

Note that Lemma A.1 gives a result about convergence on the interval $[0,1]$. It requires the following auxiliary lemma which differs from Lemma A.1 by considering the interval $[\epsilon,1]$ for arbitrary $\epsilon>0$.

Lemma A.2 *Under H_0 and Assumptions (A1)–(A4), for arbitrary $\epsilon>0$, in $l_\infty([\epsilon,1],\mathbb{R})$,*

$$\hat{D}\frac{\tau(\cdot)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(\cdot)}-\sigma^2\right)\to_d W_1(\cdot)$$

where $\sigma^2=\mathsf{E}(X_1^2)-(\mathsf{E}(X_1))^2$ *and* $W_1(z)$ *is a one-dimensional Brownian Motion.*

Lemma A.2 is proved with a basic theorem on a modified functional delta method given in Subsect. A.2 which is also used in Wied et al. (2011). This method is only applicable on the interval $[\epsilon, 1]$ and not on $[0, 1]$, because $\sup_{z\in[\epsilon,1]}\left|\frac{\sqrt{T}}{\tau(z)}\right|\to 0$, while $\sup_{z\in[0,1]}\left|\frac{\sqrt{T}}{\tau(z)}\right|\to\infty$.

Proof of Lemma A.2 For

$$U_t=\begin{pmatrix}X_t^2-\mathsf{E}(X_1^2)\\ X_t-\mathsf{E}(X_1)\end{pmatrix}$$

we get with a common multivariate invariance principle, in $l_\infty([\epsilon,1],\mathbb{R})$,

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{\tau(\cdot)}U_t=\frac{\tau(\cdot)}{\sqrt{T}}\begin{pmatrix}\overline{X^2_{\tau(\cdot)}}-\mathsf{E}(X_1^2)\\ \overline{X_{\tau(\cdot)}}-\mathsf{E}(X_1)\end{pmatrix}\to_d D_1^{1/2}W_2(\cdot).$$

Here, $W_2(z)$ is a two-dimensional Brownian Motion with independent components and $D_1=\mathsf{E}(U_1U_1')+2\sum_{j=1}^{\infty}\mathsf{E}(U_1U_{1+j}')$.

Applying Theorem A.5 with the function $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=x-y^2$, yields

$$\frac{\tau(\cdot)}{\sqrt{T}}\left(\overline{X^2_{\tau(\cdot)}}-\left(\overline{X_{\tau(\cdot)}}\right)^2-\sigma^2\right)\to_d\left(1\ -2\mathsf{E}(X_1)\right)D_1^{1/2}W_2(\cdot)=:B\ W_2(\cdot)$$

resp.

$$\frac{\tau(\cdot)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(\cdot)}-\sigma^2\right)\to_d (BB')^{1/2}W_1(\cdot).$$

The lemma then follows with the continuous mapping theorem and the fact that D_1 can be consistently estimated with a kernel estimator from Davidson and de Jong (2000). □

Proof of Lemma A.1 With $W_T(z)=\hat{D}\frac{\tau(z)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(z)}-\sigma^2\right)$, let

$$W_T^\epsilon(z)=\begin{cases}W_T(z), & z\geq\epsilon\\ 0 & z<\epsilon\end{cases},$$

$$W^\epsilon(z)=\begin{cases}W_1(z), & z\geq\epsilon\\ 0 & z<\epsilon\end{cases}.$$

Lemma A.2 implies that

$$W_T^{\epsilon}(\cdot) \to_d W^{\epsilon}(\cdot)$$

in $l_\infty([0, 1], \mathbb{R})$ and also

$$W^{\epsilon}(\cdot) \to_d W_1(\cdot)$$

for rational $\epsilon \to 0$ in $l_\infty([0, 1], \mathbb{R})$.

The convergence of $W_T(\cdot)$ in $l_\infty([0, 1], \mathbb{R})$ follows with Theorem 4.2 in Billingsley (1968) if we can show that

$$\lim_{\epsilon\to 0} \limsup_{T\to\infty} \mathbb{P}(\sup_{z\in[0,1]} |W_T^{\epsilon}(z) - W_T(z)| \geq \eta) = \lim_{\epsilon\to 0} \limsup_{T\to\infty} \mathbb{P}(\sup_{z\in[0,\epsilon]} |W_T(z)| \geq \eta) = 0$$

for all $\eta > 0$.

For this, note that

$$\begin{aligned} W_T(z) &= \hat{D}\frac{\tau(z)}{\sqrt{T}}\left(\overline{X^2_{\tau(z)}} - \mathsf{E}(X_1^2)\right) - \hat{D}\frac{\tau(z)}{\sqrt{T}}\left(\left(\overline{X_{\tau(z)}}\right)^2 - (\mathsf{E}(X_1))^2\right) \\ &= \hat{D}\frac{\tau(z)}{\sqrt{T}}\left(\overline{X^2_{\tau(z)}} - \mathsf{E}(X_1^2)\right) - \hat{D}\frac{\tau(z)}{\sqrt{T}}\left(\overline{X_{\tau(z)}} - \mathsf{E}(X_1)\right)\left(\overline{X_{\tau(z)}} + \mathsf{E}(X_1)\right). \end{aligned}$$

We can deduce that

$$\begin{aligned} \limsup_{T\to\infty} \mathbb{P}(\sup_{z\in[0,\epsilon]} |W_T(z)| \geq \eta) \leq &\ \mathbb{P}(\sup_{z\in[0,\epsilon]} C_1|W_1^*(z)| \geq \eta) \\ &+ \mathbb{P}(\sup_{z\in[0,\epsilon]} C_2|W_1^{**}(z)| \geq \eta), \end{aligned}$$

where C_1 and C_2 are two constants and $W_1^*(z)$ and $W_1^{**}(z)$ are two Brownian motions, respectively. This sum becomes arbitrarily small for $\epsilon \to 0$ and so the lemma is proved. □

Proof of Theorem 2.1 We have

$$\begin{aligned} &\hat{D}\frac{\tau(z)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(z)} - [\mathsf{Var}X]_T\right) \\ &= \hat{D}\frac{\tau(z)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(z)} - \sigma^2\right) + \hat{D}\frac{\tau(z)}{\sqrt{T}}\left(\sigma^2 - [\mathsf{Var}X]_T\right) \\ &= \hat{D}\frac{\tau(z)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(z)} - \sigma^2\right) - \frac{\tau(z)}{T}\hat{D}\frac{\tau(1)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(1)} - \sigma^2\right) \end{aligned}$$

and thus get

$$\hat{D}\frac{\tau(\cdot)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(\cdot)} - [\mathsf{Var}X]_T\right) \to_d A(\cdot)$$

with $A(z) = W_1(z) - zW_1(1)$. This is a representation of a one-dimensional Brownian Bridge. Now, the theorem follows with the continuous mapping theorem. □

We now prove Theorem 3.1 for the local power properties with essentially the same techniques as Theorem 2.1.

Lemma A.3 *Under Assumptions (A2)–(A5), in* $l_\infty([0, 1], \mathbb{R})$,

$$\hat{D}\frac{\tau(\cdot)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(\cdot)} - \sigma^2\right) \to_d W_1(\cdot) + D^*(\cdot)$$

where $\sigma^2 = m_x^2 - (\mathsf{E}(X_1))^2$, $W_1(z)$ *is a one-dimensional Brownian Motion and*

$$D^*(z) = C\int_0^z g(u)du$$

with a positive constant C.

Lemma A.3 requires

Lemma A.4 *Under Assumptions (A2)–(A5), for arbitrary* $\epsilon > 0$, *in* $l_\infty([\epsilon, 1], \mathbb{R})$,

$$\hat{D}\frac{\tau(\cdot)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(\cdot)} - \sigma^2\right) \to_d W_1(\cdot) + D^*(\cdot)$$

where $\sigma^2 = m_x^2 - (\mathsf{E}(X_1))^2$, $W_1(z)$ *is a one-dimensional Brownian Motion and* $D^*(z)$ *is the same as in Lemma A.3.*

Proof of Lemma A.4 For

$$U_t = \begin{pmatrix} X_t^2 - m_x^2 - g\left(\frac{t}{T}\right) \\ X_t - \mathsf{E}(X_1) \end{pmatrix}$$

we get as above in Lemma A.4

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{\tau(\cdot)} U_t = \frac{\tau(\cdot)}{\sqrt{T}}\begin{pmatrix} \overline{X_{\tau(\cdot)}^2} - m_x^2 \\ \overline{X_{\tau(\cdot)}} - \mathsf{E}(X_1) \end{pmatrix} - \begin{pmatrix} \frac{1}{T}\sum_{t=1}^{\tau(\cdot)} g\left(\frac{t}{T}\right) \\ 0 \end{pmatrix} \to_d D_1^{1/2} W_2(\cdot).$$

Another application of the modified functional delta method yields with $B = \left(1 \; -2\mathsf{E}(X_1)\right) D_1^{1/2}$ and D_1 from Assumption (A2)

$$\frac{\tau(\cdot)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(\cdot)} - \sigma^2\right) \to_d (BB')^{1/2} W_1(\cdot) + \int_0^{\cdot} g(u)du.$$

The continuous mapping theorem yields

$$\hat{D}\frac{\tau(\cdot)}{\sqrt{T}}\left([\mathsf{Var}X]_{\tau(\cdot)}-\sigma^2\right)\to_d W_1(\cdot)+(BB')^{-1/2}\int_0^{\cdot} g(u)du$$

which completes the proof with $C := (BB')^{-1/2}$. □

Proof of Lemma A.3 The proof is analogous to the proof of Lemma A.1 with

$$W_T(z)=\hat{D}\frac{\tau(z)}{\sqrt{T}}\left(\overline{X^2_{\tau(z)}}-m_x^2\right)-\hat{D}\frac{\tau(z)}{\sqrt{T}}\left(\left(\overline{X_{\tau(z)}}\right)^2-(\mathsf{E}(X_1))^2\right).$$

□

Proof of Theorem 3.1 The proof is analogous to the proof of Theorem 2.1. □

Proof of Corollary 3.2 We have

$$Q_T(X)\to_d \sup_{z\in[0,1]}|B(z)+MD(z)|=M\left|\frac{B(z)}{M}+D(z)\right|,$$

where $D(z)\neq 0$ for at least one z. Hence,

$$M\left|\frac{B(z)}{M}+D(z)\right|\geq MR$$

for a continuously distributed random variable R which is almost surely positive. So the test statistic becomes arbitrary large, in particular, larger than every quantile of the asymptotic distribution under the null hypothesis. □

A.2 Modified functional delta method

Theorem A.5 *Consider a sequence $(\theta_T)_T$ of functions in $l_\infty(I,\mathbb{R}^k)$ converging uniformly to a function $\theta\in l_\infty(I,\mathbb{R}^k)$. Furthermore, let $(s_T)_T$ be a sequence of functions $s_T: I\to\mathbb{R}\backslash\{0\}$ such that $||s_T^{-1}||_\infty\to 0$, and let $(M_T)_T$ be a stochastic processes on I with values in $\mathbb{R}^k$ and bounded sample paths such that*

$$||Z_T||_\infty=O_p(1)\quad with\quad Z_T:=s_T(M_T-\theta_T).$$

Furthermore, let $f:\mathbb{R}^k\to\mathbb{R}^l$ be a mapping which is continuously differentiable on an open set $\Omega\subset\mathbb{R}^k$ with derivative Df. Suppose that

$$\overline{\theta(I)}\ \textit{is a compact subset of}\ \Omega,$$

where $\overline{\theta(I)}$ stands for the closure of the set $\{\theta(i): i\in I\}$ in $\mathbb{R}^k$. Then it holds

1. $s_T(\cdot)\,(f(M_T(\cdot)) - f(\theta_T(\cdot))) = Df(\theta(\cdot))Z_T(\cdot) + R_T$
with a stochastic process such that

$$||R_T||_\infty = o_p(1).$$

2. *If Z_T even converges in distribution (in $l_\infty(I, \mathbb{R}^k)$) to a stochastic process Z, then*

$$s_T(\cdot)\,(f(M_T(\cdot)) - f(\theta_T(\cdot))) \to_d Df(\theta(\cdot))Z(\cdot).$$

Proof Assertion 2 directly follows from Assertion 1 with the usual continuous mapping theorem.

To prove the expansion from Assertion 1, note that for any $i \in I$,

$$\begin{aligned} R_T(i) &:= s_T(i)\,(f(M_T(i)) - f(\theta_T(i))) - Df(\theta(i))Z_T(i) \\ &= s_T(i)\left(f\left(\theta_T(i) + s_T^{-1}(i)Z_T(i)\right) - f(\theta_T(i))\right) - Df(\theta(i))Z_T(i) \\ &= \int_0^1 Df\left(\theta_T(i) + us_T^{-1}(i)Z_T(i)\right) Z_T(i)du - Df(\theta(i))Z_T(i) \\ &= \int_0^1 \left(Df\left(\theta_T(i) + us_T^{-1}(i)Z_T(i)\right) - Df(\theta(i))\right) du \cdot Z_T(i), \end{aligned} \tag{3}$$

provided that

$$r_n := ||\theta_T - \theta||_\infty + ||s_T^{-1}||_\infty ||Z_T||_\infty = o_p(1)$$

is smaller than

$$\rho := \inf_{x \in \overline{\theta(I)},\, y \in \mathbb{R}^k \setminus \Omega} ||x - y|| > 0.$$

The latter condition is needed such that (3) is well defined.

Hence

$$||R_T||_\infty \le \sup\left\{||Df(y) - Df(x)|| : x \in \overline{\theta(I)},\, y \in \mathbb{R}^k,\, ||y - x|| \le r_T\right\} \cdot ||Z_T||_\infty. \tag{4}$$

Here $||Df(y) - Df(x)||$ is the usual operator norm of the matrix $Df(y) - Df(x)$ in case of $y \in \Omega$. (In case of $y \notin \Omega$ define $||Df(y) - Df(x)|| = \infty$.) One can easily deduce from continuity of $Df(\cdot)$ on Ω, compactness of $\overline{\theta(I)} \in \Omega$ and $r_T = o_p(1)$ that the right hand side of (4) converges to zero in probability. □

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