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Parametric inference from system lifetime data under a proportional hazard rate model

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Abstract In this paper, we discuss the statistical inference of the lifetime distribution of components based on observing the system lifetimes when the system structure is known. A general proportional hazard rate model for the lifetime of the components is considered, which includes some commonly used lifetime distributions. Different estimation methods—method of moments, maximum likelihood method and least squares method—for the proportionality parameter are discussed. The conditions for existence and uniqueness of method of moments and maximum likelihood estimators are presented. Then, we focus on a special case when the lifetime distributions of the components are exponential. Computational formulas for point and interval estimations of the unknown mean lifetime of the components are provided. A Monte Carlo simulation study is used to compare the performance of these estimation methods and recommendations are made based on these results. Finally, an example is provided to illustrate the methods proposed in this paper.

Keywords Coherent systems · Exponential distribution · Least squares method · Maximum likelihood method · Method of moments · Order statistics

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1 Introduction

In reliability testing, experimenters are always interested in the lifetime distribution of the system as well as the lifetime distribution of the components which make-up the system. In many cases, the lifetimes of an n-component system can be observed from a life-test but not the lifetimes of the components. This problem may raise when it is not possible to put the individual components on a life-test after the n-component system is built. For instance, we will encounter this problem when the n-component systems are put on the field to work and our interest is in monitoring the reliability of the components in the systems. For fielded systems, the information on which component leads to the system failure is usually unknown because the experimenter often does not have the need or capability to identify the failed component or the whole system is discarded upon failure. In other situations, the distribution of component lifetimes may change when they are used in a specified system. In these cases, we can only observe the system lifetime but not the lifetimes of the components, and consequently the statistical inference of the lifetime distribution of the components may not be possible unless information on the system structure of the n-component system is available. On the other hand, the longest lifetime among n components is always greater than or equal to the lifetime of a n-component system with the same n components. Therefore, even when life-testing experiment of individual components are feasible to run, placing the components into n-component system and running the life-testing experiment on the n-component system possesses the advantage of saving on experimental time. However, one should keep in mind that the data observed from the two life-testing experiments are of different forms, with the former giving lifetimes of all n components while the latter giving only lifetime of the system. For this reason, the development of statistical inference for the lifetime distribution of components based on system lifetimes is of interest. In this manuscript, we discuss parametric statistical inference for the component lifetime distributions when they follow a proportional hazard rate (PHR) model in the case when the system lifetimes are observed and the systems have the same known structure.

Let T be the lifetime of a coherent system with independent and identically distributed (IID) component lifetimes $X_1, X_2, \ldots, X_n$ with common absolutely continuous cumulative distribution function (CDF) $F_X(\cdot)$, probability density function (PDF) $f_X(\cdot)$, and survival (or reliability) function (SF) $\bar{F}_X(\cdot) = 1 - F_X(\cdot)$. We denote the corresponding order statistics of the n component lifetimes as $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$. We further denote the SF of the i-th order statistic by $\bar{F}_{i:n}(\cdot)$.

Suppose m independent n-component systems with the same structure are placed on a life-test with the corresponding lifetimes $T_1, T_2, \ldots, T_m$ being identically distributed as T with CDF $F_T(\cdot)$, PDF $f_T(\cdot)$ and SF $\bar{F}_T(\cdot) = 1 - F_T(\cdot)$.

The system signature $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ of T is defined by

$$\begin{aligned} p_i &= \Pr(\text{system fails upon the failure of the } i\text{-th component}) \\ &= \Pr(T = X_{i:n}), \end{aligned}$$

where the coefficients $p_1, p_2, \ldots, p_n$ are some non-negative real numbers that do not depend on F_X and that satisfy $\sum_{i=1}^{n} p_i = 1$ (see Samaniego 1985). Actually, it is well known that the signature only depends on the structure of the system. Let us assume that the system signature is known and available. From Samaniego (1985) (see also Kochar et al. 1999; Samaniego 2007), we have the PDF and SF of the system lifetime T as

$$f_T(t) = \sum_{i=1}^{n} p_i \binom{n}{i} i f_X(t)[F_X(t)]^{i-1} \left[\bar{F}_X(t)\right]^{n-i}$$

and

$$\bar{F}_T(t) = \sum_{i=1}^{n} p_i \bar{F}_{i:n}(t), \tag{1}$$

respectively, where $\bar{F}_{i:n}(t) = \sum_{j=0}^{i-1} \binom{n}{j} [F_X(t)]^j \left[\bar{F}_X(t)\right]^{n-j}$. This representation is called Samaniego representation. Shaked and Suarez-Llorens (1993) gave the signatures of coherent systems with 3 and 4 components and compared reliability experiments based on convolution order by means of the system signature. Navarro and Rubio (2010) gave the signatures of coherent systems with 5 components. Note that the estimation and prediction of reliability of coherent systems composed of components are always of interest, see for example, Li et al. (2005).

Navarro et al. (2007) noted that the SF of the system lifetime T can be expressed as a generalized mixture of the survival functions, $\bar{F}_{1:i}(\cdot)$, of the series system lifetimes $X_{1:i} = \min(X_1, X_2, \ldots, X_i)$, $i = 1, 2, \ldots, n$, i.e.

$$\bar{F}_T(t) = \sum_{i=1}^{n} a_i \bar{F}_{1:i}(t), \tag{2}$$

for some negative and nonnegative integers $a_1, a_2, \ldots, a_n$ that do not depend on F_X and satisfy $\sum_{i=1}^{n} a_i = 1$. Navarro et al. (2007) called the vector $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ the minimal signature of the system. The minimal signature of a system can be obtained from its signature and vice versa Navarro et al. (2007, 2008). For example, from Table 2 in Navarro et al. (2007), we have that the minimal signature of a 4-components system can be obtained from $\mathbf{a} = \mathbf{p}A_4$, where

$$A_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & -3 \\ 0 & 6 & -8 & 3 \\ 4 & -6 & 4 & -1 \end{pmatrix}.$$

For instance, the system with 4 components and structure given in Fig. 1 has its lifetime as $T = \min\{X_1, \max\{X_2, X_3, X_4\}\}$ and the corresponding system signature and minimal signature are $\mathbf{p} = (1/4, 1/4, 1/2, 0)$ and $\mathbf{a} = \mathbf{p}A_4 = (0, 3, -3, 1)$.

The coherent systems are extended by the *mixed systems* which are defined as stochastic mixtures of coherent systems (Boland and Samaniego 2004; Dugas and Samaniego 2007). So, from (1), they are mixtures of $X_{1:n}$, $X_{2:n}, \ldots, X_{n:n}$. The coefficients in that representation are called the *signature* of the mixed system. Thus, a

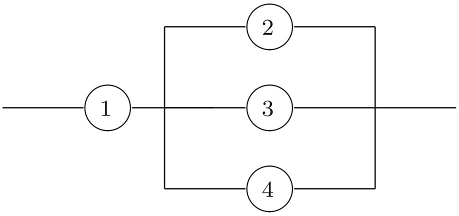

Fig. 1 Two parallel systems in series

Table 1 Variances of method of moments estimators and minimal signatures for all 3-component system

System	System lifetime (T)	Minimal signature (**a**)	$mVar(\tilde{\theta})/\theta^2$
Series	$X_{1:3}$	(0, 0, 1)	1.0000
Series-parallel	$\min\{X_1, \max\{X_2, X_3\}\}$	(0, 2, −1)	0.7500
2-out-of-3	$X_{2:3}$	(0, 3, −2)	0.5200
Parallel-series	$\max\{X_1, \min\{X_2, X_3\}\}$	(1, 1, −1)	0.6735
Parallel	$X_{3:3}$	(3, −3, 1)	0.4050

mixed system with signature $(p_1, p_2, \ldots, p_n)$ has lifetime T equal to $X_{i:n}$ with probability p_i and its SF, $\bar{F}_T(t)$, can be written as (1), where $p_1, p_2, \ldots, p_n$ are arbitrary nonnegative real numbers such that $\sum_{i=1}^n p_i = 1$. Hence, $\bar{F}_T(t)$ can also be written as (2) for some real numbers $a_1, a_2, \ldots, a_n$ such that $\sum_{i=1}^n a_i = 1$. The vector of coefficients in that representation are called *minimal signature* of the mixed system. For example, the mixed system with signature $\mathbf{p} = (0.1, 0.2, 0.3, 0.4)$ has minimal signature $\mathbf{a} = (1.6, -0.6, 0, 0)$.

Navarro et al. (2008) proved that the lifetimes of coherent systems with less than n components are equal in law to mixed systems of order n. Thus, for example, X_1 (which is a series system with one component) is equal in law to the mixed system with 4 components and signature (1/4, 1/4, 1/4, 1/4) (see Table 1 in Navarro et al. (2008)). Considering the system lifetime instead of individual component lifetime can also be viewed as an accelerated life-test when the n-component system fails before the individual components in expectation, i.e., $E(T) \leq E(X_1)$. For instance, if we consider a series system, it is equivalent to having a right-censored sample with total sample size n (effective sample size = 1 and number of components right censored = $n - 1$). Based on the signature of the system, we can determine if the inequality $E(T) \leq E(X_1)$ holds when we know the distribution of component lifetimes (Navarro and Rubio 2010; Navarro and Rychlik 2007).

In recent years, many authors have discussed theory and applications of system signatures; see, for example Arcones et al. (2002), Boland et al. (2003), Navarro (2007, 2008), Navarro et al. (2005, 2007, 2008), Navarro and Rychlik (2010), Navarro et al. (2010), Navarro and Shaked (2010). A comprehensive discussion on system signatures and their applications in engineering reliability can be found in a recent book by Samaniego (2007). Although extensive work has been carried out in reliability engineering based on system signatures, parametric inference of system lifetime data, with signature being available, has not been studied much (see Gåsemyr and Natvig 1998, 2001; Meilijson 1991). The purpose of this paper is to fill this gap by discussing the statistical inference of the lifetime distribution of components based on observing the coherent (or mixed) system lifetimes when the system structure is known.

The rest of this paper is organized as follows. Section 2 gives details of the model of the component and system lifetimes and the problem of interest. We also discuss in this section the estimation methods for the parameter using the method of moments, maximum likelihood method and least squares method. The conditions for existence and uniqueness of method of moments and maximum likelihood estimators are presented. In Sect. 3, we consider a special case when the lifetime distributions of the components are exponential. Different methods for point and interval estimation of the unknown mean lifetime of the components are discussed. A Monte Carlo simulation study is used to compare the performance of these estimation methods and recommendations are made based on these results in Sect. 4. An illustrative example is given in Sect. 5. Finally, some conclusions are provided in Sect. 6.

2 Model and parametric statistical estimation

In this paper, we consider the popular proportional hazard rate (PHR) model for the common distribution of the IID lifetimes of the components, i.e., we assume that the SF of X_i is

$$\bar{F}_X(t) = \left[\bar{G}(t)\right]^{\alpha} \tag{3}$$

for $i = 1, 2, \ldots, n$, where $\alpha > 0$ is the unknown parameter and $\bar{G}(t)$ is the baseline SF of a lifetime distribution with support $[0, \infty)$, whose form is completely specified and it does not depend on α. The PHR model covers some commonly used statistical lifetime distributions which are applicable to model component lifetimes. The following are some examples:

Exponential distribution: Suppose $\bar{G}(t) = e^{-t}$. Then, under the model in (3), we have $\bar{F}_X(t) = e^{-\alpha t}$, which is equivalent to assuming the lifetime of the components to be exponentially distributed with constant hazard rate α.

Pareto distribution: Suppose $\bar{G}(t) = (1+t)^{-1}$. Then, under the model in (3), we have

$$\bar{F}_X(t) = (1+t)^{-\alpha}, \tag{4}$$

which is equivalent to assuming the lifetime of the components to be standard Pareto type II distributed with shape parameter α (see Johnson et al. 1994, Chap. 20).

Weibull distribution with known shape parameter: Suppose $\bar{G}(t) = e^{-t^{\beta}}$ with known β. Then, under the model in (3), we have $\bar{F}_X(t) = e^{-\alpha t^{\beta}}$, which is equivalent to assuming the lifetime of the components to be Weibull distributed with hazard rate function $\alpha\beta t^{\beta-1}$.

Based on model (3) and using the fact that $X_1, X_2, \ldots, X_n$ are IID, we have $\bar{F}_{1:i}(t) = \bar{G}^{i\alpha}(t)$. Hence, from (2), the PDF and SF of the system lifetime can be expressed as

$$f_T(t) = \alpha g(t) \sum_{i=1}^{n} i a_i \bar{G}^{i\alpha-1}(t) \tag{5}$$

and

$$\bar{F}_T(t) = \sum_{i=1}^{n} a_i \bar{G}^{i\alpha}(t),$$

respectively, where $g(t) = -d\bar{G}(t)/dt$ is the baseline PDF.

Suppose m independent n-component systems with the same distribution as T are placed on a life-test and that the corresponding lifetimes $T_1, T_2, \ldots, T_m$ are observed. We are interested in estimating the parameter α based on the system lifetimes when the signature (or, equivalently, the minimal signature) of the system is available.

2.1 Method of moments estimation

The method of moments equates sample moments to parameter estimates. The method of moments estimators always have the advantage of simplicity. In our case, when the first moment of the system lifetime exists, it can be obtained as

$$\begin{aligned} E(T) &= \int_0^\infty \bar{F}_T(t)dt \\ &= \sum_{i=1}^{n} a_i \int_0^\infty \bar{G}^{i\alpha}(t)dt. \end{aligned}$$

By equating the first moment with the sample moment, we can obtain the method of moments estimator by solving the following equation

$$\frac{1}{m}\sum_{k=1}^{m} T_k = \sum_{i=1}^{n} a_i \int_0^\infty \bar{G}^{i\alpha}(t)dt. \tag{6}$$

Proposition 1 *If* $\lim_{\alpha\to\infty} \int_0^\infty \bar{G}^\alpha(t)dt = 0$, *then Eq.* (6) *has a unique nonnegative solution.*

Proof From Kochar et al. (1999), it is known that the lifetimes of two systems with the same structure and IID components are stochastically ordered if so are the respective component distributions. In other words, if $X_1, X_2, \ldots, X_n$ are IID components from distribution F and $Y_1, Y_2, \ldots, Y_n$ are IID components from distribution F^*, F is stochastically ordered with respect to F^* and $T = \varphi(X_1, X_2, \ldots, X_n)$ is the lifetime a coherent system with minimal signature **a**, then T is stochastically ordered with respect to $T^* = \varphi(Y_1, Y_2, \ldots, Y_n)$. Since in the PHR model in (3), it is evident that for $\alpha_1 < \alpha_2$, $\left[\bar{G}(t)\right]^{\alpha_1} > \left[\bar{G}(t)\right]^{\alpha_2}$ for all t, by using the above fact, we readily have the function

$$\phi(\alpha) = \sum_{i=1}^{n} a_i \int_0^{\infty} \bar{G}^{i\alpha}(t)dt$$

to be strictly decreasing in α and such that $\phi(\infty) = \lim_{\alpha\to\infty} \phi(\alpha) = 0$. Let us show that $\phi(0+) = \lim_{\alpha\to 0^+} \phi(\alpha) = \infty$. First, note that

$$\phi(\alpha) = E(T) \geq E(X_{1:n}) = \int_0^{\infty} \bar{G}^{n\alpha}(t)dt.$$

Hence, for a fixed positive real number c, if we choose α such that $\bar{G}^{n\alpha}(2c) \geq 1/2$, then

$$\phi(\alpha) \geq \int_0^{\infty} \bar{G}^{n\alpha}(t)dt \geq \int_0^{2c} \bar{G}^{n\alpha}(t)dt \geq \int_0^{2c} (1/2)dt = c.$$

Therefore, $\phi(0+) = \infty$. Hence, as $\frac{1}{m}\sum_{k=1}^{m} T_k > 0$, (6) has a unique nonnegative solution. □

The next example shows that this method can be used even when the common mean of the component lifetimes does not exist.

Example 1 Let us consider the system given in Fig. 1 with minimal signature $\mathbf{a} = (0, 3, -3, 1)$ and let us assume a standard Pareto type II baseline SF in (4). Then (6) reduces to $\phi(\alpha) = \frac{1}{m}\sum_{k=1}^{m} T_k$, where

$$\phi(\alpha) = 3\left(\frac{1}{2\alpha - 1}\right) - 3\left(\frac{1}{3\alpha - 1}\right) + \frac{1}{4\alpha - 1}.$$

It is easy to see that $\phi(\alpha)$ is a strictly decreasing function in $(1/2, \infty)$, with $\phi(1/2+) = \infty$ and $\phi(\infty) = 0$. Hence (6) has a unique solution in $(1/2, \infty)$. Note $E(T)$ does not exist for $0 < \alpha \leq 1/2$. Also note that $E(X) = 1/(\alpha - 1)$ and that it does not exist for $0 < \alpha \leq 1$. Hence, the method of moments, from a sample of system lifetimes, can be applied for $\alpha \in (1/2, 1]$ while it cannot be applied from a component lifetimes sample. In this example, one can also see that the system lifetime is shorter than individual component in expectation since $E(T) < E(X)$, for $\alpha > 1$.

2.2 Maximum likelihood estimation

Based on a complete sample with observed lifetimes $t_1, t_2, \ldots, t_m$, the likelihood function is

$$\begin{aligned} L(\alpha) &= \prod_{k=1}^{m} f_T(t_k) \\ &= \left\{ \frac{\prod_{k=1}^{m} g(t_k)}{\prod_{k=1}^{m} \bar{G}(t_k)} \right\} \alpha^m \prod_{k=1}^{m} \sum_{i=1}^{n} i a_i \bar{G}^{i\alpha}(t_k) \end{aligned}$$

and the log-likelihood function is

$$\ln L(\alpha) = c + m \ln \alpha + \sum_{k=1}^{m} \ln \left\{ \sum_{i=1}^{n} i a_i \bar{G}^{i\alpha}(t_k) \right\},$$

where c is a constant which does not depend on α. Therefore, the likelihood equation is

$$\frac{d \ln L(\alpha)}{d\alpha} = \frac{m}{\alpha} + \sum_{k=1}^{m} \left\{ \frac{\sum_{i=1}^{n} i^2 a_i \bar{G}^{i\alpha}(t_k)}{\sum_{i=1}^{n} i a_i \bar{G}^{i\alpha}(t_k)} \right\} \ln \bar{G}(t_k) = 0 \tag{7}$$

and the maximum likelihood estimator of α, $\hat{\alpha}$, can be obtained by solving the above non-linear equation for α.

Proposition 2 *If $q(x)$ is a strictly decreasing function in* $(0, 1)$*, where*

$$q(x) = \frac{\sum_{i=1}^{n} i^2 a_i x^i}{\sum_{i=1}^{n} i a_i x^i},$$

then (7) *has a unique positive solution, and $L(\alpha)$ attains a maximum at that point.*

Proof Note that (7) can be written as $\psi(\alpha) = 0$, where

$$\psi(\alpha) = \frac{m}{\alpha} + \sum_{k=1}^{m} q\left(\bar{G}^{\alpha}(t_k)\right) \ln \bar{G}(t_k),$$

$\ln \bar{G}(t_k) < 0$ and $0 < \bar{G}^{\alpha}(t_k) < 1$. Hence, if $q(x)$ is strictly decreasing in $(0, 1)$, then $\psi(\alpha)$ is strictly decreasing in $(0, \infty)$. Moreover, as $q(0) = i_1 > 0$ where $i_1 = \min\{i : a_i \neq 0\}$, and

$$q(1) = \frac{\sum_{i=1}^{n} i^2 a_i}{\sum_{i=1}^{n} i a_i}$$

is a real number or $-\infty$, then $\psi(0+) = \infty$ and $\psi(\infty) = i_1 \sum_{k=1}^{m} \ln \bar{G}(t_k) < 0$. Therefore, (7) has a unique positive solution and $L(\alpha)$ attains a maximum at that point. □

It is easy to see that the sign of the derivative of $q(x)$ is equal to the sign of

$$\sum_{i<j} ija_i a_j (j-i)^2 x^{i+j}.$$

Using this property and the minimal signatures given in Navarro et al. (2007), we have proved that $q(x)$ is a decreasing function for all coherent systems with 4 or less components. However, we do not know if this property holds for any mixed system or for coherent systems with more than 4 components. But, this property is easy to check in practice when the signature is available.

From the asymptotic theory of maximum likelihood estimator, the observed Fisher information can be computed as

$$\begin{aligned} I(\hat{\alpha}) &= -\frac{d^2 \ln L(\alpha)}{d\alpha^2}\bigg|_{\alpha=\hat{\alpha}} \\ &= \frac{m}{\hat{\alpha}^2} - \sum_{k=1}^{m} \left\{ \frac{\sum_{i=1}^{n} ia_i \bar{G}^{i\hat{\alpha}}(t_k)[\ln \bar{G}(t_k)]^2}{\sum_{i=1}^{n} ia_i \bar{G}^{i\hat{\alpha}}(t_k)} - \left[\frac{\sum_{i=1}^{n} ia_i \bar{G}^{i\hat{\alpha}}(t_k) \ln \bar{G}(t_k)}{\sum_{i=1}^{n} ia_i \bar{G}^{i\hat{\alpha}}(t_k)} \right]^2 \right\}. \end{aligned} \tag{8}$$

The variance of $\hat{\alpha}$ can be approximated by the inverse of the observed Fisher information, i.e.,

$$\widehat{Var}(\hat{\alpha}) = I^{-1}(\hat{\alpha}),$$

and an asymptotic $100(1-\gamma)\%$ confidence interval for α is

$$\hat{\alpha} \pm z_{1-\gamma/2} \sqrt{\widehat{Var}(\hat{\alpha})},$$

where z_q is the q-th upper percentile of the standard normal distribution.

2.3 Least squares estimation

If $T_{1:m} < T_{2:m} < \cdots < T_{m:m}$ are the ordered system lifetimes, then

$$E[F_T(T_{k:m})] = \frac{k}{m+1}$$

and

$$Var[F_T(T_{k:m})] = \frac{k(m-k+1)}{(m+1)^2(m+2)}$$

are both independent of the parameter α. The least squares estimator of α then can be obtained by minimizing

$$G_{LS}(\alpha) = \sum_{k=1}^{m} \left[F_T(T_{k:m}) - \frac{k}{m+1} \right]^2.$$

Taking the derivative of $G_{LS}(\alpha)$ with respect to α and setting it to zero, we have

$$\frac{dG_{LS}(\alpha)}{d\alpha} = 2 \sum_{k=1}^{m} \left[F_T(T_{k:m}) - \frac{k}{m+1} \right] \frac{dF_T(T_{k:m})}{d\alpha} = 0,$$

and the least squares estimator of α, $\hat{\alpha}_{LS}$, can then be obtained by solving

$$\frac{dG_{LS}(\alpha)}{d\alpha} = -2 \sum_{k=1}^{m} \left[\frac{m+1-k}{m+1} - \sum_{i=1}^{n} a_i \bar{G}^{i\alpha}(T_{k:m}) \right] \left[\sum_{i=1}^{n} i a_i \bar{G}^{i\alpha}(T_{k:m}) \right] \ln \bar{G}(T_{k:m}) = 0 \tag{9}$$

for α. To solve this equation, we must study the polynomials

$$p_k(x) = \left[\frac{m+1-k}{m+1} - \sum_{i=1}^{n} a_i x^i \right] \left[\sum_{i=1}^{n} i a_i x^i \right]$$

for $x \in [0, 1]$ and $k = 1, 2, \ldots, m$, where $p_k(0) = 0$ and $p_k(1) = \frac{-k}{m+1} \sum_{i=1}^{n} i a_i \leq 0$ from (5). Moreover, note that $p_k(x) > 0$ for positive numbers x in the neighborhood of 0 and $p_k(x) < 0$ for x in the neighborhood of 1 such that $x < 1$ (since from (5) we have $\sum_{i=1}^{n} i a_i x^i \geq 0$ for all $x \in [0, 1]$). Therefore, $G_{LS}(\alpha)$ is decreasing for α close to 0 and increasing for large values of α. However, (9) does not necessarily have a unique positive solution. The weighted least squares estimator of α can be obtained in a similar manner by using the weight function $w_k = 1/Var[F_T(T_{k:m})] = \frac{(m+1)^2(m+2)}{k(m-k+1)}$ for $k = 1, \ldots, m$.

3 Systems with exponentially distributed components

We assume that the lifetime of the n components in a system are IID with constant hazard rate, say $1/\theta$, i.e., they are exponentially distributed with PDF and CDF

$$f_X(x) = \frac{1}{\theta} \exp\left(-\frac{x}{\theta}\right) \quad \text{and} \quad F_X(x) = 1 - \exp\left(-\frac{x}{\theta}\right), \quad x > 0, \quad \theta > 0,$$

respectively. The readers can refer to the volume by Balakrishnan and Basu (1995) (see also Johnson et al. 1994, Chap. 19) for an extensive review of the genesis of this distribution and its properties. As mentioned in Sect. 2, this model is equivalent to setting $\bar{G}(t) = e^{-t}$ and $\alpha = 1/\theta$ in (3). In this section, we discuss different methods of point and interval estimation for the parameter α (or, equivalently, θ).

3.1 Method of moments

From Eq. (6), the first moment of the system lifetime with exponentially distributed components is

$$E(T) = \sum_{i=1}^{n} \frac{a_i}{i\alpha} > 0$$

for all $\alpha > 0$. From (6), we have the method of moments estimator of α, $\tilde{\alpha}$, as

$$\tilde{\alpha} = \frac{m \sum_{i=1}^{n} \frac{a_i}{i}}{\sum_{k=1}^{m} T_k}. \tag{10}$$

Since it is easier to work with the method of moments estimate of $\theta = 1/\alpha$ in this case, we will study the properties of the method of moments estimator of θ which is simply given by

$$\tilde{\theta} = \frac{\sum_{k=1}^{m} T_k}{m \sum_{i=1}^{n} \frac{a_i}{i}}.$$

This estimator is unbiased since

$$E(\tilde{\theta}) = \frac{\sum_{k=1}^{m} E(T_k)}{m \sum_{i=1}^{n} \frac{a_i}{i}} = \frac{E(T)}{\sum_{i=1}^{n} \frac{a_i}{i}} = \frac{1}{\alpha} = \theta$$

and the variance of $\tilde{\theta}$ is

$$\begin{aligned} Var(\tilde{\theta}) &= \frac{1}{m^2 \left[\sum_{i=1}^{n} \frac{a_i}{i}\right]^2} \sum_{k=1}^{m} Var(T_k) \\ &= \frac{\theta^2}{m \left[\sum_{i=1}^{n} \frac{a_i}{i}\right]^2} \left\{ 2 \sum_{i=1}^{n} \frac{a_i}{i^2} - \left[\sum_{i=1}^{n} \frac{a_i}{i}\right]^2 \right\} \\ &= \frac{\theta^2}{m} \left\{ \frac{2 \sum_{i=1}^{n} \frac{a_i}{i^2}}{\left[\sum_{i=1}^{n} \frac{a_i}{i}\right]^2} - 1 \right\}. \end{aligned}$$

Table 2 Variances of method of moments estimators and minimal signatures for all 4-component system

System lifetime (T)	Minimal signature (**a**)	$mVar(\tilde{\theta})/\theta^2$
$X_{1:4}$	(1, 0, 0, 0)	1.0000
$\max\{\min\{X_1, X_2, X_3\}, \min\{X_2, X_3, X_4\}\}$	(0, 0, 2, −1)	0.8400
$\min\{X_1, \max_{2\leq i\leq j\leq 4}\{X_i, X_j\}\}$	(0, 0, 3, −2)	0.6667
$\min\{X_1, \max\{X_2, X_3\}, \max\{X_2, X_4\}\}$	(0, 1, 1, −1)	0.7551
$\min\{X_1, \max\{X_2, X_3, X_4\}\}$	(0, 3, −3, 1)	0.7037
$X_{2:4}$	(0, 0, 4, −3)	0.5102
$\max\{\min\{X_1, X_2\}, \min\{X_1, X_3, X_4\}, \min\{X_2, X_3, X_4\}\}$	(0, 1, 2, −2)	0.5625
$\max\{\min\{X_1, X_2\}, \min\{X_3, X_4\}\}$	(0, 2, 0, −1)	0.5556
$\max\{\min\{X_1, X_2\}, \min\{X_1, X_3\}, \min\{X_2, X_3, X_4\}\}$	(0, 2, 0, −1)	0.5556
$\max\{\min\{X_1, X_2\}, \min\{X_2, X_3\}, \min\{X_3, X_4\}\}$	(0, 3, −2, 0)	0.5200
$\max\{\min\{X_1, \max\{X_2, X_3, X_4\}, \min\{X_2, X_3, X_4\}\}$	(0, 3, −2, 0)	0.5200
$\max\{\min\{X_1, \max\{X_2, X_3, X_4\}, \min\{X_2, X_3\}\}$	(0, 4, −4, 1)	0.4711
$\min\{\max\{X_1, X_2\}, \max\{X_3, X_4\}\}$	(0, 4, −4, 1)	0.4711
$\min\{\max\{X_1, X_2\}, \max\{X_1, X_3, X_4\}, \max\{X_2, X_3, X_4\}\}$	(0, 5, −6, 2)	0.4167
$X_{3:4}$	(0, 6, −8, 3)	0.3609
$\max\{X_1, \min\{X_2, X_3, X_4\}\}$	(1, 0, 1, −1)	0.7870
$\max\{X_1, \min\{X_2, X_4\}, \min\{X_3, X_4\}\}$	(1, 2, −3, 1)	0.5733
$\max\{X_1 \max_{2\leq i\leq j\leq 4} \min\{X_i, X_j\}\}$	(1, 3, −5, 2)	0.4844
$\max\{X_1, X_2, \min\{X_3, X_4\}\}$	(2, 0, −2, 1)	0.4681
$X_{4:4}$	(4, −6, 4, −1)	0.3280

The exact distribution of the method of moments estimator $\tilde{\theta}$ can be shown to be a generalized mixture of gamma distributions. As mentioned earlier in Sect. 1, the performance of the statistical inference procedure can be used as a criteria to compare systems with different structures. Since the variance of the method of moments estimators does not depend on the observed data, it can be used to compare systems with different signatures. In Tables 1 and 2, we have provided the variances of $\tilde{\theta}$ for all 3- and 4-component systems, respectively. From these results, we can compare systems with the same or different number of components in terms of the performance of point estimate of the parameter θ. For example, we can see that the performance of point estimate of θ based on a parallel 3-component system ($mVar(\tilde{\theta})/\theta^2 = 0.4050$) is better than a 2-out-of-4 system ($mVar(\tilde{\theta})/\theta^2 = 0.5102$).

Since $\tilde{\theta}$ is a linear function of a sum of independent random variables T_k, $k = 1, 2, \ldots, m$, based on central limit theorem, we have

$$\frac{\tilde{\theta} - \theta}{\sqrt{Var(\tilde{\theta})}} \xrightarrow{D} N(0, 1).$$

As a result, we have two options to construct confidence intervals for θ: (i) Replacing θ by $\tilde{\theta}$ in $Var(\tilde{\theta})$ to estimate the variance of $\tilde{\theta}$ and obtain an asymptotic confidence interval of θ (namely, AMOM1) as

$$\tilde{\theta} \pm z_{1-\gamma/2} \sqrt{\frac{\tilde{\theta}^2}{m} \left\{ \frac{2\sum_{i=1}^{n} \frac{a_i}{i^2}}{\left[\sum_{i=1}^{n} \frac{a_i}{i}\right]^2} - 1 \right\}}; \tag{11}$$

and (ii) Solving the equation

$$-z_{1-\gamma/2} < \frac{\tilde{\theta} - \theta}{\sqrt{\frac{\theta^2}{m} \left\{ \frac{2\sum_{i=1}^{n} \frac{a_i}{i^2}}{\left[\sum_{i=1}^{n} \frac{a_i}{i}\right]^2} - 1 \right\}}} < z_{1-\gamma/2}$$

for θ to obtain an asymptotic confidence interval of θ (namely, AMOM2) as

$$\frac{\tilde{\theta}}{1 \pm z_{1-\gamma/2} \sqrt{\frac{1}{m} \left\{ \frac{2\sum_{i=1}^{n} \frac{a_i}{i^2}}{\left[\sum_{i=1}^{n} \frac{a_i}{i}\right]^2} - 1 \right\}}}. \tag{12}$$

It should be noted that the confidence interval for α can be obtained simply by inverting the confidence limits for θ.

3.2 Maximum likelihood estimation

For systems with exponentially distributed components, the likelihood equation in (7) is

$$\frac{m}{\alpha} + \sum_{k=1}^{m} t_k \left\{ \frac{\sum_{i=1}^{n} i^2 a_i e^{-\alpha i t_k}}{\sum_{i=1}^{n} i a_i e^{-\alpha i t_k}} \right\} = 0$$

and the maximum likelihood estimator of α, $\hat{\alpha}$, can be obtained by solving the above non-linear equation for α (recall that we know that it has a unique positive solution for all coherent systems with 4 or less components). Numerical methods, such as the Newton-Raphson method, can be used for this purpose and the method of moments estimator $\tilde{\alpha}$ given in (10) can be used as an initial value in the iterative procedure.

The observed Fisher information in (8) is

$$I(\hat{\alpha}) = \frac{m}{\hat{\alpha}^2} - \sum_{k=1}^{m} t_k^2 \left\{ \frac{\sum_{i=1}^{n} i^3 a_i e^{-\hat{\alpha} i t_k}}{\sum_{i=1}^{n} i a_i e^{-\hat{\alpha} i t_k}} - \left[\frac{\sum_{i=1}^{n} i^2 a_i e^{-\hat{\alpha} i t_k}}{\sum_{i=1}^{n} i a_i e^{-\hat{\alpha} i t_k}} \right]^2 \right\}.$$

and an estimate of the variance of $\hat{\alpha}$ is $\widehat{Var}(\hat{\alpha}) = 1/I(\hat{\alpha})$. An asymptotic $100(1-\gamma)\%$ confidence interval for α (namely AMLE) is then

$$\hat{\alpha} \pm z_{1-\gamma/2}\sqrt{\widehat{Var}(\hat{\alpha})}. \tag{13}$$

3.3 Least squares estimation

If $t_{1:m} < t_{2:m} < \cdots < t_{m:m}$ are the observed values of the ordered system lifetimes, from the results in Sect. 2.3, the least squares estimate of α, $\hat{\alpha}_{LS}$, can be obtained by solving

$$\sum_{k=1}^{m} t_{k:m}\left[\frac{m+1-k}{m+1} - \sum_{i=1}^{n} a_i e^{-\alpha i t_{k:m}}\right]\left[\sum_{i=1}^{n} i a_i e^{-\alpha i t_{k:m}}\right] = 0$$

for α. Similarly, the weighted least squares estimate of α, $\hat{\alpha}_{WLS}$, can be obtained by solving

$$\sum_{k=1}^{m} w_k t_{k:m}\left[\frac{m+1-k}{m+1} - \sum_{i=1}^{n} a_i e^{-\alpha i t_{k:m}}\right]\left[\sum_{i=1}^{n} i a_i e^{-\alpha i t_{k:m}}\right] = 0$$

for α. Numerical methods are required to solve the above non-linear equation and the method of moments estimator can be used as an initial estimate in the iterative procedures.

Since the asymptotic properties of the least squares and weighted least squares estimators are intractable in this situation, we construct confidence intervals using the parametric percentile bootstrap method with bias-correction and acceleration (BCA) based on these estimates (Efron and Tibshirani 1993). To obtain the BCA percentile bootstrap confidence intervals for α based on the least squares estimator, namely BLSE, we use the following algorithm:

Parametric bootstrap:

1. Based on the original observed system lifetimes $t_1, t_2, \ldots, t_k$, obtain $\hat{\alpha}_{LS}$, the least squares estimate of α.
2. Simulate m system lifetimes with minimal signature **a** with the components following an exponential distribution with mean $1/\hat{\alpha}_{LS}$, T_1^*, T_2^*, ..., T_m^*, and put them in order $T_{1:m}^* < T_{2:m}^* < \cdots < T_{m:m}^*$.
3. Compute the least squares estimate of α based on $T_{1:m}^* < T_{2:m}^* < \cdots < T_{m:m}^*$, say $\hat{\alpha}_{LS}^{(1)}$.
4. Repeat Steps 2–3 B times and obtain $\hat{\alpha}_{LS}^{(1)}$, $\hat{\alpha}_{LS}^{(2)}$, ..., $\hat{\alpha}_{LS}^{(B)}$.
5. Arrange $\hat{\alpha}_{LS}^{(1)}$, $\hat{\alpha}_{LS}^{(2)}$, ..., $\hat{\alpha}_{LS}^{(B)}$ in ascending order and obtain $\hat{\alpha}_{LS}^{[1]} < \hat{\alpha}_{LS}^{[2]} < \cdots < \hat{\alpha}_{LS}^{[B]}$.
6. A two-sided $100(1-\gamma)\%$ BCA bootstrap confidence interval of α based on least squares estimators, say $[\alpha_{LS}^L, \alpha_{LS}^U]$, is then given by

$$\alpha_{LS}^L = \hat{\alpha}_{LS}^{[B\gamma_1]}, \qquad \alpha_{LS}^U = \hat{\alpha}_{LS}^{[B\gamma_2]},$$

where

$$\gamma_1 = \Phi\left[\hat{z}_0 + \frac{\hat{z}_0 - z_{1-\gamma/2}}{1 - \hat{c}(\hat{z}_0 - z_{1-\gamma/2})}\right]$$

and

$$\gamma_2 = \Phi\left[\hat{z}_0 + \frac{\hat{z}_0 + z_{1-\gamma/2}}{1 - \hat{c}(\hat{z}_0 + z_{1-\gamma/2})}\right].$$

Here, $\Phi(\cdot)$ is the standard normal CDF and the value of bias correction $\hat{z}_0$ is

$$\hat{z}_0 = \Phi^{-1}\left(\frac{\text{number of } \hat{\alpha}_{LS}^{(j)} < \hat{\alpha}_{LS}}{B}\right),$$

where $\Phi^{-1}(\cdot)$ is the inverse function of the standard normal CDF; the acceleration factor $\hat{c}$ is

$$\hat{c} = \frac{\sum_{k=1}^{m}\left[\hat{\alpha}_{LS(\cdot)} - \hat{\alpha}_{LS(k)}\right]^3}{6\left\{\sum_{k=1}^{m}\left[\hat{\alpha}_{LS(\cdot)} - \hat{\alpha}_{LS(k)}\right]^2\right\}^{3/2}},$$

where $\hat{\alpha}_{LS(k)}$ is the least squares estimate of α, computed based on the original sample with the k-th observation deleted (i.e., the jackknife method, see Efron (1982)), $k = 1, 2, \ldots, m$ and $\hat{\alpha}_{LS(\cdot)} = \sum_{k=1}^{m} \hat{\alpha}_{LS(k)}/m$.

The bootstrap confidence interval based on weighted least squares estimator, namely BWLS, can be constructed in a similar manner using the above algorithm. Note that the parametric bootstrap method can also be applied to the method of moments and the maximum likelihood estimators. The performance of these interval estimation methods will be evaluated by means of a Monte Carlo simulation study in the next section.

4 Monte carlo simulation study

To evaluate the performance of all the methods of point and interval estimation described in Sect. 3 for systems with exponentially distributed components, we carried out a Monte Carlo simulation study. Six different four-component systems are considered. The following algorithm is used to generate a system lifetime T with system signature $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and with exponentially distributed components:

1. Generate a random number v from a discrete n-point distribution with $\Pr(V = v) = p_v$, $v = 1, 2, \ldots, n$.
2. Generate y_1, y_2, …, y_v from IID exponential distribution with parameter α.

Table 3 System signatures and minimal signatures of the 4-component systems considered in the simulation study

System no.	System lifetime (T)	Minimal signature (**a**)
1	$X_{1:4}$	(1, 0, 0, 0)
2	$\min\{X_1, \max\{X_2, X_3, X_4\}\}$	(0, 3, −3, 1)
3	$\max\{X_1, \min\{X_2, X_3, X_4\}\}$	(1, 0, 1, −1)
4	$\min\{X_1, \max\{X_2, X_3\}, \max\{X_2, X_4\}\}$	(0, 1, 1, −1)
5	$\max\{\min\{X_1, X_2\}, \min\{X_1, X_3, X_4\}, \min\{X_2, X_3, X_4\}\}$	(0, 1, 2, −2)
6	$X_{3:4}$	(0, 6, −8, 3)

3. From the properties of order statistics from exponential distribution (see, for example, Arnold et al. 1992), the system lifetime T is

$$T = \sum_{l=1}^{v} \frac{y_l}{n - l + 1}.$$

For a general component lifetime distribution with CDF F_X, the system lifetime T can be generated by replacing Step 3 with

3′. From the properties of order statistics, we generate the v-th order statistic from uniform distribution, $U_{(v)}$, using the fact that $U_{(v)}$ follows a beta distribution with parameter v and $m + 1 - v$. Then, by using inverse transform method, the system lifetime T is

$$T = F_X^{-1}(U_{(v)}),$$

where F_X^{-1} is the inverse function of F_X defined by $F_X^{-1}(u) = \inf\{x : F_X(x) \geq u\}$.

For notational convenience, Table 3 lists the different systems and their minimal signatures used in the simulation study. Note that the first mixed system is a series system equal in law to the minimum of n component lifetimes thus resulting in least informative data for inference, and so it can be used as a basis for the comparison of precision of estimates for the component lifetimes based on data of lifetimes from other coherent systems. Without loss of generality, we took $\alpha = \theta = 1$.

For different choices of sample sizes (m) and systems with signatures available, we generated 1,000 sets of system lifetimes, in order to obtain the estimated bias and mean squares errors (MSEs) for the point estimates as well as the estimated coverage probabilities and average conditional width (width of confidence intervals which contained the true value of the parameter) for 95% confidence intervals.

For the point estimation, we considered the method of moments estimator (MOM), the maximum likelihood estimator (MLE), the least squares estimator (LSE), and the weighted least squares estimator (WLSE). The simulated bias and mean squares errors (MSEs) of these point estimates of α are tabulated in Table 4 for $m = 5$, 10, 15 and 25. Since the numerical computation of the LSE and WLSE was unstable due to the

Table 4 Simulated biases and mean squares errors (MSEs) of method of moments estimators (MOM), maximum likelihood estimators (MLE), least squares estimators (LSE) and weighted least squares estimators (WLSE) of α for different 4-component systems with $m = 5, 10, 15$ and 25

m	System no.	Bias				MSE			
		MOM	MLE	LSE	WLSE	MOM	MLE	LSE	WLSE
5	1	0.062	0.062	0.187	0.195	0.184	0.184	0.310	0.303
	2	0.044	0.029	0.118	0.127	0.140	0.133	0.203	0.207
	3	0.007	0.031	0.068	0.070	0.158	0.160	0.207	0.204
	4	0.011	0.012	0.086	0.093	0.155	0.157	0.219	0.217
	5	0.002	0.011	0.047	0.049	0.122	0.123	0.161	0.158
	6	0.009	0.013	0.032	0.034	0.076	0.076	0.097	0.097
10	1	−0.008	−0.008	0.060	0.060	0.102	0.102	0.140	0.131
	2	−0.008	−0.017	0.037	0.038	0.066	0.063	0.095	0.090
	3	0.003	0.013	0.051	0.038	0.082	0.078	0.100	0.104
	4	0.000	0.002	0.055	0.051	0.081	0.081	0.099	0.102
	5	0.011	0.018	0.024	0.016	0.061	0.061	0.069	0.072
	6	0.016	0.018	0.014	0.013	0.040	0.040	0.046	0.045
15	1	0.005	0.005	0.044	0.037	0.065	0.065	0.092	0.089
	2	−0.004	−0.008	0.034	0.028	0.048	0.046	0.063	0.064
	3	−0.004	0.007	0.029	0.025	0.055	0.054	0.062	0.062
	4	0.000	0.001	0.026	0.021	0.050	0.050	0.066	0.066
	5	−0.002	0.002	0.026	0.023	0.036	0.036	0.048	0.047
	6	0.002	0.003	0.013	0.012	0.025	0.025	0.028	0.026
25	1	−0.004	−0.004	0.018	0.016	0.042	0.042	0.055	0.050
	2	0.002	−0.002	0.017	0.014	0.030	0.029	0.039	0.036
	3	−0.002	0.007	0.016	0.015	0.032	0.030	0.035	0.033
	4	−0.002	−0.002	0.023	0.023	0.029	0.029	0.040	0.037
	5	0.004	0.006	0.014	0.012	0.022	0.022	0.029	0.027
	6	0.001	0.002	0.009	0.008	0.014	0.014	0.017	0.016

present of multiple roots, especially in case of small sample sizes (say, $m = 5$ and 10), the values of the LSE and WLSE could not be obtained in some cases (about 5–10% of times) and so these cases were not taken into account in the simulation results.

For the interval estimation, we considered the two asymptotic confidence intervals based on MOM in Eq. (11) (AMOM1) and Eq. (12) (AMOM2), the asymptotic confidence interval based on MLE in Eq. (13) (AMLE), the BCA percentile bootstrap method applied to MOM (BMOM), MLE (BMLE), LSE (BLSE) and WLSE (BWLS). The confidence level was set to be 95% and for the BCA percentile bootstrap confidence intervals, we set the number of bootstrapped samples as $B = 200$. The simulated coverage probabilities and average conditional widths of 95% confidence intervals for all these methods are tabulated in Tables 5 and 6, respectively, for $m = 5, 10, 15$ and 25.

Table 5 Simulated coverage probabilities of 95% confidence intervals of α for different 4-component systems with $m = 5$, 10, 15 and 25

m	System no.	AMLE	AMOM1	AMOM2	BMOM	BMLE	BLSE	BWLS
5	1	95.5	80.2	95.7	86.7	86.7	95.8	96.9
	2	95.1	82.8	95.3	87.4	86.6	92.7	94.2
	3	96.4	87.6	96.5	91.4	92.4	94.0	93.4
	4	95.0	83.4	94.2	88.1	89.1	92.6	94.2
	5	96.1	89.5	95.5	91.8	92.5	92.1	92.7
	6	94.8	89.3	94.8	89.3	90.0	93.0	93.8
10	1	99.3	90.1	95.5	92.0	92.4	92.6	93.3
	2	99.1	91.2	95.7	92.9	92.9	92.2	92.5
	3	98.2	92.3	94.3	93.5	94.2	94.4	93.6
	4	98.7	91.2	94.5	92.6	91.9	93.1	93.0
	5	97.7	93.4	94.8	93.2	93.9	94.7	93.8
	6	96.6	93.0	94.5	92.9	93.5	92.9	92.1
15	1	99.9	92.4	96.0	93.7	93.2	94.4	93.6
	2	99.3	92.1	94.9	93.1	93.7	93.9	93.4
	3	99.2	92.1	95.1	93.2	94.7	94.3	94.2
	4	99.5	92.4	95.4	92.4	93.3	93.3	92.9
	5	98.8	93.6	95.7	93.5	94.0	93.7	94.4
	6	97.7	94.1	95.1	93.3	92.7	94.3	94.3
25	1	100.0	93.0	95.0	93.1	93.0	93.2	93.4
	2	99.5	93.1	94.5	93.1	93.7	94.0	93.9
	3	99.9	94.3	94.8	94.1	94.6	94.9	94.9
	4	99.9	93.4	94.5	93.5	93.0	94.0	94.5
	5	99.2	94.8	95.3	94.5	93.6	93.9	93.7
	6	98.2	94.3	95.1	94.1	93.9	94.4	94.4

4.1 Comparison of point estimators

From Table 4, it is not surprising to see that the MSEs of the estimators are decreasing with increase sample size (m), and the performance of the estimators depend on the structure of the system. As mentioned before, System No. 1 in Table 3 is equal in law to the minimum of n component lifetimes, and therefore serves as a basis for the comparison of precision of estimates for the component lifetimes based on data of lifetimes from other coherent systems. We can see that the inference based on samples from different systems (System No. 2–6) provide better results (smaller MSE) compared to the inference based on the minimum of n component lifetimes (System No. 1). This suggests that it would be more efficient for the estimation of the mean lifetimes of components if we choose a suitable system structure and place the components in the system to run the life-testing experiment. For instance, the comparison between systems with different signatures can be done in terms of the variance of the MOM as illustrated in Sect. 3.1.

Table 6 Simulated averaged conditional lengths of 95% confidence intervals of α for different 4-component systems with $m = 5$, 10, 15 and 25

m	System no.	AMLE	AMOM1	AMOM2	BMOM	BMLE	BLSE	BWLS
5	1	2.513	7.855	2.243	2.246	2.251	3.890	3.655
	2	1.867	3.260	1.763	1.880	1.741	3.248	3.285
	3	1.942	4.063	1.876	1.792	1.902	2.372	2.361
	4	1.960	3.735	1.836	1.889	1.866	3.162	3.058
	5	1.544	2.385	1.485	1.499	1.548	2.040	2.010
	6	1.188	1.475	1.148	1.160	1.154	1.460	1.418
10	1	2.205	2.069	1.399	1.383	1.373	2.032	1.959
	2	1.446	1.445	1.132	1.150	1.103	1.668	1.621
	3	1.576	1.606	1.205	1.157	1.144	1.460	1.364
	4	1.576	1.538	1.183	1.171	1.156	1.560	1.520
	5	1.188	1.200	0.987	0.970	0.960	1.113	1.089
	6	0.863	0.856	0.767	0.761	0.749	0.850	0.836
15	1	2.086	1.373	1.083	1.087	1.087	1.461	1.400
	2	1.252	1.047	0.899	0.913	0.881	1.211	1.152
	3	1.386	1.143	0.960	0.937	0.917	1.063	1.042
	4	1.375	1.098	0.931	0.928	0.923	1.183	1.143
	5	1.019	0.895	0.792	0.783	0.777	0.872	0.849
	6	0.724	0.673	0.623	0.620	0.613	0.679	0.662
25	1	2.054	0.937	0.824	0.815	0.829	0.985	0.950
	2	1.023	0.738	0.680	0.682	0.664	0.825	0.783
	3	1.155	0.799	0.724	0.709	0.700	0.762	0.743
	4	1.152	0.774	0.704	0.694	0.700	0.813	0.781
	5	0.816	0.644	0.601	0.597	0.596	0.662	0.646
	6	0.571	0.499	0.478	0.474	0.475	0.518	0.504

In this simulation study we observe that the MOM has the smallest bias among all the estimators considered here. The performance of MOM and MLE are quite similar and they out perform the LSE and WLSE in terms of MSE. Additionally, the LSE and WLSE are numerically unstable and sometimes there is no unique solution. Since the MOM has a closed-form solution which makes it easier to calculate, and that the MLE requires the solution of a non-linear equation numerically, we recommended the MOM for use in this situation. Note that the MOM may not have the advantage of having a closed-form solution when the component lifetime distributions are not exponential. In these cases, we can either use MOM or MLE.

4.2 Comparison of interval estimates

From Tables 5 and 6, once again, we observe that the average widths of the confidence intervals are decreasing with increasing sample size (m) and the performance of the confidence intervals do depend on the structure of the system.

Table 7 Dataset for illustrative example based on system number 2 in Table 3 with exponential components, signature $\mathbf{p} = (1/4, 1/4, 1/2, 0)$ and $m = 15$

k	1	2	3	4	5	6	7	8
$t_{k:15}$	0.00904	0.01088	0.13532	0.15275	0.17916	0.18682	0.23807	0.28067
k	9	10	11	12	13	14	15	
$t_{k:15}$	0.32102	0.33895	0.37707	0.46568	0.52325	0.63889	1.02907	

When comparing the confidence intervals in terms of coverage probabilities (see Table 5), we find that the AMLE always maintains coverage probabilities above the nominal level and also AMOM2 has coverage probabilities above the desired nominal level in most of the cases. The coverage probabilities of AMOM1 are often smaller than the nominal level, especially for small sample sizes (say, $m = 5$ and 10). The coverage probabilities of the bootstrap confidence intervals (BMOM, BMLE, BLSE and BWLS) are all quite similar and that they are all below the nominal level. In general, we observe that the order of performance in terms of coverage probabilities is AMOM1 $<$ BMOM, BMLE, BLSE, BWLS $<$ AMOM2 $<$ AMLE.

The AMLE always have coverage probabilities above the nominal level, but the trade-off is the average conditional width is the largest in most of the situations except for a few cases in small sample sizes. When comparing the confidence intervals in terms of the average conditional widths (see Table 6), we observe that the order of widths, in general, is AMOM2, BMOM, BMLE $<$ AMOM1 $<$ BWLS $<$ BLSE $<$ AMLE.

Taking the coverage probabilities as well as the average conditional widths into account, the asymptotic confidence interval based on method of moments estimator (AMOM2) in Eq. (12) provide a good balance between the two even when sample size is as small as $m = 5$. Therefore, we would recommend the use of AMOM2 for interval estimation of the mean lifetime of components.

5 Illustrative example

To illustrate all the methods presented in the preceding section, a sample of size $m = 15$ is generated from a 4-component system with system signature $\mathbf{p} = (1/4, 1/4, 1/2, 0)$ and minimal signature $\mathbf{a} = (0, 3, -3, 1)$ (see Fig. 1), and with components following the exponential distribution with hazard rate $\alpha = 2$. The data are presented in Table 7.

With these data, we obtain the point and interval estimates and these are summarized in Table 8. We find that all the confidence intervals do indeed here cover the true value of the parameter α. We also observe that AMOM2 provids the shortest confidence interval while BLSE, BWLS and AMLE all yield wider confidence intervals, which are consistent with the findings based on simulations.

6 Conclusions

In this paper, statistical inference on the lifetime distribution of components is discussed based on data of lifetimes of systems with the same structure for which the

Table 8 Point and interval estimates of α for illustrative example based on system number 2 in Table 3 with exponential components, signature $\mathbf{p} = (1/4, 1/4, 1/2, 0)$ and $m = 15$

Point estimates		95% Confidence intervals	
Method	Estimate	Method	Estimate
MOM	2.302	AMOM1	(1.616, 4.000)
		AMOM2	(1.325, 3.280)
		BMOM	(1.509, 3.790)
MLE	2.340	AMLE	(0.981, 3.699)
		BMLE	(1.570, 3.569)
LSE	2.200	BLSE	(1.437, 4.115)
WLSE	2.193	BWLS	(1.530, 4.226)

signature is known. Different estimators for the proportionality parameter in a general proportional hazard rate model are derived and their properties are examined. When the lifetime distribution of the components is exponential, point and interval estimation of the unknown mean lifetime of the components are studied as well. Comparison of systems with different structures based on the performance of estimation procedure is discussed. An extensive Monte Carlo simulation study carried out shows the merit of placing components in systems with known structure (i.e., signature) and then observing the lifetimes of those systems rather than conducting a life-testing experiment directly on individual components. Based on the simulation results, we recommend the point estimation based on either the method of moments or the maximum likelihood (with the former being simpler in form for the exponential case) and the asymptotic confidence intervals based on them for the purpose of interval estimation.

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