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A dynamic measure of inaccuracy between two past lifetime distributions

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Abstract In the present communication we introduce a dynamic measure of inaccuracy between two past lifetime distributions over the interval $(0, t)$. Based on proportional reversed hazard rate model (PRHRM), a characterization problem for this dynamic inaccuracy measure has been studied. An upper bound to the dynamic measure of inaccuracy $H^*(f, g; t)$ has also been derived.

Keywords Kerridge inaccuracy · Past entropy · Measure of discrimination · Reversed hazard rate function · Proportional reversed hazards model

1 Introduction

Let X and Y be two non-negative random variables representing time to failure of two systems with p.d.f., respectively, $f(x)$ and $g(x)$. Let $F(x) = P(X \leq x)$ and $G(y) = P(Y \leq y)$ be failure distributions, $\mu_F(x) = \frac{f(x)}{F(x)}$ and $\mu_G(x) = \frac{g(x)}{G(x)}$ be reversed hazard rates, and $\bar{F}(x) = 1 - F(x)$ and $\bar{G}(x) = 1 - G(x)$ be survival functions of X and Y, respectively.

Shannon's (1948) measure of uncertainty associated with the random variable X, Kullback's (1959) measure of discrimination of X about Y, and Kerridge (1961) measure of inaccuracy are given by

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$$H(f) = -\int_0^\infty f(x) log f(x) \mathrm{d}x, \tag{1}$$

$$H(f/g) = \int_0^\infty f(x) log \frac{f(x)}{g(x)} \mathrm{d}x \tag{2}$$

and

$$H(f, g) = -\int_0^\infty f(x) log\ g(x) \mathrm{d}x \tag{3}$$

respectively.

Since these measures are not applicable to a system which has survived for some unit of time, thus the concept of residual measures have been developed in the literature. A direct approach to measure uncertainty in the residual lifetime distribution was initiated by Ebrahimi (1996) and explored further by Ebrahimi and Pellerey (1995), Ebrahimi and Kirmani (1996a,b), and various others.

Given that the system has survived up to time t, the corresponding dynamic measures of uncertainty, refer Ebrahimi (1996), discrimination of X about Y , refer Ebrahimi and Kirmani (1996a,b) and of inaccuracy, refer Taneja et al. (2009) are given by

$$H(f; t) = -\int_t^\infty \frac{f(x)}{\bar{F}(t)} log \frac{f(x)}{\bar{F}(t)} \mathrm{d}x \tag{4}$$

$$H(f/g; t) = \int_t^\infty \frac{f(x)}{\bar{F}(t)} log \frac{f(x)/\bar{F}(t)}{g(x)/\bar{G}(t)} \mathrm{d}x \tag{5}$$

and

$$H(f, g; t) = \int_t^\infty \frac{f(x)}{\bar{F}(t)} log \frac{g(x)}{\bar{G}(t)} \mathrm{d}x \tag{6}$$

respectively.

When $t = 0$, then (4),(5) and (6) reduce respectively to (1), (2) and (3).

However, in many realistic situations, uncertainty is not necessarily related to the future but can also refer to the past. For instance if at time t, a system which is observed only at certain preassigned inspection times, is found to be down, then the uncertainty of the system's life relies on the past, that is, at which instant in (0, t) the system has failed. To be more specific, in a periodic replacement policy where the system is observed at times $T, 2T, 3T, \ldots$ for some preassigned time T, it is possible that at time $(n-1)T$ the system is functioning, but at time nT the system is found to be down, where n is a positive integer. Then, if X is the failure time of the system, the variable of interest is $[nT - X|X \leq nT]$.

By writing $nT = t$, we have the random variable $X_t = [t - X|X \leq t]$, known as the inactivity time. This is because, once at time X the system fails, and at time t it is observed to be in a failure state, the random time for which the system was down is X_t.

Based on this idea, Di Crescenzo and Longobardi (2002, 2004) have studied measures of entropy and discrimination based on past entropy over $(0, t)$ given respectively as

$$H^*(f;t) = -\int_0^t \frac{f(x)}{F(t)} log \frac{f(x)}{F(t)} \mathrm{d}x \tag{7}$$

and

$$H^*(f/g;t) = \int_0^t \frac{f(x)}{F(t)} log \frac{f(x)/F(t)}{g(x)/G(t)} \mathrm{d}x \tag{8}$$

In sequel to these measures of entropy and discrimination based on the past entropy over $(0, t)$, we propose

$$H^*(f,g;t) = -\int_0^t \frac{f(x)}{F(t)} log \frac{g(x)}{G(t)} \mathrm{d}x \tag{9}$$

as *a dynamic measure of inaccuracy based on the past entropy over* $(0, t)$.

Here we observe that

$$H^*(f;t) + H^*(f/g;t) = H^*(f,g;t) \tag{10}$$

in confirmation with the result

$$H(f) + H(f/g) = H(f,g),$$

in the literature, refer Kerridge (1961) inaccuracy.

In case we have a system with true distribution function F and the reference distribution function G, then the measure $H^*(f,g;t)$ can be interpreted as a measure of inaccuracy associated with the probability density functions f_t and g_t, where $f_t = \frac{f(x)}{F(t)}$ and $g_t = \frac{g(x)}{G(t)}$.

When $t = \infty$, then (7), (8) and (9) reduce respectively to (1) (2) and (3).

When $g(x) = f(x)$, then (9) becomes (7), the dynamic measure of past entropy given by Di Crescenzo and Longobardi (2002).

In Sect. 2, we characterize the dynamic inaccuracy measure (9) under proportional reversed hazard rate model (PRHRM). In Sect. 3, it is shown that under PRHRM the uniform distribution is characterized by $H^*(f,g;t)$. In Sect. 4, we obtained the upper bound to the dynamic inaccuracy measure $H^*(f,g;t)$.

2 Characterization problem

In this section, we study characterization problem for the past dynamic inaccuracy measure (9) under the assumption of PRHRM.

First we define the *reversed hazard rate*. Let X be a non- negative random variable, with distribution function $F(x)$, denoting the lifetime of a component, then the reversed hazard rate of X is given by

$$\mu_X(x) = \frac{\mathrm{d}}{\mathrm{d}x} log F(x) = \frac{f(x)}{F(x)},$$

where $f(x)$ is the probability density function (p.d.f.) of X.

Here $\mu_X(x)\mathrm{d}x$ provides the probability of failing in the interval $(x - \mathrm{d}x, x)$, when a unit has been found failed at time x. For example, if lifetime of a component is uniformly distributed in interval (a, b), then the reversed hazard rate is, $\mu_X(x) = \frac{f(x)}{F(x)} = \frac{1}{x-a}$.

Next, two random variables X and Y satisfy the proportional reversed hazard rate model (PRHRM), refer Gupta et al. (1998) with proportionality constant $\beta(> 0)$, if

$$\mu_Y(x) = \beta\mu_X(x) \quad \beta > 0. \tag{11}$$

The PRHRM is equivalent to the model

$$G(x) = [F(x)]^\beta, \tag{12}$$

where $F(x)$ is the baseline distribution function and $G(x)$ can be considered as some reference distribution function. This model was proposed by Gupta et al. (1998) in contrast to the celebrated proportional hazard model (PHM). This model is flexible enough to accommodate both monotonic as well as non-monotonic failure rates even though the baseline failure rate is monotonic.

As an example, for some positive integer value of β, if $X_1, X_2, \ldots, X_\beta$ are independent and identically distributed (i.i.d.) random variables with distribution function $F(x)$ representing the lifetime of components, in a β-component parallel system, then the lifetime of the system is given by

$Y = max(X_1, X_2, \ldots, X_\beta)$ with distribution function $G(x)$ given by (12).

Consider

$$\begin{aligned} H^*(f, g; t) &= -\int_0^t \frac{f(x)}{F(t)} log \frac{g(x)}{G(t)} \mathrm{d}x \\ &= log\ G(t) - \int_0^t \frac{f(x)}{F(t)} log\ g(x)\mathrm{d}x \qquad (13) \\ &= log\ G(t) - \int_0^t \frac{f(x)}{F(t)} log\ \mu_G(x)\mathrm{d}x - \frac{1}{F(t)} \int_0^t f(x) log\ G(x)\mathrm{d}x \qquad (14) \end{aligned}$$

Using (11) and (12), in (14) we obtain

$$H^*(f, g; t) = \beta - log\beta - \int_0^t \frac{f(x)}{F(t)} log\ \mu_F(x)\mathrm{d}x \tag{15}$$

When $\beta = 1$, that is, $G(x) = F(x)$, then (15) becomes the past entropy given by Di Crescenzo and Longobardi (2002).

Remark 2.1 We observe the following relation between the three inaccuracy measures considered above.

$$H(f; g) = \overline{F}(t)H(f, g; t) + F(t)H^*(f, g; t) + H\left[F(t), G(t)\right]; \tag{16}$$

where $H\left[F(t), G(t)\right] = -F(t)\ log\ G(t) - [1 - F(t)]\ log\ [1 - G(t)]$, corresponds to the Kerridge (1961) inaccuracy. When $g = f$, then (16) reduces to

$$H(f) = H\left[F(t), \overline{F}(t)\right] + F(t)H^*(t) + \overline{F}(t)H(t),$$

a result obtained by Di Crescenzo and Longobardi (2002); where $H[p, 1 - p] = -p\ log\ p - (1 - p)log\ (1 - p)$ is the entropy of a Bernoulli distribution.

3 Characterization result

In this section, we characterize uniform distribution in terms of the past dynamic inaccuracy measure under the assumption that X and Y satisfy the proportional reversed hazard rate model (PRHRM).

Differentiating (13) w.r.t. t and using $\mu_G(x) = \beta\mu_F(x)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}H^*(f, g, t) = \mu_G(t) - \mu_F(t)log\ g(t) + \mu_F(t)\int_0^t \frac{f(x)}{F(t)} log\ g(x)\mathrm{d}x \tag{17}$$

$$\begin{aligned}
&= \mu_F(t)\left[\beta - log\ g(t) - \int_0^t \frac{f(x)}{F(t)} log\ g(x)\mathrm{d}x\right] \\
&= \mu_F(t)\left[\beta - log\ g(t) + logG(t) + \int_0^t \frac{f(x)}{F(t)} log\frac{g(x)}{G(t)}\mathrm{d}x\right] \\
&= \mu_F(t)[\beta - log\ \mu_G(t) - H^*(f, g; t)] \\
&= \mu_F(t)[\beta - log\beta - log\mu_F(t) - H^*(f, g; t)]
\end{aligned} \tag{18}$$

This gives

$$\frac{\mathrm{d}}{\mathrm{d}t}H^*(f, g; t) - \mu_F(t)[\beta - log\beta - log\mu_F(t) - H^*(f, g; t)] = 0.$$

Hence for a fixed $t > 0$, $\mu_F(t)$ is a solution of $g_1(x) = 0$, where

$$g_1(x) = \frac{\mathrm{d}}{\mathrm{d}t} H^*(f, g; t) - x[\beta - log\beta - logx - H^*(f, g; t)]. \tag{19}$$

Differentiating both side of (19) with respect to x, we get

$$g_1^{'}(x) = [1 - \beta + log\ \beta + log\ x + H^*(f, g; t)]$$

Thus $g_1^{'}(x) = 0$ gives $x = exp\ [\beta - 1 - log\ \beta - H^*(f, g; t)] = x_0$, say.

Next, we give a theorem which characterizes uniform distribution in terms of increasing past inaccuracy measure $H^*(f, g; t)$.

Theorem 3.1 *If random variables X and Y satisfy the proportional reversed hazard rate model (PRHRM) with proportionality constant β (> 0), then random variable X over (a, b) $a < b$ has uniform distribution if, and only if*

$$H^*(f, g; t) = \beta - log\ \beta - 1 + log\ (t - a), \quad a < t < b \tag{20}$$

Proof The "only if"; part of the theorem is straightforward since in case of uniform distribution of X over (a, b)

$$F(x) = \frac{x - a}{b - a} \quad \text{and} \quad f(x) = \frac{1}{b - a}$$

Hence, under PRHRM, $G(x) = \left[\frac{x-a}{b-a}\right]^\beta$ which gives $g(x) = \frac{\beta(x-a)^{\beta-1}}{(b-a)^\beta}$. Substituting these in (13) and simplifying, we get

$$H^*(f, g; t) = \beta - log\ \beta - 1 + log\ (t - a)$$

To prove the 'if part' let (19) be valid. Then from (20), we have $g_1(0) = \frac{\mathrm{d}}{\mathrm{d}t} H^*(f, g; t) > 0$. Also we can show that $g_1(x)$ is a convex function with minimum occurring at $x = x_0$. So $g_1(x) = 0$ has unique solution if $g_1(x_0) = 0$. We have

$$x_0 = exp[\beta - 1 - log\ \beta - H^*(f, g; t)]$$

Using (20), we get $x_0 = \frac{1}{t-a}, \quad t > a$ and

$$g_1(x_0) = \frac{\mathrm{d}}{\mathrm{d}t} H^*(f, g; t) - x_0[\beta - log\ \beta - log\ x_0 - H^*(f, g; t)] = 0.$$

Thus, $g_1(x) = 0$ has unique solution given by $x = x_0$. But $\mu_F(t)$ is a solution to (19). Hence $\mu_F(t) = x_0 = (t - a)^{-1}$, $t > a$ is the unique solution to $g_1(x) = 0$. Thus the distribution is uniform, and this proves the result.

To illustrate the characterization results obtained above we consider the following example:

Example 3.1 Consider a parallel system of n components having i.i.d lifetime $X_i's$, $i = 1, 2, \ldots n$, where $X_i's$ are exponentially distributed with same parameter θ, and let $Y = max\{X_1, X_2, \ldots, X_n\}$ be the lifetime of the system.

Denoting by $F(x)$ the c.d.f. and by $f(x)$ the p.d.f. of X_i If G is the distribution function for Y; then under PRHRM, the c.d.f. of Y is $G(x) = [F(x)]^n$ and its p.d.f. is $g(x) = n[F(x)]^{n-1} f(x)$.

Here

$$f(x) = \theta\ e^{-\theta x}$$
$$F(x) = 1 - e^{(-\theta x)}$$
$$g(x) = n\theta\ e^{-\theta x}[1 - e^{(-\theta x)}]^{n-1}$$
$$H^*(f, g; t) = log\ G(t) - \int_0^t \frac{f(x)}{F(t)} log\ g(x)\mathrm{d}x$$

Substituting for G, F, f and g, this gives

$$H^*(f, g; t) = n - log\ n\theta + log(1 - e^{-\theta t}) - \frac{\theta t e^{-\theta t}}{1 - e^{-\theta t}} \tag{21}$$

Taking limit as $t \to \infty$, we obtain

$$lim_{t\to\infty} H^*(f, g; t) = n - log\ n\theta \tag{22}$$

a result in confirmative the inaccuracy measure $H(f, g)$ under PRHRM for

$$f(x) = \theta e^{-\theta x}$$

Figure 1 suggests that when n, the number of components increases in a parallel system then the magnitude of past inaccuracy measure $H^*(f, g; t)$ increases. Otherwise, also, for fix n, $H^*(f, g; t)$ is an increasing function of t. The curve in Fig. 1 have been plotted for $t \in [0, 2]$.

Next, we consider another example where $F(x)$ and $G(x)$ does not satisfy proportional reversed hazard rate model.

Example 3.2 Let X and Y be two nonnegative random variables having distribution function

$$F(x) = \begin{cases} \frac{x^2}{2}, & \text{for } 0 \le x < 1 \\ \frac{x^2+2}{6}, & \text{for } 1 \le x < 2 \\ 1 & \text{for } x \ge 2 \end{cases}$$

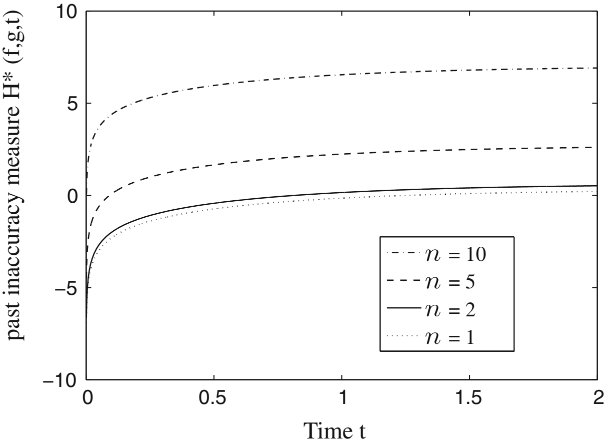

Fig. 1 Plot of $H^*(f, g; t)$ (Example 3.1) versus t for different values of n

and

$$G(x) = \begin{cases} \frac{x^2+x}{4}, & \text{for } 0 \le x < 1 \\ \frac{x}{2}, & \text{for } 1 \le x < 2 \\ 1 & \text{for } x \ge 2 \end{cases}$$

Then the past inaccuracy measure is given by

$$H^*(f, g; t) = \begin{cases} \frac{1}{2} - \frac{1}{2t} + \frac{1}{4t^2} log(2t+1) + log\frac{t^2+t}{2t+1}, & \text{for } 0 < t < 1 \\ log\frac{t}{2} + \left(\frac{t^2-1}{t^2+2}\right) log2 + \frac{6}{t^2+2} log2 - \frac{9}{4(t^2+2)} log3, & \text{for } 1 \le t < 2 \\ \frac{3}{2} log2 - \frac{3}{8} log3 & \text{for } t \ge 2 \end{cases}$$

For $t \in [0, 1)$, the past inaccuracy measure $H^*(f, g; t)$ is increasing, and it is shown in Fig. 2.

4 An upper bound to $H^*(f, g; t)$

We prove the following result

Theorem 4.1 *If $\bar{F}$ and $\bar{G}$ satisfy the proportional reversed hazard rate model and $\mu_F(t)$ is decreasing, then*

$$H^*(f, g, t) \le \beta - log\beta - log\ \mu_F(t) \tag{23}$$

where $\mu_F(t)$ is the reversed failure rate function.

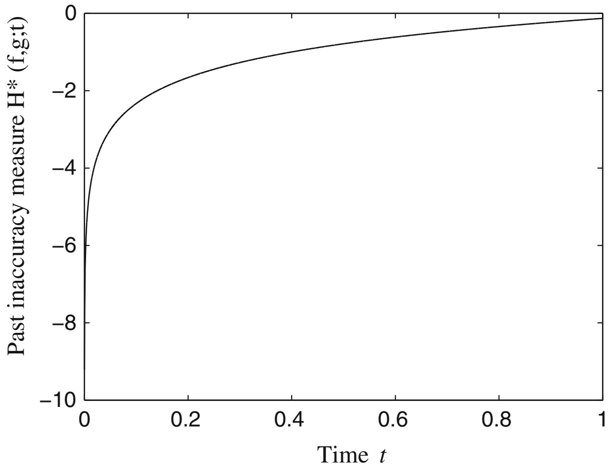

Fig. 2 Plot of $H^*(f, g; t)$ (Example 3.2) against $t \in [0, 1]$

Proof The dynamic measure of inaccuracy is

$$H^*(f, g; t) = -\int_0^t \frac{f(x)}{F(t)} log \frac{g(x)}{G(t)} \mathrm{d}x$$

Using proportional reversed hazard rate model, we have

$$\begin{aligned} H^*(f, g; t) &= \beta - log\beta - \int_0^t \frac{f(x)}{F(t)} log\ \mu_F(x)\mathrm{d}x, \\ &= (\beta - log\beta - 1) + 1 - \int_0^t \frac{f(x)}{\bar{F}(t)} log\ \mu_F(x)\mathrm{d}x \end{aligned} \tag{24}$$

Also,

$$H^*(f; t) = -\int_0^t \frac{f(x)}{F(t)} log \frac{f(x)}{F(t)} \mathrm{d}x \le 1 - log\ \mu_F(t) \tag{25}$$

refer, Di Crescenzo and Longobardi (2002). Using this in (24), we get (23).

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