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ORIGINAL ARTICLE

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Goodness-of-fit criteria for the Adams and Jefferson rounding methods and their limiting laws

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Abstract Multiplier methods are used to round probabilities on finitely many categories to rational proportions. Focusing on the classical methods of Adams and Jefferson, we investigate goodness-of-fit criteria for this rounding process. Assuming that the given probabilities are uniformly distributed, we derive the limiting laws of the criteria, first when the rounding accuracy increases, and then when the number of categories grows large.

Keywords Apportionment method · q-Stationary multiplier method · Rounding error analysis · Sainte–Laguë divergence · Convergence in distribution · Gaussian limit law

1 Introduction

Let $W = (W_1, \ldots, W_c)$ be some "arbitrary" probability vector, for a fixed number of *categories* c. We model "arbitrariness" by assuming W to be random and, moreover, to follow a continuous distribution on the probability simplex

$$\mathcal{S}_c = \left\{(w_1, \ldots, w_c) \in [0, 1]^c : w_1 + \cdots + w_c = 1\right\}.$$

Rounding methods are used to round the continuous *weights* W_i to rational proportions of the form N_i/n, for some prescribed integer *accuracy* n. In order that the proportions N_i/n again form a valid probability vector, the numerators $N_1, \ldots, N_c$ must of course sum to n.

The idea underlying q-stationary multiplier methods (for some fixed $q \in [0, 1]$) relies on the rounding function $r_q(x)$, which rounds down to the next integer if the

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fractional part of the number x is less than q, and up to the next integer if it is greater than or equal to q. More formally,

$$r_q(x) := \lfloor x + 1 - q \rfloor = \begin{cases} \lfloor x \rfloor & \text{for } x - \lfloor x \rfloor < q, \\ \lfloor x \rfloor + 1 & \text{for } x - \lfloor x \rfloor \geq q, \end{cases}$$

where $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$, i.e. $x - 1 < \lfloor x \rfloor \leq x$.

Now, given a set $W_1, \ldots, W_c$ of weights, each weight is first scaled by some global *multiplier* $\nu > 0$ and then rounded as $N_i = r_q(\nu W_i)$, where the multiplier ν is adjusted such as to achieve the correct total

$$\sum_{i \leq c} N_i = n.$$

Note that individual rounding of each of the scaled weights nW_i to integers $r_q(nW_i)$ (for some $q \in [0, 1]$) does *not* guarantee that these integers achieve the correct total n. The rounding errors connected with q-stationary multiplier methods are described by the *discrepancy*

$$D_c^{(n)}(q, \nu) = \sum_{i \leq c} r_q(\nu W_i) - n \quad \text{for } \nu \geq 0,$$

see L. Heinrich et al. (submitted) and references therein. For the global multiplier $\nu_n = n + c(q - 1/2)$ the discrepancy

$$D_{q,c}^{(n)} := D_c^{(n)}(q, \nu_n) = \sum_{i \leq c} \left(r_q(\nu_n W_i) - \nu_n W_i \right) + c \left(q - \frac{1}{2} \right) \tag{1}$$

attains the integer values $-\lfloor \frac{c-1}{2} \rfloor, \ldots, 0, \ldots, \lfloor \frac{c-1}{2} \rfloor$ for all $W \in \mathcal{S}_c \setminus \mathcal{N}_{q,c}^{(n)}$, where $\mathcal{N}_{q,c}^{(n)}$ is a subset of $\mathcal{S}_c$ with Lebesgue measure zero for all $n \geq c$. For distributional properties of $D_{q,c}^{(n)}$ we refer the reader to Happacher (2001).

Since in general the above discrepancies do not disappear, many statistical publications include a salvatory clause, often hidden away in some small-print footnote, that "percentages may not sum to 100 due to rounding errors".

There are other spheres of life, though, that do not tolerate such a liberal attitude towards rounding errors. Most noticeably this concerns apportionment methods for proportional representation in electoral systems. There, the c categories signify the political bodies participating in the apportionment process, and the accuracy n is the number of seats to be apportioned among them. For instance, the $n = 435$ seats of the US House of Representatives are apportioned to the $c = 50$ States, proportionally to their population, or the $n = 598$ seats in the German Bundestag are apportioned among $c = 5$ eligible parties, proportionally to their electoral votes. In the political arena it is plainly *not* acceptable that an apportionment procedure would terminate with a nonvanishing discrepancy, leaving some seats unaccounted for "due to rounding errors". In fact, the field of politics abounds with *apportionment methods* properly partitioning the total n into integers $N_1, \ldots, N_c$. The seminal monograph of Balinski and Young (2001) is an excellent source for the political history of proportional representation, as well as for the mathematical theory of apportionment methods that flows from the historical experience.

Because each apportionment method includes a rounding process, inevitable gaps arises between the ideal seat allocations nW_i based on continuous fractions and the actual seat allocations N_i based on the accuracy given by the size n of the parliament. It is generally accepted that these gaps should be minimized simultaneously. However, there are many ways of carrying out the optimization. For example, minimizing the so-called Sainte–Laguë divergence

$$\sum_{i \le c} \frac{(N_i - nW_i)^2}{W_i} \quad \text{with} \quad N_1 + \cdots + N_c = n,$$

gives rise to the multiplier method with standard rounding. Heinrich et al. (2004) study the limiting law of this criterion for an increasing rounding accuracy n as well as for a large number of categories. Minimizing the Sainte–Laguë divergence is motivated as follows: if party i gains weight $W_i = V_i/V$, calculated from the vote count V_i and the vote total V, then

$$\sum_{i \le c} \frac{(N_i - nW_i)^2}{W_i} = \frac{n^2}{V} \sum_{i \le c} V_i \left(\frac{N_i/n}{V_i/V} - 1 \right)^2$$

is the sum, for each of the V_i voters, of the squared differences between the realized success values of the voters $\frac{N_i/n}{V_i/V}$ and the ideal success value 1.

However, other functions seem likewise worth to be optimized. From the parties' point of view, it is desirable that the maximal discrepancy between the ideal and actual seat allocations becomes as small as possible. To achieve this objective we consider the goodness-of-fit criteria

$$S^J_{c,n} = \max_{i \le c} \left(\frac{N_i}{W_i} - n \right), \tag{2}$$

and

$$S^A_{c,n} = \min_{i \le c} \left(\frac{N_i}{W_i} - n \right), \tag{3}$$

in both cases conditional on $N_1 + \cdots + N_c = n$. Minimizing the Jefferson criterion $S^J_{c,n}$,

$$\min \left\{ \max_{i \le c} \left(\frac{n_i}{W_i} - n \right) : n_1, \ldots, n_c \in \{0, 1, \ldots, n\}, \sum_{i \le c} n_i = n \right\},$$

makes the most advantaged party as little advantaged as possible, and leads to the multiplier method with rounding down ($q = 1$), the method of Jefferson (Balinski and Young 2001, Proposition 3.10). Similarly, maximizing the Adams criterion $S^A_{c,n}$,

$$\max \left\{ \min_{i \le c} \left(\frac{n_i}{W_i} - n \right) : n_1, \ldots, n_c \in \{0, 1, \ldots, n\}, \sum_{i \le c} n_i = n \right\},$$

makes the least advantaged party as advantaged as possible, and leads to the multiplier method with rounding up ($q = 0$), the method of Adams (Balinski and Young 2001, Proposition 3.10).

In section 2 we present a slightly modified form of the *Adjustment Algorithm* for q-stationary multiplier methods (see L. Heinrich et al. (2005)) to find the apportionment vector $(N_{q,1}^{(n)}, \dots, N_{q,c}^{(n)})$ and summarize some asymptotic results for $S_{c,n}^J$ and $S_{c,n}^A$ as the accuracy n tends to infinity. The corresponding weak limits S_c^J and S_c^A turn out to be approximately Gaussian distributed for large c. These and related results are formulated in section 3. The proofs are deferred to the sections 4 and 5. The final section 6 rounds off the paper with a brief discussion about possible extensions and potential applications of the obtained results.

2 Some preliminary results

To be mathematically precise, let $[\Omega, \mathfrak{A}, \mathsf{P}]$ be a common probability space on which all random elements in this paper will be defined. Slightly stronger than in section 1, we assume that the random vector $W_{(c-1)} := (W_1, \dots, W_{c-1})$ takes values only in the $(c-1)$-dimensional open unit simplex

$$\mathcal{T}_{c-1} = \left\{(w_1, \dots, w_{c-1}) \in (0,1)^{c-1} : w_1 + \cdots + w_{c-1} < 1\right\},$$

according to some absolutely continuous distribution, and let

$$W_c = 1 - W_1 - \cdots - W_{c-1}. \tag{4}$$

Next, we shall rewrite the discrepancy $D_{q,c}^{(n)}$ defined by (1) in terms of the q-stationary residuals $\nu_n W_i - r_q(\nu_n W_i)$ for $i = 1, \dots, c-1$. For this, we introduce the sequence of random vectors $U_{(c-1)}^{(n)}(q) := (U_{q,1}^{(n)}, \dots, U_{q,c-1}^{(n)})$ taking values in the half-open cube $(-1/2\,,\ 1/2]^{c-1}$ with components

$$U_{q,i}^{(n)} = r_q(\nu_n\, W_i) - \nu_n\, W_i + q - \frac{1}{2} \qquad \text{for} \quad i = 1, \dots, c-1 \quad \text{and } i = c.$$

From (4) it is easy to see that

$$\begin{aligned} U_{q,c}^{(n)} &= \left\lfloor (c-1)\left(q - \frac{1}{2}\right) - \sum_{i<c} \nu_n\, W_i + \frac{1}{2} \right\rfloor - (c-1)\left(q - \frac{1}{2}\right) + \sum_{i<c} \nu_n\, W_i \\ &= \left\lfloor \sum_{i<c} U_{q,i}^{(n)} + \frac{1}{2} \right\rfloor - \sum_{i<c} U_{q,i}^{(n)} \quad \left(\in \left(-\frac{1}{2}, \frac{1}{2}\right]\right), \end{aligned}$$

which, by (1), gives

$$D_{q,c}^{(n)} = \sum_{i \le c} U_{q,i}^{(n)} = \left\lfloor \sum_{i<c} U_{q,i}^{(n)} + \frac{1}{2} \right\rfloor.$$

Next, for fixed $q \in [0, 1]$, define a double sequence of (measurable) random functions $\{ X_{q,c}^{(n)}(\tau, \cdot), \ \tau \in [0, c/2] \}$ over $[\Omega, \mathfrak{A}, \mathsf{P}]$ by

$$X_{q,c}^{(n)}(\tau, \omega) := \sum_{i \le c} \left\lfloor \tau \, W_i(\omega) + \operatorname{sgn}(D_{q,c}^{(n)}(\omega)) \, U_{q,i}^{(n)}(\omega) + \frac{1}{2} \right\rfloor \quad \text{for } \omega \in \Omega.$$

To ensure that each of the piecewise constant, nondecreasing, and right-continuous functions $\tau \mapsto X_{q,c}^{(n)}(\tau, \omega)$ possesses only upward-jumps of magnitude 1 we have to "clean" Ω from a P-nullset $\Omega \setminus \Omega_c^*$ (due to the distributional assumption on $W_{(c-1)}$), where

$$\Omega_c^* = \bigcap_{n \ge c} \left\{ \omega \in \Omega : X_{q,c}^{(n)}(0, \omega) = 0 \, , \ X_{q,c}^{(n)}(\tau, \omega) - X_{q,c}^{(n)}(\tau - 0, \omega) \le 1 \, , \ \forall \tau > 0 \right\}.$$

Hence, the values of the random variable

$$\tau_{q,c}^{(n)}(\omega) := \begin{cases} \min \left\{ \tau > 0 \ : \ X_{q,c}^{(n)}(\tau, \omega) \ = \ | \, D_{q,c}^{(n)}(\omega) \, | \right\}, & D_{q,c}^{(n)}(\omega) \neq 0 \\ 0, & D_{q,c}^{(n)}(\omega) = 0 \end{cases}$$

are uniquely defined for $\omega \in \Omega_c^*$. For completeness, set $\tau_{q,c}^{(n)}(\omega) = 0$ otherwise. By introducing the non-negative integers

$$m_i^{(n)}(D_{q,c}^{(n)}) := \left\lfloor \tau_{q,c}^{(n)} \, W_i + \ \operatorname{sgn}(D_{q,c}^{(n)}) \, U_{q,i}^{(n)} + \frac{1}{2} \right\rfloor \quad \text{for } i = 1, \ldots, c, \qquad (5)$$

which coincide with the adjustment terms $m_i^{(n)}(D_{q,c}^{(n)})$ obtained by the Adjustment Algorithm in L. Heinrich et al. (2005) [Lemma 2.1], it is immediately seen that

$$\sum_{i \le c} \ \operatorname{sgn}(D_{q,c}^{(n)}) \, m_i^{(n)}(D_{q,c}^{(n)}) = D_{q,c}^{(n)} \quad (\text{on } \Omega),$$

which yields the final apportionment vector $(N_{q,1}^{(n)}, \ldots, N_{q,c}^{(n)})$ with

$$N_{q,i}^{(n)} = r_q(\nu_n \, W_i) - \ \operatorname{sgn}(D_{q,c}^{(n)}) \, m_i^{(n)}(D_{q,c}^{(n)}) \quad i = 1, \ldots, c. \qquad (6)$$

The following result, which has been established in L. Heinrich et al. (2005) (see Heinrich et al. 2004 for the case of standard rounding $q = 1/2$), gives the key to determine the limiting behaviour of the Sainte–Laguë divergence (for $q \in [0, 1]$, see L. Heinrich et al. (2005)) as well as of the criteria $S_{c,n}^J$ (for $q = 1$) and $S_{c,n}^A$ (for $q = 0$) of Jefferson and Adams, respectively, as $n \to \infty$.

Theorem 1 *If the weight vector* $W_{(c-1)} := (W_1, \ldots, W_{c-1})$ *has a Riemann integrable Lebesgue density on* $\mathcal{T}_{c-1}$, *then*

$$\left(U_{(c-1)}^{(n)}(q) \, , \ W_{(c-1)} \right) \ \xrightarrow[n \to \infty]{\mathrm{d}} \ \left(U_{(c-1)} \, , \ W_{(c-1)} \right), \qquad (7)$$

for any fixed $q \in [0, 1]$, *where the random vector* $U_{(c-1)} := (U_1, \ldots, U_{c-1})$ *is uniformly distributed on the* $(c - 1)$*-dimensional cube* $(-1/2, 1/2)^{c-1}$ *(with independent components) and stochastically independent of* $W_{(c-1)}$.

Multiple application of the Continuous Mapping Theorem (see e.g. Billingsley 1999, p 21) reveals that the above random sequences $D_{q,c}^{(n)}$, $U_{q,c}^{(n)}$, and $X_{q,c}^{(n)}(\tau,\cdot)$ for $\tau \in [0, c/2]$ converge in distribution (as $n \to \infty$) to random variables U_c, D_c, and $X_c(\tau,\cdot)$ (defined on $[\Omega, \mathfrak{A}, \mathsf{P}]$ and having distributions not depending on q) defined by

$$D_c := \left\lfloor \sum_{i<c} U_i + \frac{1}{2} \right\rfloor, \quad U_c := D_c - \sum_{i<c} U_i \quad \left(\in \left(-\frac{1}{2}, \frac{1}{2} \right] \right),$$

and

$$X_c(\tau,\omega) := \sum_{i\le c} \left\lfloor \tau\, W_i(\omega) + \operatorname{sgn}(D_c(\omega))\, U_i(\omega) + \frac{1}{2} \right\rfloor \quad \text{for } \omega \in \Omega, \quad 0 \le \tau \le \frac{c}{2}.$$

Moreover, again using the Continuous Mapping Theorem, we get

$$\tau_{q,c}^{(n)} \xrightarrow[n\to\infty]{\mathrm{d}} \tau_c \quad \text{and} \quad \left(m_i^{(n)}(D_{q,c}^{(n)}) \right)_{i=1}^{c} \xrightarrow[n\to\infty]{\mathrm{d}} (m_i(D_c))_{i=1}^{c}.$$

The weak limits of the adjustment terms (5) are expressible by

$$m_i(D_c) := \left\lfloor \tau_c\, W_i + \operatorname{sgn}(D_c)\, U_i + \frac{1}{2} \right\rfloor \quad \text{for } i = 1, \ldots, c, \tag{8}$$

with

$$\tau_c(\omega) := \begin{cases} \min\{\tau > 0 : X_c(\tau,\omega) = |\, D_c(\omega)\,|\}, & \text{if } D_c(\omega) \neq 0, \quad \omega \in \Omega^* \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

where the event

$$\Omega^* = \bigcap_{c\ge 2} \{\omega \in \Omega : X_c(0,\omega) = 0,\, X_c(\tau,\omega) - X_c(\tau - 0,\omega) \le 1,\ \forall \tau > 0\}$$

differs from Ω by a P-nullset in view of the distributional properties of $W_{(c-1)}$ and $U_{(c-1)}$.

It should be noticed that, by using simply the properties of the function $\lfloor\cdot\rfloor$, the vector of the "limit adjustment terms" (8) is uniquely determined (at least on Ω^*) by the min–max-inequalities:

$$\max_{i\le c} \left(\frac{m_i(D_c) - \operatorname{sgn}(D_c)\, U_i - \frac{1}{2}}{W_i} \right) \le \tau_c < \min_{i\le c} \left(\frac{m_i(D_c) - \operatorname{sgn}(D_c)\, U_i + \frac{1}{2}}{W_i} \right).$$

Rewriting the apportionment numbers (6) in terms of the random variables $U_{q,i}^{(n)}$, $i = 1, \ldots, c$, gives

$$N_{q,i}^{(n)} - n\, W_i = U_{q,i}^{(n)} - \operatorname{sgn}(D_{q,c}^{(n)})\, m_i^{(n)}(D_{q,c}^{(n)}) + (c\, W_i - 1) \left(q - \frac{1}{2} \right).$$

In the extremal cases $q = 1$ (rounding down) and $q = 0$ (rounding up), the latter leads to the following expressions for the minimized Jefferson criterion (2) and the maximized Adams criterion (3):

$$S^J_{c,n} = \max_{i\le c}\left(\frac{U^{(n)}_{1,i} - \operatorname{sgn}(D^{(n)}_{1,c})\, m^{(n)}_i(D^{(n)}_{1,c}) - \frac{1}{2}}{W_i} + \frac{c}{2}\right),$$

and

$$S^A_{c,n} = \min_{i\le c}\left(\frac{U^{(n)}_{0,i} - \operatorname{sgn}(D^{(n)}_{0,c})\, m^{(n)}_i(D^{(n)}_{0,c}) + \frac{1}{2}}{W_i} - \frac{c}{2}\right).$$

Theorem 1 and its above-stated consequences combined with the Continuous Mapping Theorem can be summarized in

Theorem 2 *Under the assumptions of Theorem 1 we have*

$$S^J_{c,n} \xrightarrow[n\to\infty]{\text{d}} S^J_c := \max_{i\le c}\left(\frac{U_i - \operatorname{sgn}(D_c)\, m_i(D_c) - \frac{1}{2}}{W_i} + \frac{c}{2}\right),$$

and

$$S^A_{c,n} \xrightarrow[n\to\infty]{\text{d}} S^A_c := \min_{i\le c}\left(\frac{U_i - \operatorname{sgn}(D_c)\, m_i(D_c) + \frac{1}{2}}{W_i} - \frac{c}{2}\right).$$

For the simplest case of two categories with uniformly distributed weights (W_1, W_2), a short calculation confirms that the weak limits S^J_c and S^A_c are uniformly distributed on $(0, 1)$ and $(-1, 0)$, respectively. How the distributions of these random variables behave for large c is answered in the next section.

3 Limit theorems for large numbers of categories

In this section we formulate our main results on the normal approximation of some of the weak limits obtained in section 2 (as $n \to \infty$) when additionally the number c of categories increases unboundedly. To begin with, observe that Lévy's central limit theorem applied to the identity

$$\operatorname{sgn}(D_c)\sum_{i\le c} m_i(D_c) = D_c = \left\lfloor \sum_{i<c} U_i + 1/2 \right\rfloor, \tag{10}$$

yields

$$\frac{\operatorname{sgn}(D_c)}{\sqrt{c}}\sum_{i\le c} m_i(D_c) \xrightarrow[c\to\infty]{\text{d}} N\left(0, \frac{1}{12}\right), \tag{11}$$

regardless which probability density $W_{(c-1)}$ has. As usual $N(0,\sigma^2)$ denotes a mean zero Gaussian random variable with variance $\sigma^2 > 0$ having the distribution function $\Phi(x/\sigma)$, where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\,\mathrm{d}t \quad \text{for } x \in \mathbb{R}.$$

The proofs of the subsequent results need the additional assumption that $W = (W_1,\ldots,W_c)$ is uniformly distributed on $\mathcal{S}_c$.

Theorem 3 *Assume that the distribution $W_{(c-1)}$ is uniform on $\mathcal{T}_{c-1}$ (i.e. W is uniformly distributed on $\mathcal{S}_c$), and let the random variables $U_1,\ldots,U_{c-1}$ be independent and uniformly distributed on $(-1/2,1/2)$ and independent of $W_{(c-1)}$. Then we have*

$$\frac{sgn(D_c)\,\tau_c}{\sqrt{c}} \xrightarrow[c\to\infty]{\mathrm{d}} N\left(0,\frac{1}{12}\right), \tag{12}$$

i.e.

$$\mathsf{P}\left(\tau_c > x\sqrt{c}\right) \xrightarrow[c\to\infty]{} 2\left(1-\Phi(2\sqrt{3}\,x)\right) \quad \textit{for all } x \geq 0. \tag{13}$$

The asymptotic normality of $\tau_c/\sqrt{c}$ is essential to derive the Gaussian limits in Theorems 4 and 5.

Theorem 4 *Under the assumption of Theorem 1 we have*

$$\frac{sgn(D_c)}{c^{3/2}} \sum_{i\leq c} \frac{m_i(D_c)}{W_i} \xrightarrow[c\to\infty]{\mathrm{d}} N\left(0,\frac{1}{12}\right). \tag{14}$$

As an immmediate consequence of Theorem 4 we get

Corollary 1

$$\frac{1}{c^2} \sum_{i\leq c} \frac{m_i(D_c)}{W_i} \xrightarrow[c\to\infty]{\mathsf{P}} 0, \tag{15}$$

where $\xrightarrow[c\to\infty]{\mathsf{P}}$ means "convergence in probability".

It should be mentioned that the summands in (14) are neither independent nor have finite expectation. Corollary 1 turns out to be indispensible in proving the exact asymptotics of Sainte–Laguë's divergence in particular in the case of non-standard rounding, see Theorem 2 in L. Heinrich et al. (2005).

Finally, the suitably centered and normalized weak limits S_c^J and S_c^A of Theorem 2 can be shown to have Gaussian limits. At the first glance this result is somewhat surprising since both functionals are defined as extreme value statistics. On the other hand, both limiting relations reveal that the lower and upper bound of τ_c in the above min–max-inequality differ only by a stochastic term of order $o(\sqrt{c})$ as $c \to \infty$.

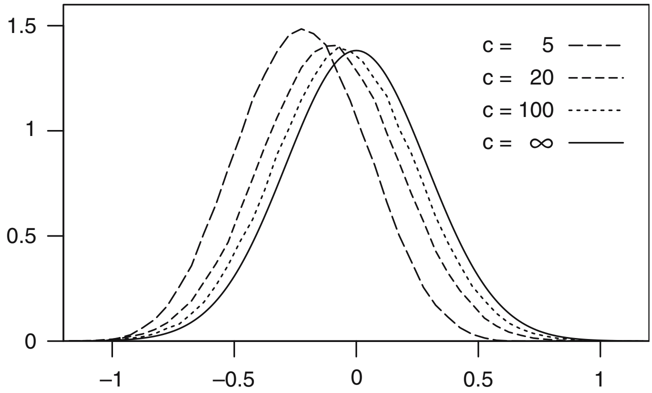

Fig. 1 Simulated probability density functions of $(S_c^J - \frac{c}{2})/\sqrt{c}$ (dashed curves) based on 200.000 realizations of the random vectors $U_{(c-1)}$ and $W_{(c-1)}$. The plotted empirical densities illustrate the rate of convergence to the limit probability density function $\sqrt{\frac{6}{\pi}}\,\exp\{-6\,x^2\}$, $x \in \mathbb{R}$ (solid curve)

Theorem 5 *Let the assumption of Theorem 1 be satisfied. Then*

$$\frac{S_c^J - c/2}{\sqrt{c}} = \max_{i \le c}\left(\frac{U_i - \ sgn(D_c)\, m_i(D_c) - \frac{1}{2}}{\sqrt{c}\ W_i}\right) \xrightarrow[c\to\infty]{\mathrm{d}} N\left(0, \frac{1}{12}\right)$$

and

$$\frac{S_c^A + c/2}{\sqrt{c}} = \min_{i \le c}\left(\frac{U_i - \ sgn(D_c)\, m_i(D_c) + \frac{1}{2}}{\sqrt{c}\ W_i}\right) \xrightarrow[c\to\infty]{\mathrm{d}} N\left(0, \frac{1}{12}\right).$$

Figure 1 gives an impression of a local version of Theorem 5 which does not directly follow from the pointwise convergence of distribution functions.

4 Proofs of Theorems 3 and 4

The ω-wise definition (9) of the random variable τ_c implies that, for any $x \ge 0$, the identity

$$\left\{\frac{\tau_c}{\sqrt{c}} > x\right\} \cap \Omega^* = \left\{\sum_{i \le c}\left\lfloor x\sqrt{c}\,W_i + \ \mathrm{sgn}(D_c)\,U_i + \frac{1}{2}\right\rfloor < |\,D_c\,|\right\} \cap \Omega^*$$

holds. Together with $(-U_1, \ldots, -U_{c-1}) \overset{\mathrm{d}}{=} (U_1, \ldots, U_{c-1})$ and

$$\left\lfloor \sum_{i<c} -U_i + \frac{1}{2}\right\rfloor = -\left\lfloor \sum_{i<c} U_i + \frac{1}{2}\right\rfloor \quad \mathsf{P}\text{-a.s.},$$

we get

$$\mathsf{P}\left(\frac{\tau_c}{\sqrt{c}} > x\right) = \mathsf{P}\left(\sum_{i\le c}\lfloor x\sqrt{c}\,W_i + V_i\rfloor < \left|\left\lfloor \sum_{i<c} U_i + \frac{1}{2}\right\rfloor\right|\right),$$

where $V_i = U_i + \frac{1}{2}$ for $i = 1, \ldots, c$. By definition, each V_i is uniformly distributed on (0, 1), where V_c depends on the independent random variables $V_1, \ldots, V_{c-1}$. Further, using the well-known distributional identity

$$(W_1, \ldots, W_c) \overset{\mathrm{d}}{=} \left(\frac{E_1}{S_c}, \ldots, \frac{E_c}{S_c}\right) \quad \text{with } S_c = \sum_{i\le c} E_i, \tag{16}$$

where $E_1, \ldots, E_c$ are independent, exponentially distributed random variables with unit mean, see Aitchison (1986), we may write

$$\mathsf{P}\left(\frac{\tau_c}{\sqrt{c}} > x\right) = \mathsf{P}\left(\sum_{i\le c}\left\lfloor \frac{x\sqrt{c}\,E_i}{S_c} + V_i\right\rfloor < \left|\left\lfloor \sum_{i<c} U_i + \frac{1}{2}\right\rfloor\right|\right). \tag{17}$$

Now, define $\overline{S}_c := S_c/c$ and let $\varepsilon \in (0, 1)$ be fixed. By the decomposition

$$\Omega = \left\{\frac{1}{1+\varepsilon} \le \frac{1}{\overline{S}_c} \le \frac{1}{1-\varepsilon}\right\} \cup \left\{\,|\,\overline{S}_c - 1\,| > \varepsilon\right\}, \tag{18}$$

and the elementary estimate $|\lfloor y + 1/2\rfloor| \le |y| + 1/2$ for all $y \in \mathbb{R}$, we conclude from (17) that

$$\mathsf{P}\left(\frac{\tau_c}{\sqrt{c}} > x\right) \le \mathsf{P}(|\overline{S}_c - 1| > \varepsilon) + \mathsf{P}\left(\sum_{i<c}\left\lfloor \frac{x\,E_i}{(1+\varepsilon)\sqrt{c}} + V_i\right\rfloor < \left|\sum_{i<c} U_i\right| + \frac{1}{2}\right).$$

We need the following auxiliary result.

Lemma 1 *Let V be uniformly distributed on* (0, 1) *and independent of the exponentially distributed random variable E with* $\mathsf{E}(E) = 1$. *Then*

$$\mathsf{E}\left(\lfloor \gamma\,E + V\rfloor\right) = \gamma \quad \textit{and}\; \mathsf{E}\left(\lfloor \gamma\,E + V\rfloor^2\right) = |\gamma| \coth\left(\frac{1}{2|\gamma|}\right) \tag{19}$$

for any $\gamma \in \mathbb{R}$, where $\coth(x) = (1 + e^{-2x})/(1 - e^{-2x})$.

The proof of Lemma 1 is shifted to the end of this section.

Making use of (19) with γ replaced by the null sequence $\gamma_c = x/(1+\varepsilon)\sqrt{c}$ (as $c \to \infty$), we find that

$$\frac{1}{\sqrt{c-1}} \sum_{i<c} \mathsf{E}\left(\lfloor \gamma_c\,E_i + V_i\rfloor\right) = \sqrt{c-1}\,\gamma_c = \frac{x}{1+\varepsilon} - \frac{\gamma_c}{\sqrt{c} + \sqrt{c-1}}$$

and

$$\mathsf{Var}\left(\frac{1}{\sqrt{c-1}} \sum_{i<c} \lfloor \gamma_c E_i + V_i \rfloor \right) = \gamma_c \left(\coth\left(\frac{1}{2\gamma_c} \right) - \gamma_c \right) \xrightarrow[c\to\infty]{} 0.$$

Hence, by means of Chebyshev's inequality, it follows that

$$R_c(x,\varepsilon) := \frac{1}{\sqrt{c-1}} \sum_{i<c} \lfloor \gamma_c E_i + V_i \rfloor - \frac{x}{1+\varepsilon} - \frac{1}{2\sqrt{c-1}} \xrightarrow[c\to\infty]{\mathsf{P}} 0$$

for any $x \geq 0$ und $0 < \varepsilon < 1$. Applying Lévy's central limit theorem to the i.i.d. sequence $U_1, \ldots, U_{c-1}$ combined with Slutzky's theorem yields

$$\frac{1}{\sqrt{c-1}} \left| \sum_{i<c} U_i \right| - R_c(x,\varepsilon) \xrightarrow[c\to\infty]{\mathrm{d}} \left| N\left(0, \frac{1}{12}\right) \right|,$$

which is the same as

$$\mathsf{P}\left(\frac{1}{\sqrt{c-1}} \left| \sum_{i<c} U_i \right| > \frac{1}{\sqrt{c-1}} \sum_{i<c} \lfloor \gamma_c E_i + V_i \rfloor - \frac{1}{2\sqrt{c-1}} \right)$$
$$\xrightarrow[c\to\infty]{} \mathsf{P}\left(\left| N\left(0, \frac{1}{12}\right) \right| \geq \frac{x}{1+\varepsilon} \right) = 2\left(1 - \Phi\left(\frac{2\sqrt{3}\,x}{1+\varepsilon} \right) \right).$$

Since, by the weak law of large numbers, $\mathsf{P}(\,|\,\overline{S}_c - 1\,| > \varepsilon\,) \xrightarrow[c\to\infty]{} 0$, we obtain

$$\limsup_{c\to\infty} \mathsf{P}\left(\frac{\tau_c}{\sqrt{c}} > x \right) \leq 2\left(1 - \Phi\left(\frac{2\sqrt{3}\,x}{1+\varepsilon} \right) \right), \tag{20}$$

for any $x \geq 0$ and $0 < \varepsilon < 1$.

In the next step we use (17) to estimate $\mathsf{P}(\,\tau_c > x\,\sqrt{c}\,)$ from below. For doing this, let U_c^* be an additional random variable chosen independently of $U_1, \ldots, U_{c-1}$ with $U_c^* \stackrel{\mathrm{d}}{=} U_c$. Clearly, the estimates

$$\left\lfloor \frac{x\,E_c}{\sqrt{c}\,S_c} + U_c + \frac{1}{2} \right\rfloor < \left\lfloor \frac{x\,E_c}{\sqrt{c}\,S_c} + U_c^* + \frac{1}{2} \right\rfloor + 1,$$

and

$$\left| \left\lfloor \sum_{i<c} U_i + \frac{1}{2} \right\rfloor \right| \geq \left| \sum_{i<c} U_i + U_c^* \right| - 1$$

hold ω-wise. To simplify notation we write again U_c instead of U_c^* so that $U_1, \ldots,$ U_c i.i.d. uniformly on $(-1/2\,,\ 1/2)$. Hence, the previous estimates and (17) imply

$$\mathsf{P}\left(\frac{\tau_c}{\sqrt{c}} > x \right) \geq \mathsf{P}\left(\sum_{i<c} \left\lfloor \frac{x\,E_i}{\sqrt{c}\,\overline{S}_c} + V_i \right\rfloor + 2 < \left| \sum_{i<c} U_i \right| \right).$$

Using this and the inclusion $\Omega \supseteq \{\overline{S}_c \geq 1-\varepsilon\} \supseteq \Omega \setminus \{|\overline{S}_c - 1| > \varepsilon\}$, we can deduce the inequality

$$\mathsf{P}\left(\frac{\tau_c}{\sqrt{c}} > x\right) \geq \mathsf{P}\left(\frac{1}{\sqrt{c}} \sum_{i\leq c} \left\lfloor \frac{x\,E_i}{\sqrt{c}\,(1-\varepsilon)} + V_i \right\rfloor + \frac{2}{\sqrt{c}} < \frac{1}{\sqrt{c}} \left|\sum_{i\leq c} U_i\right|\right) \\ -\mathsf{P}(\,|\overline{S}_c - 1| > \varepsilon\,).$$

By repeating almost verbatim the arguments used to prove (20) we can derive from the latter inequality that

$$\liminf_{c\to\infty} \mathsf{P}\left(\frac{\tau_c}{\sqrt{c}} > x\right) \geq 2\left(1 - \Phi\left(\frac{2\sqrt{3}\,x}{1-\varepsilon}\right)\right)$$

for any $x \geq 0$ and $0 < \varepsilon < 1$. This combined with (20) proves the desired limiting relation (13).

Having in mind the symmetry of the cases $D_c > 0$ and $D_c < 0$ together with $\{\tau_c = 0\} \cap \Omega^* = \{D_c = 0\} \cap \Omega^*$, it is clear that $\mathsf{P}(\tau_c > (\geq)\, x\sqrt{c}) = 2\,\mathsf{P}(\tau_c > (\geq)\, x\sqrt{c},\, D_c > 0)$ for $x \geq (>)\, 0$ and $-\mathrm{sgn}(D_c)\,\tau_c \stackrel{\mathrm{d}}{=} \mathrm{sgn}(D_c)\,\tau_c$.

Hence, in view of (13),

$$\mathsf{P}\left(\frac{\mathrm{sgn}(D_c)\,\tau_c}{\sqrt{c}} \leq x\right) = \begin{cases} 1 - \mathsf{P}(\tau_c > x\sqrt{c},\, D_c > 0) & \text{for } x \geq 0 \\ \mathsf{P}(\tau_c \geq -x\sqrt{c},\, D_c > 0) & \text{for } x < 0 \end{cases} \\ \underset{c\to\infty}{\longrightarrow} \Phi(\,2\sqrt{3}\,x\,) \quad \text{for all } x \in \mathbb{R}, \tag{21}$$

which proves (12). Thus, the proof of Theorem 3 is completed. □

The proof of Theorem 4 relies on the identity

$$\left\{\frac{\tau_c}{\sqrt{c}} > x\right\} \cap \Omega^* = \left\{\sum_{i\leq c} \frac{m_i(D_c)}{W_i} > \sum_{i\leq c} \frac{\lfloor x\sqrt{c}\,W_i + \mathrm{sgn}(D_c)\,U_i + \frac{1}{2}\rfloor}{W_i}\right\} \cap \Omega^*$$

for all $x \geq 0$, whence, due to the symmetry $-U_{(c-1)} \stackrel{\mathrm{d}}{=} U_{(c-1)}$ and (16), we obtain

$$\mathsf{P}\left(\frac{\tau_c}{\sqrt{c}} > x\right) = \mathsf{P}\left(\frac{1}{c^{3/2}} \sum_{i\leq c} \frac{m_i(D_c)}{W_i} > \frac{\overline{S}_c}{\sqrt{c}} \sum_{i\leq c} \frac{\lfloor x\,E_i/\sqrt{c}\,\overline{S}_c + V_i\rfloor}{E_i}\right).$$

Hence, relation (13) and Slutzky's theorem imply

$$\mathsf{P}\left(\frac{1}{c^{3/2}} \sum_{i\leq c} \frac{m_i(D_c)}{W_i} > x\right) \underset{c\to\infty}{\longrightarrow} 2\left(1 - \Phi(2\sqrt{3}\,x)\right), \tag{22}$$

for $x \geq 0$, if we are able to prove

$$\frac{\overline{S}_c}{\sqrt{c}} \sum_{i\leq c} \frac{\lfloor x\,E_i/\overline{S}_c\sqrt{c} + V_i\rfloor}{E_i} \xrightarrow[c\to\infty]{\mathsf{P}} x$$

or, equivalently,

$$\frac{1}{\sqrt{c}} \sum_{i \le c} \left(\frac{\lfloor x\, E_i/\sqrt{c} + V_i \rfloor}{E_i} - \frac{x}{\sqrt{c}} \right) \xrightarrow[c\to\infty]{\mathsf{P}} 0. \tag{23}$$

The equivalence of the latter two limiting relations seems to be evident in view of $\overline{S}_c \xrightarrow[c\to\infty]{\mathsf{P}} 1$. A rigorous proof is based on lower and upper bounds of the terms $\lfloor x\, E_i/\overline{S}_c\sqrt{c} + V_i \rfloor$ according to the decomposition (18) for arbitrarily small $\varepsilon > 0$. The details are left to the reader.

To verify (23) we consider only the sum of the first $c - 1$ i.i.d. summands, calculate its expectation, and apply Chebychev's inequality. This requires to check the relations

$$\sqrt{c}\,\mathsf{E}\left(\frac{\lfloor x\, E_1/\sqrt{c} + V_1 \rfloor}{E_1} \right) \xrightarrow[c\to\infty]{} x \quad \text{and} \quad \mathsf{Var}\left(\frac{\lfloor x\, E_1/\sqrt{c} + V_1 \rfloor}{E_1} \right) \xrightarrow[c\to\infty]{} 0,$$

which are immediately seen from the following

Lemma 2 *For random variables V and E as defined in Lemma 1 and* $0 \le \gamma \le 1/2$, *we have*

$$\left| \mathsf{E}\left(\frac{\lfloor \gamma\, E + V \rfloor}{E} \right) - \gamma \right| \le \gamma\, e^{-1/\gamma} \quad \textit{and} \quad \left| \mathsf{E}\left(\frac{\lfloor \gamma\, E + V \rfloor}{E} \right)^2 - \gamma \right| \le 2\,\gamma\, e^{-1/\gamma}.$$

The proof of Lemma 2 relies on similar calculations as carried out in the proof of Lemma 1. Some additional estimates and the technical details are left to the reader.

To accomplish the proof of (14) we make use of (22) and repeat the symmetry arguments showing the equivalence of relations (13) and (21). This completes the proof of Theorem 4. □

Proof of Lemma 1 The relations (19) are obviously valid for $\gamma = 0$. Since $\lfloor -x \rfloor = -\lfloor x + 1 \rfloor$ for any non-integer $x \in \mathbb{R}$ and $V \overset{d}{=} 1 - V$, it is clear that

$$\mathsf{E}\left(\lfloor -\gamma\, E + V \rfloor\right) = -\mathsf{E}\left(\lfloor \gamma\, E - V + 1 \rfloor\right) = -\mathsf{E}\left(\lfloor \gamma\, E + V \rfloor\right),$$

as well as

$$\mathsf{E}\left(\lfloor -\gamma\, E + V \rfloor^2\right) = \mathsf{E}\left(\lfloor \gamma\, E + V \rfloor^2\right).$$

Therefore, it suffices to show (19) for $\gamma > 0$. By definition of the integer part $\lfloor \cdot \rfloor$ we obtain

$$\begin{aligned}
\mathsf{E}\left(\lfloor \gamma\, E + V \rfloor\right) &= \sum_{n\ge 1} n\, \mathsf{P}(\,(n - V)/\gamma \le E < (n + 1 - V)/\gamma\,) \\
&= \sum_{n\ge 1} n \int_0^1 \left(e^{(v-n)/\gamma} - e^{(v-n-1)/\gamma} \right) \mathrm{d}v \\
&= \int_0^1 e^{v/\gamma}\, dv\, \frac{e^{-1/\gamma}}{1 - e^{-1/\gamma}} \; = \; \gamma,
\end{aligned}$$

where we have used that $\sum_{n\ge 1} n(z^n - z^{n+1}) = z/(1 - z)$ for $|z| < 1$.

Likewise, applying the formula $\sum_{n\geq 1} n^2(z^n - z^{n+1}) = z(1+z)/(1-z)^2$ for $z = e^{-1/\gamma}$, we get

$$\mathsf{E}\left(\lfloor \gamma\, E + V \rfloor^2\right) = \sum_{n\geq 1} n^2 \int_0^1 \left(e^{(v-n)/\gamma} - e^{(v-n-1)/\gamma}\right) \mathrm{d}v$$

$$= \int_0^1 e^{v/\gamma}\, dv\, \frac{e^{-1/\gamma}\,(1+e^{-1/\gamma})}{(1-e^{-1/\gamma})^2} \;=\; \gamma\, \frac{1+e^{-1/\gamma}}{1-e^{-1/\gamma}},$$

which is just the second relation of (19) for $\gamma > 0$. □

5 Proof of Theorem 5

We introduce the abbreviation $\{x\} := x - \lfloor x \rfloor$ for the *fractional part* of $x \in \mathbb{R}$. By definition (8), assuming $D_c \neq 0$, we first rewrite the maximum-term $(S_c^J - c/2)/\sqrt{c}$ as

$$\frac{S_c^J - c/2}{\sqrt{c}} = -\min_{i\leq c}\left(\frac{\frac{1}{2} - U_i + \operatorname{sgn}(D_c)\, m_i(D_c)}{\sqrt{c}\; W_i}\right) = -\operatorname{sgn}(D_c)\,\frac{\tau_c}{\sqrt{c}}$$
$$-\min_{i\leq c}\left(\frac{\frac{1}{2}(1+\operatorname{sgn}(D_c)) - \operatorname{sgn}(D_c)\{\tau_c\, W_i + \operatorname{sgn}(D_c)\, U_i + \frac{1}{2}\}}{\sqrt{c}\; W_i}\right)$$

and the minimum-term $(S_c^A + c/2)/\sqrt{c}$ as

$$\frac{S_c^A + c/2}{\sqrt{c}} = \min_{i\leq c}\left(\frac{\frac{1}{2} + U_i - \operatorname{sgn}(D_c)\, m_i(D_c)}{\sqrt{c}\; W_i}\right) = -\operatorname{sgn}(D_c)\,\frac{\tau_c}{\sqrt{c}}$$
$$+\min_{i\leq c}\left(\frac{\frac{1}{2}(1-\operatorname{sgn}(D_c)) + \operatorname{sgn}(D_c)\{\tau_c\, W_i + \operatorname{sgn}(D_c)\, U_i + \frac{1}{2}\}}{\sqrt{c}\; W_i}\right).$$

The remaining case $D_c = \operatorname{sgn}(D_c) = 0$ can be neglected by noting that

$$\mathsf{P}(D_c = 0) = \mathsf{P}\left(-\frac{1}{2} < \sum_{i<c} U_i \leq \frac{1}{2}\right) \xrightarrow[c\to\infty]{} 0.$$

The above identities reveal that both assertions of Theorem 5 follow immediately from (12) combined with Slutzky's theorem whenever each of the sequences

$$\min_{i\leq c}\left(\frac{1 - \{\tau_c\, W_i \pm U_i + \frac{1}{2}\}}{\sqrt{c}\; W_i}\right) \text{ and } \min_{i\leq c}\left(\frac{\{\tau_c\, W_i \pm U_i + \frac{1}{2}\}}{\sqrt{c}\; W_i}\right)$$

converges to zero in probability as $c \to \infty$. Note that both minima are taken over dependent random variables having infinite expectation. This requires particularly careful estimates which are demonstrated in case of the left-hand sequence. The other sequence can be treated analogously. As in section 4, we use the symmetry $-U_{(c-1)} \overset{\mathrm{d}}{=} U_{(c-1)}$ and (16). Therefore, in view of $\overline{S}_c \xrightarrow[c\to\infty]{\mathsf{P}} 1$, our problem reduces to show

$$\sqrt{c}\,\min_{i<c}\left(\frac{1-\{\frac{\tau_c}{S_c}\,E_i+V_i\}}{E_i}\right)\xrightarrow[c\to\infty]{\mathsf{P}} 0, \tag{24}$$

where $V_i = U_i + \frac{1}{2}$, $i = 1, \ldots, c-1$, are independent and uniformly distributed on (0, 1).

Let $\varepsilon \in (0, 1)$ be fixed. Define the events $\Gamma_{c,1} = \left\{\max_{i<c} E_i \le c^{1/4}\right\}$ and

$$\Gamma_{c,2} = \left\{\min_{i<c}\left(\frac{1-V_i}{E_i}\right) \le \frac{\varepsilon}{\sqrt{c}}\right\},\quad \Gamma_{c,3} = \left\{0 < c^{1/4}\,\frac{\tau_c}{S_c} \le \varepsilon\right\},$$
$$\Delta_i(\varepsilon) = \left\{1 - V_i - \frac{\tau_c}{S_c}\,E_i + \left\lfloor \frac{\tau_c}{S_c}\,E_i + V_i\right\rfloor \ge \frac{\varepsilon\,E_i}{\sqrt{c}}\right\}\quad \text{for } i = 1, \ldots, c-1.$$

Theorem 2 implies $\tau_c/c^{3/4} \xrightarrow[c\to\infty]{\mathsf{P}} 0$ so that $\mathsf{P}\left(\Gamma^{\mathbf{c}}_{c,3}\right) \xrightarrow[c\to\infty]{} 0$.

Furthermore, we get

$$\mathsf{P}\left(\Gamma^{\mathbf{c}}_{c,1}\right) \le (c-1)\,\mathsf{P}\left(E_1 > c^{1/4}\right) = (c-1)\,e^{-c^{1/4}} \xrightarrow[c\to\infty]{} 0$$

and

$$\begin{aligned}\mathsf{P}\left(\Gamma^{\mathbf{c}}_{c,2}\right) &= \left(\mathsf{P}\left(1 - V_1 > \frac{\varepsilon}{\sqrt{c}}\,E_1\right)\right)^{c-1}\\ &= \left(1 - \frac{\varepsilon}{\sqrt{c}}\left(1 - e^{-\sqrt{c}/\varepsilon}\right)\right)^{c-1} \xrightarrow[c\to\infty]{} 0.\end{aligned}$$

Now, we are in a position to bound the probability of the event

$$\left\{\min_{i<c}\left(\frac{1 - V_i - \frac{\tau_c}{S_c}\,E_i + \lfloor \frac{\tau_c}{S_c}\,E_i + V_i\rfloor}{E_i}\right) \ge \frac{\varepsilon}{\sqrt{c}}\right\} = \bigcap_{i<c} \Delta_i(\varepsilon),$$

by sequences tending to zero for any $\varepsilon > 0$.

Observe that, for $\omega \in \Gamma_{c,1} \cap \Gamma_{c,3}$, the term $\frac{\tau_c(\omega)}{S_c(\omega)}\,E_i(\omega) + V_i(\omega)$ lies in the interval (0, 2) so that the integer $\lfloor \frac{\tau_c(\omega)}{S_c(\omega)}\,E_i(\omega) + V_i(\omega)\rfloor$ takes only the values 0 or 1 according to $\frac{\tau_c(\omega)}{S_c(\omega)}\,E_i(\omega) + V_i(\omega) < 1$ or $\frac{\tau_c(\omega)}{S_c(\omega)}\,E_i(\omega) + V_i(\omega) \ge 1$, respectively.

By definition of the above events, the following inclusions are valid for each $i = 1, \ldots, c-1$:

$$\begin{aligned}
&\Delta_i(\varepsilon) \cap \bigcap_{k=1}^{3} \Gamma_{c,k} \\
&\quad \subseteq \Gamma_{c,2} \cap \Gamma_{c,3} \cap \left(\left\{ 1 - V_i - \frac{\tau_c}{S_c} E_i \geq \frac{\varepsilon\, E_i}{\sqrt{c}} \right\} \cup \left\{ \frac{\tau_c}{S_c} E_i + V_i \geq 1 \right\} \right) \\
&\quad = \Gamma_{c,2} \cap \Gamma_{c,3} \cap \left(\left\{ \frac{1 - V_i}{E_i} \geq \frac{\tau_c}{S_c} + \frac{\varepsilon}{\sqrt{c}} \right\} \cup \left\{ \frac{1 - V_i}{E_i} \leq \frac{\tau_c}{S_c} \right\} \right) \\
&\quad \subseteq \Gamma_{c,3} \cap \left\{ \frac{1 - V_i}{E_i} \leq \frac{\tau_c}{S_c} \right\} \subseteq \left\{ 1 - V_i \leq \frac{\varepsilon}{c^{1/4}} E_i \right\}.
\end{aligned}$$

Hence,

$$\bigcap_{i<c} \Delta_i(\varepsilon) \cap \bigcap_{k=1}^{3} \Gamma_{c,k} \subseteq \left\{ \min_{i<c} \left(\frac{1 - V_i}{E_i} \right) \leq \frac{\varepsilon}{c^{1/4}} \right\}.$$

Finally, by standard arguments combined with the above limits, we arrive at

$$\mathsf{P}\left(\bigcap_{i<c} \Delta_i(\varepsilon) \right) \leq \sum_{k=1}^{3} \mathsf{P}\left(\Gamma_{c,k}^{\mathbf{c}}\right) + \left(\frac{\varepsilon}{c^{1/4}} \left(1 - e^{-c^{1/4}/\varepsilon} \right) \right)^{c-1} \underset{c\to\infty}{\longrightarrow} 0,$$

which is equivalent to (24). Thus, the proof of Theorem 5 is complete. □

6 Concluding remarks

Although the proofs of the Theorems 3, 4, and 5 heavily depend on the uniform distribution of W, there are some reasons which give rise to the conjecture that all above results will remain true under milder distributional assumptions on W. This conjecture is mainly supported by the fact that the Gaussian limits in (12), (14), and in Theorem 5 coincide with the weak limit of $Z_c := (U_1 + \cdots + U_{c-1})/\sqrt{c}$ as $c \to \infty$, see (10) and (11). The uniform distribution of W is essentially used to approximate the left-hand sides of (12), (14), and of both relations in Theorem 5 by Z_c up to a stochastic null sequence. For making this remainder term arbitrarily small (as c becomes large), for example, it suffices to choose weights of the form $W_i = X_i/(X_1 + \cdots + X_c)$ for $i = 1, \ldots, c$, where $X_1, X_2, \ldots$ are independent (not necessarily identically distributed) positive random variables satisfying suitable moment assumptions.

In the light of such robustness of our results against the assumed uniform distribution of W, Theorem 5 can be used for testing two hypotheses, namely, that an apportionment vector $(N_1, \ldots, N_c)$ (provided n and c are large enough) is obtained either by minimizing the Jefferson criterion $S^J_{c,n}$, see (2), or by maximizing the Adams criterion $S^A_{c,n}$, see (3). Given a level $\alpha > 0$ of significance, the null hypothesis that the apportionment vector $(N_1, \ldots, N_c)$ minimizes the Jefferson criterion $S^J_{c,n}$, i.e. it is obtained by the multiplier method with rounding down $(q = 1)$, is rejected at level α when

$$\sqrt{\frac{3}{c}}\, |\, 2\, S^J_{c,n} - c\, | \geq \frac{z_\alpha}{2}, \quad \text{where } \Phi(z_{\alpha/2}) = 1 - \frac{\alpha}{2}.$$

Analogously, the inequality $\sqrt{3}\, |\, 2\, S^A_{c,n} + c\, |/\sqrt{c} \geq z_{\alpha/2}$ contradicts the hypothesis (at level α) that $(N_1, \ldots, N_c)$ maximizes the Adams criterion $S^A_{c,n}$.

Furthermore, for small sizes of categories c (with large n), the two limiting relations of Theorem 2 suggest corresponding tests based on simulated values of the weak limits. The hypothesis that an apportionment vector results from a q-stationary multiplier method [for some $q \in (0, 1)$] can be checked using the non-normal limits of the Sainte–Laguë divergence obtained in Heinrich et al. (2004), L. Heinrich et al. (submitted).

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