Metrika (2006) 63: 145–168
DOI 10.1007/s00184-005-0008-9

ORIGINAL ARTICLE

Marie Hušková · Simos G. Meintanis

Change point analysis based on empirical characteristic functions

Empirical characteristic functions

Received: 30 April 2004 / Accepted: 3 January 2005 / Published online: 25 January 2006

Abstract Test procedures for detection of a change in the distribution of a sequence of independent observations based on empirical characteristic functions are developed and their limit properties are studied. Theoretical results are accompanied by a simulation study.

Mathematics Subject Classification (2000) 62G20 · 62E20 · 60F17

Keywords Empirical characteristic function · Change point analysis

1 Introduction and main results

Let $Y_1, \ldots, Y_n$ be independent random variables, Y_j having a distribution function F_j, $j = 1, \ldots, n$. We consider the testing problem

$$H_0 : F_1 = \cdots = F_n \tag{1}$$

against

$$H_1 : F_1 = \cdots = F_m \neq F_{m+1} = \cdots = F_n \qquad for \quad m < n, \tag{2}$$

The work of the first author was partially supported by grants GAČR 201/03/0945 and MSM 113200008.

M. Hušková (✉)
Charles University of Prague, Department of Statistics,
Sokolovská 83, CZ–186 75 Praha 8, Czech Republic

M. Hušková
ÚTIA Czech Academy of Sciences, Pod vodárenskou věží 4,
18000 Praha 8, Czech Republic
E-mail: marie.huskova@karlin.mff.cuni.cz

S.G. Meintanis
National and Kapodistrian University of Athens, Department of Economics,
8 Pesmazoglou Street, 105 59 Athens, Greece
E-mail: simosmei@econ.uoa.gr

where m, F_1 and F_n are unknown. Most of the test procedures for this problem assume that the change concerns only a change in the parameters like mean, regression parameters, variance etc., for a comprehensive treatment see, e.g., the book by Csörgő and Horváth (1997) and Antoch et al (2001). The nonparametric type procedures considered in the literature are based either on empirical distribution functions, quantile functions or U-statistics, for a survey of the recent results see again the book by Csörgő and Horváth (1997).

In the present paper test procedures based on empirical characteristic functions for testing H_0 against H_1 are considered. Particularly, we study the following class of test statistics:

$$T_{n,\gamma}(w) = \max_{1\le k<n} \Big(\frac{k(n-k)}{n^2}\Big)^{\gamma} \frac{k(n-k)}{n} \int_{-\infty}^{\infty} |\phi_k(t) - \phi_k^0(t)|^2 w(t)\mathrm{d}t, \tag{3}$$

where $w(\cdot)$ is a nonnegative weight function, $\phi_k(t)$ and $\phi_k^0(t)$ are empirical characteristic functions based on $Y_1, \dots, Y_k$ and $Y_{k+1}, \dots, Y_n$, respectively, i.e.

$$\phi_k(t) = \frac{1}{k}\sum_{j=1}^{k} \exp\{itY_j\}, \quad k = 1, \dots, n, \tag{4}$$

$$\phi_k^0(t) = \frac{1}{n-k}\sum_{j=k+1}^{n} \exp\{itY_j\}, \quad k = 1, \dots, n, \tag{5}$$

and $\gamma \in [0, 1]$. Empirical characteristic functions have proved a useful tool in a variety of estimation and testing problems. Some earlier works include, among others, Press (1972), Heathcote (1972), Koutrouvelis (1980a,b), Epps and Pulley (1983) and Csörgő (1985a,b). For recent applications the reader is referred to Epps (1999), Koutrouvelis and Meintanis (1999), Gürtler and Henze (2000) and Kankainen and Ushakov (1998). A large part of the literature on the empirical characteristic function is covered in Ushakov (1999).

Letting

$$c_{k,n}(\gamma) = \Big(\frac{k(n-k)}{n^2}\Big)^{\gamma} \frac{k(n-k)}{n}, \quad k = 1, \dots, n-1, \tag{6}$$

straightforward algebra yields that

$$T_{n,\gamma}(w) = \max_{1\le k<n} c_{k,n}(\gamma)\, V_{k,n}, \tag{7}$$

with

$$V_{k,n}(w) = \frac{1}{k^2}\sum_{l,m=1}^{k} h_w(Y_l - Y_m) + \frac{1}{(n-k)^2}\sum_{l,m=k+1}^{n} h_w(Y_l - Y_m) - \frac{2}{k(n-k)}\sum_{l=1}^{k}\sum_{m=k+1}^{n} h_w(Y_l - Y_m), \tag{8}$$

where

$$h_w(\beta) = \int_{-\infty}^{\infty} \cos(\beta t) w(t) \mathrm{d}t. \tag{9}$$

The choice of the weight function w and of the tuning parameter γ influence the limit behavior of the considered test statistic. Most of the results presented below hold true for any nonnegative weight function w with the property

$$0 < \int w(t)\mathrm{d}t < \infty.$$

However, typical choices used in practice are either

$$w_a^{(1)}(t) = \exp(-a|t|), \quad t \in R^1,\ a > 0, \tag{10}$$

or

$$w_a^{(2)}(t) = \exp(-at^2), \quad t \in R^1,\ a > 0. \tag{11}$$

Then the corresponding functions h in (8) have the form

$$h_w^{(1)}(\beta) = 2a/(a^2 + \beta^2), \quad \beta \in R^1. \tag{12}$$

and

$$h_w^{(2)}(\beta) = \sqrt{\pi/a}\ \exp(-\beta^2/4a), \quad \beta \in R^1. \tag{13}$$

The respective test statistics are denoted by $T_{n,\gamma}^{(1)}(a)$ and $T_{n,\gamma}^{(2)}(a)$. The role of the weight parameter $a > 0$ is to control the rate of decay of the weight function. With small (resp. large) values of a, the weight function decreases slowly (resp. rapidly). In this connection, it may be shown that the family of test statistics $\{T_{n,\gamma}^{(1)}(a), 0 < a < \infty\}$ and $\{T_{n,\gamma}^{(2)}(a), 0 < a < \infty\}$ are closed at the boundary $a = \infty$. Particularly, by the arguments in Meintanis (2004) we have,

$$\lim_{a\to\infty} a^3\ T_{n,\gamma}^{(1)}(a) = 4 \max_{1\le k<n} c_{k,n}(\gamma)\ (\bar{Y}_k - \bar{Y}_k^o)^2,$$

$$\lim_{a\to\infty} a^{3/2}\ T_{n,\gamma}^{(2)}(a) = \frac{\sqrt{\pi}}{2} \max_{1\le k<n} c_{k,n}(\gamma)\ (\bar{Y}_k - \bar{Y}_k^o)^2,$$

where $\bar{Y}_k = k^{-1}\sum_{l=1}^{k} Y_l$ and $\bar{Y}_k^o = (n-k)^{-1}\sum_{l=k+1}^{n} Y_l$. Hence apart from irrelevant constant factors, the test statistics when suitably normalized approach limit values. These values correspond to procedures appropriate for detecting changes in location similar to the tests considered by Antoch and Hušková (2001), among others.

Concerning the choice of the tuning parameter γ, it would be natural in accordance with other test procedures for detection of changes to choose $\gamma = 0$. Then our test statistic is the maximum of the standardized test statistics for the two sample problem, where the observations are split into two groups with k and $n - k$

observations. However, the disadvantage of this test statistic is that it tends to infinity even under the null hypothesis. More precisely, under H_0 and mild assumptions on w, as $n \to \infty$,

$$T_{n,\gamma}(w) \to \infty, \quad \gamma = 0$$

in probability, while

$$T_{n,\gamma}(w) = O_P(1), \quad \gamma > 0.$$

We focus here on $\gamma \in (0, 1]$ and make some comments for the case $\gamma = 0$.

Large values of the test statistics indicate that the null hypothesis is not true. Hence the null hypothesis is rejected when the critical value is exceeded, where the critical value is determined in such a way that the test has level α. Generally, to find reasonable approximations for critical values one can either use the limit distribution under the null hypothesis or use some resampling methods. Concerning the limit distribution, unfortunately, it depends on the unknown parameters (see Theorem A below) and hence this approach does not provide proper approximations for critical values. Concerning resampling methods they provide good approximation when the data follow the null hypothesis or local alternatives (see Theorem C below). Particularly, bootstrap without replacement leads to a test with level α. Bootstrap without replacement leads to tests with asymptotic level α. In case of the so called fixed alternatives, resampling methods do not lead to a reasonable approximation to the critical values, however the resulting tests are consistent. Such results are discussed in more detail in section 4.

The application of the bootstrap without replacement can be interpreted as an application of the permutation principle. We will use the later term in the most of the rest of the paper. The permutation version $T_{n,\gamma}(w, \boldsymbol{R})$ of $T_{n,\gamma}(w)$ is defined by (3) or equivalently by (7) with $Y_1, \dots, Y_n$ being replaced by $Y_{R_1}, \dots, Y_{R_n}$, where $R_1, \dots, R_n$ is a random permutation of $1, \dots, n$. The critical value $d_{n,\gamma}(\alpha, \boldsymbol{Y})$ is obtained as the $100(1-\alpha)\%$ quantile of the conditional distribution of $T_{n,\gamma}(w, \boldsymbol{R})$ given $Y_1, \dots, Y_n$. The resulting tests then reject H_0 on the level α if

$$T_{n,\gamma}(w, \boldsymbol{R}) \geq d_{n,\gamma}(\alpha, \boldsymbol{Y}). \tag{14}$$

Since the conditional distribution function of $T_{n,\gamma}(w, \boldsymbol{R})$ is discrete the exact level α need not be attained. However, it can be reached using classical randomization, (see e.g. Lehmann 1991).

It is known (see, e.g. Meintanis 2004) that the limit distribution of functionals of characteristic functions are neither exactly nor asymptotically distribution free under H_0. This is an unpleasant property. However we will see that despite this, the tests work reasonably well. We need not even assume that the underlying distribution function is continuous.

To assess the performance of the proposed tests we have conducted a simulation study which includes a variety of sampling situations, both under H_0 and under alternatives. The results show that the tests based on $T^{(1)}_{n,\gamma}(a)$ and $T^{(2)}_{n,\gamma}(a)$, defined below (8), keep the actual level of significance close to its nominal value, while at the same time they are very sensitive in detecting departures from the null hypothesis.

The rest of the paper is organized as follows. Section 2 studies distributional properties of the considered test statistics under H_0, while section 3 investigates properties of their distributional behavior under alternatives. Section 4 discusses properties of the permutation versions. The results of a simulation study are presented in section 5. The essence of the proofs is contained in section 6.

2 Results under H_0

Here theoretical properties of $T_{n,\gamma}(w)$ are studied under the null hypothesis, i.e. distributional properties of $T_{n,\gamma}(w)$ are investigated under the assumptions that $Y_1, \dots, Y_n$ are i.i.d. random variables with common distribution function F. The main results of the section are formulated in Theorem A at the end of the section.

In order to have a picture of the limit behavior of $T_{n,\gamma}(w)$ we decompose $V_{k,n}(w)$ into a few transparent terms. Towards this end,

$$h(x, y) = h_w(x - y) = \int \cos(t(x - y))\, w(t)\, \mathrm{d}t, \tag{15}$$

$$\tilde{h}(x, y) = h(x, y) - E(h(x, Y_s)) - E(h(Y_r, y)) - Eh(Y_r, Y_s), \quad r \neq s. \tag{16}$$

Hence

$$E(\tilde{h}(Y_r, Y_s)|Y_r) = E(\tilde{h}(Y_r, Y_s)|Y_s) = E\tilde{h}(Y_r, Y_s) = 0, \quad r \neq s. \tag{17}$$

Then $V_{k,n}(w)$ can be decomposed as follows:

$$V_{k,n} = V_{k,n}(w) = A_{k1} + A_{k2} + A_{k3}, \quad k = 1, \dots, n-1, \tag{18}$$

where

$$A_{k1} = \frac{n}{k(n-k)} \left(\frac{1}{k} \sum_{v=1}^{k} \sum_{s=1, v \neq s}^{k} \tilde{h}(Y_v, Y_s) + \frac{1}{(n-k)} \sum_{v=k+1}^{n} \sum_{s=k+1, v \neq s}^{n} \tilde{h}(Y_v, Y_s) - \frac{1}{n} \sum_{v=1}^{n} \sum_{s=1, v \neq s}^{n} \tilde{h}(Y_v, Y_s) \right), \tag{19}$$

$$A_{k2} = \frac{n}{k(n-k)} \Big(\int w(t)\, \mathrm{d}t - Eh(Y_1, Y_2) \Big), \tag{20}$$

$$A_{k3} = -\frac{2}{k^2} \sum_{r=1}^{k} (E(h(Y_r, Y_s)|Y_r) - Eh(Y_1, Y_2)) - \frac{2}{(n-k)^2} \sum_{r=k+1}^{n} (E(h(Y_r, Y_s)|Y_r) - Eh(Y_1, Y_2)). \tag{21}$$

Clearly, A_{k1} is a function of degenerate U-statistics, $A_{k2} = E\, V_{k,n}(w)$ is a nonrandom term and A_{k3} is the sum of independent random variables with zero mean. Direct calculations give that under H_0

$$\operatorname{var} A_{k1} \approx 2 \Big(\frac{n}{k(n-k)} \Big)^2 E\tilde{h}^2(Y_1, Y_2)$$

and

$$\operatorname{var} A_{k3} \approx 4\Big(\frac{1}{k^3} + \frac{1}{(n-k)^3}\Big) \operatorname{var}\{E(h(Y_1, Y_2)|Y_1)\},$$

where $a_n \approx b_n$ means that $a_n/b_n \to 1$. Hence under the null hypothesis the terms A_{k3} are negligible and do not influence the limit distribution, while A_{k2} and $\sqrt{\operatorname{var} A_{k1}}$ are of the same order and both influence the limit behavior.

Since the function $\tilde{h}(x, y)$ is symmetric in its arguments, by (17) and

$$E\,\tilde{h}^2(Y_1, Y_2) = \int \tilde{h}^2(x, y)\, \mathrm{d}F(x)\, \mathrm{d}F(y) < \infty, \tag{22}$$

there exist orthogonal eigenfunctions $\{g_j(t),\ j = 1, 2, \dots\}$ and eigenvalues $\{\lambda_j,\ j = 1, 2, \dots\}$ such that (see, e.g., Serfling 1980)

$$\lim_{K\to\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big(\tilde{h}(x, y) - \sum_{j=1}^{K} \lambda_j g_j(x) g_j(y)\Big)^2 dF(x) dF(y) = 0, \tag{23}$$

$$\int_{-\infty}^{\infty} g_j^2(x)\, \mathrm{d}F(x) = 1, \quad j = 1, 2, \dots, \tag{24}$$

$$\int_{-\infty}^{\infty} g_j(x) g_i(x)\, \mathrm{d}F(x) = 0, \quad i \neq j = 1, 2, \dots \tag{25}$$

and

$$E\,\tilde{h}^2(Y_1, Y_2) = \int \tilde{h}^2(x, y)\, \mathrm{d}F(x)\, \mathrm{d}F(y) = \sum_{j=1}^{\infty} \lambda_j^2. \tag{26}$$

Next, we formulate the assertions on the limit behavior of the test statistic $T_{n,\gamma}(w)$ under H_0.

Theorem A *Let $Y_1, \dots, Y_n$ be i.i.d. random variables with common distribution function F. Let $\gamma \in (0, 1]$ and let w be a symmetric nonnegative function such that*

$$0 < \int w(t)\ dt < \infty. \tag{27}$$

Then the limit behavior of $T_{n,\gamma}(w)$ is the same as that of

$$\sup_{t\in(0,1)} \Big(t(1-t)\Big)^{\gamma} \Big| \Big(\int w(u) du - E\, h(Y_1, Y_2)\Big) + \sum_{j=1}^{\infty} \lambda_j \Big\{\frac{B_j^2(t)}{(1-t)t} - 1\Big\}\Big|, \tag{28}$$

where $\{B_{j,n}(t),\ t \in (0, 1)\},\ j = 1, 2 \dots,$ are independent Brownian bridges, *Proof* It is postponed to section 6. □

Remark 2.1 The explicit distribution of (28) is unknown. By properties of Brownian bridges the random variable in (28) is bounded in probability.

Remark 2.2 Since the eigenvalues $\{\lambda_j\}$ and eigenfunctions $\{g_j\}$ depend on the underlying distribution function F which is unknown, the limit distribution of (28) depends on the unknown parameters and unknown functions so that the limit distribution does not provide a useful approximation for the critical values.

Remark 2.3 By results in chapter 4 of de la Peña and Giné (1999) on the law of the iterated logarithm for degenerate U-statistics,

$$T_{n,0}(w) = O_P(\log\,\log\,n).$$

There is an open question what is the limit distribution of $T_{n,0}(w)(\log\,\log\,n)^{-1}$.

3 Behavior of $T_{n,\gamma}(w)$ under alternatives

We investigate the limit behavior of $T_{n,\gamma}(w)$ under alternatives and as a consequence we obtain the consistency of the proposed tests.

Throughout the section we assume that there is $m(< n)$ such that $Y_1, \dots, Y_m$ are i.i.d. with d.f. F_1 and $Y_{m+1}, \dots, Y_n$ are i.i.d. r.v.'s with d.f. F_n, where all Y_i's are independent and m and F_1, F_n satisfy

$$m = m_n \in [\kappa_1 n, \kappa_2 n] \quad \text{for some} \quad 0 < \kappa_1 \le \kappa_2 < 1 \tag{29}$$

and, as $n \to \infty$,

$$n\Delta_{n,1} \to \infty, \quad \gamma \in (0, 1], \tag{30}$$

$$(n\Delta_{n,1})/(\log\log n) \to 0, \quad \gamma = 0$$

and

$$\frac{\Delta_{n,2}}{n\Delta_{n,1}^2} \to 0, \tag{31}$$

where

$$\begin{aligned}\Delta_{n,1} &= E(h(Y_1, Y_2) - 2h(Y_1, Y_n) + Eh(Y_n, Y_{n-1})) && (32)\\ &= \int \Big\{\Big(\int \cos(tx)\,\mathrm{d}(F_1(x) - F_n(x))\Big)^2 && (33)\\ &\quad + \Big(\int \sin(tx)\,\mathrm{d}(F_1(x) - F_n(x))\Big)^2\Big\} w(t)\,\mathrm{d}t, && (34)\\ \Delta_{n,2} &= E\big(E(h(Y_1, Y_2) - h(Y_1, Y_n))|Y_1)\big)^2 && (35)\\ &\quad + E\big(E(h(Y_1, Y_2) - h(Y_n, Y_{n-1}))|Y_n)\big)^2. && (36)\end{aligned}$$

Clearly, $\Delta_{n,j}$, $j = 1, 2$, describe the considered class of alternatives. It covers both local and fixed alternatives. Typically these assumptions are satisfied if

$$\sqrt{n} \sup_x |F_1(x) - F_n(x)| \to \infty.$$

The assumptions (30) and (31) are satisfied, e.g., if $F_1 = F$ and $F_n = G$ with F and G being fixed (i.e. not changing with n), $F \neq G$ and $\Delta_{n,1} \neq 0$.

In order to get a picture of the behavior of $T_{n,\gamma}(w)$ we proceed similarly as in the previous section. Particularly, we use the decomposition (18), where A_{k2} and A_{k3} have to be modified as follows:

$$A_{k2} = EV_{k,n}(w), \tag{37}$$

$$A_{k3} = B_{k1} + B_{k2} + B_{k3}, \tag{38}$$

where

$$B_{k1} = \frac{1}{k^2} \sum_{r=1}^{k} \sum_{s=1, s\neq r}^{k} \Big(E(h(Y_r, Y_s)|Y_r) + E(h(Y_r, Y_s)|Y_s) - 2Eh(Y_r, Y_s) \Big), \tag{39}$$

$$B_{k2} = \frac{2}{(n-k)^2} \sum_{r=k+1}^{n} \sum_{s=k+1, s\neq r}^{n} \big(E(h(Y_r, Y_s)|Y_r) + E(h(Y_r, Y_s)|Y_s) - 2Eh(Y_r, Y_s) \big), \tag{40}$$

$$B_{k3} = -\frac{2}{(n-k)k} \sum_{r=1}^{k} \sum_{s=k+1}^{n} (E(h(Y_r, Y_s)|Y_r) + E(h(Y_r, Y_s)|Y_s)). \tag{41}$$

Here E denotes the expectation under the setup considered in this section. Similarly as under H_0 the terms A_{k1} are functions of degenerate U-statistics, A_{k2} are non-random terms and A_{k3} are sums of independent random variables with zero mean. It will be shown in section 6 that in our present setup the terms A_{k2} dominate A_{k1} and A_{k3}. Therefore we will have a closer look at A_{k2}. Tedious but straightforward calculations give uniformly for $k \leq m$:

$$EV_{k,n}(w) = \frac{n}{k(n-k)} \Big(\int w(t)\, dt - Eh(Y_1, Y_2) \Big) + \Big(\frac{n-m}{n-k} \Big)^2 \Delta_{n,1} \Big(1 + O((n-m)^{-1}) \Big) \tag{42}$$

and, uniformly for $k > m$:

$$EV_{k,n}(w) = \frac{n}{k(n-k)} \Big(\int w(t)\, dt - Eh(Y_1, Y_2) \Big) + \Big(\frac{m}{k} \Big)^2 \Delta_{n,1} \Big(1 + O(m^{-1}) \Big). \tag{43}$$

Notice that the first terms in both (42) and (43) are identical and are equal to the expectation of $V_{n,k}(w)$ under H_0. The second terms are the main ones reflecting the alternatives. Moreover, they are monotone for $1 \leq k \leq m$ and $m < k \leq n$, respectively, the maximum of their absolute values is reached for $k = m$ and its order for $k = m$ is $O(\Delta_{n,1})$.

This implies the following theorem.

Theorem B *Let $Y_1, \ldots, Y_n$ be independent random variables, $Y_1, \ldots, Y_m$ have common d.f. F_1 and $Y_{m+1}, \ldots, Y_n$ have common distribution function F_n. Let m satisfy (29) and let $\gamma \in [0, 1]$. If $\Delta_{n,j}$, $j = 1, 2$, satisfy (30) and (31) then, as $\to \infty$,*

$$T_{n,\gamma}(w) = \left(\frac{m(n-m)}{n^2}\right)^{1+\gamma} \left(n\Delta_{n,1} + O_P\left(\sqrt{n|\Delta_{n,2}|} + 1 + \mathbf{I}\{\gamma = 0\} \log \log n\right)\right). \tag{44}$$

Proof is postponed to section 6. □

With a larger effort one could get deeper results but they would not provide substantially more information and the proofs would be much more technical. Clearly, the limit behavior of $T_{n,\gamma}(w)$ under alternatives and the null hypothesis differs considerably. Under the respective assumptions of Theorem B, as $n \to \infty$,

$$T_{n,\gamma}(w) \to \infty,$$

in probability. Since under H_0

$$T_{n,\gamma}(w) = O_P(1),$$

then the test with critical regions

$$T_{n,\gamma}(w) \geq t_n$$

for any $t_n \to \infty$ leads to the consistent tests with asymptotic level $\alpha_n \to 0$. However we can get better results through resampling methods as we will see in the next section.

In case of $\gamma = 0$ we have that the test with critical region

$$\frac{1}{\log \log n} T_{n,\gamma}(w) \geq t_n$$

for any $t_n \to \infty$ leads to consistent tests with asymptotic level $\alpha_n \to 0$.

4 Properties of permutation versions

Here we investigate properties of the conditional distribution of the permutational version of the statistic $T_{n,\gamma}(w, \boldsymbol{R})$ given $Y_1, \ldots, Y_n$ with the aim to get a picture about the approximate behavior of the critical values $d_{n,\gamma}(\alpha, \boldsymbol{Y})$.

We assume that the random variables $Y_1, \ldots, Y_n$ are independent random variables such that for $m (\leq n)$ $Y_1, \ldots, Y_m$ have common distribution function F_* and $Y_{m+1}, \ldots, Y_n$ have common distribution function F^*, where F_* and F^* do not depend on n, while m satisfies

$$m = m_n = [n\kappa] \quad \text{for some} \quad \kappa \in (0, 1]. \tag{45}$$

We set

$$F_\kappa(x) = \kappa F_*(x) + (1-\kappa) F^*(x), \quad x \in R^1. \tag{46}$$

Notice that Y_{R_i} has the distribution function

$$P(Y_{R_i} \le x) = \frac{m}{n} F_*(x) + \frac{n-m}{n} F^*(x) = F_\kappa(x) + O(n^{-1}), \quad x \in R^1.$$

Recall that the permutation version of $T_{n,\gamma}(w, \boldsymbol{R})$ of $T_{n,\gamma}(w)$ is defined as

$$T_{n,\gamma}(w, \boldsymbol{R}) = \max_{1 \le k < n} c_{k,n}(\gamma) \big| V_{k,n}(w, \boldsymbol{R}) \big|, \tag{47}$$

with

$$V_{k,n}(w, \boldsymbol{R}) = \frac{1}{k^2} \sum_{s,r=1}^{k} h(Y_{R_s}, Y_{R_r}) + \frac{1}{(n-k)^2} \sum_{s,r=k+1}^{n} h(Y_{R_s}, Y_{R_r}) - \frac{2}{k(n-k)} \sum_{s=1}^{k} \sum_{r=k+1}^{n} h(Y_{R_s}, Y_{R_r}), \tag{48}$$

where $\boldsymbol{R} = (R_1, \ldots, R_n)$ is a random permutation of $(1, \ldots, n)$, independent of $Y_1, \ldots, Y_n$.

We decompose $V_{k,n}(w, \boldsymbol{R})$ similarly as $V_{k,n}(w)$:

$$V_k(w, \boldsymbol{R}) = A_{k1}(\boldsymbol{R}) + A_{k2}(\boldsymbol{R}) + A_{k3}(\boldsymbol{R}) \tag{49}$$

where

$$A_{k1}(\boldsymbol{R}) = \frac{n}{k(n-k)} \Big(\frac{1}{k} \sum_{v=1}^{k} \sum_{s=1, v \ne s}^{k} \widehat{h}(Y_{R_v}, Y_{R_s}) + \frac{1}{(n-k)} \sum_{v=k+1}^{n} \sum_{s=k+1, v \ne s}^{n} \widehat{h}(Y_{R_v}, Y_{R_s}) \Big), \tag{50}$$

$$A_{k2}(\boldsymbol{R}) = \frac{n}{k(n-k)} \Big(\int w(t)\, \mathrm{d}t - \overline{h}_n(\boldsymbol{Y}) \Big), \tag{51}$$

$$A_{k3}(\boldsymbol{R}) = -\frac{2}{n-1} \Big(\frac{1}{k^2} \sum_{j=1}^{k} \sum_{s=1, s \ne R_j}^{n} \big(h(Y_{R_j}, Y_s) - \overline{h}_n(\boldsymbol{Y}) \big) + \frac{1}{(n-k)^2} \sum_{j=k+1}^{n} \sum_{s=1, s \ne R_j}^{n} \big(h(Y_{R_j}, Y_s) - \overline{h}_n(\boldsymbol{Y}) \big) \Big), \tag{52}$$

where

$$\overline{h}_n(\boldsymbol{Y}) = \frac{1}{n(n-1)} \sum_{v=1}^{n} \sum_{s=1, s \ne v}^{n} h(Y_s, Y_v) \tag{53}$$

$$\widehat{h}(Y_{R_v}, Y_{R_s}) = h(Y_{R_v}, Y_{R_s}) - E(h(Y_{R_v}, Y_{R_s}) | \boldsymbol{Y}, R_v) - E(h(Y_{R_v}, Y_{R_s}) | \boldsymbol{Y}, R_s) + E(h(Y_{R_v}, Y_{R_s}) | \boldsymbol{Y}), \quad v \ne s. \tag{54}$$

Notice that

$$E(h(Y_{R_v}, Y_{R_s})|\boldsymbol{Y}, R_s) = \frac{1}{n-1} \sum_{j=1, j\neq R_s} h(Y_j, Y_{R_s})$$

and

$$E\widehat{h}(Y_{R_v}, Y_{R_s})|\boldsymbol{Y}, R_s) = 0, \quad s \neq v.$$

The similarity with $A_{kj},\ j = 1, 2, 3$, is obvious. The term $A_{k1}(\boldsymbol{R})$ is a functional of U-statistics in $R_1, \ldots, R_n$, $A_{k2}(\boldsymbol{R})$ does not depend on $R_1, \ldots, R_n$ and $A_{k3}(\boldsymbol{R})$ can be viewed as simple linear rank statistics. It can be shown that A_{k3} will not influence the limit behavior of $T_{n,\gamma}(w, \boldsymbol{R})$.

Before we formulate the main assertion of the present section we introduce a representation of the function

$$\widetilde{h}_\kappa(x, y) = h(x, y) - E_\kappa h(x, Z_2) - E_\kappa h(Z_1, y) + E_\kappa h(Z_1, Z_2), \tag{55}$$

where Z_1, Z_2 are independent random variables with distribution function F_κ and E_κ denotes the expectation w.r.t. F_κ.

By Serfling (1980) for each $\kappa \in (0, 1]$ there are orthogonal eigenfunctions $\{g_j(y;\kappa), y \in R^2\}_{j=1}^\infty$ and eigenvalues $\{\lambda_j(\kappa)\}_{j=1}^\infty$ such that

$$\lim_{K\to\infty} \int\int \Big(\widetilde{h}_\kappa(x, y) - \sum_{s=1}^{K} \lambda_s(\kappa) g_s(x;\kappa) g_s(y;\kappa)\Big)^2 dF_\kappa(x) dF_\kappa(y) = 0, \tag{56}$$

$$\int g_s(x;\kappa) dF_\kappa(x) = 0, \quad , s = 1, \ldots, \tag{57}$$

$$\int g_s(x;\kappa) g_j(y;\kappa) dF_\kappa(x) = \delta_{j,s}, \quad j, s = 1, \ldots, \tag{58}$$

where $\delta_{j,s} = 1$ if $s = j$ and $\delta_{j,s} = 0$ if $s \neq j$.

Here is the main assertion on the limit behavior of the permutational distribution of $T_{n,\gamma}(w, \boldsymbol{R})$.

Theorem C *Let $Y_1, \ldots, Y_n$ be independent random variables such that $Y_1, \ldots, Y_m$ have common distribution function F_* and $Y_{m+1}, \ldots, Y_n$ have common distribution function F^*, where m satisfies (45). Then, for $\gamma \in (0, 1]$, as $n \to \infty$,*

$$P(T_{n,\gamma}(w, \boldsymbol{R}) \le x|\boldsymbol{Y}) - P\Big(\sup_{t\in(0,1)} \Big(t(1-t)\Big)^\gamma \Big| \Big(\int w(z)\, dz - E_\kappa\, h(Z_1, Z_2)\Big) + \sum_{j=1}^\infty \lambda_j(\kappa) \Big\{ \frac{B_j^2(t)}{(1-t)t} - 1 \Big\} \Big| \le x\Big) \to^P 0, \quad x \in R^1, \tag{59}$$

where Z_1, Z_2 are independent random variables with distribution function F_κ.

Proof The proof is postponed to section 6. □

Noticing that, under $H_0, \lambda_j = \lambda_j(1),\ j = 1, \dots$, and comparing the assertions of Theorem A and Theorem C we see that under H_0 the limit distribution of $T_{n,\gamma}(w)$ and the limiting conditional distribution of $T_{n,\gamma}(w, \boldsymbol{R})$ coincide as expected. Unfortunately, this is not true under alternatives with $\kappa \in (0, 1)$ because the eigenvalues $\{\lambda_j\}$ and $\{\lambda_j(\kappa)\}$ are generally different. However, in any case, $T_{n,\gamma}(w, \boldsymbol{R})$ is bounded in probability while under alternatives (section 3) $T_{n,\gamma}(w) \to \infty$ in probability. Therefore despite that $d_{n,\gamma}(\alpha, \boldsymbol{Y})$ need not provide a reasonable approximation for critical values, its application leads to consistent tests.

Concerning the limit behavior of $T_{n,\gamma}(w, \boldsymbol{R})$ with $\gamma = 0$ we can only show, using properties of rank statistics and results in Chapter 4 of de la Peňa and Giné (1999), that

$$\frac{1}{\log\log n} T_{n,\gamma}(w, \boldsymbol{R}) = O_P(1),$$

which in combination with the results in Theorems A and B implies that the test with the critical region (14) is consistent. To obtain deeper results is an open problem. Nevertheless, even in this case the permutation principle provides a consistent test with level α.

5 Simulations

In order to investigate the finite-sample behavior of the test statistics we perform a simulation experiment. For sample sizes $n = 40$ and $n = 100$, 2,000 samples are generated. First the test statistic is calculated based on the original sample $Y_1, \ldots, Y_n$. Then B permutations of $(1, 2, \ldots, n)$ are chosen at random from all $n!$ total number of permutations. For each permutation $(R_1, \ldots, R_n)$ the test statistic is computed based on $Y_{R_1}, \ldots, Y_{R_n}$. The $(1 - \alpha)100\%$ quantile of this ‘subset’ permutation distribution is used as the critical point for the current sample.

Before proceeding to the simulation results some computational guidelines are given. In particular, the test statistics may conveniently be written as,

$$T^{(j)}_{n,\gamma}(a) = \psi^{(j)}_a \max_{1\le k\le n-1} c_{k,n}(\gamma)\left[\frac{1}{k^2}S_{1,k} + \frac{1}{(n-k)^2}S_{2,k} - \frac{2}{k(n-k)}S_{3,k}\right],$$

with

$$S_{1,k} = \sum_{l,m=1}^{k} \rho_{l,m}, \quad S_{2,k} = \sum_{l,m=k+1}^{n} \rho_{l,m}, \quad S_{3,k} = \sum_{l=1}^{k}\sum_{m=k+1}^{n} \rho_{l,m}.$$

If $w^{(1)}_a(t) = \exp(-a|t|)$, then $\psi^{(1)}_a = 2a$, and $\rho_{l,m} = (a^2 + (Y_l - Y_m)^2)^{-1}$. Consequently the test statistic is written as

$$T^{(1)}_{n,\gamma}(a) = 2a \max_{1\le k\le n-1} c_{k,n}(\gamma)\left[\frac{1}{k^2}\sum_{l,m=1}^{k}\frac{1}{a^2+Y^2_{lm}} + \frac{1}{(n-k)^2}\sum_{l,m=k+1}^{n}\frac{1}{a^2+Y^2_{lm}}\right.$$
$$\left. - \frac{2}{k(n-k)}\sum_{l=1}^{k}\sum_{m=k+1}^{n}\frac{1}{a^2+Y^2_{lm}}\right],$$

with $Y_{lm} = Y_l - Y_m$, and $c_{k,n}(\gamma)$ given by (6).

Likewise, if $w_a^{(2)}(t) = \exp(-at^2)$, then $\psi_a^{(2)} = \sqrt{\pi/a}$, and $\rho_{l,m} = \exp(-Y_{lm}^2/4a)$. Consequently,

$$T_{n,\gamma}^{(2)}(a) = \sqrt{\frac{\pi}{a}} \max_{1\le k\le n-1} c_{k,n}(\gamma) \times\left[\frac{1}{k^2}\sum_{l,m=1}^{k}\exp\left(-\frac{Y_{lm}^2}{4a}\right) + \frac{1}{(n-k)^2}\sum_{l,m=k+1}^{n}\exp(-\frac{Y_{lm}^2}{4a}) - \frac{2}{k(n-k)}\sum_{l=1}^{k}\sum_{m=k+1}^{n}\exp\left(-\frac{Y_{lm}^2}{4a}\right)\right].$$

Recursive formulas for the computation of $S_{j,k}$, $j = 1, 2, 3$, are,

$$S_{1,k+1} = S_{1,k} + S_{1,1} + 2\sum_{l=1}^{k}\rho_{k+1,l}, \quad S_{2,k+1} = S_{2,k} - S_{1,1} - 2\sum_{l=k+2}^{n}\rho_{k+1,l},$$

$$S_{3,k+1} = S_{3,k} + \sum_{l=k+2}^{n}\rho_{k+1,l} - \sum_{l=1}^{k}\rho_{k+1,l}.$$

In Table 1, level results are shown (percentage of rejection rounded to the nearest integer), for nominal level $\alpha = 0.05$ and $\alpha = 0.10$, and $B = 200$. These results correspond to the test statistics $T_{n,\gamma}^{(1)}(a)$ (top entry) and $T_{n,\gamma}^{(2)}(a)$ (bottom entry), for $\gamma = 0, 0.5, 1.0$. The distributions included are: The normal (N), the uniform (U), the double exponential (DE), the logistic (L), the gamma distribution with shape parameter equal to one ($\Gamma 1$), and the gamma distribution with shape parameter equal to two ($\Gamma 2$). All distributions considered are in standard form. From this table it is evident that even with such a small subset of the permutation distribution, the test statistics capture the nominal level of significance to a satisfactory degree, and are fairly robust with respect to the value of the weight parameter a. We have also obtained power results, under alternatives with $F_1(x)$ being any of the distributions referred to above, $F_n(x) = F_1[(x-\delta)/b]$, for $\delta = 1$, $b = 1$ or $\sqrt{2}$, and $m = n/2$ or $m = n/4$. The results are shown in Table 2 ($\gamma = 0$), Table 3 ($\gamma = 0.5$) and Table 4 ($\gamma = 1.0$). An asterisk denotes power 100%. For the uniform distribution the power was in all cases 100%. These results indicate that the tests based on $T_{n,\gamma}^{(1)}(a)$ and $T_{n,\gamma}^{(2)}(a)$ are considerably powerful under most alternative situations considered.

6 Proofs

The proofs of our theorems are rather technical. We focus on the main steps of the proofs only.

Proof of Theorem A In order to prove the assertion it suffices [confer (18)] to derive a representation of $A_{k,1}$, defined by (19), in terms of functionals of Brownian bridges and to show that the A_{k3}, defined by (21), do not influence the limit behavior of $T_{n,\gamma}(w)$.

Table 1 Rejection rate under H_0 corresponding to 5% and 10% nominal level for the test statistics $T_{n,\gamma}^{(1)}(a)$ (top entry) and $T_{n,\gamma}^{(2)}(a)$ (bottom entry)

	$n = 40$										$n = 100$									
a	1.0		1.5		2.0		3.0		4.0		1.0		1.5		2.0		3.0		4.0	
$\gamma = 0$																				
N	4	9	5	10	5	10	5	10	5	10	6	10	6	11	6	11	5	11	6	10
	5	10	5	10	6	10	6	10	6	10	6	11	5	11	5	10	6	11	6	11
U	5	10	5	10	5	10	6	10	6	10	6	11	6	11	6	10	6	10	6	11
	6	10	6	10	6	10	6	10	6	10	6	11	6	11	5	10	6	10	6	10
DE	4	8	4	9	5	10	4	9	4	9	6	11	5	11	5	11	6	11	6	11
	4	9	4	9	5	10	4	10	5	10	5	11	6	11	5	10	6	11	6	11
L	4	8	4	9	5	9	5	10	5	9	5	10	6	10	5	10	5	11	5	11
	4	9	5	10	5	9	5	9	5	10	6	10	6	11	5	10	5	11	5	11
Γ1	5	9	5	9	5	9	4	9	4	10	6	11	5	11	5	11	6	11	6	11
	4	10	4	9	5	9	5	9	5	9	6	11	6	11	5	11	5	11	5	11
Γ2	5	10	5	10	5	10	5	10	5	10	4	9	4	9	4	10	4	10	4	9
	5	10	5	10	5	10	5	10	5	10	4	9	4	9	5	10	5	10	5	9
$\gamma = 0.5$																				
N	5	10	5	10	5	10	5	11	5	10	5	10	6	11	6	11	5	10	5	11
	5	10	5	11	5	10	5	10	5	10	6	11	5	10	5	11	6	11	6	11
U	6	10	6	10	6	10	6	10	6	10	6	11	6	11	5	11	5	10	6	11
	6	10	6	10	6	10	6	10	6	10	6	11	6	11	5	10	5	11	6	11
DE	5	10	5	10	5	10	5	10	5	10	6	11	6	10	6	12	6	10	5	12
	5	10	5	10	5	10	5	10	5	10	6	11	5	10	5	11	6	10	6	12
L	5	10	5	10	5	10	5	10	5	11	6	10	5	10	6	11	6	12	6	11
	5	11	5	10	5	10	5	11	5	10	5	11	6	10	5	10	6	11	6	12
Γ1	5	10	5	10	5	10	5	10	5	10	5	10	5	10	6	11	6	11	6	11
	5	10	5	10	5	10	5	10	5	10	6	10	5	11	5	11	6	11	6	12
Γ2	5	10	5	10	5	10	5	10	5	10	5	10	5	9	4	10	4	9	4	9
	5	9	5	10	5	10	5	10	5	10	5	9	5	10	5	10	4	9	4	9
$\gamma = 1.0$																				
N	5	11	5	10	5	10	5	10	5	10	6	10	6	11	6	11	6	11	6	10
	5	10	5	10	5	10	5	10	5	10	6	11	6	11	6	11	6	10	5	11
U	5	11	6	10	5	11	5	10	5	11	6	11	6	10	5	10	5	11	6	11
	5	10	5	10	5	11	5	11	5	11	6	11	6	11	5	11	6	10	6	11
DE	5	10	5	10	5	10	5	10	5	10	6	11	6	11	5	11	6	11	5	11
	5	10	5	10	5	10	5	10	5	10	6	11	6	11	5	11	5	10	6	11
L	4	10	5	11	5	10	5	10	5	10	5	11	6	11	6	11	5	11	6	10
	5	11	5	10	5	10	5	10	5	10	5	11	6	11	5	11	5	11	5	10
Γ1	5	11	5	10	5	10	5	10	5	10	6	10	6	11	5	11	6	11	6	11
	5	10	5	10	5	10	5	10	5	10	6	10	5	11	6	11	6	11	5	11
Γ2	5	10	5	10	5	9	5	9	5	9	5	10	5	10	5	10	5	10	5	9
	5	9	5	9	5	9	5	9	5	10	5	10	5	10	4	10	5	10	5	9

We start with the later part. Clearly,

$$Z_r = E(h(Y_r, Y)|Y_r) - Eh(Y_1, Y_2), \quad r = 1, \ldots, n, \tag{60}$$

are i.i.d. random variables with zero mean and finite variance. Here Y is independent of Y_r and has the distribution F. Then by the Hájek-Rényi inequality (see, e.g. Chow and Teicher 1988, Theorem 8, p. 247) for any $A > 0$ and $\gamma \in [0, 1]$

Table 2 Rejection rate under H_1 corresponding to 5 and 10% nominal level for the test statistic $T_{n,0}^{(1)}(a)$ (upper part) and $T_{n,0}^{(2)}(a)$ (lower part) when $Y_n = Y_1 + 1$ (top entry) or $Y_n = \sqrt{2}Y_1 + 1$ (bottom entry)

	$n = 40$										$n = 100$									
a	1.0		1.5		2.0		3.0		4.0		1.0		1.5		2.0		3.0		4.0	
N	56	68	61	73	64	75	66	76	67	76	95	98	97	98	97	99	98	99	98	99
$m = n/2$	44	56	48	60	50	63	51	64	51	63	87	92	90	95	92	95	92	96	92	96
	42	54	46	58	48	60	50	63	50	64	84	91	88	93	91	95	92	96	93	96
$m = n/4$	29	41	31	43	32	44	31	44	30	43	71	82	76	85	77	86	77	87	77	87
DE	49	61	49	61	48	61	44	57	40	54	90	95	90	95	90	94	88	93	85	91
$m = n/2$	39	52	40	53	39	52	35	47	31	44	82	89	83	90	82	89	79	87	75	83
	37	48	37	48	36	47	32	44	29	41	77	85	77	86	77	86	75	83	71	81
$m = n/4$	25	38	25	38	24	36	21	32	18	28	65	77	66	77	65	75	59	71	53	66
L	19	30	22	32	24	34	25	35	24	34	44	56	50	62	53	64	56	66	56	67
$m = n/2$	19	28	21	31	21	31	21	32	21	31	43	55	47	59	50	62	51	64	50	63
	14	24	16	26	17	27	19	28	19	28	34	45	38	50	40	52	42	54	42	54
$m = n/4$	12	21	13	22	13	21	13	21	13	20	30	43	34	46	35	46	34	46	32	45
Γ1	93	96	90	94	85	91	76	86	67	81	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆
$m = n/2$	94	97	92	96	90	95	84	92	79	89	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆
	77	85	73	81	67	77	57	70	50	66	99	⋆	99	99	98	99	96	98	94	97
$m = n/4$	77	86	75	84	72	82	62	75	55	72	99	⋆	99	⋆	99	⋆	98	99	97	99
Γ2	43	56	43	56	42	56	40	54	38	51	89	94	89	94	88	94	86	92	83	91
$m = n/2$	68	78	71	81	73	82	73	81	71	81	99	⋆	99	⋆	⋆	⋆	⋆	⋆	99	⋆
	32	45	33	46	32	44	29	41	26	39	74	81	75	82	74	83	71	81	67	79
$m = n/4$	49	60	50	63	50	63	50	62	48	61	93	96	94	97	95	97	95	98	94	98
N	63	75	66	76	66	76	66	76	67	77	97	99	98	99	98	99	98	99	98	99
$m = n/2$	50	63	51	64	51	64	51	64	50	63	91	95	93	96	92	96	92	96	92	96
	48	60	49	62	50	63	50	64	51	64	90	95	92	95	93	96	93	96	93	96
$m = n/4$	32	44	32	44	31	44	31	42	30	41	77	86	77	87	77	87	77	87	76	86
DE	47	60	46	58	44	57	41	54	39	53	89	94	88	94	87	93	85	91	83	90
$m = n/2$	38	51	37	50	35	48	32	45	30	43	82	89	81	88	79	88	76	84	73	83
	35	47	34	45	33	44	30	41	28	39	76	85	75	85	74	83	72	81	69	79
$m = n/4$	24	36	23	34	22	32	20	29	18	27	64	75	62	74	60	72	56	68	52	64
L	22	32	23	35	24	35	24	35	24	34	51	62	54	65	55	66	56	66	56	67
$m = n/2$	20	31	21	31	22	32	22	32	21	31	48	60	50	63	51	64	50	64	50	63
	16	26	17	27	18	28	19	28	19	27	38	51	40	52	41	53	42	54	42	54
$m = n/4$	13	22	13	22	13	21	13	21	12	20	34	46	35	46	34	47	34	46	32	45
Γ1	84	90	78	88	74	85	66	80	62	78	⋆	⋆	⋆	⋆	⋆	⋆	99	⋆	99	⋆
$m = n/2$	89	94	87	92	84	91	79	89	75	87	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆
	66	76	60	72	55	69	49	65	46	63	97	99	96	98	95	98	93	97	92	96
$m = n/4$	71	81	66	78	62	75	56	71	51	69	98	99	98	99	98	99	97	99	96	99
Γ2	42	55	41	54	40	53	38	51	36	50	87	93	86	93	85	92	83	91	81	89
$m = n/2$	70	80	72	82	72	82	71	80	70	79	99	⋆	99	⋆	99	⋆	99	⋆	99	⋆
	32	45	31	43	30	41	27	39	25	38	72	82	72	82	71	81	68	79	66	77
$m = n/4$	49	62	50	63	50	63	49	62	48	60	94	97	94	97	94	97	94	98	94	98

$$P_{H_0}\left(\max_{1\le k<n} n\left(\frac{k(n-k)}{n^2}\right)^{1+\gamma}\frac{1}{k^2}\left|\sum_{r=1}^{k} Z_r\right| \ge A\right)$$

$$\le \frac{\mathrm{var}\, Z_1}{A^2 n^{2\gamma}}\sum_{k=1}^{n}\frac{1}{k^{2(1-\gamma)}} \le \frac{D_1}{A^2}n^{-\min(2\gamma,1)}(1+\mathbf{I}\{\gamma=1/2\}\log n) \qquad (61)$$

Table 3 Rejection rate under H_1 corresponding to 5 and 10% nominal level for the test statistic $T^{(1)}_{n,0.5}(a)$ (upper part) and $T^{(2)}_{n,0.5}(a)$ (lower part) when $Y_n = Y_1 + 1$ (top entry) or $Y_n = \sqrt{2}Y_1 + 1$ (bottom entry)

	$n = 40$										$n = 100$									
a	1.0		1.5		2.0		3.0		4.0		1.0		1.5		2.0		3.0		4.0	
N	64	74	69	79	72	81	74	84	75	84	98	99	98	99	99	99	99	⋆	99	⋆
$m = n/2$	51	63	56	68	59	71	61	73	61	73	92	96	94	97	95	98	96	98	96	98
	41	55	47	60	50	64	52	67	54	68	85	92	90	95	92	96	94	97	94	97
$m = n/4$	29	42	32	46	34	48	34	50	34	49	75	83	79	87	81	89	82	91	81	91
DE	57	68	58	70	59	70	58	70	57	68	95	98	95	98	95	98	95	98	94	97
$m = n/2$	47	59	49	61	49	62	48	61	47	60	88	94	90	94	90	94	89	94	88	93
	36	50	39	52	40	53	40	52	38	51	78	87	80	89	81	89	81	89	79	88
$m = n/4$	26	39	28	40	28	40	27	39	26	38	68	79	70	80	70	79	68	78	65	77
L	23	34	27	39	29	41	31	43	32	44	53	65	58	69	62	72	65	76	67	77
$m = n/2$	22	33	24	37	25	38	27	38	27	38	51	65	57	70	60	73	62	75	63	75
	14	24	17	27	18	29	20	30	21	31	34	47	40	52	43	55	46	58	47	60
$m = n/4$	14	21	14	23	15	24	16	25	15	25	32	45	36	48	38	50	39	52	39	51
Γ1	95	98	94	97	93	96	89	94	87	92	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆
$m = n/2$	97	98	96	98	95	98	94	97	93	96	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆
	78	85	77	84	74	82	68	80	65	77	99	⋆	99	⋆	99	99	98	99	97	99
$m = n/4$	78	86	77	86	76	86	74	84	72	82	⋆	⋆	99	⋆	99	⋆	99	⋆	99	⋆
Γ2	52	65	54	66	53	67	53	66	52	65	94	98	95	98	94	97	93	97	92	96
$m = n/2$	75	84	78	86	80	87	81	89	82	89	99	⋆	99	⋆	⋆	⋆	⋆	⋆	99	⋆
	31	44	33	46	34	47	33	46	31	45	75	82	76	85	76	85	76	86	76	85
$m = n/4$	46	60	51	63	53	65	54	68	55	69	93	96	95	97	96	98	96	99	97	99
N	71	81	74	83	74	84	75	84	76	85	99	99	99	99	99	⋆	98	⋆	98	⋆
$m = n/2$	58	70	60	72	60	73	61	73	61	73	95	98	96	98	96	98	96	98	96	98
	50	64	52	66	53	67	54	68	54	68	92	96	93	96	94	97	94	97	94	97
$m = n/4$	34	38	35	49	34	49	34	48	34	47	80	89	81	89	82	91	81	91	81	91
DE	57	69	58	69	57	69	56	68	56	67	95	97	94	97	94	97	94	97	93	97
$m = n/2$	47	59	47	61	48	60	47	60	45	59	89	94	89	94	89	94	88	93	87	93
	38	52	39	52	39	51	39	50	37	50	80	88	80	89	80	88	79	88	78	87
$m = n/4$	28	39	27	39	27	39	26	38	25	37	69	78	69	78	68	78	66	77	63	75
L	28	39	30	41	30	43	31	44	32	44	59	70	62	73	64	74	66	76	67	77
$m = n/2$	24	36	26	38	26	38	27	39	27	39	57	70	60	73	62	74	63	75	63	75
	17	27	18	29	19	30	20	31	21	31	40	53	43	56	45	57	47	59	48	60
$m = n/4$	14	24	14	24	15	24	15	24	15	25	36	49	38	50	39	51	39	52	39	51
Γ1	92	95	89	94	88	94	86	92	84	90	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆
$m = n/2$	94	97	94	97	93	96	92	96	91	96	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆
	72	82	69	80	67	79	65	76	63	75	98	99	98	99	98	99	97	98	97	98
$m = n/4$	75	85	73	84	73	83	71	82	70	81	98	⋆	99	⋆	99	⋆	99	⋆	99	99
Γ2	52	66	52	65	53	65	52	64	51	64	94	97	93	96	93	96	92	96	92	96
$m = n/2$	78	85	79	87	80	88	81	88	81	88	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆
	33	46	33	46	33	46	31	45	31	45	75	85	76	85	76	85	76	85	75	84
$m = n/4$	51	63	53	65	53	66	54	68	54	69	95	97	95	98	96	98	96	99	97	99

and

$$P_{H_0}\Big(\max_{1\le k<n} n\Big(\frac{k(n-k)}{n^2}\Big)^{1+\gamma}\frac{1}{(n-k)^2}\Big|\sum_{r=k+1}^{n} Z_r\Big| \ge A\Big)$$
$$\le \frac{D_1}{A^2} n^{-\min(2\gamma,1)}(1+\mathbf{I}\{\gamma = 1/2\}\log n). \tag{62}$$

Table 4 Rejection rate under H_1 corresponding to 5 and 10% nominal level for the test statistic $T^{(1)}_{n,1.0}(a)$ (upper part) and $T^{(2)}_{n,1.0}(a)$ (lower part) when $Y_n = Y_1 + 1$ (top entry) or $Y_n = \sqrt{2}Y_1 + 1$ (bottom entry)

	$n = 40$										$n = 100$									
a	1.0		1.5		2.0		3.0		4.0		1.0		1.5		2.0		3.0		4.0	
N	66	77	71	81	74	83	77	85	77	86	98	99	99	99	99	99	99	⋆	99	⋆
$m = n/2$	54	65	59	70	61	73	64	75	64	75	93	96	96	98	96	98	97	98	97	99
	38	51	43	57	46	61	48	64	50	65	82	90	88	93	90	95	92	96	93	97
$m = n/4$	28	40	30	45	32	47	33	47	32	47	71	82	76	86	78	88	79	90	79	90
DE	59	70	61	72	62	73	62	72	60	72	96	98	96	98	96	98	96	98	95	97
$m = n/2$	49	61	51	63	52	64	52	63	50	63	90	95	91	96	91	95	91	95	90	95
	34	46	36	49	36	50	37	50	35	49	75	84	77	86	79	87	78	87	77	86
$m = n/4$	24	37	26	39	28	38	27	38	25	37	63	76	66	77	66	77	65	77	62	75
L	25	36	28	40	31	43	33	45	34	47	56	67	60	71	64	74	68	78	70	79
$m = n/2$	24	34	26	38	27	39	28	41	29	42	54	68	60	73	63	75	66	77	66	77
	14	23	15	26	16	28	18	29	19	29	30	44	35	50	39	53	43	56	45	57
$m = n/4$	12	21	13	22	14	23	15	24	14	24	29	42	33	47	35	48	37	50	36	50
Γ1	97	99	95	98	94	97	91	95	89	94	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆
$m = n/2$	97	99	96	99	96	98	95	98	94	97	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆
	74	83	72	82	70	80	65	77	63	75	99	⋆	99	⋆	98	99	98	99	97	98
$m = n/4$	73	83	73	83	73	84	71	82	69	81	99	⋆	99	⋆	99	⋆	99	99	99	⋆
Γ2	55	68	57	70	57	70	57	69	56	68	96	98	96	98	95	98	95	98	94	97
$m = n/2$	77	85	80	88	82	89	83	90	84	90	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆
	26	40	30	43	31	43	30	44	29	44	70	80	72	82	72	83	73	84	73	84
$m = n/4$	42	55	46	60	48	63	51	66	53	68	91	95	93	97	94	97	95	98	96	98
N	74	83	76	85	77	86	78	86	78	86	99	99	99	⋆	99	⋆	98	⋆	99	⋆
$m = n/2$	61	72	63	75	64	75	64	75	64	75	96	98	97	98	97	98	97	98	97	98
	46	61	48	63	49	65	50	65	50	66	90	95	91	96	93	96	94	97	94	97
$m = n/4$	32	46	33	47	33	47	32	47	32	46	77	80	79	89	79	90	79	90	78	90
DE	60	71	61	72	61	72	60	71	59	70	95	98	95	98	95	98	95	97	94	97
$m = n/2$	50	63	50	64	51	63	50	62	49	61	90	95	91	95	90	94	90	94	89	94
	36	49	36	49	36	49	35	49	34	48	77	86	77	87	77	86	77	86	76	85
$m = n/4$	26	38	26	38	26	38	25	38	25	37	65	76	65	76	65	76	62	75	61	74
L	29	41	31	43	32	45	34	46	34	47	62	71	65	75	66	76	68	78	70	79
$m = n/2$	26	38	27	39	28	40	29	41	29	42	60	73	63	75	65	76	66	77	66	77
	16	26	17	28	18	29	19	30	19	29	37	50	40	53	42	55	44	57	45	58
$m = n/4$	13	22	14	23	14	24	15	24	15	24	33	47	35	49	36	49	37	50	36	50
Γ1	93	96	91	95	90	95	88	93	87	92	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆
$m = n/2$	95	98	95	97	94	97	94	97	93	97	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆
	68	80	65	78	65	77	63	75	61	74	98	99	98	99	98	99	97	98	97	98
$m = n/4$	72	82	70	82	70	82	69	81	68	80	99	⋆	99	⋆	99	⋆	98	99	98	99
Γ2	56	69	56	69	56	68	55	68	55	68	95	98	94	97	94	97	94	97	93	97
$m = n/2$	80	87	81	89	82	89	83	90	84	90	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆
	30	43	30	43	30	44	30	44	29	43	72	82	73	83	72	83	73	83	72	83
$m = n/4$	46	61	49	63	49	65	52	66	52	68	93	97	94	97	95	98	95	98	96	98

with some $D_1 > 0$. Therefore

$$\max_{1\le k<n} n\Big(\frac{k(n-k)}{n^2}\Big)^{1+\gamma}|A_{k3}| = O_P\big(n^{-\min(\gamma,1/2)}(1+\mathbf{I}\{\gamma=1/2\}\log n)\big). \quad (63)$$

Hence the terms A_{k3} do not influence the limit behavior of the considered statistic.

Next we investigate the terms A_{k1}. Towards this end, we consider the auxiliary statistics

$$S_k(\widetilde{h}) = \sum_{1\le i<j\le k} \widetilde{h}(Y_i, Y_j), \quad k = 1, \dots n,$$

where $\widetilde{h}$ is defined by (16). Since

$$E_{H_0}(S_{k+1}|Y_1, \dots, Y_k) = S_k + \sum_{i=1}^{k} E(\tilde{h}(Y_i, Y_{k+1})|Y_1, \dots, Y_k) = S_k,$$

$k = 1, \dots, n-1$, $\{S_k(\widetilde{h}), \sigma(Y_1, \dots, Y_k);\ k = 1 \dots, n\}$ is a martingale. Here $\sigma(Y_1, \dots, Y_k)$ denotes the σ-field generated by $Y_1, \dots, Y_k$. Then by the Hájek-Rényi inequality (see, e.g., Chow and Teicher 1988)

$$P\Big(\max_{1\le k<n} n\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma+1}\frac{1}{k^2}|S_k| \ge A)$$
$$\le A^{-2}\frac{1}{n^{2\gamma}}\sum_{k=1}^{n-1}\frac{1}{k^{2-2\gamma}}E(S_k - S_{k-1})^2 \le \frac{D_2}{A^2}E\tilde{h}^2(Y_1, Y_2)$$

with some $D_2 > 0$. Proceeding similarly with $\sum_{k+1\le i<j\le n}\widetilde{h}(Y_i, Y_j)$ we get after some standard steps that

$$P\Big(\max_{1\le k<n} n\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma+1}|A_{k1}| \ge A\Big) \le \frac{D_3}{A^2}E\tilde{h}^2(Y_1, Y_2) \quad (64)$$

with some $D_3 > 0$. The last inequality holds true for any function $\widetilde{h}$ satisfying (17). Therefore replacing $\widetilde{h}$ by the function $\widetilde{h} - \widetilde{h}_K$ with

$$\widetilde{h}_K(x, y) = \sum_{s=1}^{K}\lambda_s g_s(x) g_s(y), \quad (65)$$

where $\{\lambda_j, g_j\}$ are eigenvalues and eigenfunctions defined in (23)–(25) and K is an arbitrary natural number, we obtain

$$P\Big(\max_{1\le k<n} n\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma+1}|A_{k1} - A_{k1}(K)| \ge A)$$
$$\le \frac{D_4}{A^2}E(\widetilde{h}(Y_1, Y_2) - \widetilde{h}_K(Y_1, Y_2))^2$$
$$= \frac{D_4}{A^2}\sum_{j=K+1}^{\infty}\lambda_j^2 \quad (66)$$

with some $D_4 > 0$ and for any natural $n \geq 2$ and any natural K. Here $A_{k1}(K)$ is defined by (16) with $\widetilde{h}$ replaced by $\widetilde{h}_K$ (see (65)). Notice that

$$\frac{k(n-k)}{n} A_{k1}(K) = \frac{k(n-k)}{n} \sum_{s=1}^{K} \lambda_j \times \Big\{ \Big(\frac{1}{k} \sum_{j=1}^{k} g_s(Y_j) - \frac{1}{n-k} \sum_{j=k+1}^{n} g_s(Y_j) \Big)^2$$

$$- \Big(\frac{1}{k^2} \sum_{j=1}^{k} g_s^2(Y_j) + \frac{1}{(n-k)^2} \sum_{j=k+1}^{n} g_s^2(Y_j) \Big) \Big\}$$

$$= \sum_{s=1}^{K} \lambda_j \Big\{ \frac{n}{k(n-k)} \Big(\sum_{j=1}^{k} g_s(Y_j) - \frac{k}{n} \sum_{j=k+1}^{n} g_s(Y_j) \Big)^2$$

$$- \frac{1}{n} \Big(\frac{n-k}{k} \sum_{j=1}^{k} g_s^2(Y_j) + \frac{k}{n-k} \sum_{j=1}^{n} g_s^2(Y_j) \Big) \Big\}.$$

Since $Y_1, \ldots, Y_n$ are i.i.d. random variables and by (24) and (25), we have as $n \to \infty$,

$$\Big\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \boldsymbol{g}_K(Y_i),\ t \in (0,1) \Big\} \to^{D((0,1))} \{ \boldsymbol{W}_K(t),\ t \in (0,1) \},$$

where $\boldsymbol{g}_K = (g_1, \ldots, g_K)$ with components defined in (24) and (25) and $\{\boldsymbol{W}_K(t) = (W_1(t), \ldots, W_K(t)),\ t \in (0,1)\}$ is a K-dimensional Wiener process with independent components. Then for $\gamma \in (0,1]$, $\max_{1 \leq k < n} \Big(\frac{k(n-k)}{n^2} \Big)^{\gamma+1} \times A_{k1}(K)$ has the same limit distribution as

$$\max_{1 \leq k < n} n \Big(\frac{k(n-k)}{n^2} \Big)^{\gamma} \times \sum_{s=1}^{K} \lambda_j \Big\{ \frac{n^2}{k(n-k)} \Big(W_s(k/n) - \frac{k}{n} W_s(1) \Big)^2 - 1 \Big\}. \quad (67)$$

Moreover, by the properties of Wiener processes, for arbitrary $A > 0$ and arbitrary natural number K,

$$P\Big(\max_{1 \leq k < n} n \Big(\frac{k(n-k)}{n^2} \Big)^{\gamma} \times \Big| \sum_{s=K+1}^{\infty} \lambda_j \times \Big\{ \frac{n^2}{k(n-k)} \Big(W_s(k/n) - \frac{k}{n} W_s(1) \Big)^2 - 1 \Big\} \Big| \geq A \Big)$$

$$\leq D_4 A^{-2} \sum_{s=K+1}^{\infty} \lambda_j^2. \quad (68)$$

Combining properties of Wiener processes, (66)–(68) and letting $K \to \infty$ we can conclude that the limit distribution of

$$\max_{1 \leq k < n} \Big(\frac{k(n-k)}{n^2} \Big)^{\gamma+1} A_{k1}$$

is the same as that of

$$\max_{t\in(0,1)} n\Big(t(1-t)\Big)^{\gamma} \sum_{s=1}^{\infty} \lambda_j \Big\{\frac{(W_s(t)-tW_s(1))^2}{t(1-t)} - 1\Big\}.$$

Now, regarding (20), (63) and going carefully once more through the proof we infer that the assertion of Theorem A holds true. □

Proof of Theorem B Due to (42) and (43) it suffices to show that under the setup considered in section 3

$$\max_{1\le k<n} n\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma+1} |A_{k1}| = O_P(1) \tag{69}$$

and

$$\max_{1\le k<n} n\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma+1} |A_{k3}| = O_P(\sqrt{n\Delta_{n,2}}). \tag{70}$$

To get the former relation we proceed as in treating A_{k1} under the null hypothesis and we get in the very same way that

$$P\left(\max_{1\le k<n} n\left(\frac{k(n-k)}{n^2}\right)^{\gamma+1} |A_{k1}| \ge A\right) \quad \le \frac{D_5}{A^2}$$
$$\times (Eh^2(Y_1,Y_2) + Eh^2(Y_1,Y_n) + Eh^2(Y_{n-1},Y_n))$$

with some $D_5 > 0$, which immediately implies (69). Concerning (70) we obtain, by tedious but straightforward calculations, that for $k \le m$

$$A_{k3} = 2\frac{n-m}{n-k}\left(\frac{1}{k}\sum_{j=1}^{k}(G_1(Y_j)-G_n(Y_j)) - \frac{1}{n-k}\sum_{j=k+1}^{n}(G_1(Y_j)-G_n(Y_j))\right)$$
$$-2\left(\frac{1}{k^2}\sum_{j=1}^{k}G_1(Y_j) + \frac{1}{(n-k)^2}\left(\sum_{j=k+1}^{m}G_1(Y_j) + \sum_{j=m+1}^{n}G_n(Y_j)\right)\right) \tag{71}$$

and for $k > m$

$$A_{k3} = 2\frac{m}{k}\left(\frac{1}{k}\sum_{j=1}^{k}(G_1(Y_j)-G_n(Y_j)) - \frac{1}{n-k}\sum_{j=k+1}^{n}\big(G_1(Y_j)-G_n(Y_j)\big)\right)$$
$$-2\left(\frac{1}{k^2}\left(\sum_{j=1}^{m}G_1(Y_j) + \sum_{j=m+1}^{k}G_n(Y_j)\right) + \frac{1}{(n-k)^2}\sum_{j=k+1}^{n}G_n(Y_j)\right) \tag{72}$$

with

$$G_1(Y_j) = E(h(Z_1,Y_j)|Y_j) - E\,h(Z_1,Y_j),$$
$$G_n(Y_j) = E(h(Z_n,Y_j)|Y_j) - E\,h(Z_n,Y_j),$$

where Z_1 and Z_n have distribution functions F_1 and F_n, respectively, and are independent of $Y_1, \dots, Y_n$. Standard methods give

$$\max_{1\le k<n} n\Big(\frac{k(n-k)}{n^2}\Big)^{1+\gamma}|A_{k3}| = O_P\Big(\sqrt{n}\Big(\operatorname{var}(G_1(Y_1)-G_n(Y_1)) + \operatorname{var}(G_1(Y_n)-G_n(Y_n))\Big)^{1/2}\Big) = O_P(\sqrt{n\Delta_{n,2}}).$$

This immediately implies (70). Theorem B is proved. □

Proof of Theorem C The proof is somewhat parallel to the proof of Theorem A but instead of independent random variables we have ranks so that we are loosing the advantage of working with i.i.d. random variables. However, the statistics that have to be treated can be viewed as rank statistics that, when properly standardized, are martingales and hence again the Hájek-Rényi inequality can be employed giving the desired results. In the final step the theorems of convergence of simple linear statistics to a Wiener processes are utilized. This holds conditionally, given the original observations $Y_1, \dots, Y_n$, and the results hold true if some limit properties of statistics in $Y_1, \dots, Y_n$ are satisfied.

At first we prove an auxiliary lemma on rank statistics.

Lemma *Let $R_1, \dots, R_n$ be the ranks corresponding to the sample from uniform distribution on $(0, 1)$ of size n. Let $\gamma \in [0, 1]$ and let $a_n(1), \dots, a_n(n)$ and $b_n(i, j)$, $i, j = 1, \dots, n$ be scores such that*

$$\sum_{i=1}^{n} a_n(i) = 0,$$

and

$$b_n(i,j) = b_n(j,i), \quad i, j = 1, \dots, n, \qquad \sum_{i=1, i\ne j}^{n} b_n(i,j) = 0, \quad j = 1, \dots, n.$$

Put

$$L_k = \sum_{i=1}^{k} a_n(R_i), \quad k = 1, \dots, n$$

and

$$Q_k = \sum_{1\le i<j\le k}^{n} b_n(R_i, R_j), \quad Q_k^0 = \sum_{k<i<j\le n} b_n(R_i, R_j), \qquad k = 2, \dots, n.$$

Then for any $A > 0$ there is a positive constant D_A such that

$$P\Big(\max_{1\le k<n}\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma}\frac{k(n-k)}{n}\Big(\frac{1}{k^2}|L_k| + \frac{1}{(n-k)^2}|L_n - L_k)|\Big) \ge A\Big) \le D_A A^{-2} n^{\min(2\gamma,1)}\Big(\frac{1}{n}\sum_{i=1}^{n} a_n^2(i)\Big)\Big(1 + \mathbf{I}\{\gamma = 1/2\}\log n\Big) \tag{73}$$

and

$$P\Big(\max_{2\le k<n}\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma}\Big(\frac{1}{k}|Q_k|+\frac{1}{n-k}|Q_k^0|\Big)\ge A\Big)$$
$$\le D_A A^{-2}\Big(\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n} b_n^2(i,j)\Big)\Big(1+\mathbf{I}\{\gamma=0\}\log n\Big). \qquad (74)$$

Proof of Lemma It can be easily checked that $\{\frac{1}{n-k}L_k, \sigma(R_1,\dots,R_k),\ k=1,\dots,n-1\}$, $\{\frac{1}{k}(L_n-L_k), \sigma(R_{k+1},\dots,R_n),\ k=1,\dots,n-1\}$, $\{\frac{1}{(n-k)(n-k+1)}L_k, \sigma(R_1,\dots,R_k),\ k=1,\dots,n-1\}$ and $\{\frac{1}{k(k-1)}Q_k^0, \sigma(R_{k+1},\dots,R_n),\ k=2,\dots,n-1\}$ are martingales. Applying the Hájek-Rényi inequality we receive similarly as in the proof of Theorem A the assertion of our lemma. □

Continuation of the proof of Theorem C Applying Lemma with

$$a_n(i)=\frac{1}{n-1}\sum_{j=1,j\neq i}^{n}(h(Y_i,Y_j)-\overline{h}_n(\boldsymbol{Y})),\quad i=1,\dots,n,$$

we obtain that after some standard steps that for any $A>0$ there is a $D_{A1}>0$

$$P\Big(\max_{1\le k<n}\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma}\frac{k(n-k)}{n}|A_{k3}(\boldsymbol{R})|\ge A\Big)$$
$$\le D_{A1}A^{-2}n^{-\min(2\gamma,1)}\frac{1}{n}\sum_{i=1}^{n}\Big(\frac{1}{n-1}\sum_{j=1,j\neq i}^{n}(h(Y_i,Y_j)-\overline{h}_n(\boldsymbol{Y}))\Big)^2. \quad (75)$$

Applying classical theorems on the weak law of large numbers we find that, as $n\to\infty$,

$$\frac{1}{n}\sum_{i=1}^{n}\Big(\frac{1}{n-1}\sum_{j=1,j\neq i}^{n}(h(Y_i,Y_j)-\overline{h}_n(\boldsymbol{Y}))\Big)^2\to^P 0. \qquad (76)$$

Therefore the terms $A_{k3}(\boldsymbol{R})$, $k=1,\dots,n$, do not influence the conditional limit distribution on $T_{n,\gamma}(w,\boldsymbol{R})$.

Next we treat $A_{k1}(\boldsymbol{R})$, $k=1,\dots,n$, defined by (50). We will stress the dependence of $A_{k1}(\boldsymbol{R})$ on the function h writing $A_{k1}(h;\boldsymbol{R})$. Applying our Lemma with $b_n(i,j)=\widehat{h}(Y_i,Y_j)$, $i,j=1,\dots,n$, we receive that for any A there exists $D_{A2}>0$ such that

$$P\Big(\max_{1\le k<n}\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma}\frac{k(n-k)}{n}|A_{k3}(h,\boldsymbol{R})|\ge A\Big)$$
$$\le D_{A2}A^{-2}\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\widehat{h}^2(Y_i,Y_j) \qquad (77)$$

Noticing that

$$\widehat{h}(Y_{R_i}, Y_{R_j}) - \widehat{\widetilde{h}}_\kappa(Y_{R_i}, Y_{R_j}) = \frac{1}{n-1}\Big\{ \int \Big(h(Y_j, y) - \frac{1}{n}\sum_{v=1}^{n} h(Y_v, y)\Big)\, \mathrm{d}F_\kappa(y)$$
$$+ \int \Big(h(x, Y_i) - \frac{1}{n}\sum_{v=1}^{n} h(x, Y_v)\Big)\mathrm{d}F_\kappa(x)\Big\}, \quad i \neq j \tag{78}$$

we receive by (77) that for any A there exists $D_{A3} > 0$ such that

$$P\Big(\max_{1\le k<n}\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma}\frac{k(n-k)}{n}|A_{k3}(h - \widetilde{h}_\kappa, \boldsymbol{R})| \ge A\Big)$$
$$\le D_{A3}A^{-2}\frac{1}{n^3}\sum_{i=1}^{n}\Big(\int h(Y_i, y)\,\mathrm{d}F_\kappa(y)\Big)^2 \tag{79}$$

Letting

$$\widetilde{h}_{\kappa,K}(x, y) = \sum_{s=1}^{K}\lambda_s(\kappa)g_s(x;\kappa)g_s(y;\kappa) \tag{80}$$

we get again by (77) that for any A there exists $D_{A4} > 0$ such that

$$P\Big(\max_{1\le k<n}\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma}\frac{k(n-k)}{n}|A_{k3}(\widetilde{h}_\kappa - \widetilde{h}_{\kappa,K}, \boldsymbol{R})| \ge A\Big)$$
$$\le D_{A4}A^{-2}\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}\Big(\widetilde{h}_\kappa(Y_i, Y_j) - \widetilde{h}_{\kappa,K}(Y_i, Y_j)\Big)^2 \tag{81}$$

where D_{A4} does not depend on K, $K = 1, \ldots$. By standard tools we get that for any natural K conditional limit distribution (given $\boldsymbol{Y}$) of

$$\max_{1\le k<n}\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma}\frac{k(n-k)}{n}A_{k3}(\widetilde{h}_{\kappa,K}, \boldsymbol{R}) \tag{82}$$

is the same as that of

$$\max_{1\le k<n}\Big(\frac{k(n-k)}{n^2}\Big)^{\gamma}\sum_{s=1}^{K}\lambda_s(\kappa)\Big\{\frac{n}{k(n-k)}\Big(\sum_{j=1}^{k}(g_s(Y_{R_j};\kappa) - \frac{1}{n}\sum_{v=1}^{n} g_s(Y_v;\kappa))\Big)^2$$
$$-\frac{1}{n}\Big(\frac{n-k}{k}\sum_{j=1}^{k} g_s^2(Y_{R_j};\kappa) + \frac{k}{n-k}\sum_{j=k+1}^{n} g_s^2(Y_{R_j};\kappa)\Big)\Big\} \tag{83}$$

In comparison to the proof of Theorem A we have here, given $\boldsymbol{Y}$, processes based on rank statistics. By standard results on processes related to simple linear rank statistics we infer that if for some sequence of $y_1, y_2, \ldots$, as $n \to \infty$,

$$\frac{\max_{1\le i\le n} g_s^2(y_i,\kappa)}{\sum_{j=1}^{n} g_s^2(y_j,\kappa)} \to 0, \quad s = 1, \ldots, K \tag{84}$$

the processes

$$\left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \boldsymbol{g}_K(y_{R_i}, \kappa),\ t \in (0,1) \right\} \to^{D(0,1)} \{(B_1(t), \ldots, B_K(t)), \quad t \in (0,1)\}$$

where $\{B_j(t),\ t \in (0,1)\},\ j = 1, \ldots, K$ are independent Brownian bridges.

The rest of the proof is quite analogous to that of Theorem A. Going once more through the proof of the present theorem and recalling that we treated everything conditionally given $\boldsymbol{Y}$ we may conclude the assertion of Theorem C holds true. □

Acknowledgement The authors wish to express their sincere thanks to the referees for carefully reading the paper and for a number of valuable comments.

References

Antoch J, Hušková M (2001) Permutation tests for change point analysis. Stat Probab Lett 53:37–46

Antoch J, Hušková M, Jarušková D (2001) Off-line quality control. In: Lauro NC et al. (eds) Multivariate total quality control: foundation and recent advances. Springer, Berlin Heidelberg New York, pp 1–86

Chow YS, Teicher H (1988) Probability theory. Springer, Berlin Heidelberg, New York

Csörgő M, Horváth L (1997) Limit theorems in change-point analysis. J. Wiley, New York

Csörgő S (1985a) Testing for independence by the empirical characteristic function. J Multiv Anal 16:290–299

Csörgő S (1985b) Testing for linearity. Stat Probab Lett 3:45–49

de la Peňa VH, Giné E (1999) Decoupling. From dependence to independence. Springer, Berlin Heidelberg New York

Epps T, Pulley L (1983) A test for normality based on the empirical characteristic function. Biometrika 70:723–726

Epps T (1999) Limit behaviour of the ICF test for normality under Gram–Charlier alternatives. Stat Probab Lett 42:175–184

Heathcote CR (1972) A test for goodness of fit for symmetric random variables. Aust J Stat 14:172–181

Hušková M (2003) Permutation principle and bootstrap in change point analysis. In: Horváth L, Szyszkowicz B (eds) Asymptotic methods in stochastics, vol 44. Fields Institute Communications, pp 273– 291

Gürtler N, Henze N (2000) Goodness–of–fit tests for the Cauchy distribution based on the empirical characteristic function. Ann Inst Stat Math 52:267–286

Kankainen A, Ushakov NG (1998) A consistent modification of a test for independence based on the empirical characteristic function. J Math Sci 89:1486–1494

Koutrouvelis IA (1980a) Regression–type estimation of the parameters of stable laws. J Amer Stat Assoc 75:918–928

Koutrouvelis IA (1980b) A goodness of fit test for simple hypotheses based on the empirical characteristic function. Biometrika 67:238–240

Koutrouvelis IA, Meintanis S (1999) Testing for stability based on the empirical characteristic function with applications to financial data. J Stat Comput Simul 64:275–300

Lehmann EL (1991) Testing statistical hypotheses. Wadsworth & Brooks/Cole, California

Meintanis S (2005) Permutation tests for homogeneity based on the empirical characteristic function. J Nonparametric Stat 17: 583–592

Press SJ (1972) Estimation in univariate and multivariate stable distributions. J Amer Stat Assoc 67:842–846

Serfling R (1980) Approximation theorems of mathematical statistics. Wiley, New York

Ushakov NG (1999) Selected topics in characteristic functions. VSP, Utrecht