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# Network formation and anti-coordination games*

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**Abstract.** We study a setting in which individual players choose their partners as well as a mode of behavior in $2 \times 2$ anti-coordination games – games where a player's best response is to choose an action unlike that of her partner. We characterize the equilibrium networks as well as study the effects of network structure on individual behavior. Our analysis shows that both network architecture and induced behavior crucially depend on the value of the cost of forming links. In general, equilibrium configurations are found to be neither unique nor efficient.

**Key words:** networks, links, anti-coordination games, efficiency.

## 1. Introduction

In the past few years, there has been an extensive literature on social networks which shows that the structure of interaction between individuals can be decisive in determining the nature of the outcomes. In much of this literature, the structure of interaction is exogenously specified and the nature of the outcome under different specifications is examined (see e.g. Anderlini and Ianni (1996), Ellison (1993), Morris (2000)).

Recently, interest has grown in understanding the process through which the interaction structure itself develops. The earlier part of this literature (e.g. Aumann and Myerson (1989), Jackson and Wolinsky (1996), Dutta, van den Nouweland and Tijs (1998), Bala and Goyal (2000), among others) has

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focused on contexts where players choose links with others and there is no additional strategic dimension (i.e. there is no explicit game being played among connected players). Later contributions, such as Goyal and Vega-Redondo (2000) and Jackson and Watts (2000) have studied settings in which each agent plays a game with each of her 'partners' and therefore (in addition to connecting decisions) has to choose a mode of behavior in that game. This research has focused on a class of games where individuals have an incentive to choose the same action as their partners; these games are referred to as coordination games.

In the present paper, we wish to consider the role of network formation in the opposite case, where individuals prefer to choose an action unlike that chosen by their partners. We shall refer to these interactions as games of anti-coordination[1] Many interesting situations can be conceived in this fashion, e.g. when the successful completion of a task requires that the individuals involved adopt complementary actions (or skills), or when a meaningful interaction can only be conducted when the agents adopt different roles (say, buyers and sellers), or when in the contest for a certain resource, an optimal response is not to choose the same behavior (aggressive or peaceful, as in the Hawk-Dove game) as one's opponent.

We consider a model where each individual can form pair-wise links on her own initiative, i.e. link formation is one-sided. In addition, each player also chooses which of two actions to play in the interaction with her partners. Each bilateral interaction provides some gross return to the players involved, depending on the actions chosen. On the other hand, links are costly, with the player initiating each link paying for it. Thus, overall, the total net payoff earned by a player consists of the sum of the gross returns obtained from each of the pair-wise interactions in which she involves minus the cost of the links she initiates. For simplicity, we assume that the gross return accruing from each link is non-negative, so that no link initiated by an agent is ever refused by her partner.

We first characterize the strict Nash equilibria of the static game (Propositions 1 and 2). We find that, as the cost of link formation increases, the equilibrium network becomes more sparse. For a low cost it is complete, for a high cost empty, while for a moderate cost it is a bipartite graph (i.e. a network "split" in two disjoint sets of nodes with all links going across these sets). The cost of link formation also has a profound impact on the number of players who choose the two actions. In particular, for a low cost, the number of players choosing the two actions roughly corresponds to the proportions that would arise in the mixed strategy equilibrium of the two-person anti-coordination game, while for a moderate cost a wide range of proportions can be sustained in equilibrium. The intuition for this latter multiplicity is as follows: consider the class of games with symmetric payoffs and suppose a player wishes only to form a link with a player who is performing the other action. In this setting, a player has an incentive to be on the 'short-side', i.e. in the group that chooses the less popular action, since in this way she plays the largest number of games. However, a player has to balance these considerations with the fact that costly links have to be formed in order to play the

[1] Bramoullé (2001) analyzes anti-coordination games played on a *fixed* structure. He shows that the structure has a much stronger impact on the equilibria than in the case of coordination games.

game. This argument also suggests that as the cost of forming links increases, the distribution of links can have a bigger influence on the incentives to switch actions. Thus for a larger cost, a player may be induced to choose an action that is relatively popular, because the players choosing the other action are supporting all the links with her in equilibrium.

We then study the efficiency of different network structures (Propositions 3 and 4). In general, the architecture of efficient networks becomes less dense as the cost of link formation increases. For a low cost, typically, the efficient network is complete, while for a moderate cost the efficient network is bipartite. The cost of forming links also has an impact on the proportions of players choosing different actions. For instance, when the links are only worthwhile between players choosing different actions, efficient profiles have roughly equal proportions of players choosing the two actions. A comparison of efficient and equilibrium networks thus suggests that equilibrium and efficient networks are very different.

The above results are in contrast to the findings on coordination games reported in Droste, Gilles and Johnson (2000), Goyal and Vega-Redondo (2000) and Jackson and Watts (2000). Droste, Gilles and Johnson (2000) consider spatially located players whose linking cost depends on relative distance. This induces an interplay between the (endogenous) social network and the (exogenous) spatial structure that is absent from our model and the other two papers just mentioned. These latter papers find that, for all interesting values of the cost, the complete network is the unique non-empty stochastically stable network. By contrast, here we conclude that, in anti-coordination games, the selected network architectures are generally incomplete and their qualitative structure depends in interesting ways on the underlying payoffs and the linking cost. They also find that there is a certain threshold for the linking cost below which risk dominance is selected, while in the present paper the relationship is much richer and, in some cases exactly the reverse: efficient outcomes are only guaranteed for a low linking cost.

The rest of the paper is organized as follows: In Section 2 we set up the model. In Section 3 we discuss the Nash equilibrium results. The welfare analysis is also taken up in Section 3. Section 4 briefly discusses some extensions while Section 5 summarizes the results and concludes.

## 2. The Model

### *2.1 Link formation*

Let $N = \{1, 2, \ldots, n\}$ be a set of players where, for simplicity, $n(\geq 2)$ is assumed even. We are interested in modeling a situation where each of these players can choose the subset of other players with whom to play a fixed bilateral game. Formally, let $g_i = (g_{i1}, \ldots g_{i,i-1}, g_{i,i+1}, \ldots g_{in})$ be the set of links formed by player $i$. We choose $g_{ij} \in \{1, 0\}$, and say that player $i$ forms a link with player $j$ if $g_{ij} = 1$. The set of link options is denoted by $\mathscr{G}_i$. Any player profile of link decisions $g = (g_1, g_2 \ldots g_n)$ defines a directed graph, called a *network*.

Specifically, the network $g$ has the set of players $N$ as its set of *vertices* and its set of arrows, $\Gamma \subset N \times N$, defined as follows, $\Gamma = \{(i, j) \in N \times N : g_{ij} = 1\}$. Graphically, the link $(i, j)$ may be represented as an edge between $i$ and $j$, a filled circle lying on the edge near agent $i$ indicating that this agent has formed (or supports) that link. Every link profile $g \in \mathcal{G}$ has a unique representation in this manner. Figure 1 below depicts an example: player 1 has formed links with players 2 and 3, player 3 has formed a link with player 1, while player 2 has formed no links.[2]

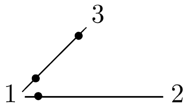


**Fig. 1.**

Given a network $g$, we say that a pair of players $i$ and $j$ are *linked* if at least one of them has established a link with the other one, i.e. if $\max\{g_{ij}, g_{ji}\} = 1$. To describe the pattern of players' links, it is useful to define a modified version of $g$, denoted by $\bar{g}$, that is defined as follows: $\bar{g}_{ij} = \max\{g_{ij}, g_{ji}\}$ for each $i$ and $j$ in $N$. Note that $\bar{g}_{ij} = \bar{g}_{ji}$ so that the index order is irrelevant.

A network $g$ is said to be *bipartite* if there exists a partition of the players into two mutually exclusive and exhaustive sets, $N_1$ and $N_2$, such that $\bar{g}_{ij} = 1$ only if $i \in N_1$ and $j \in N_2$. A bipartite network is *complete* if $\bar{g}_{ij} = 1$ for every pair of players $i \in N_1$ and $j \in N_2$.

We denote by $N(i; g) \equiv \{j \in N : g_{ij} = 1\}$ the set of players in network $g$ with whom player $i$ has established links, while $\nu(i; g) \equiv |N(i; g)|$ stands for its cardinality. Similarly, we let $N(i; \bar{g}) \equiv \{j \in N : \bar{g}_{i,j} = 1\}$ be the set of players in network $g$ with whom player $i$ is linked.

### *2.2 Social game*

Individuals located in a social network play a $2 \times 2$ symmetric game in strategic form with a common action set. The set of partners of player $i$ depends on her location in the network. We assume that two individuals can play a game if and only if they have a link between them. Thus, player $i$ plays a game with all other players in the set $N(i; \bar{g})$.

We now describe the two-person game that is played between any two partners. The set of actions is $A = \{\alpha, \beta\}$. For each pair of actions $a, a' \in A$,

| $i$ \ $j$ | $\alpha$ | $\beta$ |
|---|---|---|
| $\alpha$ | $d$ | $e$ |
| $\beta$ | $f$ | $b$ |

Table I

[2]Since agents choose strategies independently of each other, two agents may simultaneously initiate a link, as seen in Figure 1.

the payoff $\pi(a, a')$ earned by player $i$ choosing $a$ when the partner $j$ plays $a'$ is given by the following table:

We assume that it is one of anti-coordination with two pure strategy equilibria, $(\alpha, \beta)$ and $(\beta, \alpha)$. In other words we consider the following restrictions on the payoffs:

$$d < f \text{ and } b < e. \tag{1}$$

Players choose links and actions in the anti-coordination game simultaneously.[3] We assume that every player $i$ is obliged to choose the same action in the (generally) several bilateral games that she is engaged in. This assumption is natural in the present context; if players were allowed to choose a different action for every two-person game this would make the behavior of players in any particular game insensitive to the network structure.[4] Therefore the strategy space of a player can be identified with $S_i = \mathcal{G}_i \times A$, where $\mathcal{G}_i$ is the set of possible link decisions by $i$ and $A$ is the common action space of the underlying bilateral game.

Now we define the payoffs of the social game. These reflect the following two important features of the link formation mechanism. First, links are *costly*. Specifically, every agent who establishes a link with some other player incurs a fixed cost $c > 0$. (Thus, the cost of forming a link is independent of the number of links being established and is the same across all players.) The second important feature of the model is that links are *one-sided*. That is, an individual can form a link with another player on her own initiative, and no consent of the other player is required. This aspect of the model allows us to use standard solution concepts from non-cooperative game theory in addressing the mechanism of link formation. It raises, however, the issue of whether a proposal to form a link might not be accepted by the player who receives it (even though she would bear no linking cost). In the present paper, we abstract from these complications by simply positing that the payoffs of the bilateral game are non-negative and, therefore, no player has any incentives to refuse forming a proposed link.

In view of the former considerations, the payoff to a player $i$ from playing some strategy $s_i = (g_i, a_i)$ when the strategies of other players are $s_{-i} = (s_1, \dots s_{i-1}, s_{i+1}, \dots s_n)$ can be written as follows:

$$\Pi_i(s_i, s_{-i}) = \sum_{j \in N(i;\bar{g})} \pi(a_i, a_j) - v(i; g) \cdot c \tag{2}$$

We note that the individual payoffs are aggregated across the games played by him. In our framework, the number of games an individual plays is endogenous, and we want to explicitly account for the influence of the size of the neighborhood. This motivates the aggregate payoff formulation. As indicated, the above payoff expression allows us to particularize the standard

[3]An alternative would be to think of actions and link decisions as sequential. We have analyzed games with such a sequential order of moves, the results being briefly summarized in Section 4.

[4]Thus, our setup would be best suited to model those cases where action versability is too costly to be worthwile (e.g. the choice of a profession). A more general formulation would allow individuals to choose different actions with different partners. For a study of the role of costs of flexibility in coordination games with local interaction, see Goyal and Janssen (1997).

notion of Nash equilibrium as follows. A strategy profile $s^* = (s_1^*, \ldots s_n^*)$ is said to be a *Nash equilibrium* for the game if, for all $i \in N$,

$$\Pi_i(s_i^*, s_{-i}^*) \geq \Pi_i(s_i, s_{-i}^*), \forall s_i \in S_i. \tag{3}$$

A Nash equilibrium is said to be *strict* if every player gets a strictly higher payoff with his current strategy than she would with any other strategy.

## 3. Analysis

This section contains our results on strict Nash equilibria and socially efficient strategy profiles. We start by characterizing the set of strict Nash equilibria of the social game. First, we describe the types of Nash networks and how they depend on the anti-coordination game and on the cost of link formation. Second, we characterize the range of possible values for the number of agents playing each action ($\alpha$ or $\beta$) in equilibrium.

Anti-coordination games have different possible payoffs configurations and we see that they also lead to different types of Nash networks. Without loss of generality, assume that $f > e$, i.e. when two players anti-coordinate, $\beta$-players (i.e. players who choose action $\beta$) earn a higher payoff than $\alpha$-players (i.e. those who choose action $\alpha$). If all the parameters are distinct (the non degenerate cases), there are three possible orderings of the parameters:

Case 1: $b < e < d < f$

Case 2: $b < d < e < f$

Case 3: $d < b < e < f$

Each ordering corresponds to a different type of anti-coordination game. In Case 1, the payoff of coordinating on $\alpha$ is higher than the payoff of an $\alpha$-player in (anti-coordination) equilibrium. Therefore, Case 1 represents exploitation games akin to the Hawk-Dove game. In Cases 2 and 3, the equilibrium payoffs of the anti-coordination bilateral game are higher than any other payoffs. Cases 2 and 3 represent situations of complementarity, in which both players earn higher payoffs in equilibrium than out of it. In Case 2 the payoff of coordinating on $\alpha$ is higher than the payoff of coordinating on $\beta$,

| $i$ \ $j$ | $\alpha$ | $\beta$ |
|---|---|---|
| $\alpha$ | 3 | 2 |
| $\beta$ | 4 | 1 |

Case 1

| $i$ \ $j$ | $\alpha$ | $\beta$ |
|---|---|---|
| $\alpha$ | 2 | 3 |
| $\beta$ | 4 | 1 |

Case 2

| $i$ \ $j$ | $\alpha$ | $\beta$ |
|---|---|---|
| $\alpha$ | 1 | 3 |
| $\beta$ | 4 | 2 |

Case 3

Payoff tables illustrating the three cases considered

while the situation is reversed in Case 3. The following three tables illustrate payoffs configurations corresponding to each of the three cases.

Since link formation is one-sided, the cost of any link at equilibrium is supported by only one agent and Nash networks involve no redundant links. The pattern and number of links, on the other hand, depend on how the cost of link formation compares with the parameters of the game. For example, when $c > b$, $\beta$-players do not have an incentive to form links with other $\beta$-players and so, there is no link among $\beta$-players in equilibrium. Instead, when $c < b$, the $\beta$-players are willing to form links and to support the cost of link formation with any other agent playing $\beta$. In equilibrium, therefore, all the $\beta$-players are linked with all the other $\beta$-players and the network of links among $\beta$-players is complete.[5]

The following shorthand notation allows us to refer to all the possible types of Nash networks.

- $\beta \emptyset \alpha$: the empty network
- $\beta \rightarrow \alpha$: all $\beta$-players are linked to all $\alpha$-players, but no $\alpha$-player is linked to a $\beta$-player
- $\beta \rightleftharpoons \alpha$: all $\beta$-players are linked with all $\alpha$-players
- $\beta \rightarrow \vec{\alpha}$: all $\beta$-players are linked to all $\alpha$-players, all $\alpha$-players are linked with all $\alpha$-players but no $\alpha$-player is linked to a $\beta$-player
- $\beta \rightleftharpoons \vec{\alpha}$: all $\alpha$-players are linked with all $\alpha$-players and with all $\beta$-players
- $\vec{\beta} \rightleftharpoons \vec{\alpha}$: the complete network

Hence, $\beta \rightarrow \alpha$ and $\beta \rightleftharpoons \alpha$ represent complete and bipartite networks, while $\beta \rightarrow \vec{\alpha}$ and $\beta \rightleftharpoons \vec{\alpha}$ are what we call (complete) *semi-bipartite networks*, i.e. networks that can be partitioned into two exclusive and comprehensive parts with internal links (connecting nodes of the same part) only existing within one of the two parts. Using the above notation, the following result describes how the cost of link formation determines the type of Nash networks.

**Proposition 1.** *Suppose (1) holds. Then Nash equilibria exhibit the following pattern of link formation:*

| *Case* 1 | | *Case* 2 | | *Case* 3 | |
|---|---|---|---|---|---|
| $0 < c < b$ | $\vec{\beta} \rightleftharpoons \vec{\alpha}$ | $0 < c < b$ | $\vec{\beta} \rightleftharpoons \vec{\alpha}$ | $0 < c < d$ | $\vec{\beta} \rightleftharpoons \vec{\alpha}$ |
| $b < c < e$ | $\beta \rightleftharpoons \vec{\alpha}$ | $b < c < d$ | $\beta \rightleftharpoons \vec{\alpha}$ | $d < c < b$ | $\vec{\beta} \rightleftharpoons \alpha$ |
| $e < c < d$ | $\beta \rightarrow \vec{\alpha}$ | $d < c < e$ | $\beta \rightleftharpoons \alpha$ | $b < c < e$ | $\beta \rightleftharpoons \alpha$ |
| $d < c < f$ | $\beta \rightarrow \alpha$ | $e < c < f$ | $\beta \rightarrow \alpha$ | $e < c < f$ | $\beta \rightarrow \alpha$ |
| $f < c$ | $\beta \emptyset \alpha$ | $f < c$ | $\beta \emptyset \alpha$ | $f < c$ | $\beta \emptyset \alpha$ |

The proof of this Proposition is straightforward and omitted. However, a number of interesting points follow from the above statements. *Firstly*, they show that (except for a very low cost) the nature of links is quite complicated, with the link initiation (and hence the network architecture) depending very much on the game that is being played. For instance, if the game is one of exploitation (Case 1) and $e < c < d$, its Nash networks are of the form $\beta \rightarrow \vec{\alpha}$. The reason is that $\alpha$-players are then willing to support the cost of link formation with themselves but not with $\beta$-players, while $\beta$-players are willing to support the cost of link formation with $\alpha$-players but not with themselves. On the other hand, if the game is one of strict complementarity (Cases 2 and 3) and the linking cost is between the coordi-

[5]Of course, it is our assumption that payoffs depend linearly on the number of social neighbors playing a strategy that causes this 'all or nothing' result.

nation and anti-coordination payoffs, Nash outcomes induce bipartite networks of the form $\beta \rightleftharpoons \alpha$. In this case, both $\alpha$ and $\beta$-players have an interest to be linked to players choosing the other action, while they do not wish to be linked with players choosing the same action. *Secondly*, the above proposition also shows that, in accord with intuition, increasing the linking cost has a negative effect on network density. That is, as the cost of link formation rises, the possible types of Nash networks become more sparse, going from the complete network to the empty network through three intermediary cases.

We now analyze how the number of players choosing each of the different actions in equilibrium depends on the linking cost $c$. Given a strategy profile $s$, denote by $n_\beta$ the number of $\beta$-players in $s$ and $n_\alpha = n - n_\beta$ the number of $\alpha$-players in $s$. Our next result derives the lower and upper bounds for $n_\beta$ (hence for $n_\alpha$) in equilibrium. These bounds are obtained by examining the best-responses for every possible case.[6] Define $p_\beta = \frac{f-d}{f-d+e-b}$. Notice that $p_\beta$ is the probability of playing $\beta$ in the mixed strategy equilibrium of the anti-coordination game. It is useful to introduce two auxiliary functions $\varphi$ and $\psi$ as follows:

$$\varphi(c) = \begin{cases} p_\beta & \text{if } c < b\frac{f-d}{f-d+e-c} \\ & \text{if } b < c < e \\ 1 & \text{if } e < c \end{cases} \tag{4}$$

and

$$\psi(c) = \begin{cases} p_\beta & \text{if } c < d \\ \frac{f-c}{f-c+e-b} & \text{if } d < c < f \\ 0 & \text{if } f < c \end{cases} \tag{5}$$

Note that $\varphi$ and $\psi$ are continuous, $\varphi$ is increasing and $\psi$ is decreasing. These functions bound the relative sizes of the different $\alpha$- and $\beta$-parts of the network, as established by the following result.

**Proposition 2.** *Suppose (1) holds. Then there exists a strict Nash equilibrium with $n_\beta$ players choosing $\beta$ iff $(n-1)\psi(c) < n_\beta < (n-1)\varphi(c) + 1$. If the inequalities hold weakly, the characterization applies to all Nash (possibly non-strict) equilibria.*

The proof of this result is given in the appendix. This result and Figure 2 illustrate the precise relationship between the linking cost and the range of equilibrium behavior in the respective game. In particular, it states that for a low cost of forming links, the proportion of players choosing actions $\alpha$ and $\beta$ corresponds (roughly) to the mixed-strategy of the two-person anti-coordination game. This simply follows from the fact that, for a low

[6] The best-response equations do not depend on the particular payoffs and cost configuration, but only on the type of Nash architecture to which this configuration leads, as established by Proposition 1. For example, situations where the payoffs correspond to Case 1 and $b < c < e$, and where the payoffs correspond to Case 2 and $b < c < d$ both support $\beta \rightleftharpoons \vec{\alpha}$ as Nash networks. Hence, both cases can be analyzed as one. This reduces the number of domains to analyze from 16 to 6.

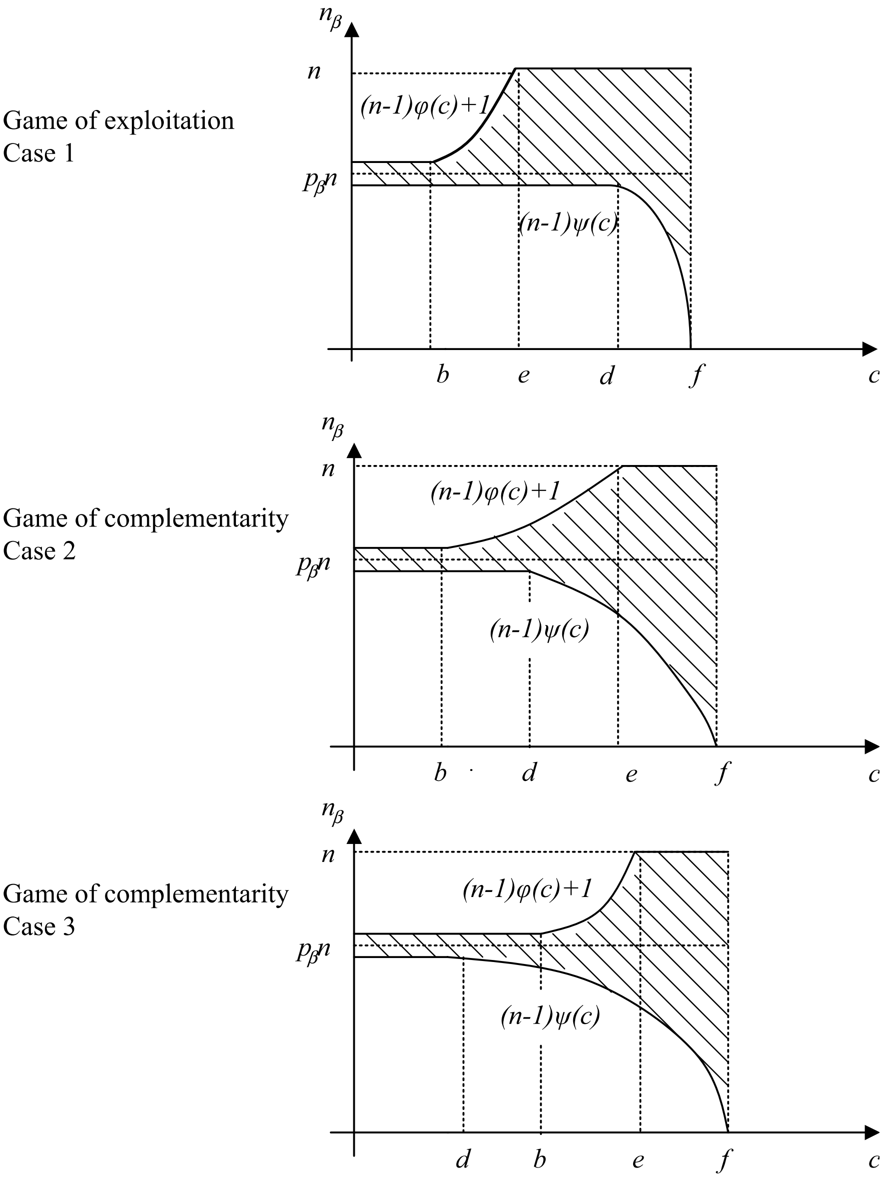


**Fig. 2.** Number of $\beta$-players in equilibrium

linking cost, players have incentives to form the complete network and hence the link formation mechanism has no particular influence on individual behavior. However, beyond this low range, the cost of link formation has a profound impact on individual choice of actions. In particular, a broader range of proportions of players choosing actions $\alpha$ and $\beta$ becomes possible.

The intuition behind the latter conclusion is best explained in the context of strict complementarity, where a player wishes to form a link only with a partner choosing a different action. In this setting, if both actions are symmetric, the player has an incentive to be on the 'short-side', i.e. in the group that chooses the less popular action. In adjusting her behavior, however, she has to take into account that the creation of any new link on her part involves a cost. This implies that, for a fixed configuration of actions, the incentives for any given player to keep doing what she currently does are maximized when she is the "passive recipient" of all links to the players who are choosing the other action. This argument allows us to compute the bounds on the maximum and minimum number of players who can be playing each action at equilibrium. It also suggests that, as the cost of forming links increases, its distribution can have a bigger influence on the incentives to switching actions. In particular, for a large cost level, a player may be induced to choose an action that is relatively popular, because the players choosing the other action are supporting all the links with her.

Propositions 1 and 2 characterize the Nash equilibria of the social game. However, they do not typically provide information on either the direction of the links or the payoff distribution among the agents at equilibrium. Take for example the case where actions are symmetric and the equilibrium network is of the kind $\beta \rightleftharpoons \alpha$. Then, the direction of links formed between $\alpha$ and $\beta$-players is not determined. Indeed, the above discussion precisely highlights the sort of trade-off that we observe at equilibrium, i.e. when agents of a certain type are more scarce than those of the other type, they must bear, on average, a greater share of the cost of link formation. In this way, the benefits of being on the short-side are balanced by the cost of supporting the links. This, of course, does not apply when the equilibrium network is balanced and bipartite (i.e. $\frac{n_\beta}{n} = \frac{1}{2}$), in which case *all* possible distributions of active and passive links among $\alpha$ and $\beta$-players are possible. This, in fact, highlights the additional important insight that payoff distribution among agents at equilibrium is *not* determined either. Balanced networks, for example, can sustain symmetric payoff distribution where all agents support the same number of active links as well as a very asymmetric payoff distribution where only agents of a certain type incur the cost of the links and thus have significantly lower payoffs than agents of the other type. Thus, even though it is precisely the interplay of network architecture and suitable distribution of the linking cost that supports much of our equilibrium multiplicity, this is far from determining in a precise fashion the payoff distribution among the different agents – i.e. sharp payoff asymmetries, both across and within types can prevail at equilibrium.

We now study welfare properties. There are many ways to measure the social welfare of a network structure. Here, we identify welfare with the sum of individuals' payoffs. More precisely, the welfare of a strategy profile $s = (s_1, \ldots, s_n)$, denoted as $W(s)$, is set equal to the sum of the individuals' payoffs,

$$W(s) = \sum_{i=1}^{n} \Pi_i(s).$$

Furthermore, we say that a state $s$ is *efficient* if and only if $W(s) \geq W(s')$, for all $s' \in S$.

First of all, notice that the welfare contribution of a link is $2b - c$ in the case of two $\beta$-players, $2d - c$ in the case of two $\alpha$-players, and $e + f - c$ in the case of an $\alpha$-player linked to a $\beta$-player.[7] This implies that the appropriate classification of payoffs configuration for welfare analysis is different from the classification used for equilibrium analysis. It is important to keep this in mind concerning the ensuing results on welfare. It is also worth noting that, given any particular pattern of connections, the division between passive and active links does not affect its welfare. Therefore, in order to characterize an efficient profile, it is enough to focus on the undirected counterpart of the network, and consider only the number of players choosing each action.

First, we consider the case where $2b < e + f < 2d$. Then, a link involving two players choosing $\alpha$ provides the highest payoff, which easily leads to the following result:

**Proposition 3.** *Suppose $2b < e + f < 2d$. Then if $c < 2d$ a strategy profile is efficient iff its induced network is complete, there are no redundant links, and all players choose $\alpha$. If $c > 2d$, then every efficient strategy profile yields an empty network.*

The proof is given in the appendix. The above class of games exhibit a severe form of "exploitation" in which the welfare of the anti-coordination game is highest off-equilibrium. The other two possible parameter configurations are given by the inequalities $2b < 2d < e + f$ and $2d < 2b < e + f$. They lead to a more complicated analysis, which is taken up next. Since these two latter cases are symmetric across actions, we merely focus here on the first case. To state the result, it is useful to introduce an auxiliary correspondence $g$ as follows:

$$g(c) = \begin{cases} \left\lceil \frac{e+f-2d+\frac{d-b}{n}}{e+f-(d+b)} \right\rfloor & \text{if } c < 2b \\ \left\lceil \frac{e+f-2d+\frac{d-c/2}{n}}{e+f-d-c/2} \right\rfloor & \text{if } 2b < c < 2d \\ 1 & \text{if } 2d < c, \end{cases}$$

where $\lceil a \rfloor$ refers to the (at most two) integers closest to $a$ (Note that, for almost all $c$, $\lceil a \rfloor$ is a singleton and therefore $g(c)$ is single-valued). It is straightforward to see that $g(c)$ is piece-wise linear and increasing.

**Proposition 4.** *Suppose (1) holds and in addition $2b < 2d < e + f$. Then the following statements hold: (i) If $c < 2b$ then a profile is efficient iff its induced network is of type $\vec{\beta} \rightleftharpoons \vec{\alpha}$, and $n^*_\beta = g(c)\frac{n}{2}$. (ii) If $2b < c < 2d$ a profile is efficient iff its induced network is of type $\beta \rightleftharpoons \vec{\alpha}$, and $n^*_\beta = g(c)\frac{n}{2}$. (iii) If $2d < c < e + f$ a profile is efficient iff its induced network is of type $\beta \rightleftharpoons \alpha$, and*

[7] Since the cost of a link is incurred only by one of the agents forming the link, the formation of it can be optimal in terms of welfare, yet not feasible at equilibrium. This occurs, for example, between two $\beta$ -players if $b < c < 2b$.

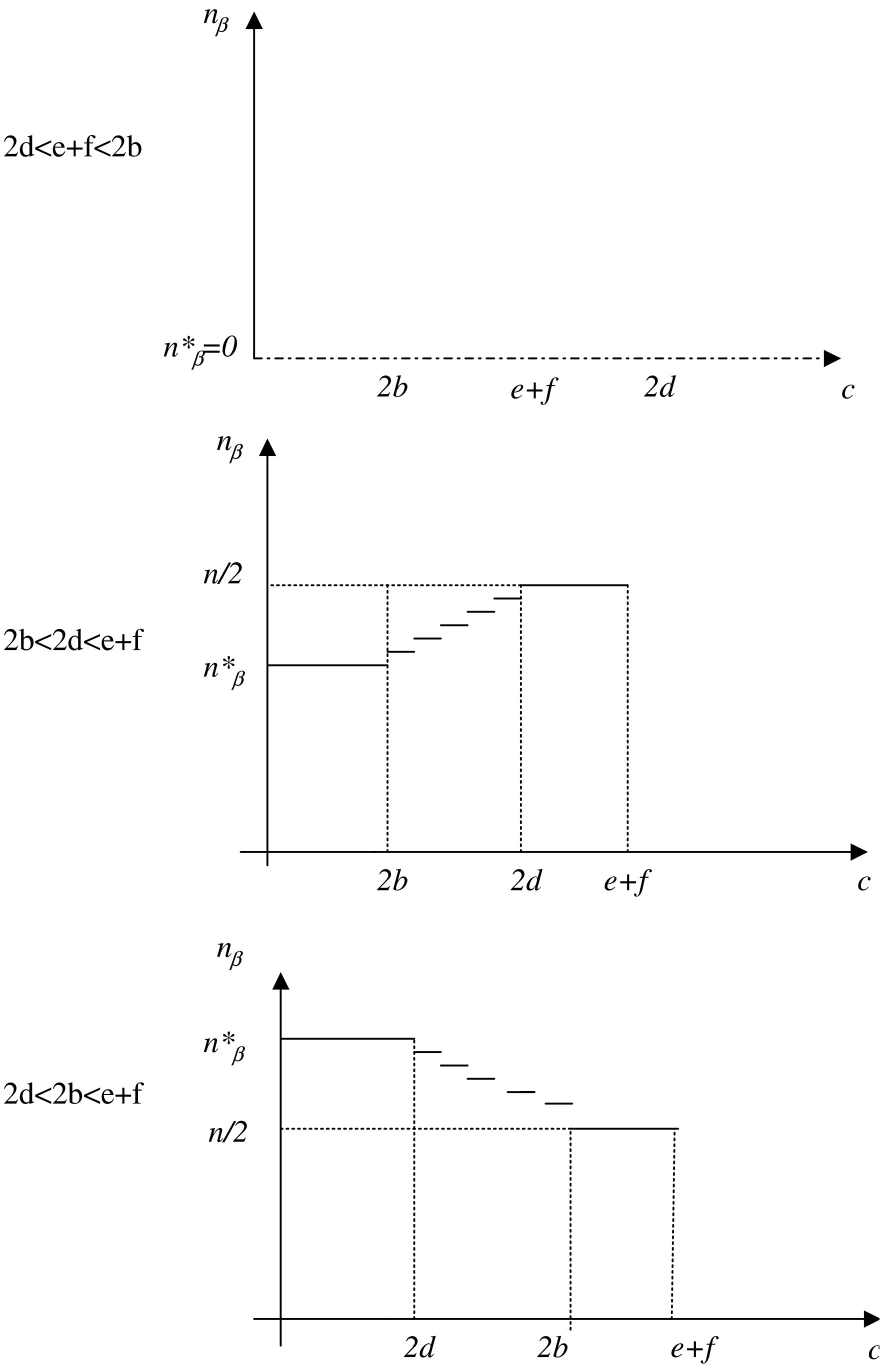


**Fig. 3.** Number of $\beta$-players in an efficient profile

$n_\beta^* = g(c)\frac{n}{2}$. *(iv) If* $e + f < c$ *a profile is efficient iff its induced network is empty.*

The proof is given in the appendix. Proposition 4 tells us that, as the linking cost increases, efficient networks become less connected going from the complete network to the empty network through two intermediary cases. Moreover, efficiency generally selects a unique relative size of the two parts, which become of equal size (i.e., $n_\beta = n_\alpha = \frac{n}{2}$) when the efficient network is bipartite. The reason for the latter conclusion is that, when the efficient network is bipartite, each link provides the same welfare contribution $e + f - c$. Therefore, in order to maximize welfare, the number of links must be maximized, which occurs when the two groups of players have the same size. Figure 3 illustrates the socially efficient number of players choosing different actions as a function of the cost of forming links.

If we compare Propositions 3 and 4 with Propositions 1 and 2 we conclude that, in general, Nash profiles are not efficient and vice versa. There are two related reasons for this negative conclusion.

1. First, let us consider the effect induced by the fact that the mechanism of link formation is one-sided. This implies that a link can be welfare improving even if no agent wants to form it – e.g. if $b < c < 2b$, $\beta$-players do not form a link among themselves, even though this would clearly increase collective welfare. This problem could be somewhat alleviated under alternative assumptions on link formation or if we allowed, say, for some agent bargaining that might lead to the sharing of the cost. Apart from this consideration, the cost of link formation also has implications over the distribution of passive and active links sustained in equilibrium. As the cost increases, the range of possible sizes in equilibrium extends. This is because when the cost is high the positive externalities induced by passive links are higher. Nevertheless, passive and active links have *no role* in welfare analysis. This is why, typically, there is just a *single* relative size of the two parts in efficient profiles.
2. Another reason for the discrepancy between efficiency and equilibrium is the fact that actions in the anti-coordination game are typically asymmetric. To distinguish this most clearly from previous considerations, it is useful to consider a situation where the cost of link formation is low. Thus assume that $c$ is close to 0. Then, both equilibrium and efficient networks are complete. Equilibrium requires that $\frac{n_\beta}{n_\alpha} \approx \frac{f-d}{e-b}$ in every case while, in contrast, efficiency requires that $\frac{n_\beta}{n_\alpha} = 0$ when $2b < e + f < 2d$ and $\frac{n_\beta}{n_\alpha} \approx \frac{e+f-2d}{e+f-2b}$ otherwise. In the first case, efficiency and equilibrium requirements can never be reconciled, while in the other cases, efficiency and equilibrium are compatible only when $f - e$ is close to 0.[8]

[8]One way out is to consider repeated relationships where the sharing of costs over time helps to smooth the asymmetries. This appears to be a natural way to tackle such problems in some contexts but leads to a very different framework than the one in this paper.

The above discussion leads us to the following question: what are the strategy profiles that, among all Nash equilibria, yield the highest welfare? We find it useful to distinguish between two cases here. The first case arises when individually rational links are the same as the collectively rational links. This happens, for instance, when $2b, 2d < c < e, f$. In this class of games, an efficient network is a complete bipartite network with some specific $n/2$ players choosing $\beta$. Then, it is easy to see that we can rank equilibria in terms of the number of players choosing $\beta$, and the equilibrium which has $n_\beta$ closer to $n/2$ has the highest welfare. The second case arises when efficient networks have a pattern of links that is qualitatively different from equilibrium networks. This happens for example when $b, d < c < e, f, 2b, 2d$, in which case an efficient network is complete while every equilibrium network is bipartite. At equilibrium, therefore, the gross welfare attained is simply proportional to the number of links between $\alpha$ and $\beta$-players (in the present case, they are the only existing links at equilibrium). Since the payoff per each of these links is constant, welfare at an equilibrium is maximized when their number is maximized as well, i.e. when $n_\beta$ is the closest possible to $n/2$. Social welfare, however, need not be maximized in this way – recall that efficient networks are complete in this case and therefore the efficient value of $n_\beta$ generally depends on the relative magnitudes of $b$ and $d$, the payoffs obtained by agents choosing the same action.

## 4. Extensions

We have studied a number of extensions and variations of the model. Here, we briefly outline the results obtained along two different lines. First, we discuss the implications of embedding the present static model in a dynamic learning context. Second, we consider the implications of assuming that the linking decisions and actions are not simultaneous but sequential. In this respect, we have addressed the two possibilities: actions are the initial decisions (then followed by links), or vice versa. The results on the dynamic model can be found in our working paper Bramoullé *et al.* (2002), while those on the model with sequential decisions are available upon request.

### *4.1 Dynamics*

As customary in the evolutionary-learning literature, let us posit that the social game is played repeatedly and, at each point in time, players adjust their decisions myopically (i.e. taking the current behavior of others as given). Specifically, suppose that each player is given an adjustment opportunity every period with a given independent probability $p \in (0, 1)$. Then, it can be shown that, starting from arbitrary initial conditions, the induced process converges almost surely to a strict Nash equilibrium of the social game, as characterized by Propositions 1 and 2. Since all strict Nash equilibria are stationary for the adjustment dynamics, we enrich the model by introducing some perturbation and then evaluate the different robustness of each of them. Specifically, we follow the common practice of supposing that, at each point in time, every individual is subject to a small independent probability $\varepsilon$ of "mutation", i.e. arbitrarily changing her strategy. Then, by studying the long-run dynamics of the induced (ergodic) stochastic process as $\varepsilon \to 0$, we find that *all* strict Nash equilibria of the social game are stochastically stable (i.e. they are played a significant fraction of time in the long run, independently of initial conditions). This confirms that the equilibrium multiplicity arising from our static analysis is indeed a robust phenomenon that remains in place even if "tested" against dynamic adjustment and behavioral perturbations.

### *4.2 Sequential decisions*

Consider first the case where the action decisions are taken first. Then, we find that the qualitative nature of the analysis is not affected, at least if the linking cost $c$ is lower than the anti-coordination payoffs (i.e. $c < e,f$). Specifically, there are still counterparts $\tilde{\varphi}$ and $\tilde{\psi}$ of the functions given in (4)–(5) that bound the size of the $\alpha$- and $\beta$ -parts of the networks prevailing at sub-game perfect equilibria of the social game. Interestingly, we find that $\tilde{\varphi}(\cdot) > \varphi(\cdot)$ and $\tilde{\psi}(\cdot) < \psi(\cdot)$, so that the range of possibilities is wider than in the simultaneous case. The intuition here is that when other players can adjust their link decisions in a second stage of the game the scope for profitable deviations decreases, which in turn entails a larger set of Nash equilibria.

Analogous increase in the range of equilibrium outcomes obtains if linking is instead the initial decision taken by players. In that case, if $c$ continues to be lower than the anti-coordination payoffs (but higher than $b$ and $d$), it can be shown that all Nash outcomes of the simultaneous game are attainable through subgame perfect equilibria of the sequential setup. When the linking cost is larger (e.g. $e < c < f$) such a widening of equilibrium outcomes tends to be reinforced. To illustrate this note that the empty network is always an equilibrium outcome in this latter case if, after any deviation by a single player, all other players (credibly) threaten to play $\beta$ against the deviator. In a sense, this reflects considerations analogous to those arising in received game-theoretic models of network formation (see e.g. Dutta *et al.* (1998)), where one must typically rely on coalition-based refinements of Nash equilibrium (Pairwise Stability, Strong equilibrium, etc.) to escape the perverse effects of possible mis-coordination in the agents' linking decisions.

## 5. Concluding remarks

We study a setting where individuals choose partners as well as actions in the games they play with partners. In this paper, the focus is on anti-coordination games, where players prefer to choose actions other than the action their partners choose. We assume that partnerships are formed via investments in links. As the cost of linking varies, we find that there is a wide range of (strict) Nash architectures: complete networks, semi-bipartite networks, bipartite networks and empty networks. The relative numbers of individuals taking each action depends crucially on the cost of forming links. More specifically we observe that, for a low cost, the only stable network is complete, with the proportion of individuals taking each action coinciding with the mixed strategy equilibrium proportions of the anti-coordination game. As the cost of link formation increases, a wider set of relative proportions become sustainable in equilibrium. This effect arises due to the trade-off faced by any player between the advantages of cheap passive links and the gains from being on the shorter side of the population. In addition, the payoffs in an anti-coordination game are such that players have an incentive to be on the short side of the 'market' even if aggregate welfare is enhanced when all players choose the same action. This strategic conflict is a second source of inefficiency. These two considerations imply that efficiency and equilibrium requirements typically conflict in anti-coordination games.

## 6. Appendix

*Proof of Proposition 2:* We proceed by successive examination of all the possible domains, focusing for concreteness on strict Nash equilibria. For each domain, the first step is to derive the two strict best-response equations, one for the $\alpha$-players, denoted by BR$\alpha$, and one for the $\beta$-players, denoted by BR$\beta$. The second step is to understand how the best response equations allow one to compute the lower and upper bounds on $n_\beta$. In general, BR$\beta$ leads to the upper bound, whereas BR$\alpha$ leads to the lower bound. The reason is intuitive: for anti-coordination games, the higher the number of people playing $\beta$, the lower the utility of playing $\beta$ compared to the utility of playing $\alpha$. Hence, when $\beta$ players are too numerous, BR$\beta$ does not hold. Given any strategy profile $s$, let $q_i^{s,k}$ denote the number of active links supported by some given player with individuals choosing some action $k$, where $k \in \{\alpha, \beta\}$. Depending on the action chosen by the player in question ($\alpha$ or $\beta$), the best-response condition applicable for each parameter configuration may be identified as follows. (We avoid the superscript $s$ in $q_i^{s,k}$ when there is no risk of ambiguity.)

**1**: $c < b, d, e, f$. Nash networks are complete.

$$BR\alpha \Leftrightarrow (n_\alpha - 1)d + n_\beta e - c(q_i^\alpha + q_i^\beta) > (n_\alpha - 1)f + n_\beta b - c(q_i^\alpha + q_i^\beta)$$

The left term of the inequality is the utility obtained by an agent playing $\alpha$. The right term is the utility that an agent playing $\alpha$ would obtain if he changed to $\beta$. Through elementary algebraic manipulations, we obtain

$$\begin{aligned} BR\alpha \quad &\Leftrightarrow \quad n_\beta(e - b) > (n_\alpha - 1)(f - d) \\ BR\alpha \quad &\Leftrightarrow \quad n_\beta(e - b + f - d) > (n - 1)(f - d) \end{aligned}$$

Similarly, we show that BR$\beta$ is as follows:

$$BR\beta \quad \Leftrightarrow \quad n_\beta(e - b + f - d) < (n - 1)(f - d) + (e - b + f - d).$$

**2**: $b < c < d, e, f$. Nash networks are of the type $\beta \rightleftharpoons \vec{\alpha}$. The $\alpha$-players are linked (actively or passively) with every other agent. Thus, they obtain $e$

with $\beta$ players and $d$ with all $\alpha$ players, while they have to pay for all the links they support. Hence, the utility of an $\alpha$ player is $n_\beta e + (n_\alpha - 1)d - c(q_i^\alpha + q_i^\beta)$. If she changed to $\beta$, she would sever her active links with $\beta$ players, but keep her active links with $\alpha$ players. She would still be linked (actively or passively) with all the $\alpha$ players, but would now only be passively linked with $\beta$ players. The number of passive links she has with $\beta$ players is equal to $n_\beta$ minus the number of active links she has with them. Therefore,

$$BR\alpha \Leftrightarrow n_\beta e + (n_\alpha - 1)d - c(q_i^\alpha + q_i^\beta) > (n_\alpha - 1)f + (n_\beta - q_i^\beta)b - cq_i^\alpha$$

which yields:

$$BR\alpha \Leftrightarrow n_\beta(e - b + f - d) > (n-1)(f-d) + q_i^\beta(c-b)$$

Similarly, we can show that

$$BR\beta \quad \Leftrightarrow \quad n_\beta(e - c + f - d) < (n-1)(f-d) + (e - c + f - d)$$

Hence, BR$\beta$ directly gives the upper bound for $n_\beta$. To find the lower bound for $n_\beta$, first note that the lowest possible value of $n_\beta$ is $(n-1)p_\beta$, and that it is attained for a state such that $\forall i \in N_\alpha, q_i^\beta = 0$. Second, let us show that this state indeed leads to the lower bound. By definition, this state satisfies BR$\alpha$. This state satisfies BR $\beta$ iff

$$(n-1)\frac{f-d}{f-d+e-b} < (n-1)\frac{f-d}{f-d+e-c} + 1$$

Since $b < c$, we have $e - b > e - c$ and $\frac{f-d}{f-d+e-b} < \frac{f-d}{f-d+e-c}$ . Thus, the state leading to the lowest possible lower bound is a strict Nash equilibrium.
**3**: $d < c < b, e, f$. Nash networks are of the type $\vec{\beta} \rightleftharpoons \alpha$.
By symmetry, we can apply the previous result to $n_\alpha$ by exchanging $f$ with $e$ and $d$ with $b$. This leads to

$$(n-1)p_\alpha < n_\alpha < (n-1)\frac{e-b}{e-b+f-c} + 1$$

Since $n_\beta = n - n_\alpha$, we obtain that, in this case, there exists a strict Nash equilibrium iff

$$(n-1)\frac{f-c}{f-c+e-b} < n_\beta < (n-1)p_\beta + 1$$

**4**: $b, d < c < e, f$. Nash networks are of the type $\beta \rightleftharpoons \alpha$.

As in part 2, the proof for the upper bound unfolds in three steps. First, as usual, we derive the best-response equations for $\alpha$ and $\beta$. After simplification, both equations depend on the number of active links of the agent. Second, we use BR$\beta$ to show that the highest possible upper bound for $n_\beta$ is obtained for a state where no $\beta$-player has an active link (all the active links are supported by $\alpha$-players). Third, we show that this state satisfies BR$\alpha$, hence is a valid

Nash equilibrium, and thus leads to the actual upper bound for $n_\beta$. The lower bound of $n_\beta$ can be computed in a similar fashion.

Finally, it remains to check that all the values between these two bounds can be sustained as a Nash equilibrium. To show this, we prove that the ranges of values of $n_\beta$ that sustain the two most asymmetric states overlap. This means that any $n_\beta$ between the two bounds can sustain one of these two states, which completes the proof. The details of the computations are omitted due to space constraints.

Finally, we note that the analysis of cases $b, e < c < f, d$ and $b, d, e < c < f$ uses arguments similar to the ones above and is omitted. ■

*Proof of Proposition 3:* Given a particular state $s$, denote by $n_{\alpha\alpha}$ the number of links in $s$ between two $\alpha$-players, $n_{\beta\beta}$ to be the number of links in $s$ between two $\beta$-players, and $n_{\alpha\beta}$ to be the number of links in $s$ between players choosing different actions $(\alpha, \beta)$. The welfare of a complete network with no redundant link and every agent choosing $\alpha$ is $\binom{n}{2}(2d-c)$. Any other possible profile would provide a lower welfare because $\binom{n}{2}(2d-c) \geq n_{\alpha\alpha}(2d-c) + n_{\beta\beta}(2d-c) + n_{\alpha\beta}(2d-c) \geq n_{\alpha\alpha}(2d-c) + n_{\beta\beta}(2b-c) + n_{\alpha\beta}(f+e-c)$ given that $n_{\alpha\alpha} + n_{\beta\beta} + n_{\alpha\beta} \leq \binom{n}{2}$. Thus, if $c < 2d$ the efficient profile is a complete network of agents choosing $\alpha$. ■

*Proof of Proposition 4:* (i) If $c < 2b$ then all links are profitable and therefore the efficient network must be complete. In order to obtain the size of $n_\beta$ that maximizes welfare we must work out the following maximization problem:

$$\max_{0 \leq n_\beta \leq n} n_{\alpha\alpha}(2d-c) + n_{\beta\beta}(2b-c) + n_{\alpha\beta}(f+e-c).$$

Taking into account that in a complete network with no redundant links

$$n_{\alpha\alpha} = \binom{n-n_\beta}{2} = \frac{(n-n_\beta)(n-n_\beta-1)}{2},$$

$$n_{\beta\beta} = \binom{n_\beta}{2} = \frac{(n_\beta)(n_\beta-1)}{2},$$

$$n_{\alpha\beta} = (n_\beta)(n-n_\beta).$$

the above expression reaches the maximum at,

$$n_\beta^* = \lceil \frac{e+f-2d+\frac{d-b}{n}}{e+f-(d+b)} \rfloor \frac{n}{2} = g(c)\frac{n}{2}.$$

(ii) If $2b < c < 2d$ then the links between two players choosing $\beta$ are not profitable, which implies that $n_{\beta\beta} = 0$. Apart from these links all other links are formed. The maximization problem we have to solve is the following:

$$\max_{0 \leq n_\beta \leq n} n_{\alpha\alpha}(2d-c) + n_{\alpha\beta}(f+e-c),$$

It is easily seen that:

$$n_{\alpha\alpha} = \binom{n - n_\beta}{2} = \frac{(n - n_\beta)(n - n_\beta - 1)}{2},$$

and

$$n_{\alpha\beta} = (n_\beta)(n - n_\beta).$$

Thus, the solution is:

$$n^*_\beta = \lceil \frac{e + f - 2d + \frac{d - c/2}{n}}{e + f - d - c/2} \rfloor \frac{n}{2} = g(c)\frac{n}{2},$$

where $g(\cdot)$ is piece-wise linear and increasing.
(iii) If $2d < c < e + f$ then the only links that are profitable are the ones between players choosing different actions. Thus $n_{\alpha\alpha} = n_{\beta\beta} = 0$ and the maximization problem we have to solve is the following:

$$\max_{0 \le n_\beta \le n} n_{\alpha\beta}(f + e - c),$$

It follows that $n_{\alpha\beta} = (n_\beta)(n - n_\beta)$, and so the solution is $n^*_\beta = n/2$. ■

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