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Existence and uniqueness of a Nash equilibrium feedback for a simple nonzero-sum differential game

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Abstract. Existence and uniqueness of a Nash equilibrium feedback is established for a simple class nonzero-sum differential games on the line.

Key words: Nash equilibrium, nonzero-sum differential game.

1 Introduction

The object of this paper is the study of Nash equilibria for some nonzero-sum two players differential games on the line. The dynamics of the differential game is

$$x' = u + v, \text{ with } u \in [-1, 1], \quad v = [-1, 1]. \tag{1}$$

The payoff is the terminal payofff $g(x(T)) = (g_1(x(T)), g_2(x(T)))$, for some function $g : \mathbb{R} \to \mathbb{R}^2$. Player I, playing with u, wants to maximize $g_1(x(T))$, while player II, playing with v, wants to maximize $g_2(x(T))$. Our goal is to investigate closed-loop Nash equilibrium feedbacks for this game.

For *zero-sum* differential games, i.e., when $g_2 = -g_1$, the notion of Nash equilibrium is replaced by the notion of value of the game. The existence and the characterization of the value for such a game is now well-known (see in particular, for the existence, [4, 5], and for the characterization [3]). For the dynamics (1), the value function $V(t, x)$ (for Player I) of the game is simply given by

$$\forall (t, x) \in [0, T] \times \mathbb{R}, \quad V(t, x) = g_1(x).$$

Moreover, if g_1 is sufficiently regular (for instance, if $g_1 = -g_2$ is $\mathscr{C}^1$ and has a finite number of local extrema), we can define explicitely *optimal feedback strategies* $u^*(t, x)$ and $v^*(t, x)$ for player I and II given respectively by:

$$u^*(t, x) = sgn(g_1'(x)) \text{ and } v^*(t, x) = -sgn(g_1'(x)),$$

where $sgn(s) = 1$ if $s > 0$, -1 if $s < 0$, and 0 if $s = 0$.

In conclusion, for the dynamics (1), the solution of the *zero-sum* differential game is completely understood and essentially trivial. Surprizingly, this is not at all the case for *nonzero-sum* differential games, even for dynamics as simple as (1).

For nonzero-sum differential games, played with closed loop strategies or with strategies with memory, one can find in the litterature merely two approaches:

The first one is inspired by Isaacs work for zero-sum differential games. Its main goal is the explicit computation of the Nash equilibrium payoff as a function of the time and the space, i.e., in our example, as a function $E : [0, T] \times \mathbb{R} \to \mathbb{R}^2$, which associates with any initial condition (t_0, x_0) "some" Nash equilibrium payoff $E(t_0, x_0)$ (see in particular [2], [9] and the references therein). The key idea is that the payoff E should satisfy a system of Hamilton-Jacobi equations given for our game (with dynamics (1)) by:

$$\begin{cases} \frac{\partial E_1}{\partial t} + \left|\frac{\partial E_1}{\partial x}\right| + \frac{\partial E_1}{\partial x} sgn(\frac{\partial E_2}{\partial x}) = 0 \\ \frac{\partial E_2}{\partial t} + \left|\frac{\partial E_2}{\partial x}\right| + \frac{\partial E_2}{\partial x} sgn(\frac{\partial E_1}{\partial x}) = 0 \end{cases} \tag{2}$$

with terminal condition $E(T, \cdot) = g$. Unfortunately, there is up to now no global theory for such a system. In particular, it does not fit with the assumptions required in the theory of viscosity solutions for systems of first order PDEs. A local theory exists - the so-called characteristic method - but it is only applicable in a neighbourhood of the points (T, x) where $g_1'(\cdot)$ and $g_2'(\cdot)$ are well defined and do not vanish. Therefore this method says very little in the case we are studying.

However when this method is applicable, it enjoys several interesting properties. First the solution E satisfies a dynamic programming property (also called "time consistency" in [2]): For any time $T_0 \in (0, T)$, the restriction of E to the interval $[0, T_0]$ is a Nash equilibrium payoff for the game with horizon T_0 and terminal payoff $E(T_0, \cdot)$. Second this method provides Nash equilibrium feedbacks, i.e., strategies which only depend on the current position of the player and on the current time, and which ensure Nash equilibrium payoffs whatever the initial position of the game.

Another theory for the nonzero-sum differential games has been developped by Kononenko in [7], by Kleimenov in [6] and by Tołwiński, Haurie and Leitmann in [10]. This theory is the counterpart of the so-called "Folk Theorem" for repeated (discrete) games. Its main result is the characterization of Nash equilibrium payoffs when the game is played with memory strategies. The basic idea is that memory strategies incorporate a threat which will be used if the opponent does not observe the agreement, memory allowing each player to recall a possible deviation from the agreement. In [6], [7], [10], the Nash equilibrium payoffs for such strategies are completely characterized and, under the well-known Isaacs conditions, are proved to exist for any initial position.

Unfortunately, there are in general infinitely many such Nash equilibrium payoffs for a given initial position. Thus the question of selecting "good" Nash equilibrium payoffs arises naturally. Kleimenov gives in [6] several selection methods, but none makes the connection with the feedback strategies suggested by the first theory described above. In particular, none enjoys the dynamic programming property.

The main objective of this paper is to make a link between the two approaches for the game with dynamics (1) and a terminal payoff g. For this we construct, for the dynamics (1) and for a large class $\mathcal{G}$ of terminal payoffs g, a pair (u^*, v^*) of feedback strategies, depending only on the current position of the player and on the current time, such that, for any initial position of the game, the associated payoff is, on the one hand, a selection of the set of Nash equilibrium payoffs in the sense of [6], [7], [10] and, on another hand, satisfies the dynamic programming property. We call such a pair of strategies a Nash equilibrium feedback.

It turns out that this notion of Nash equilibrium feedback is not enough discriminating: Indeed, even for elementary examples, there are in general many "uninteresting" Nash equilibrium feedbacks (see Example 2.16 below). So we are led to introduce an additional requirement of being "completely maximal".

We say that a *payoff* $(e_1, e_2) \in \mathbb{R}^2$ is *maximal* for a given initial position (t_0, x_0) if there is some solution $\bar{x}(\cdot)$ of the controlled system (1) (for some time-measurable control $(\bar{u}(\cdot), \bar{v}(\cdot)))$ such that: $(e_1, e_2) = g(\bar{x}(T))$ and both e_1 and e_2 are the maximum of g_1 and g_2 among the values that can be reached by the controlled system (1) starting from (t_0, x_0). Of course, such a maximal payoff seldom exists, but when it exists, it is reasonable to think that both players should prefer it. We say that a *Nash equilibrium feedback* is *maximal* if, at any point (t_0, x_0) where such a maximal payoff exists, the payoff of the feedback is equal to this maximal payoff. It is *completely maximal*, if its restriction to any subinterval $[0, T_0]$, with $T_0 < T$ is also maximal. Note that the condition of being completely maximal is not very restrictive, because there are in general few point for which a maximal payoff exists.

Our main result (Theorem 2.11) is that such a completely maximal Nash equilibrium feedback exists. Its associated payoff is even unique—in the sense that any two completely maximal Nash equilibrium feedbacks have the same payoff. Moreover, at each point (t_0, x_0), this payoff is a Nash equilibrium payoff in the sense of [6], [7], [10]. It is even a Pareto one in the set of these payoffs (Theorem 2.15). This holds true for a large class of terminal payoffs g, and, in particular, when g is continuous and g_1 and g_2 have a finite number of local extrema. To the best of our knowledge, this result is the first of this nature.

However, it is quite difficult to prove, and this is the reason why we had to restrict our study to the dimension one and to the particular dynamics $x' = u + v$. We have actually faced two main issues:

First we were unable to find a general existence argument for a Nash equilibrium feedback even for a small interval of time. Hence we had to compute *explicitely* the solution for such a small interval. For this, we had to restrict ourself to the dynamics (1) and a particular class of terminal payoffs (the class $\mathcal{G}$). There are up to now very few complete solutions of nonzero-sum games with constraints on the controls (apart [9] and some references therein). We think that some examples treated here are completely new and present unexpected features: See for instance the solution of the game where g_1 and g_2 are continuous, decreasing on $(-\infty, 0)$ and increasing on $(0, +\infty)$ (case III-III in part 3). Let us point out that, although some parts of the construction could probably be extended to more general dynamics on the line, we have absolutely no idea how to do it in higher space dimensions.

The second difficulty we have met is that the payoff associated with a completely maximal Nash equilibrium feedback turns out to be discontinuous in general. Worse, at these points of discontinuity, such a payoff cannot be defined univoquely in a natural way. This has several consequences, the main one being that the concatenation of two Nash equilibrium feedbacks, defined on some time intervals $[T_0, T_1]$ and $[T_1, T_2]$, is not necessarily a Nash equilibrium feedback on the time interval $[T_0, T_2]$. So we have to be very carefull when constructing our feedbacks, in order to guaranty such a concatenation property to hold. Here again, the choice of a suitable class $\mathcal{G}$ of terminal payoffs plays a crucial role.

We are aware that the conditions under which we are working are extremely restrictive: We only consider an example on the line, with a dynamics (1) which is the simplest possible one. For the moment we are unable to say in what extent our results could be generalized to more general dynamics and terminal payoffs. In particular, it is possible that the notion of maximal payoff is only interesting in the framework of dynamics (1); it has certainly to be adapted in higher dimensions of space.

However we would like to point out that the class of examples we are studying is not trivial. It is sufficiently rich to allow investigations on some basic properties of the payoffs of the completely maximal Nash equilibrium feedbacks, in particular, on the stability property of these payoffs, and on the connections between these payoffs and the system of Hamilton-Jacobi Equations (2). These question are currently under study. Moreover, even such simple class of examples shows very interesting and new features: It shows that dynamic programming is not sufficient for selecting interesting Nash equilibrium payoffs (see example 2.16). It also shows that there are some points in the time-space at which one cannot define Nash equilibrium payoffs (at least in a simple way) in order to guaranty the dynamic programming. We do not think that these two points have ever been noticed.

Let us now explain how this paper is organized. In Section 2, we introduce the different notions used in the paper, state the main results (Theorem 2.11 and Theorem 2.15) and give several examples. The rest of the paper is devoted to proofs: in Section 3 we construct the solution for small intervals of times, while, in Section 4, we show that the construction can be extended to the full interval $[0, T]$.

2 Nash equilibrium feedbacks

In this section we introduce the main definitions of this paper as well as the main result (Theorem 2.11).

2.1 Feedbacks and payoffs

The dynamics of the differential game is (1):

$$x'(t) = u(t) + v(t) \quad u(t) \in [-1, 1] \text{ and } v(t) \in [-1, 1].$$

The game is of fixed duration and the terminal time is denoted by T. The terminal payoff is a function $g = (g_1, g_2) : \mathbb{R} \to \mathbb{R}^2$: Player I wants to maximize $g_1(x(T))$ while Player II wants to maximize $g_2(x(T))$.

We denote by $\mathcal{U}$ the set of functions $u : [0, T] \times \mathbb{R} \to [-1, 1]$, interpreted as **strategies** for Player I, and by $\mathcal{V}$ the set of functions $v : [0, T] \times \mathbb{R} \to [-1, 1]$ interpreted as **strategies** for Player II. We call a pair $(u, v) \in \mathcal{U} \times \mathcal{V}$ a **feedback**.

As usual we have to give a sense to the equation

$$x'(t) = u(t, x(t)) + v(t, x(t))$$

for discontinuous feedbacks $(u, v) \in \mathcal{U} \times \mathcal{V}$. This has already been done before (see for instance [8]), and we follow more or less the same method.

For any $(t_0, x_0) \in [0, T] \times \mathbb{R}$ and any feedback $(u, v) \in \mathcal{U} \times \mathcal{V}$, we denote by $\mathcal{X}(t_0, x_0, u, v)$ the set of solutions of the differential inclusion

$$\begin{cases} \dot{x} \in \tilde{f}(t, x, u, v) & \text{on } [t_0, T] \\ x(t_0) = x_0 \end{cases}$$

where $\tilde{f}(t, x, u, v)$ is the smallest upper semi-continuous (usc) convex and compact set-valued map containing the map $(t, x) \to u(t, x) + v(t, x)$. It is well-known that this set of solutions is compact for the uniform convergence, and that it has a closed graph (see [1]).

We also denote by $\mathcal{X}(t_0, x_0)$ the set of all the solutions of (1).

The **lower and upper payoffs** of the strategies (u, v) for the initial position (t_0, x_0) are respectively given by: For $j = 1, 2$, the lower payoff of Player j, denoted by $J_j^\flat(t_0, x_0, u, v)$, is

$$J_j^\flat(t_0, x_0, u, v) = \inf_{x \in \mathcal{X}(t_0, x_0, u, v)} (g_j)_*(x(T))$$

while the upper payoff $J_j^\sharp(t_0, x_0, u, v)$ is

$$J_j^\sharp(t_0, x_0, u, v) = \sup_{x \in \mathcal{X}(t_0, x_0, u, v)} (g_j)^*(x(T)).$$

Here $(g_j)_*$ and $(g_j)^*$ are respectively the lower semi-continuous and the upper semi-continuous envelopes of g_j (i.e., respectively the largest lower semi-continuous (lsc) function which is not larger than g_j and the the smallest upper semi-continuous (usc) function which is not smaller than g_j).

Proposition 2.1. *If g_1 and g_2 are usc and $(u, v) \in \mathcal{U} \times \mathcal{V}$, then, for $j = 1$ or $j = 2$, there is some $x_j(\cdot) \in \mathcal{X}(t_0, x_0, u, v)$ such that*

$$J_j^\sharp(t_0, x_0, u, v) = g_j(x_j(T)).$$

Proof: The proof is a straightforward consequence of the compactness of the set $\mathcal{X}(t_0, x_0, u^*, v^*)$ for the topology of the uniform convergence. □

We use below the notations

$$J^\sharp(t_0, x_0, u, v) = (J_1^\sharp(t_0, x_0, u, v), J_2^\sharp(t_0, x_0, u, v)) \text{ and } J^\flat(t_0, x_0, u, v) = (J_1^\flat(t_0, x_0, u, v), J_2^\flat(t_0, x_0, u, v)).$$

2.2 Definition of the Nash equilibrium feedbacks

Definition 2.2. *A Nash equilibrium feedback on the time interval* $[T_0, T]$ *for the terminal time* T *and the terminal payoff* g *is a feedback* $(u^*, v^*) \in \mathcal{U} \times \mathcal{V}$ *such that for all* $t_0 \in [T_0, T]$ *there exists a set* $S_{t_0} \subset \mathbb{R}$ *of zero measure such that for any* $x_0 \in \mathbb{R}\backslash S_{t_0}$*, we have*

$$\forall u \in \mathcal{U}, \inf_{x \in \mathcal{X}(t_0, x_0, u^*, v^*)} (g_1)_*(x(T)) \geq \sup_{x \in \mathcal{X}(t_0, x_0, u, v^*)} (g_1)^*(x(T))$$

and

$$\forall v \in \mathcal{V}, \inf_{x \in \mathcal{X}(t_0, x_0, u^*, v^*)} (g_2)_*(x(T)) \geq \sup_{x \in \mathcal{X}(t_0, x_0, u^*, v)} (g_2)^*(x(T)).$$

Remarks:

1. Using the notations of the previous section, the two above inequalities can be rewritten

$$\forall (u, v) \in \mathcal{U} \times \mathcal{V}, J_1^\flat(t_0, x_0, u^*, v^*) \geq J_1^\sharp(t_0, x_0, u, v^*) \text{ and } J_2^\flat(t_0, x_0, u^*, v^*) \geq J_2^\sharp(t_0, x_0, u^*, v).$$

2. Setting $(u, v) = (u^*, v^*)$ in the previous inequality shows that, for $j = 1, 2$, we have

$$\forall x_0 \notin S_{t_0}, J_j^\flat(t_0, x_0, u^*, v^*) = J_j^\sharp(t_0, x_0, u^*, v^*).$$

In order to simplify the notations, we denote by $J_1(t_0, x_0, u^*, v^*)$ (resp. $J_2(t_0, x_0, u^*, v^*)$) this common value. We say that $J(t_0, x_0, u^*, v^*) = (J_1(t_0, x_0, u^*, v^*), J_2(t_0, x_0, u^*, v^*))$ is the payoff of the Nash equilibrium feedback (u^*, v^*) at the point (t_0, x_0). It is defined for any $t_0 \in [0, T]$ and for any $x_0 \in \mathbb{R}\backslash S_{t_0}$ where S_{t_0} has a zero measure in $\mathbb{R}$.

3. From standard arguments, it is easy to check that the map $x \to J^\sharp(t, x, u^*, v^*)$ is continuous at each point $(t, x) \in \mathbb{R}$ for which the following equality holds:
$J^\sharp(t, x, u^*, v^*) = J^\flat(t, x, u^*, v^*)$.

We also need below the following remark:

Lemma 2.3. *Let* $(u^*, v^*) \in \mathcal{U} \times \mathcal{V}$ *be a Nash equilibrium feedback on the time interval* $[T_0, T]$ *for the game with terminal time* T *and terminal payoff* g *and let* (S_t) *be its associated set of zero measure. Then for any* $t_0 \in [T_0, T]$, *any* $x_0 \notin S_{t_0}$ *and any* $x^*(\cdot) \in \mathcal{X}(t_0, x_0, u^*, v^*)$, *we have*

$$J^\sharp(t, x^*(t), u^*, v^*) = J^\flat(t, x^*(t), u^*, v^*) = J(t_0, x_0, u^*, v^*) = g(x^*(T)) \quad \forall t \in [t_0, T].$$

In particular, the map $J^\sharp(t, \cdot, u^*, v^*)$ *is continuous at* $x^*(t)$.

Proof: Since (u^*, v^*) is a Nash equilibrium feedback and $x_0 \notin S_{t_0}$, the last equality holds and g is continuous at $x^*(T)$. Let $t \in [t_0, T]$ and $x(\cdot) \in \mathcal{X}(t, x^*(t), u^*, v^*)$. It is enough to prove g is continuous at $x(T)$ and that $g(x(T)) = g(x^*(T))$. For this, let us define the solution $x_1(\cdot)$ by setting $x_1(\cdot) = x^*(\cdot)$ on $[t_0, t]$ and $x_1(\cdot) = x(\cdot)$ on $[t, T]$. Then $x_1(\cdot)$ belongs to $\mathcal{X}(t_0, x_0, u^*, v^*)$. Therefore, since (u^*, v^*) is a Nash equilibrium feedback and $x_0 \notin S_{t_0}$, g is continuous at $x_1(T) = x(T)$ and

$$g(x^*(T)) = J(t_0, x_0, u^*, v^*) = g(x_1(T)) = g(x(T)). \qquad \square$$

Before giving the next definition, we introduce a notation : For a map $g : \mathbb{R} \to \mathbb{R}^2$ and a point $x \in \mathbb{R}$, we set

$$ess - \limsup_{x' \to x} g(x) = \{a \in \mathbb{R}^2 \mid \forall \epsilon > 0, \forall r > 0, \mid g^{-1}(B_\epsilon(a)) \cap]x - r, x + r[\mid > 0\},$$

where $|A|$ denotes the outward Lebesgue measure of a subset A of $\mathbb{R}$ and where $B_\epsilon(a)$ denotes the ball of center a and radius ϵ in $\mathbb{R}^2$.

Definition 2.4. [*Maximal payoff*] *Let* $(t_0, x_0) \in [0, T] \times \mathbb{R}$ *and* $(e_1, e_2) \in \mathbb{R}^2$. *We say that* (e_1, e_2) *is a maximal payoff at the point* (t_0, x_0) *for the game with terminal time T and terminal payoff g if there is some* $x_0(\cdot) \in \mathcal{X}(t_0, x_0)$ *such that* $(e_1, e_2) \in ess - \limsup_{x' \to x_0(T)} g(x')$ *and such that, for any* $x(\cdot) \in \mathcal{X}(t_0, x_0)$, *we have*

$$\text{for } j = 1, 2, \quad e_j \geq (g_j)^*(x(T)).$$

Remarks:

1. Loosely speaking, this only means that both functions g_1 and g_2 reach a maximum at the same point $x_0(T)$ among the points $x(T)$ that one can reach starting from (t_0, x_0).
2. For a given point (t_0, x_0) such a maximal equilibrium payoff does not necessarily exist (and in fact seldom exists). However, if it exists, it is clearly unique.
3. If g is continuous, we have of course $(e_1, e_2) = g(x_0(T))$. However, for later use, we need to have this definition at points where the function g can be discontinuous.

A typical example of what we want to avoid is the following situation. Let $g : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$\forall x \in \mathbb{R}, g(x) = \begin{cases} (-1, 1) & \text{if } x < 0 \\ (1, -1) & \text{if } x > 0 \\ (1, 1) & \text{if } x = 0 \end{cases}$$

Then the value $(1, 1)$ does not belong to $ess - \limsup_{x' \to 0} g(x')$, although it could be a good candidate for being a maximal payoff, since at this point both functions g_1 and g_2 have a maximum. But examples of differential games can be given for which this would lead to some contradictions (see example 2.18).

For later use, we need the following remark:

Lemma 2.5. *Let* $(t_0, x_0) \in [0, T] \times \mathbb{R}$. *Suppose that at* (t_0, x_0) *there is some maximal payoff* $(e_1, e_2) \in \mathbb{R}^2$ *for the game with terminal time T and terminal payoff g. Let* $x_0(\cdot) \in \mathcal{X}(t_0, x_0)$ be *such that*

$$(e_1, e_2) \in \ ess - \limsup_{x' \to x_0(T)} g(x').$$

Then, for any $t_1 \in [t_0, T)$, (e_1, e_2) *is the maximal payoff at the point* $(t_1, x_0(t_1))$ *for the game with terminal time T and terminal payoff g.*

Proof: It is a straightforward application of the concatenation property of the solutions of system (1). □

Definition 2.6. [*Maximal Nash equilibrium feedback*] *Let* $(u^*, v^*) \in \mathcal{U} \times \mathcal{V}$ *be a Nash equilibrium feedback on some time interval* $[T_0, T]$ (*with* $T_0 < T$) *for the game with terminal time T and terminal payoff g. We say that* (u^*, v^*) *is* **maximal** *on* $[T_0, T]$ *for the game with terminal time T and terminal payoff g, if* (u^*, v^*) *is a Nash equilibrium feedback on* $[T_0, T]$ *and if* (u^*, v^*) *satisfies the additional requirement: For any* $(t_0, x_0) \in [T_0, T) \times \mathbb{R}$, *if there is some maximal payoff* (e_1, e_2) *for the game with terminal time T and terminal payoff g at the point* (t_0, x_0), *then*

$$\text{for } j = 1, 2, \quad e_j = J_j^\sharp(t_0, x_0, u^*, v^*).$$

Remark. *The requirement of* (u^*, v^*) *to be maximal is somehow the weakest requirement we could make because there are in general few initial conditions at which such a maximal payoff exists. Let us point out that, in general, Nash equilibrium feedbacks are not maximal* (*see example* 2.16 *below*).

Before introducing the last—and main—definition of this section, we need the following remark:

Proposition 2.7. *Let us assume that* g_1 *and* g_2 *are usc. Let* (u^*, v^*) *be a Nash equilibrium feedback on the time interval* $[T_0, T]$. *Then, for any* $T_1 \in (T_0, T)$, *the restriction of* (u^*, v^*) to $[T_0, T_1] \times \mathbb{R}$ *is a Nash equilibrium feedback for the horizon* T_1 *and the terminal payoff* $J^\sharp(T_1, \cdot, u^*, v^*)$.

Remark. *This result is nothing but the well-known dynamic programming principle* (*or time consistency*). *Although its proof follows standard arguments, we give it nevertheless since some attention has to be paid to the payoff of the feedback* (u^*, v^*) *which is not well defined at time* T_1 *on the set* S_{T_1}.

Proof: We only prove that, for any $t_0 \in [T_0, T_1)$, there is some set S_{t_0} of zero measure such that, for any $u \in \mathcal{U}$, for any $x_0 \in \mathbb{R} \backslash S_{t_0}$, we have

$$\inf_{x^*(\cdot) \in \mathcal{X}(t_0, x_0, u^*, v^*)} (J_1^\sharp)_*(T_1, x^*(T_0), u^*, v^*) \geq \sup_{x(\cdot) \in \mathcal{X}(t_0, x_0, u, v^*)} (J_1^\sharp)^*(T_1, x(T_0), u^*, v^*). \tag{3}$$

The inequality for J_2 can be obtained in a symmetric way.

Let (S_t) be the set of zero measure associated with (u^*, v^*). Then, for any $x_0 \in \mathbb{R}\backslash S_{t_0}$, for any $u \in \mathscr{U}$, let us fix $x^*(\cdot) \in \mathscr{X}(t_0, x_0, u^*, v^*)$ and $x(\cdot) \in \mathscr{X}(t_0, x_0, u, v^*)$. Let us recall that $J_1^\sharp$ is usc. Hence $(J_1^\sharp)^* = J_1^\sharp$. From Lemma 2.3, we know that $J^\sharp(T_1, \cdot, u^*, v^*)$ is continuous at $x^*(T_0)$ and that we have the equality

$$(J_1^\sharp)_*(T_1, x^*(T_0), u^*, v^*) = J_1(T_1, x^*(T_0), u^*, v^*) = g_1(x^*(T)). \tag{4}$$

Since g_1 is usc, Proposition 2.1 states that there is some solution $x_1(\cdot) \in \mathscr{X}(T_1, x(T_1), u^*, v^*)$ such that

$$g_1(x_1(T)) = J_1^\sharp(T_1, x(T_1), u^*, v^*).$$

Let us now define

$$\tilde{u}(t,x) = \begin{cases} u(t,x) & \text{if } t \in [t_0, T_1) \\ u^*(t,x) & \text{if } t \in [T_1, T] \end{cases} \text{ and } \tilde{x}(t) = \begin{cases} x(t) & \text{if } t \in [t_0, T_1) \\ x_1(t) & \text{if } t \in [T_1, T] \end{cases}$$

Then we have clearly $\tilde{x}(\cdot) \in \mathscr{X}(t_0, x_0, \tilde{u}, v^*)$. Therefore, since (u^*, v^*) is a Nash equilibrium feedback, and since $x_0 \in \mathbb{R}\backslash S_{t_0}$, we have

$$g_1(x^*(T)) \geq g_1(\tilde{x}(T)). \tag{5}$$

Moreover, from the construction of $\tilde{x}(\cdot)$, we have

$$g_1(\tilde{x}(T)) = g_1(x_1(T)) = J_1^\sharp(T_1, x(T_1), u^*, v^*). \tag{6}$$

Then combining (4), (5) and (6) together gives (3). □

Definition 2.8. *Let* $(u^*, v^*) \in \mathscr{U} \times \mathscr{V}$ *be a Nash equilibrium feedback on some time interval* $[T_0, T]$ (where $T_0 < T$) *for the game with terminal time* T *and terminal payoff* g. *We say that* (u^*, v^*) *is* **completely maximal** *on the time interval* $[T_0, T]$ *if, for any* $T_1 \in (T_0, T]$, (u^*, v^*) *is a maximal Nash equilibrium feedback for the game with terminal time* T_1 *and terminal payoff* $J^\sharp(T_1, \cdot, u^*, v^*)$.

Remark. *This assumption is the weakest one if one want that the Nash equilibrium payoff is maximal and satisfies some time consistency.*

2.3 The main Theorem

In order to state the main results of this paper, we have to define the class $\mathscr{G}$ of terminal payoffs for which we can solve the problem.

Definition 2.9. *The class of admissible payoffs* $\mathscr{G}$ *is the set of maps* $g = (g_1, g_2) : \mathbb{R} \to \mathbb{R}^2$ *for which there is a partition* $\sigma_0 = -\infty < \sigma_1 < \cdots < \sigma_k < \sigma_{k+1} = +\infty$ *such that*

1. *for any* $i = 0, \ldots, k$,
   – *either* g_1 (*resp.* g_2) *is continuous and* (*strictly*) *increasing or decreasing on* (σ_i, σ_{i+1}),

– *or* g_1 *and* g_2 *are simultaneously constant on the interval* (σ_i, σ_{i+1}),

2. g_1 *and* g_2 *are usc on* $\mathbb{R}$, *and, for* $i = 1, \dots, k$ *and* $j = 1, 2, g_j$ *is right or left continuous at* σ_i.
3. *if* g_1 (*resp.* g_2) *has a strict local maximum at the point* σ_i, *then* g_1 *and* g_2 *are continuous at* σ_i.

Terminology. We say below that $\Sigma = \{\sigma_0 < \sigma_1 < \cdots < \sigma_k < \sigma_{k+1}\}$ is the partition associated with the map g.

Remark. *In particular, if* $g = (g_1, g_2)$ *is continuous and* g_1 *and* g_2 *have a finite number of local extrema, then* g *belongs to* $\mathcal{G}$.

For later use, let us introduce another class of terminal payoffs:

Definition 2.10. *The subclass of admissible payoffs* $\widetilde{\mathcal{G}}$ *is the set of maps* $g = (g_1, g_2) \in \mathcal{G}$ *which satisfy the additional requirement: If* g_1 (*resp.* g_2) *has a strict local maximum at some point* σ_i *belonging to the partition associated with* g, *then* g_2 (*resp.* g_1) *has a local minimum at* σ.

We are now ready to give the main result of this paper:

Theorem 2.11. *Assume that the terminal payoff* g *belongs to the class* $\mathcal{G}$. *Then there exists a completely maximal Nash equilibrium feedback for the game with terminal time* T *and terminal payoff* g *on the time interval* $[0, T]$.

Moreover, the payoffs of any two completely maximal Nash equilibrium feedbacks coincide almost everywhere.

Remark

1. The uniqueness part of the result means the following: Let (u^*, v^*) be some completely maximal Nash equilibrium feedback on the time interval $[0, T]$, and let (u_1^*, v_1^*) be another completely maximal Nash equilibrium feedback on some time interval $[T_1, T]$ with $T_1 \in [0, T)$. Then

   $$\forall t \in [T_1, T], \quad J(t, \cdot, u^*, v^*) = J(t, \cdot, u_1^*, v_1^*) \text{ a.e. .}$$

2. We prove below that, for any $t \in [0, T)$, the function $J(t, \cdot, u^*, v^*)$ belongs to $\widetilde{\mathcal{G}}$ in some sense.

The proof of Theorem 2.11 is given in the last two parts of the paper. Roughly speaking, we mimic the proof of existence of solutions of ordinary differential equations: We first prove existence and uniqueness of a completely maximal Nash equilibrium feedback on a small interval of time (Proposition 3.1). Then we show that—under suitable conditions—the concatenation of two completely maximal Nash equilibrium feedbacks is still a completely maximal Nash equilibrium feedback (Proposition 4.5). From this we deduce that there is a maximal interval $(T_0, T]$ of existence for the feedback. In order to prove that this interval is in fact the full interval $[0, T]$, we have to show that, otherwise, one could enlarge again the interval of definition of the feedback. For this we prove an extension result, which roughly states that, at time T_0, the payoff of the completely maximal Nash equilibrium feedback still belongs to the class $\mathcal{G}$ of admissible terminal payoffs: This is the aim of Proposition 4.6.

2.4 Link with Nash equilibrium payoffs for memory-strategies

We now make the link between the Nash equilibrium feedbacks defined above and the Nash equilibrium payoffs for memory strategies as defined in [6], [7], [10].

We only recall here the characterization of these payoffs, since their exact definition would require the introduction of memory strategies, which we prefer to avoid for sake of shortness. For the original definition and the interpretation, see [6], [7], [10].

Proposition 2.12. *Let us consider the game with dynamics* (1) *and with terminal payoff* $g = (g_1, g_2)$ *where* $g : \mathbb{R} \to \mathbb{R}^2$ *is a continuous function. Let* $(t_0, x_0) \in [0, T) \times \mathbb{R}$ *be a fixed initial position. A pair* $(e_1, e_2) \in \mathbb{R}^2$ *is a Nash equilibrium payoff for this game played with memory strategies* (*in short* **Nash equilibrium payoff**) *for the point* (t_0, x_0), *if and only if there is some solution* $x(\cdot) \in \mathcal{X}(t_0, x_0)$ *such that*

$$\forall t \in [t_0, T], \text{ for } j = 1, 2, \quad g_j(x(t)) \le e_j = g_j(x(T)).$$

We call the solution $x(\cdot)$ a **Nash trajectory**.

Remarks:

1. The exact statement of the characterization result of [6], [7], [10] is the following: A pair $(e_1, e_2) \in \mathbb{R}^2$ is a Nash equilibrium payoff for the point (t_0, x_0), if and only if there is some solution $x(\cdot) \in \mathcal{X}(t_0, x_0)$ such that

   $$\forall t \in [t_0, T], \text{ for } j = 1, 2, V_j(t, x(t)) \le e_j = g_j(x(T)),$$

   where V_1 and V_2 are respectively the value functions of the zero-sum games where, on the one hand, Player I wants to maximize $g_1(x(T))$ while Player II wants to minimize this quantity, and where, on another hand, Player II wants to maximize $g_2(x(T))$ while Player I wants to minimize this quantity. For the game with dynamics (1), the value functions V_1 and V_2 is the unique solution (in the viscosity sense) of the Hamilton-Jacobi-Isaacs equation

   $$\begin{cases} V_t + H_j(V_x) = 0 & \text{on } [0, T) \times \mathbb{R} \\ V(T, \cdot) = g_j(\cdot) & \text{in } \mathbb{R} \end{cases}$$

   where, for $j = 1$, $H_1(p) = \sup_{u \in [-1,1]} \inf_{v \in [-1,1]} (u + v)p = 0$, while, for $j = 2$, $H_2(p) = \inf_{u \in [-1,1]} \sup_{v \in [-1,1]} (u + v)p = 0$ (see ([3]) for instance). Therefore $V_1(t, x) = g_1(x)$ and $V_2(t, x) = g_2(x)$, whence the Proposition.
2. Let us recall that the characterization result only holds for *continuous* terminal payoffs g. The discontinuous case is still an open problem.

Definition 2.13. *We say that a Nash equilibrium payoff* $(e_1, e_2) \in \mathbb{R}^2$ *is* **Pareto** *at the point* (t_0, x_0), *if, for any other Nash equilibrium payoff* $(e_1', e_2') \in \mathbb{R}^2$ *at* (t_0, x_0),

we have:

either $e_1 \geq e_1'$ or $e_2 \geq e_2'$.

In the rest of the section, we assume that g is continuous.

Proposition 2.14. *Let* (u^*, v^*) *be some Nash equilibrium feedback on the time interval* $[0, T]$ *and* $(S_t)_{t\in[0,T]}$ *be its associated set of zero measure. Then, for any* $t \in [0, T)$ *and* $x \notin S_t$, *the payoff* $(e_1, e_2) = J(t, x, u^*, v^*)$ *is a Nash equilibrium payoff and any solution* $x^*(\cdot) \in \mathcal{X}(t, x, u^*, v^*)$ *is a Nash trajectory.*

Moreover the payoff of a completely maximal Nash equilibrium feedback is Pareto at any "regular" point:

Theorem 2.15. *Let* (u^*, v^*) *be a completely maximal Nash equilibrium feedback and* (S_t) *be its associated set of zero measure. Then, for any* $t \in [0, T)$ *and for any* $x \notin S_t$, *the payoff* $J(t, x, u^*, v^*)$ *is Pareto.*

Proposition 2.14 and Theorem 2.15 are proved at the end of the paper.

2.5 Examples of Nash equilibrium feedbacks

We now give several examples of Nash equilibrium feedbacks for various terminal payoffs.

Example 2.16. *Let us assume that* g_1 *and* g_2 *are both strictly increasing and usc. Then the feedback given by*

$$\forall (t, x) \in [0, T] \times \mathbb{R},\ u^*(t, x) = v^*(t, x) = 1$$

is a completely maximal Nash equilibrium feedback. Moreover, the payoff of any other maximal Nash equilibrium feedback coincides almost everywhere with the payoff of (u^*, v^*).

However, there exists infinitely many Nash equilibrium feedbacks for which the associated payoff differs from the payoff for (u^*, v^*). *For instance, for any fixed* $a \in \mathbb{R}$ *at which* g *is continuous, the feedback* (u_a, v_a) *defined by*

$$\forall (t, x) \in [0, T] \times \mathbb{R}, \quad u_a(t, x) = v_a(t, x) = \begin{cases} 1 & \text{if } x > a \text{ or } x + 2(T - t) < a \\ -1 & \text{otherwise} \end{cases}$$

is a Nash equilibrium feedback.

Proof: Let S_T be the set of points at which g is discontinuous and, for any $t \in (0, T]$, $S_t = S_T - 2(T - t)$. Then, for any t, the set S_t has a zero measure since it is enumerable. Moreover, for any point $(t, x) \in [0, T] \times \mathbb{R}$, with $x \notin S_t$, the payoff $g(x + 2(T - t))$ is maximal because g_1 and g_2 are both increasing and continuous at $x + 2(T - t)$. This proves that the feedback (u^*, v^*) is a maximal Nash equilibrium feedback, because, for any $(t, x) \in [0, T] \times \mathbb{R}$, the unique solution $x(\cdot) \in \mathscr{X}(t,x,u^*,v^*)$ is $x(\cdot) = x + 2(\cdot - t)$. Moreover, the payoff of any other maximal Nash equilibrium payoff coincides with the payoff of (u^*, v^*) almost everywhere, because it has to coincide at the points (t,x) with $x \notin S_t$ since at these points there is a maximal payoff. Finally, the feedback (u^*, v^*) is completely maximal because, for any $T_1 \in [0,T]$, the maps $J^\sharp(T_1, \cdot, u^*, v*) = g_j(\cdot + 2(T - T_1))$ are increasing, and thus the previous arguments apply to these maps.

Let us finally prove that the feedback (u_a, v_a) is a Nash equilibrium feedback with a payoff different from the payoff of (u^*, v^*). A straightforward computation shows that $\forall (t,x), \forall x(\cdot) \in \mathscr{X}(t,x,u_a,v_a)$,

$$x(T) = \begin{cases} x + 2(T-t) & \text{if } x > a \text{ or } x \le a - 2(T-t) \\ a & \text{if } a - 2(T-t) \le x < a \end{cases}$$

(for $x = a$, then $x(T) \in [a, a + 2(T-t)]$). Thus the payoff for (u_a, v_a) differs from the payoff for (u^*, v^*) on the set $E = \{a - 2(T-t) < x < a\}$. Let us also point out that, on this set, the payoff of (u_a, v_a) is well defined (i.e., $J^\sharp = J^\flat$) because g is continuous at a. Let us set $\Sigma_t = \{a\} \cup [S_t \backslash (a - 2(T-t), a)]$.

For proving that (u_a, v_a) is a Nash equilibrium feedback with associated set of zero measure $(\Sigma_t)_{t \in [0,T]}$, let $(u,v) \in \mathscr{U} \times \mathscr{V}$. We have to prove that, for any $t \in [0,T)$ and for any $x \in \mathbb{R} \backslash \Sigma_t$, we have

$$J_1^\flat(t,x,u_a,v_a) \ge J_1^\sharp(t,x,u,v_a) \text{ and } J_2^\flat(t,x,u_a,v_a) \ge J_2^\sharp(t,x,u_a,v).$$

We only do the proof for the first inequality, the proof of the second one being symmetric.

If $x \notin (a - 2(T-t), a)$ the result is clear, because $J(t,x,u_a,v_a)$ is the maximal payoff at (t,x). Let us assume that $x \in (a - 2(T-t), a)$. Then $J_1(t,x,u_a,v_a) = g(a)$. Let now $x(\cdot) \in \mathscr{X}(t,x,u,v_a)$ and let us check that $x(T) \le a$. Indeed, let us denote by θ the first time the trajectory $x(\cdot)$ leaves the set E. As long as $x(s) \in E$, $v_a = -1$ in a neighbourhood of $(s, x(s))$. Hence $x'(s) \le 0$ and the solution $x(\cdot)$ is non increasing on this interval. Thus $x(\theta) \le x$ and $x(\theta) = a - 2(T-\theta)$ because $(\theta, x(\theta)) \in \partial E$. Therefore $x(T) \le x(\theta) + 2(T-\theta) = a$. Since g_1 is increasing, we have

$$J_1^\sharp(t,x,u,v_a) \le g_1(a) = J_1^\flat(t,x,u_a,v_a). \qquad \square$$

Next we give an example where there is no maximal payoff.

Example 2.17. *Let us assume that g_1 is strictly increasing while g_2 is strictly decreasing in $\mathbb{R}$, and that both functions are usc. Then the feedback (u^*, v^*) given by*

$$\forall (t,x) \in [0,T] \times \mathbb{R}, \quad u^*(t,x) = 1 = -v^*(t,x).$$

is a completely maximal Nash equilibrium feedback.

Moreover, for any Nash equilibrium feedback $(\bar{u}, \bar{v})$, we have

$$\forall t \in [0,T], J(t,x,\bar{u},\bar{v}) = J(t,x,u^*,v^*) = g(x) \quad \text{a.e. } x \in \mathbb{R}.$$

Remark. *Note that there is no maximal payoff at any point (t,x) because g_1 is strictly increasing while g_2 is strictly decreasing.*

Proof: Let S_T be the enumerable set of points x at which the function g is continuous and let $S_t = S_T$ for any $t \in [0,T]$. For any (t,x), there is a unique solution $x(\cdot) = \cdot$ in $\mathcal{X}(t,x,u^*,v^*)$. Then it is very easy to check that (u^*,v^*) is a Nash equilibrium feedback with associated payoff $J(t,x,u^*,v^*) = g(x)$. It it also completely maximal, since there is no maximal payoff.

Let now $(\bar{u},\bar{v})$ be any Nash equilibrium feedback and let $(\Sigma_t)_{t\in[0,T]}$ be its associated set of zero measure. Then, for any $(t,x) \in [0,T]\times\mathbb{R}$ with $x \notin S_t$, and any $x(\cdot) \in \mathcal{X}(t,x,\bar{u},\bar{v})$ we have, on the one hand:

$$g_1(x(T)) = J_1(t,x,\bar{u},\bar{v}) \geq J^\sharp(t,x,1,\bar{v}) \geq g_1(x)$$

because any solution of $\mathcal{X}(t,x,1,\bar{v})$ is non decreasing and g_1 is increasing. Inequality $g_1(x(T)) \geq g_1(x)$ implies that $x(T) \geq x$. On the other hand,

$$g_2(x(T)) = J_2(t,x,\bar{u},\bar{v}) \geq J_2^\sharp(t,x,\bar{u},-1) \geq g_2(x)$$

because any solution of $\mathcal{X}(t,x,\bar{u},-1)$ is non increasing and g_2 is decreasing. Then the inequality $g_2(x(T)) \geq g_2(x)$ implies that $x(T) \leq x$. Accordingly, $x(T) = x$, and the desired result is proved: $J(t,x,\bar{u},\bar{v}) = g(x)$. □

Next we give an example in which the payoff associated with a completely maximal Nash equilibrium feedback is discontinuous and is not well-defined at the point of discontinuity.

Example 2.18. *We assume that*

$$\forall x \in \mathbb{R}, \quad g_2(x) = g_1(-x) \text{and } g_1(x) = \begin{cases} 1 & \text{if } x \leq -2 \\ -x-1 & \text{if } -2 \leq x \leq 1 \\ x-3 & \text{if } 1 \leq x \leq 2 \\ -1 & \text{if } x \geq 2 \end{cases}$$

Then for any completely maximal Nash equilibrium feedback (u^*,v^*), *we have*

- *If* $t \in [T-1,T]$, *then, for almost all* $x \in \mathbb{R}$, $J_2(t,x,u^*,v^*) = J_1(t,-x,u^*,v^*)$ *and*

$$J_1(t,x,u^*,v^*) = \begin{cases} 1 & \text{if } x < -2+2(T-t) \\ -x-1+2(T-t) & \text{if } -2+2(T-t) \leq x < -1+(T-t) \\ -x-1 & \text{if } -1+(T-t) \leq x < 1-(T-t) \\ x-3+2(T-t) & \text{if } 1-(T-t) \leq x < 2-2(T-t) \\ -1 & \text{if } x \geq 2-2(T-t) \end{cases}$$

- *If* $t \leq T-1$, *then for almost all* $x \in \mathbb{R}$, $J_2(t,x,u^*,v^*) = J_1(t,-x,u^*,v^*)$ *and*

$$J_1(t,x,u^*,v^*) = \begin{cases} 1 & \text{if } x < 0 \\ -1 & \text{if } x \geq 0 \end{cases}$$

We shall not do the proof since existence and uniqueness for such a game are constructed in the next sections. We mainly want to point out that, at time $t = T-1$, we have

$$\forall x \in \mathbb{R}, \quad J_1^\sharp(T-1,x,u^*,v^*) = \begin{cases} 1 & \text{if } x \leq 0 \\ -1 & \text{if } x > 0 \end{cases}$$

while

$$\forall x \in \mathbb{R}, \quad J_2^\sharp(T-1, x, u^*, v^*) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

In particular, $J^\sharp(T-1, 0, u^*, v^*) = (1,1)$, and thus the point $x = 0$ is a point of maximum for both $J_1^\sharp(T-1, \cdot, u^*, v^*)$ and $J_2^\sharp(T-1, \cdot, u^*, v^*)$. However, it is not a Nash equilibrium payoff in the sense of [6], [7], [10] since there no point $x \in \mathbb{R}$ such that $g(x) = (1,1)$. Therefore the payoff $(1,1)$ cannot be considered as a maximal payoff at $(T-1, 0)$ without causing trouble. This has lead us to the above definition of maximal payoffs.

3 Local existence and uniqueness

In this section, we compute explicitely, for any terminal payoff $g \in \mathcal{G}$, a completely maximal Nash equilibrium feedback on a time interval $[T-\tau, T]$ for some $\tau > 0$. Moreover we prove that the associated payoff belongs to $\mathcal{G}$ and that the associated payoff of any other completely maximal feedback strategy coincides with it.

3.1 The local existence and uniqueness result

Proposition 3.1. *Let us assume that the terminal payoff g belongs to $\mathcal{G}$ and that Σ is an associated partition. Then:*

A) [Existence] *there is some feedback $(u^*, v^*) \in \mathcal{U} \times \mathcal{V}$ and some $\tau > 0$ such that*

i) *(u^*, v^*) is a completely maximal Nash equilibrium feedback on the time interval $[T-\tau, T]$,*

ii) *for any $t \in [T-\tau, T)$, $J^\sharp(t, \cdot, u^*, v^*)$ belongs to $\widetilde{\mathcal{G}}$,*

iii) *for any $t \in [T-\tau, T]$, there is some partition S_t associated with $J^\sharp(t, \cdot, u^*, v^*)$ such that*

$$\forall x \notin S_t, \quad J^\sharp(t, x, u^*, v^*) = J^\flat(t, x, u^*, v^*),$$

iv) *the set-valued map $t \to S_t$ is 2-Lipschitz continuous on $[T-\tau, T]$*, with *$S_T = \Sigma$, and there is some fixed M such that the cardinal of S_t is not larger than M,*

v) *for any $t_0 \in [T-\tau, T)$, for any $x_0 \notin S_{t_0}$, for any $x^*(\cdot) \in \mathcal{X}(t_0, x_0, u^*, v^*)$, we have*

$$\forall t \in [t_0, T), x^*(t) \notin S_t.$$

(*we say that (u^*, v^*) is proper with respect to the set $(S_t)_{t\in[T-\tau,T]}$*).

vi) *if moreover g belongs to $\widetilde{\mathcal{G}}$, then the cardinal of S_t is not larger than the cardinal of Σ and, for any $t_0 \in [T-\tau, T)$, for any $x_0 \notin S_{t_0}$, for any $x^*(\cdot) \in \mathcal{X}(t_0, x_0, u^*, v^*)$*, we have *$x^*(T) \notin \Sigma$* (*we say that (u^*, v^*)* is *proper with respect to the terminal set Σ*).

B) [Uniqueness] *any other completely maximal Nash equilibrium feedback* (u_1^*, v_1^*) *defined on some interval* $[T-\tau_1, T]$ *satisfies: for* $t \in [T - \min\{\tau, \tau_1\}, T]$, for $j = 1, 2$,

$$J_j(t, x, u^*, v^*) = J_j(t, x, u_1^*, v_1^*) \text{ for almost all } x \in \mathbb{R}.$$

The proof of Proposition 3.1 is splitted in several steps. We first define explicitely the feedback (u^*, v^*). Then we prove that it is indeed Nash and maximal, and that the payoff of any completely maximal Nash feedback strategy coincides with the payoff of (u^*, v^*). Next we show that condition (ii) is satisfied. We finaly establish that (u^*, v^*) is completely maximal.

3.2 Construction of the maximal Nash equilibrium feedback

In this part we explain the construction of the strategy (u^*, v^*). For doing so, we first construct some $\tau > 0$ and maps $t \to \sigma_i^+(t)$ and $t \to \sigma_i^-(t)$ in such a way that, for any $t \in [T - \tau, T]$,

$$\sigma_0(t) = -\infty < \sigma_1^-(t) \le \sigma_1^+(t) < \cdots < \sigma_k^-(t) \le \sigma_k^+(t) < \sigma_{k+1} = +\infty.$$

Then, for any $t \in [T - \tau, T]$, we define $(u^*(t,x), v^*(t,x))$ in a suitable way according to the interval $(\sigma_i^+(t), \sigma_{i+1}^-(t))$ or $(\sigma_i^-(t), \sigma_i^+(t))$ the point x belongs to. The set S_t shall be defined below as

$$S_t = \{\sigma_i^-(t), i = 1, \ldots, k\} \cup \{\sigma_i^+(t), i = 1, \ldots, k\}.$$

Let $g \in \mathcal{G}$ and $\Sigma = \{\sigma_0 = -\infty, \sigma_1, \ldots, \sigma_k, \sigma_{k+1} = +\infty\}$ be an associated partition. We divide the singularities σ_i in 4 classes. For $j = 1, 2$, and $i = 1, \ldots, k$, the point σ_i is of the type *I* (resp. *II*, *III* or *IV*) for the map g_j if:

- Type I: g_j is monotonous on $(\sigma_{i-1}, \sigma_{i+1})$.
- Type II: g_j has a strict local maximum at the point σ_i and is continuous at σ_i.
- Type III: g_j has a strict local minimum at the point σ_i and is continuous at σ_i.
- Type IV: g_j is discontinuous at σ_i and its lsc enveloppe has a strict local minimum at σ_i.

Remark. *Let us underline that a point* σ_i *is of one and only one type for* g_j ($j = 1$ or $j = 2$) *because* g *belongs to* $\mathcal{G}$.

Notation: If $\phi : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$, we denote by $\phi(x^-)$ (resp. $\phi(x^+)$) the limit, if it exists, of $\phi(s)$ when $s \to x^-$ (resp. $s \to x^+$).

Definition of $\sigma_i^+(\cdot)$ and $\sigma_i^-(\cdot)$:

For any $i = 1, \ldots, k$, the singularity σ_i propagates via two arcs, denoted $\sigma_i^+(\cdot)$ and $\sigma_i^-(\cdot)$. For defining these arcs, we have to consider all the possible cases: For $A \in \{I, II, III, IV\}$ and $B \in \{I, II, III, IV\}$, the case A–B is the case where σ_i is of type A for g_1 and of type B for g_2.

In order to restrict the (lengthy) cases discussion, we use the symmetry of the problem, which implies that the propagation of the singularity in the case (A–B) is the same as in the case (B–A).

- **Case I–I:**
  - if g_1 and g_2 are simultaneously non decreasing (resp. non increasing) on the interval $(\sigma_{i-1}, \sigma_{i+1})$ and if either g_1 or g_2 is non constant on this interval, then we set $\sigma_i^+(t) = \sigma_i^-(t) = \sigma_i - 2(T-t)$ (resp. $\sigma_i^+(t) = \sigma_i^-(t) = \sigma_i + 2(T-t)$), - otherwise, we set $\sigma_i^+(t) = \sigma_i^-(t) = \sigma_i$.
- **Case I–II:**
  - if g_1 is strictly increasing, then $\sigma_i^-(t) = \sigma_i - 2(T-t)$ and $\sigma_i^+(t) = \sigma_i$,
  - if g_1 is strictly decreasing, then $\sigma_i^-(t) = \sigma_i$ and $\sigma_i^+(t) = \sigma_i + 2(T-t)$,
- **Case I–III:**
  - if g_1 is strictly increasing, then $\sigma_i^+(t) = \sigma_i^-(t) = \sigma_i(t)$ where $\sigma_i(t)$ is the unique solution $x \in (\sigma_i - 2(T-t), \sigma_i)$ of the equation $g_2(x) = g_2(x + 2(T-t))$
  - if g_1 is strictly decreasing, then $\sigma_i^+(t) = \sigma_i^-(t) = \sigma_i(t)$ where $\sigma_i(t)$ is the unique solution $x \in (\sigma_i - 2(T-t), \sigma_i)$ of the equation $g_2(x) = g_2(x + 2(T-t))$
- **Case I–IV:**
  - if g_1 is non decreasing and $g_2(\sigma_i^-) < g_2(\sigma_i^+)$, then $\sigma_i^+(t) = \sigma_i^-(t) = \sigma_i - 2(T-t)$
  - if g_1 is non decreasing and $g_2(\sigma_i^-) > g_2(\sigma_i^+)$, then $\sigma_i^+(t) = \sigma_i^-(t) = \sigma_i$ - if g_1 is non increasing, it is the converse
- **Case II–II:** $\sigma_i^-(t) = \sigma_i - 2(T-t)$ and $\sigma_i^+(t) = \sigma_i + 2(T-t)$.
- **Case II–III:** $\sigma_i^-(t) = \sigma_i^+(t) = \sigma_i$.
- **Case III–III:** For $j = 1, 2$, let us denote by $y_j(t)$ the unique point $x \in (\sigma_i - 2(T-t), \sigma_i + 2(T-t))$ such that $g_j(x - 2(T-t)) = g_j(x + 2(T-t))$. Then $\sigma_i^+(t) = \sigma_i^-(t) = \sigma_i(t)$ where $x(\cdot) = \sigma_i(T - \cdot)$ is the unique absolutely continuous solution to:

$$\begin{cases} x'(s) \in -\partial \mathbf{1}_{I(s)}(x(s)) & \text{a.e.} \\ x(0) = \sigma_i \end{cases} \tag{7}$$

  where $I(s) = [\min\{y_1(T-s), y_2(T-s)\}, \max\{y_1(T-s), y_2(T-s)\}]$,

$$\mathbf{1}_{I(s)}(x) = \begin{cases} 0 & \text{if } x \in I(s) \\ +\infty & \text{otherwise} \end{cases} \quad \forall s \geq 0, \forall x \in \mathbb{R},$$

  and $\partial \mathbf{1}_{I(s)}(x)$ denotes the subdifferential of the convex function $\mathbf{1}_{I(s)}(\cdot)$.

- **Case III–IV:**
  - if $g_2(\sigma_i^-) < g_2(\sigma_i^+)$, then $\sigma_i^+(t) = \sigma_i^-(t) = \sigma_i(t)$ where $\sigma_i(t) = \min\{0, x(t)\}$, $x(t)$ being the unique solution in $(\sigma_i - 2(T-t), \sigma_i)$ of the equation $g_2(x) = g_2(x + 2(T-t))$
  - if $g_2(\sigma_i^-) > g_2(\sigma_i^+)$, then $\sigma_i^+(t) = \sigma_i^-(t) = \sigma_i(t)$ where $\sigma_i(t) = \max\{0, x(t)\}$, $x(t)$ being the unique solution in $(\sigma_i - 2(T-t), \sigma_i)$ of the equation $g_2(x) = g_2(x + 2(T-t))$
- **Case IV–IV:**
  - if $g_1(\sigma_i^-) < g_1(\sigma_i^+)$ and $g_2(\sigma_i^-) < g_2(\sigma_i^+)$, then $\sigma_i^-(t) = \sigma_i^+(t) = \sigma_i - 2(T-t)$.

– if $g_1(\sigma_i^-) > g_1(\sigma_i^+)$ and $g_2(\sigma_i^-) > g_2(\sigma_i^+)$, then $\sigma_i^-(t) = \sigma_i^+(t) = \sigma_i + 2(T - t)$.
– otherwise, $\sigma_i^-(t) = \sigma_i^+(t) = \sigma_i$.

Remarks.

1. Let us first point out that the above cases distinction covers all the possibilities because g belongs to $\mathscr{G}$.
2. For the cases (I–III) and (III–III), we prove below that the map $\sigma_i(\cdot)$ is indeed well defined.
3. Let us notice that $\sigma_i^-(t) = \sigma_i^+(t)$ unless we are in case (I-II) or (II, II). In particular, if $g \in \widetilde{\mathscr{G}}$, then we always have $\sigma_i^-(t) = \sigma_i^+(t)$.
4. In order to simplify the notations, we set

$$\sigma_0^+(t) = \sigma_0^-(t) = -\infty \text{ and } \sigma_{k+1}^+(t) = \sigma_{k+1}^-(t) = +\infty.$$

Definition of τ: We choose $\tau > 0$ such that

$$\forall i = 1, \ldots, k, \ \tau \leq (\sigma_{i+1} - \sigma_i)/24 \tag{8}$$

and such that, if g_j is discontinuous at σ_i (for $j \in \{1, 2\}$, $i \in \{1, \ldots, k\}$) with, for instance, $g_j(\sigma_i^-) < g_j(\sigma_i^+)$, then

$$\sup_{x \in [\sigma_i - 2\tau, \sigma_i)} g_j(x) < \inf_{x \in (\sigma_i, \sigma_i + 2\tau]} g_j(x). \tag{9}$$

Proposition 3.2. *The maps $\sigma_i^+(\cdot)$ and $\sigma_i^-(\cdot)$ defined above are well defined and Lispchitz continuous with a Lipschitz constant not larger than 2 on $[T - \tau, T]$.*

Remark. *This proves assertion (A-iv) of Proposition 3.1.*

Proof: We only do the proof for the cases (I–III) and (III–III), since the result is either obvious, or can be obtained in a similar way for the other cases.
For the case I–III, let us assume, for instance, that g_1 is increasing. Then $\sigma_i^+(t) = \sigma_i^-(t) = \sigma_i(t)$ where $\sigma_i(t)$ is the unique solution $x \in (\sigma_i - 2(T - t), \sigma_i)$ of the equation $g_2(x) = g_2(x + 2(T - t))$. There is indeed a unique solution to this equation in the interval $(\sigma_i - 2(T - t), \sigma_i)$ because the map $\phi_t : x \to g_2(x + 2(T - t)) - g_2(x)$ is increasing on this interval, and satisfies

$$\phi_t(\sigma_i - 2(T - t)) = g_2(\sigma_i) - g_2(\sigma_i - 2(T - t)) < 0$$

and

$$\phi_t(\sigma_i) = g_2(\sigma_i + 2(T - t)) - g_2(\sigma_i) > 0$$

since g_2 has a minimum on the interval $(\sigma_{i-1}, \sigma_{i+1})$ at σ_i.

For proving that the map $\sigma_i(\cdot)$ is 2-Lipschitz continuous, let us show that $\sigma_i(\cdot)$ is increasing and satisfies $\sigma_i(s) \geq \sigma_i(t) - 2(t - s)$ if $t > s$. Indeed, the map $\phi_s(\cdot)$ defined above is increasing on $(\sigma_i - 2(T - s), \sigma_i)$ and satisfies

$$\phi_s(\sigma_i(t)) = g_2(\sigma_i(t) + 2(T-s)) - g_2(\sigma_i(t))$$
$$= g_2(\sigma_i(t) + 2(T-s)) - g_2(\sigma_i(t) + 2(T-t)) > 0$$

because g_2 is increasing on (σ_i, σ_{i+1}), and

$$\phi_s(\sigma_i(t) - 2(t-s)) = g_2(\sigma_i(t) + 2(T-t)) - g_2(\sigma_i(t) - 2(t-s))$$
$$= g_2(\sigma_i(t)) - g_2(\sigma_i(t) - 2(t-s)) < 0$$

because g_2 is decreasing on (σ_{i-1}, σ_i). Therefore $\sigma_i(s)$ belongs to the interval $(\sigma_i(t) - 2(t-s), \sigma_i(t))$, which is the desired result.

Case III–III: Following the proof above, it is not difficult to check that the maps $y_j(\cdot)$ $(j = 1, 2)$ are 2-Lipschitz continuous. Let us set $z_1(s) = \min\{y_1(T-s), y_2(T-s)\}$ and $z_2(s) = \max\{y_1(T-s), y_2(T-s)\}$. We prove in the same time that a solution $x(\cdot)$ to (7) exists and is 2-Lipschitz continuous. For this, we use the so-called viability Theorem (see for instance [1]) applied to the set-valued map F and to the locally compact set K defined as follows:

$$K = \{(s,x) \in [0,\tau) \times \mathbb{R} \,|\, z_1(s) \le x \le z_2(s)\}$$

$$F(s,x) = \begin{cases} \{1\} \times \{0\} & \text{if } z_1(s) < x < z_2(s) \\ \{1\} \times [0,2] & \text{if } z_1(s) = x < z_2(s) \\ \{1\} \times [-2,0] & \text{if } z_1(s) < x = z_2(s) \\ \{1\} \times [-2,2] & \text{if } z_1(s) = x = z_2(s) \end{cases}$$

It is easy to check that F is usc with convex compact values and that K is a locally compact viability domain for F, because z_1 and z_2 are 2-Lipschitz continuous. Accordingly for the initial condition $(0, \sigma_i)$ there exists (locally) a solution $(t(\cdot), x(\cdot))$ to the differential inclusion

$$\begin{cases} (t'(s), x'(s)) \in F(t(s), x(s)) \\ (t(0), x(0)) = (0, \sigma_i) \end{cases}$$

which remains in K. Obviously $t(s) = s$ and it is easily seen that the solution can be extended to the interval $[0,\tau)$. Since $F(s,x) \subset -\partial \mathbf{1}_{I(s)}(x)$ for any (s,x), $x(\cdot)$ is also a solution to (7). Moreover, from its construction, $x(\cdot)$ is 2-Lipschitz continuous.

Let us now prove that the solution of (7) is unique. Let $x_1(\cdot)$ and $x_2(\cdot)$ be two solutions of (7). Then, for almost every s,

$$\frac{d}{ds}(x_1(s) - x_2(s))^2 = 2(x_1(s) - x_2(s))(x_1'(s) - x_2'(s)) \le 0$$

because $x_j'(s) \in -\partial \mathbf{1}_{I(s)}(x_j(s))$ for $j = 1, 2$, and because of the well-known monotonicity property of the subdifferential of convex functions. Therefore $(x_1(s) - x_2(s))^2 \le (x_1(0) - x_2(0))^2 = 0$. □

Definition of the completely maximal equilibrium feedback: Let $(u^*, v^*) \in \mathcal{U} \times \mathcal{V}$ be the pair of strategies defined by: for $i = 0, \ldots, k$ and $t \in [T-\tau, T]$,

$$\forall x \in [\sigma_i^+(t), \sigma_{i+1}^-(t)],\ u^*(t,x) = \begin{cases} 1 & \text{if } g_1 \text{ is increasing on } (\sigma_i, \sigma_{i+1}) \\ -1 & \text{if } g_1 \text{ is decreasing on } (\sigma_i, \sigma_{i+1}) \end{cases}$$

and

$$\forall x \in [\sigma_i^+(t), \sigma_{i+1}^-(t)],\ v^*(t,x) = \begin{cases} 1 & \text{if } g_2 \text{ is increasing on } (\sigma_i, \sigma_{i+1}) \\ -1 & \text{if } g_2 \text{ is decreasing on } (\sigma_i, \sigma_{i+1}) \end{cases}$$

and

$$\forall x \in (\sigma_i^-(t), \sigma_i^+(t)),\ u^*(t,x) = v^*(t,x)\begin{cases} 1 & \text{if } x \le (\sigma_i^-(t) + \sigma_i^+(t))/2 \\ -1 & \text{otherwise} \end{cases}$$

When g_1 and g_2 are constant on the interval (σ_i, σ_{i+1}), we have to be more carefull. Let $x \in [\sigma_i^+(t), \sigma_{i+1}^-(t)]$ and let us assume that g_1 and g_2 are constant on the interval (σ_i, σ_{i+1}). Then we define $u^*(t,x)$ and $v^*(t,x)$ in the following way: Let us set $a_i(t) = \frac{2}{3}\sigma_i + \frac{1}{3}\sigma_{i+1} - 2(T-t)$ and $b_i(t) = \frac{1}{3}\sigma_i + \frac{2}{3}\sigma_{i+1} +2(T-t)$. Then

$$u^*(t,x) = v^*(t,x) = \begin{cases} 1 & \text{if } x \le a_i(t) \\ -1 & \text{if } x \ge b_i(t) \\ 0 & \text{otherwise} \end{cases}$$

Remark. From assumption (8) on τ and since $\sigma_i^-(\cdot)$ and $\sigma_i^+(\cdot)$ are 2-Lipschitz, we have

$$\sigma_0(t) = -\infty < \sigma_1^-(t) \le \sigma_1^+(t) < \ldots < \sigma_k^-(t) \le \sigma_k^+(t) < \sigma_{k+1} = +\infty$$

and $a_i(t) < b_i(t)$ on $[T-\tau, T]$. Hence (u^*, v^*) is well defined.

Lemma 3.3. *Let us set*

$$\forall t \in [T-\tau, T], \quad S_t = \{\sigma_i^-(t), \quad i = 1, \ldots, k\} \cup \{\sigma_i^+(t), \quad i = 1, \ldots, k\}.$$

1) *Then for any* $t_0 \in [T-\tau, T)$, *for any* $x_0 \notin S_{t_0}$, *there is a unique solution* $x^*(\cdot) \in \mathcal{X}(t_0, x_0, u^*, v^*)$. *This solution satisfies*

   $\forall t \in [t_0, T), x^*(t) \notin S_t.$

2) *If moreover,* $g \in \widetilde{\mathcal{G}}$, *then*

   $x^*(T) \notin \Sigma.$

Remark: *In the terminology introduced in Proposition 3.1, the first statement says that* (u^*, v^*) *is proper with respect to the set* (S_t)*—which proves (A-v) of Proposition 3.1—, while the second statement says that* (u^*, v^*) *is proper with respect to the terminal set* Σ*—hence (A-vi) of Proposition 3.1 holds.*

Proof: The proof is a straightforward (but quite tedious !) verification by listing the various cases. □

3.3 The pair of strategies (u^*, v^*) is a maximal Nash equilibrium feedback

We prove here that the feedback (u^*, v^*) constructed above is a maximal Nash equilibrium feedback and that its associated payoff coincides with the payoff of any completely maximal Nash equilibrium feedback. In doing so, we prove in particular parts (A-iii) and (B) of Proposition 3.1. Let us recall that

$$\forall t \in [T-\tau, T], \quad S_t = \{\sigma_i^-(t), \quad i = 1, \dots, k\} \cup \{\sigma_i^+(t), i = 1, \dots, k\}.$$

Let (t_0, x_0) belong to $[T-\tau, T] \times \mathbb{R}$. For any feedback $(u, v) \in \mathcal{U} \times \mathcal{V}$, for any solution $x(\cdot) \in \mathcal{X}(t_0, x_0, u, v)$, we have

$$\forall t \in [t_0, T], \quad x(t) \in [x_0 - 2(t - t_0), x_0 + 2(t - t_0)].$$

Therefore the upper- and lower-values of a feedback (u, v) at (t_0, x_0) only depend on the restriction of this feedback to a neighbourhood of the set $R(t_0, x_0)$ defined by

$$R(t_0, x_0) = \bigcup_{t \in [t_0, T]} \{t\} \times [x_0 - 2(t - t_0), x_0 + 2(t - t_0)].$$

From the definition of τ (see (8)), the interval $I = [x_0 - 2(T - t_0), x_0 + 2(T - t_0)]$ contains at most one σ_i. If the interval I does not contain any σ_i, arguing as in Example 2.16 and 2.17, we can show that the restriction to a neighbourhood of $R(t_0, x_0)$ of the feedback (u^*, v^*) is a maximal Nash equilibrium feedback and that its associated payoff coincides with the payoff of any completely maximal Nash equilibrium feedback on this neighbourhood.

If the interval I contains σ_i for some $i = 1, \dots, k$, we have to discuss according to the type of σ_i. In order to avoid lengthy cases discussions, we only give a complete proof for the case III-III which is more delicate and leave the other cases to the reader.

From now on, **we assume that σ_i of type III–III**. Namely, g_1 and g_2 are continuous, are decreasing on (σ_{i-1}, σ_i) and increasing on (σ_i, σ_{i+1}). Let us set $R = \{(t, x) \in [T-\tau, T] \times \mathbb{R} \,|\, x \in [\sigma_i - 2(T - t_0), \sigma_i + 2(T - t_0)]\}$ and let us notice that, if $(t_0, x_0) \in R$, then $R(t_0, x_0) \subset R$. We are going to show that the restriction of (u^*, v^*) to the set R is a maximal Nash equilibrium feedback and that its associated payoff coincides with the payoff of any completely maximal Nash equilibrium feedback on this set.

Let us now recall the definition of (u^*, v^*) in R: For $j = 1, 2$, let us denote by $y_j(t)$ the unique point $x \in (\sigma_i - 2(T - t), \sigma_i + 2(T - t))$ such that $g_j(x - 2(T - t)) = g_j(x + 2(T - t))$. Let $\sigma_i^+(t) = \sigma_i^-(t) = \sigma_i(t)$ where $x(\cdot) = \sigma_i(T - \cdot)$ is the unique absolutely continuous solution to:

$$\begin{cases} x'(s) \in -\partial \mathbf{1}_{I(s)}(x(s)) \quad \text{a.e.} \\ x(0) = \sigma_i \end{cases}$$

Then

$$u^*(t, x,) = v^*(t, x) = \begin{cases} 1 & \text{if } x \geq \sigma(t) \\ -1 & \text{otherwise} \end{cases}$$

Let us notice that the map $\sigma_i(\cdot)$ is non decreasing on $\{\sigma_i(\cdot) > \min\{y_1(\cdot), y_2(\cdot)\}\}$ and non increasing on $\{\sigma_i(\cdot) < \max\{y_1(\cdot), y_2(\cdot)\}\}$.

We first prove that (u^*, v^*) is a Nash equilibrium feedback : For this let us fix $t_0 \in [T-\tau, T)$ and $x_0 \in [\sigma_i - 2(T - t_0), \sigma_i + 2(T - t_0)]$. We also fix some $(u, v) \in \mathcal{U} \times \mathcal{V}$.

If $x_0 < \min\{y_1(t_0), y_2(t_0)\}$, then $g(x_0 - 2(T - t_0))$ is a maximal payoff at (t_0, x_0). Since the unique solution $x^*(\cdot) \in \mathcal{X}(t, x_0, u^*, v^*)$ is $x^*(\cdot) = x_0 - 2(\cdot - t_0)$, this proves that

$$g_1(x_0 - 2(T - t_0)) = J_1(t_0, x_0, u^*, v^*) \geq J_1^\sharp(t_0, x_0, u, v^*)$$

and

$$g_2(x_0 - 2(T - t_0)) = J_2(t_0, x_0, u^*, v^*) \geq J_2^\sharp(t_0, x_0, u^*, v).$$

If $x_0 > \max\{y_1(t_0), y_2(t_0)\}$, we can argue in a symmetric way because $g(x_0 + 2(T - t_0))$ is a maximal payoff at (t_0, x_0).

Let us now assume that $x_0 \in [\min\{y_1(t), y_2(t)\}, \max\{y_1(t), y_2(t)\}]$ and $x_0 \neq \sigma_i(t)$. To fix the ideas, we consider the case $y_1(t_0) < y_2(t_0)$ and $x_0 < \sigma_i(t_0)$. In particular, these assumptions imply that $\sigma_i(t_0) > y_1(t_0)$. Let us start by proving that

$$J_2(t_0, x_0, u^*, v^*) \geq J_2^\sharp(t_0, x_0, u^*, v). \tag{10}$$

Let $x^*(\cdot) = x_0 - 2(T - \cdot)$ be the unique solution in $\mathscr{X}(t_0, x_0, u^*, v^*)$ and let us fix some $x(\cdot) \in \mathscr{X}(t_0, x_0, u^*, v)$. If on the one hand, $x(T) \geq \sigma_i$, then

$$g_2(x(T)) \leq g_2(x_0 + 2(T - t_0)) < g_2(x_0 - 2(T - t_0)) = g_2(x^*(T))$$

because g_2 is increasing on (σ_i, σ_{i+1}) and $x_0 < y_2(t_0)$. If, on another hand, $x(T) < \sigma_i$, then

$$g_2(x(T)) \leq g_2(x_0 - 2(T - t_0)) = g_2(x^*(T))$$

because g_2 is decreasing on (σ_{i-1}, σ_i). Therefore we have proved that $g_2(x(T)) \leq g_2(x^*(T))$, which entails (10).

Let us now prove that

$$J_1(t_0, x_0, u^*, v^*) \geq J_1^\sharp(t_0, x_0, u, v^*). \tag{11}$$

Let $x(\cdot)$ belong to $\mathscr{X}(t_0, x_0, u, v^*)$. Since g_1 is decreasing on (σ_{i-1}, σ_i) and since $x(T) \geq x_0 - 2(T - t_0) = x^*(T)$, if $x(T) \leq \sigma_i$ then we have $g_1(x(T)) \leq g_1(x^*(T))$. We assume from now on that $x(T) > \sigma_i$. In particular, there is some $t \in (t_0, T)$ such that $x(t) = \sigma_i(t)$. Let us set

$$\theta = \min\{t \geq t_0 \,|\, x(t) \geq \sigma_i(t)\}.$$

Since $v^* = -1$ in a neighbourhood of $x(t)$ for $t \in [t_0, \theta)$, $x(\cdot)$ is non increasing on this interval.

We claim that there is some $\theta_0 \in [t_0, \theta]$ such that $x(\theta_0) = y_1(\theta_0)$. Indeed, let us assume on the contrary that $x(t) > y_1(t)$ on $[t_0, \theta]$. This implies that $\sigma_i(t) > y_1(t)$ on $[t_0, \theta]$. Therefore, the map $s \to \sigma_i(\theta - s)$ is non increasing on the interval $[0, \theta - t_0]$ while $s \to x(\theta - s)$ is non decreasing. This implies that

$$\sigma_i(\theta) - x(\theta) \geq \sigma_i(t_0) - x(t_0) > 0,$$

which is in contradiction with the definition of θ. Therefore we have proved the existence of some $\theta_0 \in [t_0, \theta]$ such that $x(\theta_0) = y_1(\theta_0)$.

Since g_1 is increasing on (σ_i, σ_{i+1}) and since $x(T) \leq x(\theta_0) + 2(T - \theta_0) = y_1(\theta_0) + 2(T - \theta_0)$, we have that

$$g_1(x(T)) \leq g_1(y_1(\theta_0) + 2(T - \theta_0)) = g_1(y_1(\theta_0) - 2(T - \theta_0))$$

from the very definition of $y_1(\cdot)$. Moreover, we have

$$y_1(\theta_0) = x(\theta_0) \geq x_0 - 2(\theta_0 - t_0),$$

which entails that

$$g_1(y_1(\theta_0) - 2(T - \theta_0)) \leq g_1(x_0 - 2(T - t_0)) = g_1(x^*(T))$$

since g_1 is decreasing on (σ_{i-1}, σ_i). Thus $g_1(x(T)) \leq g_1(x^*(T))$ and (11) holds. So we have proved that (u^*, v^*) is a Nash equilibrium feedback.

We now prove that (u^*, v^*) **is maximal.** Let us fix $t_0 \in [T - \tau, T)$ and $x_0 \in (\sigma_i - 2(T - t_0), \sigma_i + 2(T - t_0))$. Assume first that $y_1(t_0) \neq y_2(t_0)$. Then there is a maximal payoff at (t_0, x_0) if and only if, either $x_0 < \min\{y_1(t_0), y_2(t_0)\}$—in which case $g(x_0 - 2(T - t_0))$ is this maximal payoff and we have already proved that $J^\sharp(t_0, x_0, u^*, v^*) = g(x_0 - 2(T - t_0))$—, or $x_0 > \max\{y_1(t_0), y_2(t_0)\}$—in which case $g(x_0 + 2(T - t_0))$ is this maximal payoff and we know that $J^\sharp(t_0, x_0, u^*, v^*) = g(x_0 + 2(T - t_0))$. If $y_1(t_0) = y_2(t_0)$, then $\sigma_i(t_0) = y_1(t_0) = y_2(t_0)$ and there is also a maximal payoff at the point $(t_0, \sigma_i(t_0))$. Since the trajectories $x^+(\cdot) = \sigma_i(t_0) + 2(\cdot - t_0)$ and $x^-(\cdot) = \sigma_i(t_0) - 2(\cdot - t_0)$ belong to $\mathscr{X}(t_0, \sigma_i(t_0), u^*, v^*)$, we have $J^\sharp(t_0, \sigma_i(t_0), u^*, v^*) = g(\sigma_i(t_0) + 2(T - t_0)) = g(\sigma_i(t_0) - 2(T - t_0))$, which is the maximal payoff at $(t_0, \sigma_i(t_0))$.

We finally prove that any completely maximal equilibrium feedback has the same payoff as (u^*, v^*). For this, let (u_1^*, v_1^*) be some completely maximal Nash equilibrium feedback and $(\Sigma_t)_{t\in[0,T]}$ its associated set of zero measure. Let us now define, for $t \in [T - \tau, T]$,

$$\alpha(t) = \inf\{x \in [\sigma_i - 2(T - t), \sigma_i + 2(T - t)] \mid \\ \forall x(\cdot) \in \mathscr{X}(t, x, u_1^*, v_1^*), x(T) > x - 2(T - t)\}.$$

and

$$\beta(t) = \sup\{x \in [\sigma_i - 2(T - t), \sigma_i + 2(T - t)] \mid \\ \forall x(\cdot) \in \mathscr{X}(t, x, u_1^*, v_1^*), x(T) < x + 2(T - t)\}.$$

The main part of the proof amounts to establish that $\alpha(t) = \beta(t) = \sigma_i(t)$ for $t \in [T - \tau, T]$. We split the proof of this into several claims.

Claim 1: We first prove that

$$\forall t \in [T - \tau, T], \min\{y_1(t), y_2(t)\} \leq \alpha(t) \leq \beta(t) \leq \max\{y_1(t), y_2(t)\}.$$

Proof of claim 1: Assume first that $\min\{y_1(t), y_2(t)\} > \alpha(t)$. Then there is some $x < \min\{y_1(t), y_2(t)\}$ such that, any solution $x(\cdot) \in \mathscr{X}(t, x, u_1^*, v_1^*)$ satisfies $x(T) > x - 2(T - t)$. From the definition of $y_1(t)$ and $y_2(t)$, the payoff $g(x - 2(T - t))$ is the maximal payoff at the point (t, x) and this payoff can only be attained by the trajectory $x(\cdot) = x - 2(\cdot - t)$, which does not belong to $\mathscr{X}(t, x, u_1^*, v_1^*)$. Since $\mathscr{X}(t, x, u_1^*, v_1^*)$ is compact, this implies that $J^\sharp(t, x, u_1^*, v_1^*) < g(x - 2(T - t))$, which is in contradiction with the fact that (u_1^*, v_1^*) is maximal. Hence we have proved that $\min\{y_1(t), y_2(t)\} \leq \alpha(t)$.

We can prove in a similar way that $\beta(t) \leq \max\{y_1(t), y_2(t)\}$.

Let us now show that $\alpha(t) \leq \beta(t)$ holds for all t. Indeed, if, on the contrary, we had $\beta(t) < \alpha(t)$ for some t, then, for any $x \in (\beta(t), \alpha(t))$, there would exist $x_1(\cdot) \in \mathscr{X}(t, x, u_1^*, v_1^*)$ with $x_1(t) = x + 2(T - t)$ and $x_2(\cdot) \in \mathscr{X}(t, x, u_1^*, v_1^*)$ with $x_2(t) = x - 2(T - t)$. Since $g(x_1(T)) \neq g(x_2(T))$ (unless $x = y_1(t) = y_2(t)$), there is a contradiction with the fact that

$$J^\sharp(t, x, u_1^*, v_1^*) = J^\flat(t, x, u_1^*, v_1^*) \text{ for almost all } x \in (\beta(t), \alpha(t)).$$

So we have proved the inequality $\alpha(t) \leq \beta(t)$ for all t and the proof of claim 1 is complete.

Claim 2: $\alpha(\cdot)$ is usc while $\beta(\cdot)$ is lsc, and

$$\forall s < t, \alpha(s) \geq \alpha(t) - 2(t-s), \text{ while } \beta(s) \leq \beta(t) + 2(t-s). \tag{12}$$

In particular, $\alpha(\cdot)$ and $\beta(\cdot)$ are left continuous.

Proof of claim 2: We only do the proof for $\alpha(\cdot)$, the proof for $\beta(\cdot)$ being symmetric. Since the set-valued map $(t,x) \to \mathscr{X}(t,x,u_1^*,v_1^*)$ is upper-semi continuous with compact values, the map $\alpha(\cdot)$ is clearly ucs.

Let us now prove (12). For this, we consider the game with terminal time t and terminal payoff $J^\sharp(t,\cdot,u_1^*,v_1^*)$. Since (u_1^*,v_1^*) is completely maximal, (u_1^*,v_1^*) is a maximal Nash equilibrium strategy for this game. Since both g_1 and g_2 are decreasing on (σ_{i-1},σ_i), the functions $J_j(t,\cdot,u_1^*,v_1^*)$ are decreasing on $(\sigma_i - 2(T-t), \alpha(t)]$ because they coincide with $g_j(\cdot - 2(T-t))$ on this interval. Therefore, for any $s<t$ and any $x < \alpha(t) - 2(t-s)$ the payoff $J_j(t, x-2(t-s), u_1^*, v_1^*)$ is maximal for the game with terminal time t and terminal payoff $J^\sharp(t,\cdot,u_1^*,v_1^*)$ and is only attained at the point $x-2(t-s)$. This implies that any solution $x(\cdot) \in \mathscr{X}(s,x,u_1^*,v_1^*)$ satisfies $x(t) = x-2(t-s)$. From this one derives easily the desired inequality (12). Hence the proof of claim 2 is complete.

Let us keep in mind that the previous proof shows that the maps $J_j^\sharp(t,\cdot,u_1^*,v_1^*)$ are decreasing on $(\sigma_i - 2(T-t), \alpha(t))$ for any t.

Claim 3: Let us now assume that, for some t_0, we have $\alpha(t_0) < \max\{y_1(t_0), y_2(t_0)\}$. We claim that there is some positive ϵ such that $\alpha(t) \geq \alpha(t_0)$ for $t \in [t_0-\epsilon, t_0]$.

In particular this implies that $\alpha(\cdot)$ is non increasing on the intervals where $\alpha(\cdot) < \max\{y_1(\cdot), y_2(\cdot)\}$.

Proof of claim 3: To fix the ideas, we assume that $\alpha(t_0) < y_2(t_0)$. Then $\alpha(t_0) - 2(T-t_0)$ is the unique point of maximum of g_2 on the interval $[\alpha(t_0) - 2(T-t_0), \alpha(t_0) + 2(T-t_0)]$. Since the map $x^*(\cdot) = \alpha(t_0) - 2(\cdot - t_0)$ belongs to $\mathscr{X}(t_0, \alpha(t_0), u_1^*, v_1^*)$, we have

$$J_2^\sharp(t_0, \alpha(t_0), u_1^*, v_1^*) = g_2(\alpha(t_0) - 2(T-t_0)),$$

and there is some $\gamma > 0$ such that

$$\forall x \in (\alpha(t_0), \alpha(t_0)+\gamma), J_2^\sharp(t_0,x,u_1^*,v_1^*) < J_2^\sharp(t_0,\alpha(t_0),u_1^*,v_1^*).$$

We set $\epsilon = \gamma/2$. Let us now prove the intermediate result: $\forall t \in [t_0-\epsilon, t_0]$,

$$\forall x < \alpha(t_0) \text{ with } x \notin \Sigma_t, \forall x(\cdot) \in \mathscr{X}(t,x,u_1^*,v_1^*), x(t_0) \leq \alpha(t_0). \tag{13}$$

Indeed, if on the contrary we had $x(t_0) > \alpha(t_0)$, then

$$J_2(t,x,u_1^*,v_1^*) = J_2(t_0,x(t_0),u_1^*,v_1^*) < J_2^\sharp(t_0,\alpha(t_0),u_1^*,v_1^*) < J_2^\sharp(t_0,x,u_1^*,v_1^*),$$

because $x(t_0) < \alpha(t_0) + \gamma$. This leads to a contradiction with the fact that (u_1^*,v_1^*) is Nash since the strategy

$$v(s,y) = \begin{cases} -u_1^*(s,y) & \text{if } s < t_0 \\ v_1^*(s,y) & \text{otherwise} \end{cases}$$

gives

$$J_2^\sharp(t,x,u_1^*,v) = J_2^\sharp(t_0,x,u_1^*,v_1^*) > J_2^\sharp(t,x,u_1^*,v_1^*).$$

Thus we have proved (13).

Since the minimum of two solutions is still a solution and since the functions $J_j^\sharp(t_0,\cdot,u_1^*,v_1^*)$ are decreasing on $(\sigma_i - 2(T-t_0), \alpha(t_0))$, one easily derive from (13) that, for any $t \in [t_0-\epsilon, t_0]$, the functions $J_j^\sharp(t,\cdot,u_1^*,v_1^*)$ are non increasing on the interval $[\sigma_i - 2(T-t), \alpha(t_0)]$.

For completing the proof of the claim, we argue by contradiction by assuming that there is some $t_1 \in (t_0-\epsilon, t_0)$ such that $\alpha(t_1) < \alpha(t_0)$. Then for any $t < t_1$ such that $\alpha(t_1) + 2(t_1 - t) \leq \alpha(t_0)$, and for any $x < \alpha(t_1)$, the payoff $J^\sharp(t_1, x - 2(t_1 - t), u_1^*, v_1^*)$ is the unique maximal payoff at the point (t,x) for the game with terminal time t_1 and terminal payoff $J^\sharp(t_1,\cdot,u_1^*,v_1^*)$, because the function $J_j^\sharp(t_1,\cdot,u_1^*,v_1^*)$ are non increasing on the interval $(\sigma_i - 2(T-t_1), \alpha(t_0))$ and strictly decreasing on $(\sigma_i - 2(T-t_1), \alpha(t_1))$. Therefore, for any $x(\cdot) \in \mathscr{X}(t,x,u_1^*,v_1^*)$, we have $x(t_1) = x - 2(t_1 - t)$. Hence, since $x(t_1) < \alpha(t_1)$, we have $x(T) = x - 2(T-t)$. Thus $\alpha(t) \geq \alpha(t_1)$ for any t such that $\alpha(t_1) + 2(t_1 - t) \leq \alpha(t_0)$. Since $\alpha(\cdot)$ is usc and left continuous, one derives easily a contradiction. So claim 3 is proved.

Claim 4: The next step consists in proving that $\alpha(t) \geq \sigma_i(t)$ and that $\beta(t) \leq \sigma_i(t)$ for any $t \in [T-\tau, T]$.

Proof of claim 4: We only do the proof for $\alpha(\cdot)$. Let us assume on the contrary that there is some maximal interval (t_0, t_1) on which inequality $\alpha(t) < \sigma_i(t)$ hold. Since $\alpha(\cdot)$ is left continuous, we have $\alpha(t_1) = \sigma_i(t_1)$. Since, on the one hand, $\alpha(t) < \sigma_i(t) \leq \max\{y_1(t), y_2(t)\}$, we deduce from the previous step that $\alpha(\cdot)$ is non increasing on (t_0, t_1). On the other hand, since $\min\{y_1(t), y_2(t)\} \leq \alpha(t) < \sigma_i(t)$ on (t_0, t_1), the map $\sigma_i(\cdot)$ is non decreasing on (t_0, t_1) from its construction. Therefore the map $\sigma_i(\cdot) - \alpha(\cdot)$ is non decreasing on (t_0, t_1). Since it is positive, vanishes at t_1 and is left-continuous at this point, we have found a contradiction. So claim 4 is proved.

Combining claim 4 and claim 1, we get the desired result: $\alpha(\cdot) = \beta(\cdot) = \sigma_i(\cdot)$ on $[T-\tau, T]$. From this one can prove without difficulty that the payoff for (u^*, v^*) is the same as for (u_1^*, v_1^*). Therefore the proof of the uniqueness is complete.

3.4 The payoff belongs to the class $\widetilde{\mathscr{G}}$ on $[T-\tau, T)$

The aim of this subsection is to prove part (ii) of Proposition 3.1. Namely, we want to show that, for any $t \in [T-\tau, T)$, the map $J^\sharp(t,\cdot,u^*,v^*)$ belongs to $\widetilde{\mathscr{G}}$.

A straightforward computation shows that, for any $t \in [T-\tau, T]$ and any $x \notin S_t$,

$$\text{for } j = 1,2, \quad J_j(t,x,u^*,v^*) = g_j(y(x))$$

where, if $x \in (\sigma_i^+(t), \sigma_{i+1}^-(t))$, for some $i = 0, \ldots, k$,

$$y(x) = \begin{cases} x + 2(T-t) & \text{if } g_1 \text{ and } g_2 \text{ are increasing on } (\sigma_i, \sigma_{i+1}) \\ x - 2(T-t) & \text{if } g_1 \text{and } g_2 \text{ are decreasing on } (\sigma_i, \sigma_{i+1}) \\ \min\{\max\{x, \sigma_i\}, \sigma_{i+1}\} & \text{otherwise} \end{cases}$$

and, if $x \in (\sigma_i^-(t), \sigma_i^+(t))$ for some $i = 1, \dots, k$,

$$y(x) = \sigma_i.$$

Let us point out that $y(x) \neq x^*(T)$ (where $x^*(\cdot) \in \mathscr{X}(t, x, u^*, v^*)$) only in the case g_1 and g_2 constant on the interval (σ_i, σ_{i+1}).

Therefore, for any $i = 0, \dots, k$, for $t \in [T - \tau, T]$,

- either $J_1^\sharp(t, \cdot, u^*, v^*)$ (resp. $J_2^\sharp(t, \cdot, u^*, v^*)$) is continuous and strictly increasing or decreasing on $(\sigma_{i_\sharp}^+(t), \sigma_{i+1}^-(t))$,
- or $J_1^\sharp(t, \cdot, u^*, v^*)$ and $J_2^\sharp(t, \cdot, u^*, v^*)$ are simultaneously constant on the interval $(\sigma_i(t)^+, \sigma_{i+1}^-(t))$ or on the interval $(\sigma_i(t)^-, \sigma_i^+(t))$.

Moreover $J_1^\sharp(t, \cdot, u^*, v^*)$ and $J_2^\sharp(t, \cdot, u^*, v^*)$ are usc on $\mathbb{R}$ and left- or right-continuous at each point. Finally, if for instance $J_1^\sharp(t, \cdot, u^*, v^*)$ has a strict local maximum at the point $\sigma_i^-(t)$, then we are necessarily in case (II-III), which implies that $J_1^\sharp(t, \cdot, u^*, v^*)$ and $J_2^\sharp(t, \cdot, u^*, v^*)$ are continuous at $\sigma_i^-(t)$ and that $J_2^\sharp(t, \cdot, u^*, v^*)$ has a strict local minimum. Therefore $J^\sharp(t, \cdot, u^*, v^*)$ belongs to $\widetilde{\mathscr{G}}$. □

3.5 The feedback (u^*, v^*) is completely maximal

In order to complete the proof of part (A-i) of Proposition 3.1, it only remains to prove that the feedback (u^*, v^*) is also completely maximal. To this end, let us fix $T_1 \in (T - \tau, T]$. Let (u_1^*, v^*) be the Nash equilibrium feedback constructed as above for the game with horizon T_1 and terminal payoff $J^\sharp(t, \cdot, u^*, v^*)$. It is defined (at least) on $[T - \tau, T_1]$. A straightforward (but again tedious) proof shows that we have, for any $(t, x) \in [T - \tau, T_1] \times \mathbb{R}$,

$$\sup_{x(\cdot) \in \mathscr{X}(t,x,u_1^*,v_1^*)} J^\sharp(t, x(T_1), u^*, v^*) = J^\sharp(t, x, u^*, v^*) \tag{14}$$

(the left-hand side of the equality is nothing but the upper payoff of the feedback (u_1^*, v_1^*)). We have already proved that (u_1^*, v_1^*) is maximal. Therefore, for any point $(t_0, x_0) \in [T - \tau, T_1)$ at which a maximal payoff $e \in \mathbb{R}^2$ for the game with horizon T_1 and terminal payoff $J^\sharp(t, \cdot, u^*, v^*)$ exists, we have

$$e = \sup_{x(\cdot) \in \mathscr{X}(t,x,u_1^*,v_1^*)} J^\sharp(t, x(T_1), u^*, v^*)$$

Using (14) gives $e = J^\sharp(t, x, u^*, v^*)$, which proves that (u^*, v^*) is completely maximal.

This completes the proof of Proposition 3.1. □

4 Global existence and uniqueness

In the previous section, we have obtained short time existence and uniqueness of a completely maximal Nash equilibrium feedback. We now complete the proof of Theorem 2.11 by showing that that it is possible to concatenate Nash equilibrium feedbacks (Proposition 4.5), and that a Nash equilibrium feedback defined on intervals $[t, T]$ for any $t > T_0$, can naturally be extented to the closed interval $[T_0, T]$ (Proposition 4.6). Then we finally prove Theorem 2.15.

4.1 Some preliminary results

Let us first give in this part a list of technical results which are needed later on.

Lemma 4.1. *Let* $(u,v) \in \mathscr{U} \times \mathscr{V}, (t_0,x_0) \in [0,T) \times \mathbb{R}$. *For any* $x(\cdot) \in \mathscr{X}(t_0,x_0,u,v), h > 0$, *there exists some strategy* $u_h \in \mathscr{U}$, *some trajectories* $x_h^+(\cdot) \in \mathscr{X}(t_0,x_0+h,u_h,v)$ *and* $x_h^-(\cdot) \in \mathscr{X}(t_0,x_0-h,u_h,v)$ *such that*

- *either* $x_h^+(T) = x(T)$ *or* $x_h^-(T) = x(T)$
- *or* $|x_h^+(T) - x(T)| + |x_h^-(T) - x(T)| \le 2h$.

Proof of Lemma 4.1: Let $u_h^1 \in \mathscr{U}$ be the strategy defined by

$$u_h^1(t,x) = \begin{cases} -1 & \text{if } x > x(t) \\ 1 & \text{otherwise} \end{cases}$$

Let us fix $y_h^+(\cdot) \in \mathscr{X}(t_0,x_0+h,u_h^1,v)$ and $y_h^-(\cdot) \in \mathscr{X}(t_0,x_0-h,u_h^1,v)$. We set

$$\tau_h = \inf\{t > t_0 \mid y_h^+(t) = x(t) \text{ or } y_h^-(t) = x(t)\},$$

with the convention $\tau_h = T$ if the set in the right-hand side is empty.

Let us notice that, from the definition of $u_h^1, y_h^+(\cdot)$ is non increasing on $[t_0,\tau_h]$ while $y_h^-(\cdot)$ is non decreasing on this interval. We consider now two cases:

First case: $\tau_h = T$. Then

$$\forall t \in [t_0,T], x_0 - h \le y_h^-(t) \le x(t) \le y_h^+(t) \le x_0 + h.$$

Hence, for $t = T$, the second inequality of the Lemma holds true if we set $u_h = u_h^1$, $x_h^+(\cdot) = y_h^+(\cdot)$ and $x_h^-(\cdot) = y_h^-(\cdot)$.

Second case: If, on the contrary, $\tau_h < T$, let us assume for instance that $x_h^+(\tau_h) = x(\tau_h)$. Then we define a new strategy $u_h \in \mathscr{U}$ and a new solution $x_h^+(\cdot)$ by setting

$$u_h(t,x) = \begin{cases} u_h^1(t,x) & \text{if } t \le \tau_h \\ u^*(t,x) & \text{otherwise} \end{cases}$$

and

$$x_h^+(t) = \begin{cases} y_h^+(t) & \text{if } t \le \tau_h \\ x(t) & \text{otherwise} \end{cases}$$

It is clear that $x_h^+(\cdot)$ belongs to $\mathscr{X}(t_0,x_0,u_h,v)$ and that $x_h^+(T) = x(T)$. Therefore the proof is complete. □

Corollary 4.2. *Let* $g = (g_1,g_2) : \mathbb{R} \to \mathbb{R}^2$ *be such that* g_1 *(resp.* g_2*) is, at each point, either right- or left-continuous. Let* $(\bar{u},\bar{v}) \in \mathscr{U} \times \mathscr{V}$ *some Nash equilibrium feedback on the interval* $[T-\tau,T]$ *for the terminal payoff* g *and let* (Σ_t) *be its associated set of zero measure. Then for any* $(u,v) \in \mathscr{U} \times \mathscr{V}$, *for any* $t_0 \in [T-\tau,T]$ *and any* $x_0 \in \mathbb{R}$, *we have*

$$J_1^*(t_0,x_0,\bar{u},\bar{v}) = J_1^\sharp(t_0,x_0,\bar{u},\bar{v}) \ge J_1^\sharp(t_0,x_0,u,\bar{v})$$

and

$$J_2^*(t_0,x_0,\bar{u},\bar{v}) = J_2^\sharp(t_0,x_0,\bar{u},\bar{v}) \ge J_2^\sharp(t_0,x_0,\bar{u},v).$$

where $J_1^*(t,\cdot,\bar{u},\bar{v})$ *and* $J_2^*(t,\cdot,\bar{u},\bar{v})$ *are the upper semi-continuous envelopes of the functions* $J_1(t,\cdot,\bar{u},\bar{v})$ *and* $J_2(t,\cdot,\bar{u},\bar{v})$ *(defined as functions on* $\mathbb{R}\backslash\Sigma_t$*).*

Remark *If* $g \in \mathscr{G}$*, then* g *satisfies the regularity conditions of the Corollary.*

Proof of Corollary 4.2: Let us start with proving that

$$J_1^*(t_0,x_0,\bar{u},\bar{v}) \geq J_1^\sharp(t_0,x_0,u,\bar{v}).$$

This inequality is obvious for $x_0 \notin S_{t_0}$, from the definition of a Nash equilibrium since at such a point, $J^\sharp = J$ is continuous. Let now $x_0 \in S_{t_0}$ and $x(\cdot) \in \mathscr{X}(t_0,x_0,u,\bar{v})$ be such that

$$g_1(x(T)) = J_1^\sharp(t_0,x_0,u,\bar{v}).$$

Since S_{t_0} is of zero measure, we can find $h_n \to 0^+$ such that $x_0 + h_n \notin S_{t_0}$ and $x_0 - h_n \notin S_{t_0}$. We apply Lemma 4.1 to h_n, to the feedback $(u,\bar{v})$ and to the trajectory $x(\cdot)$. Let $u_{h_n} \in \mathscr{U}$, $x_{h_n}^+(\cdot)$ and $x_{h_n}^-(\cdot)$ satisfying the conclusions of Lemma 4.1. Since g_1 is either left continuous or right continuous at the point $x(T)$, we have clearly, up to a subsequence,

$$\text{either } \lim_n g_1(x_{h_n}^-(T)) = g_1(x(T)) \text{ or } \lim_n g_1(x_{h_n}^+(T)) = g_1(x(T)).$$

Let us assume for instance that $\lim_n g_1(x_{h_n}^-(T)) = g_1(x(T))$. Then, since $x_0 - h_n \notin S_{t_0}$, we have

$$J_1(t_0,x_0-h_n,\bar{u},\bar{v}) \geq J_1^\sharp(t_0,x_0-h_n,u_{h_n},\bar{v}) \geq g_1(x_{h_n}^-(T)).$$

Taking the lim sup in this inequalities gives the desired result:

$$J_1^*(t_0,x_0,\bar{u},\bar{v}) \geq g_1(x(T)) = J_1^\sharp(t_0,x_0,u,\bar{v}).$$

The symmetric inequality for the second Player can be proven in a similar way.

This implies in particular that $J_j^* \geq J_j^\sharp$ for $j = 1,2$ (just set $(u,v) = (\bar{u},\bar{v})$). The reverse inequality $J_j^* \leq J_j^\sharp$ is also clear since the map $x_0 \to J_j^\sharp(t_0,x_0,\bar{u},\bar{v})$ is usc and coincides with J_j^* ouside of the set S_{t_0} which has a zero measure. □

Lemma 4.3. *Let* $g = (g_1,g_2) : \mathbb{R} \to \mathbb{R}^2$ *be such that* g_1 *(resp.* g_2*) is usc and is, at each point, either right- or left-continuous. If* $(\bar{u},\bar{v})$ *is a Nash equilibrium feedback on the time interval* $[T-\tau,T]$ *for some* $\tau > 0$*, then, for* $j = 1,2$*, for any* $x \in \mathbb{R}$*, the map* $t \to J_j^\sharp(t,x,\bar{u},\bar{v})$ *is non increasing.*

Proof of the Lemma 4.3: We only do the proof for $j = 1$. Let us fix t_0 and t_1 such that $T-\tau \leq t_0 < t_1 \leq T$. From Proposition 2.7, $(\bar{u},\bar{v})$ is a Nash equilibrium feedback for the time horizon t_1 and the terminal payoff $J^\sharp(t_1,\cdot,\bar{u},\bar{v})$. Hence for any $x_0 \in \mathbb{R}$, Corollary 4.2 states that the upper payoff for the feedback $(\bar{u},\bar{v})$ at (t_0,x_0)—for the game with time horizon t_1 and terminal payoff $J^\sharp(t_1,\cdot,\bar{u},\bar{v})$—is not smaller than the upper payoff for the feedback $(-\bar{v},\bar{v})$ for this game: Namely, there is some $x^*(\cdot) \in \mathscr{X}(t_0,x_0,\bar{u},\bar{v})$ such that, for any $x(\cdot) \in \mathscr{X}(t_0,x_0,-\bar{v},\bar{v})$, we have:

$$J_1^\sharp(t_1,x^*(t_1),\bar{u},\bar{v}) \geq J_1^\sharp(t_1,x(t_1),\bar{u},\bar{v}).$$

Since the set $\mathscr{X}(t_0, x_0, -\bar{v}, \bar{v})$ only contains the constant solution $x(\cdot) = x_0$, the previous inequality becomes:

$$J_1^\sharp(t_1, x^*(t_1), \bar{u}, \bar{v}) \geq J_1^\sharp(t_1, x_0, \bar{u}, \bar{v}).$$

Since

$$J_1^\sharp(t_1, x^*(t_1), \bar{u}, \bar{v}) \leq J_1^\sharp(t_0, x_0, \bar{u}, \bar{v}),$$

the two previous inequalities give the desired result:

$$J_1^\sharp(t_0, x_0, \bar{u}, \bar{v}) \geq J_1^\sharp(t_1, x_0, \bar{u}, \bar{v}).$$

□

4.2 Concatenation

Let us recall some terminology introduced in Proposition 3.1:

Definition 4.4. *Let $g \in \mathscr{G}$ and $\Sigma = \{\sigma_0 = -\infty, \sigma_1, \ldots, \sigma_k, \sigma_{k+1} = +\infty\}$ be an associated partition. Let (u^*, v^*) be a Nash equilibrium feedback defined on some time interval $[T - \tau, T]$ for some $\tau > 0$ and for the terminal payoff g, and let (S_t) be its associated set of zero-measure.*

1. *We say that (u^*, v^*) is* **proper with respect to the terminal set** Σ *if*

$$\forall t \in [T - \tau, T], \forall x \notin S_t, \forall x^*(\cdot) \in \mathscr{X}(t, x, u^*, v^*), x^*(T) \notin \Sigma.$$

2. *We say that (u^*, v^*) is* **proper with respect to the set** (S_t) *if, for any $t \in [T - \tau, T]$ and any $x \notin S_t$, we have:*

$$\forall x^*(\cdot) \in \mathscr{X}(t, x, u^*, v^*), \forall s \in [t, T], x^*(s) \notin S_s.$$

Remark. *From Proposition 3.1, if $g \in \mathscr{G}$, then there exists a completely maximal Nash equilibrium feedback which is proper with respect to its associated set of zero measure (S_t). Moreover, if $g \in \widetilde{\mathscr{G}}$ and if Σ is an associated partition, then this feedback is also proper with respect to the terminal time Σ.*

Proposition 4.5. *Let $g \in \mathscr{G}$. Let us assume that:*

- *(u_1^*, v_1^*) is a Nash equilibrium feedback on the time interval $[T - \tau_1, T]$ (for some $\tau_1 > 0$),*
- *$J^\sharp(T - \tau_1, \cdot, u_1^*, v_1^*)$ belongs to $\widetilde{\mathscr{G}}$ and there is an associated partition Σ such that, for $x \notin \Sigma$,*
  $$J^\sharp(T - \tau_1, x, u_1^*, v_1^*) = J^\flat(T - \tau_1, x, u_1^*, v_1^*),$$
- *(u_2^*, v_2^*) is a Nash equilibrium feedback on the interval $[T - \tau_2, T - \tau_1]$ (for some $\tau_2 > \tau_1$) for the terminal time $T - \tau_1$ and the terminal payoff $J^\sharp(T - \tau_1, \cdot, u_1^*, v_1^*)$, proper with respect to the terminal set Σ.*

Then

A) the feedback (u^*, v^*) defined by

$$(u^*, v^*)(t,x) = \begin{cases} (u_1^*, v_1^*)(t,x) & \text{if } t \in (T-\tau_1, T] \\ (u_2^*, v_2^*)(t,x) & \text{if } t \in [T-\tau_2, T-\tau_1] \end{cases} \tag{15}$$

is a Nash equilibrium feedback on the time interval $[T-\tau_2, T]$ *for the terminal time* T *and the terminal payoff* g.

B) *if moreover* (u_1^*, v_1^*) *and* (u_2^*, v_2^*) *are proper with respect to their associated set of zero-measure, then so is* (u^*, v^*).

C) *if* (u_1^*, v_1^*) *and* (u_2^*, v_2^*) *are completely maximal, then so is* (u^*, v^*)

Proof of Proposition 4.5: A) For any $t \in [T-\tau_2, T-\tau_1)$, let S_t be the set of zero measure associated with (u_2^*, v_2^*). We only prove that, if $u \in \mathcal{U}$, if $t_0 \in [T-\tau_2, T-\tau_1)$ and $x_0 \notin S_{t_0}$, then

$$J_1^\flat(t_0, x_0, u^*, v^*) \geq J_1^\sharp(t_0, x_0, u, v^*).$$

Indeed the corresponding inequality for $t_0 \in [T-\tau_1, T]$ is clear and the corresponding inequality for J_2 can be proved in a similar way.

Let $x^*(\cdot) \in \mathcal{X}(t_0, x_0, u^*, v^*)$ and $x(\cdot) \in \mathcal{X}(t_0, x_0, u, v^*)$. Since $(u^*, v^*) = (u_2^*, v_2^*)$ on $[T-\tau_2, T-\tau_1)$ and since (u_2^*, v_2^*) is a Nash equilibrium feedback for the terminal time $T-\tau_1$ and the terminal payoff $J^\sharp(T-\tau_1, \cdot, u_1^*, v_1^*)$ on this interval, we have

$$J_1^\sharp(T-\tau_1, x^*(T-\tau_1), u_1^*, v_1^*) \geq J_1^\sharp(T-\tau_1, x(T-\tau_1), u_1^*, v_1^*).$$

Since, from the assumption, (u_2^*, v_2^*) is proper with respect to the set Σ, the point $x^*(T-\tau_1)$ does not belong to Σ. Therefore, from the definition of Σ, we have

$$\begin{aligned} J_1^\sharp(T-\tau_1, x^*(T-\tau_1), u_1^*, v_1^*) &= J_1(T-\tau_1, x^*(T-\tau_1), u_1^*, v_1^*) = g_1(x^*(T)) \\ &= (g_1)_*(x^*(T)). \end{aligned}$$

Since (u_1^*, v_1^*) is a Nash equilibrium feedback on $[T-\tau_1, T]$ and since, from the construction of (u^*, v^*), the solution $x(\cdot)$ belongs to $\mathcal{X}(T-\tau_1, x(T-\tau_1), u, v_1^*)$, we have, from Corollary 4.2,

$$J_1^\sharp(T-\tau_1, x(T-\tau_1), u_1^*, v_1^*) \geq J_1^\sharp(T-\tau_1, x(T-\tau_1), u, v_1^*) \geq g_1(x(T)).$$

Therefore we have proved that $(g_1)_*(x^*(T)) \geq g_1(x(T))$, which is the desired result.

B) If (u_1^*, v_1^*) and (u_2^*, v_2^*) are proper with respect to their associated set of zero-measure, then it is clear that so is (u^*, v^*).

C) Let us now assume that (u_1^*, v_1^*) and (u_2^*, v_2^*) are completely maximal. We have to prove that (u^*, v^*) is also completely maximal, i.e., that, for any $T_0 \in (T-\tau_2, T]$, if the pair (e_1, e_2) is a maximal payoff for some point $(t_0, x_0) \in [T-\tau_2, T_0)$ for the game with terminal time T_0 and terminal payoff $J^\sharp(T_0, \cdot, u^*, v^*)$, then we have

$$J^\sharp(t_0, x_0, u^*, v^*) = (e_1, e_2). \tag{16}$$

Without loss of generality, we can assume that $T_0 \in (T-\tau_1, T]$ and $t_0 \in [T-\tau_2, T-\tau_1)$ since otherwise the result is a straightforward consequence of the fact that (u_1^*, v_1^*) and (u_2^*, v_2^*) are completely maximal. By using

Proposition 2.7 and again the fact that (u_1^*, v_1^*) is completely maximal, we can also assume, in order to simplify the notation, that $T_0 = T$.

In order to apply the assumption that (u_2^*, v_2^*) is completely maximal, we are going to prove that (e_1, e_2) is a maximal payoff at the point (t_0, x_0) for the differential game with terminal time $T - \tau_1$ and terminal payoff $J^\sharp(T - \tau_1, \cdot, u^*, v^*)$.

From the definition of a maximal equilibrium, there is some solution $\bar{x}(\cdot) \in \mathcal{X}(t_0, x_0)$ such that $(e_1, e_2) \in ess - \limsup_{x' \to \bar{x}(T)} g(x')$ and, for any $x(\cdot) \in \mathcal{X}(t_0, x_0)$, we have

$$\text{for } j = 1, 2,\ g_j(x(T)) \le e_j.$$

Lemma 2.5 states that (e_1, e_2) is the maximal equilibrium at any point $(t, \bar{x}(t))$. In particular, for $t = T - \tau_1$, we have, since (u_1^*, v_1^*) is maximal:

$$(e_1, e_2) = J^\sharp(T - \tau_1, \bar{x}(T - \tau_1), u_1^*, v_1^*). \tag{17}$$

For simplicity, we set $\bar{y} = \bar{x}(T - \tau_1)$. Let us now prove that

$$(e_1, e_2) \text{ belongs to } ess - \limsup_{x' \to \bar{y}} J^\sharp(T - \tau_1, x', u_1^*, v_1^*). \tag{18}$$

Proof of (18): We consider two cases.

Case 1: Let us first assume that, for any $t \in [T - \tau_1, T]$, $\bar{y} = \bar{x}(t) = \bar{x}(T)$. From Lemma 4.3, we have

$$\forall x \in \mathbb{R}, J_j^\sharp(T - \tau_1, x, u_1^*, v_1^*) \ge J_j^\sharp(T, x, u_1^*, v_1^*) = g_j(x)$$

for $j = 1, 2$. Since $(e_1, e_2) \in ess - \limsup_{x' \to \bar{x}(T)} g(x')$ and since $J_j^\sharp(T - \tau_1, \cdot, u_1^*, v_1^*)$ is usc for $j = 1, 2$, one concludes easily from (17) that (18) holds true in this case.

Case 2: Let us now assume that there is some $t \in [T - \tau_1, T]$ with $\bar{x}(t) \ne \bar{x}(T)$. We can assume for instance that there is some $\epsilon > 0$ with:

$$[\bar{y} - \epsilon, \bar{y}] \subset \bar{x}([T - \tau_1, T]). \tag{19}$$

Since, from Lemma 2.5, (e_1, e_2) is a maximal payoff at the point $(t, \bar{x}(t))$ for any $t \in [T - \tau_1, T]$ and since (u_1^*, v_1^*) is maximal, we have

$$\forall t \in [T - \tau_1, T], J^\sharp(t, \bar{x}(t), u_1^*, v_1^*) = (e_1, e_2).$$

Combining Lemma 4.3, the previous equality and inclusion (19) gives:

$$\forall x \in [\bar{y} - \epsilon, \bar{y}],\ J_j^\sharp(T - \tau_1, x, u_1^*, v_1^*) \ge J^\sharp(t, x, u_1^*, v_1^*) = e_j,$$

for $j = 1, 2$, where $t \in [T - \tau_1, T]$ is such that $\bar{x}(t) = x$. This inequality together with (17) and the fact that $J_j^\sharp(t, \cdot, u_1^*, v_1^*)$ is usc, leads to (18). Therefore (18) holds in any case.

Let $x(\cdot) \in \mathcal{X}(t_0, x_0)$. We now prove that $J_j^\sharp(T - \tau_1, x(T - \tau_1), u_1^*, v_1^*) \le e_j$ for any $j = 1, 2$. We do the proof for $j = 1$ for instance. From Corollary 4.2, there is $x_1(\cdot)$ belonging to $\mathcal{X}(T - \tau_1, x(T - \tau_1), u_1^*, v_1^*)$ such that

$$g_1(x_1(T)) = J_1^\sharp(T - \tau_1, x(T - \tau_1), u_1^*, v_1^*).$$

The trajectory $x_2(\cdot)$ defined by

$$x_2(t) = \begin{cases} x(t) & \text{if } t \in [t_0, T - \tau_1] \\ x_1(t) & \text{if } t \in [T - \tau_1, T] \end{cases}$$

belongs to $\mathscr{X}(t_0, x_0)$. Since (e_1, e_2) is the maximal payoff at (t_0, x_0), we have

$$e_1 \geq g_1(x_2(T)) = g_1(x_1(T)) = J_1^\sharp(T - \tau_1, x(T - \tau_1), u_1^*, v_1^*).$$

Thus we have proved that (e_1, e_2) is the maximal payoff at the point (t_0, x_0) for the game with terminal time $T - \tau_1$ and terminal payoff $J^\sharp(T - \tau_1, \cdot, u_1^*, v_1^*)$.

Then equality (16) holds true because (u_2^*, v_2^*) is a maximal equilibrium feedback for this game.

4.3 Extension

We now prove that, if (u^*, v^*) is a completely maximal Nash equilibrium feedback on any interval $[t, T]$ for $t \in (T_0, T]$ (where $T_0 < T$), then (u^*, v^*) is still a completely maximal Nash equilibrium feedback on the interval $[T_0, T]$. More precisely:

Proposition 4.6. *Let us assume that the terminal payoff g belongs to $\mathscr{G}$ and that there are some $(u^*, v^*) \in \mathscr{U} \times \mathscr{V}$ and some time $T_0 \in [0, T)$ such that*

i) $\forall t \in (T_0, T), (u^*, v^*)$ *is a completely maximal Nash equilibrium feedback on* $[t, T]$,
ii) *for any* $t \in (T_0, T), J^\sharp(t, \cdot, u^*, v^*)$ *belongs to* $\widetilde{\mathscr{G}}$,
iii) *for any* $t \in (T_0, T)$, *there is some partition* S_t *associated with* $J^\sharp(t, \cdot, u^*, v^*)$ *such that*

$$\forall x \notin S_t, J^\sharp(t, x, u^*, v^*) = J^\flat(t, x, u^*, v^*),$$

iv) *the set-valued map* $t \to S_t$ *is* 2*-Lipschitz continuous and there is some* M *such that the cardinal of* S_t *is not larger than* M,
v) (u^*, v^*) *is proper with respect to* $(S_t)_{t \in (T_0, T]}$, *i.e., for any* $t_0 \in (T_0, T)$, *for any* $x_0 \notin S_{t_0}$, *for any* $x^*(\cdot) \in \mathscr{X}(t_0, x_0, u^*, v^*)$, *we have*

$$\forall t \in [t_0, T], x^*(t) \notin S_t.$$

Then

a) (u^*, v^*) *is still a completely maximal Nash equilibrium feedback on* $[T_0, T]$,
b) $J^\sharp(T_0, \cdot, u^*, v^*)$ *belongs to* $\mathscr{G}$,
c) the limit S_{T_0} *of* S_t *when* $t \to T_0$ *exists and the cardinal of* S_{T_0} *is not larger than* M,
d) $\forall x \notin S_{T_0}, J^\sharp(T_0, x, u^*, v^*) = J^\flat(T_0, x, u^*, v^*)$,
e) (u^*, v^*) *is proper with respect to* $(S_t)_{t \in [T_0, T]}$ *up to time* T_0.

Remark. *We have proved in Proposition 3.1 that, if g belongs to $\mathscr{G}$, then there is some $T_0 < T$ and some feedback (u^*, v^*) for which (i), ..., (v) hold.*

Proof of Proposition 4.6: We divide the proof in two steps. In the first step, we prove that (a), (c), (d) and (e) hold. We complete the proof by establishing the more delicate point (b).

First step: Let us first notice that point (c) is immediate. We now want to prove that $(u*, v^*)$ is a Nash equilibrium feedback which is proper with respect to (S_t) (property (e)) and which satisfies (d).

For this, let $x_0 \notin S_{T_0}$ be fixed. We claim that $J(\cdot, \cdot, u^*, v^*)$ is uniformly continuous on the set $(T_0, t_0] \times (x_0 - \epsilon/2, x_0 + \epsilon/2)$ for some $t_0 \in (T_0, T)$ and $\epsilon > 0$.

Proof of the claim: From the definition of S_{T_0} we can find some positive ϵ such that

$$\forall x \in [x_0 - \epsilon, x_0 + \epsilon], \forall t \in [T_0, T_0 + \epsilon], x \notin S_t.$$

Let us set $t_0 = T_0 + \epsilon/4$. Since $J^\sharp(t_0, \cdot, u^*, v^*)$ belongs to $\widetilde{\mathscr{G}}$ and since the set S_{t_0} has an empty intersection with the interval $[x_0 - \epsilon, x_0 + \epsilon]$, the function $J^\sharp(t_0, \cdot, u^*, v^*)$ can only have two types of behaviour on $[x_0 - \epsilon, x_0 + \epsilon]$:

– either $J_1^\sharp(t_0, \cdot, u^*, v^*)$ (resp. $J_2^\sharp(t_0, \cdot, u^*, v^*)$) is continuous and strictly increasing or decreasing on $[x_0 - \epsilon, x_0 + \epsilon]$,
– or $J_1^\sharp(t_0, \cdot, u^*, v^*)$ and $J_2^\sharp(t_0, \cdot, u^*, v^*)$ are constant on $[x_0 - \epsilon, x_0 + \epsilon]$.

From Proposition 2.7, we know that (u^*, v^*) is a completely maximal Nash equilibrium feedback for the terminal time t_0 and the terminal payoff $J^\sharp(t_0, \cdot, u^*, v^*)$. Therefore we necessarily have, for any $t \in (T_0, t_0)$ and any $x \in I_\epsilon = [x_0 - \epsilon/2, x_0 + \epsilon/2]$,

$$J^\sharp(t, x, u^*, v^*) = J^\sharp(t_0, x + 2\gamma(t_0 - t), u^*, v^*)$$

where

$$\gamma = \begin{cases} 1 & \text{if } J_1^\sharp(t_0, \cdot, u^*, v^*) \text{ and } J_2^\sharp(t_0, \cdot, u^*, v^*) \text{ are increasing on } I_\epsilon \\ -1 & \text{if } J_1^\sharp(t_0, \cdot, u^*, v^*) \text{ and } J_2^\sharp(t_0, \cdot, u^*, v^*) \text{ are decreasing on } I_\epsilon \\ 0 & \text{otherwise} \end{cases}$$

This proves that the map $(t, x) \to J^\sharp(t, x, u^*, v^*)$ is uniformly continuous on $(T_0, t_0] \times I_\epsilon$ because $J^\sharp(t_0, \cdot, u^*, v^*)$ is continuous in $(x_0 - \epsilon, x_0 + \epsilon)$.

In order to prove that (u^*, v^*) is a Nash equilibrium feedback, let us now fix $u \in \mathscr{U}$, $x^*(\cdot) \in \mathscr{X}(T_0, x_0, u^*, v^*)$ and $x(\cdot) \in \mathscr{X}(T_0, x_0, u, v^*)$. Our aim is to prove that $g_1(x(T)) \leq (g_1)_*(x^*(T))$.

Let us first notice that

$$\forall t \in [T_0, T], x^*(t) \notin S_t. \tag{20}$$

Indeed, the point $x^*(t)$ does not belong to S_{t_0} for any $t \in [T_0, t_0]$ from the definition of ϵ. Since, moreover, the restriction of $x^*(\cdot)$ to $[t_0, T]$ belongs to $\mathscr{X}(t, x^*(t), u^*, v^*)$, and since the feedback (u^*, v^*) is proper, we have (20). Hence (e) is proved.

Let us now fix $\eta > 0$. Since the map $(t, x) \to J^\sharp(t, x, u^*, v^*)$ is uniformly continuous on $(t_0, T_0] \times [x_0 - \epsilon/2, x_0 + \epsilon/2]$, we can choose t sufficiently close to T_0 such that

$$(g_1)_*(x^*(T)) = g_1(x^*(T)) = J_1(t, x^*(t), u^*, v^*) \geq J_1(t, x(t), u^*, v^*) - \eta.$$

Moreover, since (u^*, v^*) is a Nash equilibrium feedback, we also have

$$J_1(t,x(t),u^*,v^*) \geq J_1^\sharp(t,x(t),u,v^*) \geq g_1(x(T)).$$

Hence inequality $(g_1)_*(x^*(T)) \geq g_1(x(T)) - \eta$ holds true for any $\eta > 0$ and thus: $(g_1)_*(x^*(T)) \geq g_1(x(T))$. This implies that:

$$\forall x_0 \notin S_{T_0},\ J_1^\flat(T_0,x_0,u^*,v^*) \geq J_1^\sharp(T_0,x_0,u,v^*).$$

We can prove in a similar way that, for any $v \in \mathscr{V}$,

$$\forall x_0 \notin S_{T_0},\ J_2^\flat(T_0,x_0,u^*,v^*) \geq J_2^\sharp(T_0,x_0,u^*,v).$$

Thus (u^*,v^*) is a Nash equilibrium feedback on $[T_0,T]$ and (d) is proved by setting $(u,v)=(u^*,v^*)$ in the two previous inequalities.

Let us finally prove that (u^*,v^*) is completely maximal on $[T_0,T]$. For simplicity we only prove that (u^*,v^*) is maximal on $[T_0,T]$, the proof for the complete maximality being the same.

Let (T_0,x_0) be a fixed point for which there is some maximal payoff $(e_1,e_2) \in \mathbb{R}^2$. There is some solution $\bar{x}(\cdot) \in \mathscr{X}(T_0,x_0)$ such that (e_1,e_2) belongs to $ess - \limsup_{x \to \bar{x}(T)} g(x)$ and for any $x(\cdot) \in \mathscr{X}(T_0,x_0)$,

$$\text{for } j=1,2, \quad g_j(x(T)) \leq e_j.$$

Since (u^*,v^*) is completely maximal on $[t,T]$ for any $t > T_0$, and since (e_1,e_2) is a maximal payoff at the point $(t,\bar{x}(t))$ from Lemma 2.5, we have

$$\forall t \in (T_0,T), J^\sharp(t,\bar{x}(t),u^*,v^*) = (e_1,e_2).$$

Hence we have

$$\text{for } j=1,2, J_j^\sharp(T_0,x_0,u^*,v^*) \geq \limsup_{t \to T_0^+} J_j^\sharp(t,\bar{x}(t),u^*,v^*) = e_j.$$

Since obviously, for $j=1,2$, we have $J_j^\sharp(T_0,x_0,u^*,v^*) \leq e_j$, the desired equality is proved: $J^\sharp(T_0,x_0,u^*,v^*) = (e_1,e_2)$.

Second step: Let us finally prove that (b) holds true, i.e., that $J^\sharp(T_0,\cdot,u^*,v^*)$ belongs to $\mathscr{G}$. Let us set as usual

$$S_{T_0} = \{\sigma_0 = -\infty, \sigma_1, \ldots, \sigma_k, \sigma_{k+1} = +\infty\}$$

From the first step, it is immediate that, on the interval (σ_i,σ_{i+1}) for $i=1,\ldots,k$, we have

- either $J_1^\sharp(T_0,\cdot,u^*,v^*)$ (resp. $J_2^\sharp(T_0,\cdot,u^*,v^*)$) is strictly increasing or strictly decreasing,
- or $J_1^\sharp(T_0,\cdot,u^*,v^*)$ and $J_2^\sharp(T_0,\cdot,u^*,v^*)$ are constant.

From Corollary 4.2, it is also clear that $J^\sharp(T_0,\cdot,u^*,v^*)$ is usc and left or right continuous at each point σ_i. It now remains to prove that, if σ_i is a strict local maximum of (say) $J_1^\sharp(T_0,\cdot,u^*,v^*)$, then $J_1^\sharp(T_0,\cdot,u^*,v^*)$ is continuous at σ_i and $J_2^\sharp(T_0,\cdot,u^*,v^*)$ has a strict local minimum and is continuous at this point.

From Corollary 4.2, there is some sequence (x_n) which converges to σ_i, with $x_n \notin S_{T_0}$ and such that the sequence $(J_1(T_0,x_n,u^*,v^*))$ converges to $J_1^\sharp(T_0,\sigma_i,u^*,v^*)$. Let $x_n(\cdot) \in \mathscr{X}(T_0,x_n,u^*,v^*)$.

From standard arguments, the sequence $(x_n(\cdot))$ converges, up to some subsequence, to some solution $x(\cdot) \in \mathscr{X}(T_0,\sigma_i,u^*,v^*)$ uniformly on $[T_0,T]$. Since $x_n \notin S_{T_0}$, Lemma 2.3 states that

$$\forall t \in [T_0, T],\ J_1^\sharp(t, x_n(t), u^*, v^*) = J_1^\sharp(T_0, x_n, u^*, v^*).$$

Taking the lim sup in this equality and using Lemma 4.3 gives

$$\forall t \in [T_0, T],\ J_1^\sharp(T_0, \sigma_i, u^*, v^*) \le J_1^\sharp(t, x(t), u^*, v^*) \le J_1^\sharp(T_0, x(t), u^*, v^*).$$

Since $J_1^\sharp(T_0, \cdot, u^*, v^*)$ has a strict local maximum at σ_i, this yields to

$$\forall t \in [T_0, T],\ x(t) = \sigma_i \text{ and } J_1^\sharp(t, \sigma_i, u^*, v^*) = J_1^\sharp(T_0, \sigma_i, u^*, v^*). \tag{21}$$

Using again Lemma 4.3 gives that, for any $x \neq \sigma_i$ sufficiently close to σ_i,

$$J_1^\sharp(T_0, \sigma_i, u^*, v^*) = J_1^\sharp(t, \sigma_i, u^*, v^*) \\ > J_1^\sharp(T_0, x, u^*, v^*) \ge J_1^\sharp(t, x, u^*, v^*). \tag{22}$$

Therefore, for any $t \in [T_0, T]$, the map $J_1^\sharp(t, \cdot, u^*, v^*)$ has a strict local maximum at σ_i.

Fix some $t_0 \in (T_0, T)$. Since the map $J_1^\sharp(t_0, \cdot, u^*, v^*)$ belongs to $\widetilde{\mathscr{G}}$ and has a strict local maximum at σ_i, it is continuous at this point. Hence, for any positive ϵ, there is some $\eta > 0$ with

$$J_1^\sharp(t_0, \sigma_i, u^*, v^*) - \epsilon \le J_1^\sharp(t_0, x, u^*, v^*) \le J_1^\sharp(t_0, \sigma_i, u^*, v^*),\quad \forall x \in (\sigma_i - \eta, \sigma_i + \eta).$$

Therefore, since $J_1^\sharp(t_0, \sigma_i, u^*, v^*) = J_1^\sharp(T_0, \sigma_i, u^*, v^*)$ and $J_1^\sharp(t_0, \cdot, u^*, v^*) \le J_1^\sharp(T_0, \cdot, u^*, v^*)$,

$$J_1^\sharp(T_0, \sigma_i, u^*, v^*) - \epsilon \le J_1^\sharp(T_0, x, u^*, v^*) \le J_1^\sharp(T_0, \sigma_i, u^*, v^*),\quad \forall x \in (\sigma_i - \eta, \sigma_i + \eta).$$

This proves that $J_1^\sharp(T_0, \sigma_i, u^*, v^*)$ is continuous at σ_i.

Let us finally establish that $J_2^\sharp(T_0, \cdot, u^*, v^*)$ is continuous at σ_i and has a strict local minimum at this point. For that purpose, let us fix some $t_0 \in (T_0, T)$. Since $J^\sharp(t_0, \cdot, u^*, v^*)$ belongs to $\widetilde{\mathscr{G}}$ and since $J_1^\sharp(t_0, \cdot, u^*, v^*)$ has a strict local maximum at σ_i, $J_2^\sharp(t_0, \cdot, u^*, v^*)$ has a strict local minimum at σ_i and it is continuous at this point.

Let $x_n \to \sigma_i$, with, for instance, $x_n < \sigma_i$. Then, for n sufficiently large, the point x_n does not belong to S_{T_0}. Let $x_n(\cdot)$ be some solution in $\mathscr{X}(T_0, x_n, u^*, v^*)$. Since (u^*, v^*) is proper, and since the point σ_i belongs to S_t for any t (because σ_i is a strict local maximum of $J_1^\sharp(t, \cdot, u^*, v^*)$) we always have $x_n(t) < \sigma_i$. Since $x_n \notin S_{T_0}$, we have, for $j = 1, 2$

$$\forall t \in [T_0, T],\quad J_j^\sharp(T_0, x_n, u^*, v^*) = J_j^\sharp(t, x_n(t), u^*, v^*).$$

Moreover, from Lemma 4.3, we have

$$\forall t \in [T_0, T],\quad J_j^\sharp(t, x_n, u^*, v^*) \le J_j^\sharp(T_0, x_n, u^*, v^*).$$

Hence, for $t = t_0$, we have:

$$\text{for } j = 1, 2,\quad J_j^\sharp(t_0, x_n, u^*, v^*) \le J_j^\sharp(t_0, x_n(t_0), u^*, v^*). \tag{23}$$

Using arguments already developed above and the fact that $J_1^\sharp(t_0, \cdot, u^*, v^*)$ is continuous at σ_i, we can prove that the sequence $x_n(\cdot)$ converges uniformly to the constant solution $x(t) = \sigma_i$. Therefore the point $x_n(t_0)$ belongs to some interval $(\sigma_i - \eta, \sigma_i)$ for n large enough. We choose $\eta > 0$ such that $J_1^\sharp(t_0, \cdot, u^*, v^*)$ is strictly decreasing and $J_2^\sharp(t_0, \cdot, u^*, v^*)$ is strictly increasing on $(\sigma_i - \eta, \sigma_i)$. Then inequality (23) imply that $x_n(t_0) = x_n$. Hence we have:

$$J_2^\sharp(T_0,x_n,u^*,v^*) = J_2^\sharp(t_0,x_n,u^*,v^*)$$
$$> J_2^\sharp(t_0,\sigma_i,u^*,v^*) = J_2^\sharp(T_0,\sigma_i,u^*,v^*)$$

and

$$\lim_n J_2^\sharp(T_0,x_n,u^*,v^*) = \lim_n J_2^\sharp(t_0,x_n,u^*,v^*)$$
$$= J_2^\sharp(t_0,\sigma_i,u^*,v^*) = J_2^\sharp(T_0,\sigma_i,u^*,v^*)$$

because $J_2^\sharp(t_0,\cdot,u^*,v^*)$ has a strict local minimum at σ_i and is continuous at this point. Therefore we have proved that $J_2^\sharp(T_0,\cdot,u^*,v^*)$ has a strict local minimum at σ_i and is continuous at this point.

This completes the proof of Proposition 4.6. □

4.4 Proof of the main Theorem

We are now ready to prove Theorem 2.11. Using Proposition 3.1 and Zorn Lemma shows that there is some maximal interval $(T_0,T]$, some feedback (u^*,v^*) and some associated set of zero measure $(S_t)_{t\in(T_0,T]}$ such that conditions (i), …, (v) of Proposition 4.6 are satisfied on any interval $[t,T]$ for $t>T_0$. Then the conclusion of Proposition 4.6 holds, which implies that the map $J^\sharp(T_0,\cdot,u^*,v^*)$ belongs to $\widetilde{\mathscr{G}}$.

Let us now assume that $T_0>0$. Then using again Proposition 3.1 shows that there is some $\tau>0$ and some feedback (u_1^*,v_1^*), and some sets of zero measure $(S_t)_{t\in[T_0-\tau,T_0]}$ satisfying the conditions (i), …, (vi) of Proposition 3.1 for the terminal time T_0 and the terminal payoff $J^\sharp(T_0,\cdot,u^*,v^*)$ on the time interval $[T_0-\tau,T_0]$. Then Proposition 4.5 implies that the new feedback

$$(u_2^*,v_2^*)(t,x) = \begin{cases} (u^*,v^*)(t,x) & \text{if } t\in(T_0,T] \\ (u_1^*,v_1^*)(t,x) & \text{if } t\in[T_0-\tau,T_0] \end{cases}$$

is a completely maximal Nash equilibrium feedback, satisfying conditions (i), …, (v) of Proposition 4.6, on the time interval $[T_0-\tau,T]$. This is in contradiction with the maximality of the interval $(T_0,T]$.

Hence we have proved that there is at least one completely maximal Nash equilibrium feedback (u^*,v^*) defined on $[0,T]$.

The fact that the payoff of any other completely maximal Nash equilibrium feedback coincides with the payoff of (u^*,v^*) is a straightforward application of the uniqueness result of Proposition 3.1 and of the fact that the map $J^\sharp(t,\cdot,u^*,v^*)$ belongs to $\widetilde{\mathscr{G}}$ for $t\in[0,T)$. □

4.5 Proof of Proposition 2.14 and of Theorem 2.15

We complete this paper by the proof of the results on the link between Nash equilibrium feedbacks and Nash equilibrium payoffs in the sense of [6], [7], [10].

Proof of Proposition 2.14: Let t, x, and $x^*(\cdot)$ be as in the Proposition. We have on the one hand that

$$\forall s\in[t,T],\quad J(s,x^*(s),u^*,v^*) = g(x^*(T)) = J(t,x,u^*,v^*)$$

and, on the other hand,

$$\forall s \in [t,T], \quad \text{for} j = 1,2, \quad J_j(s, x^*(s), u^*, v^*) \geq J_j^\sharp(T, x^*(s), u^*, v^*) = g_j(x^*(s)),$$

thanks to Lemma 4.3 applied at the times s and T. Whence the result. □

Proof of Theorem 2.15: Let $t_0 \in [0,T)$ and $x_0 \notin S_{t_0}$ be fixed. Let us assume that (e_1, e_2) is a Nash equilibrium payoff at (t_0, x_0) such that

$$\text{for } j = 1,2, \quad e_j \geq J_j(t_0, x_0, u^*, v^*).$$

We have to prove that $J(t_0, x_0, u^*, v^*) = (e_1, e_2)$.

From the characterization of Nash equilibrium payoffs, there is some $x(\cdot) \in \mathscr{X}(t_0, x_0, u^*, v^*)$ such that

$$\forall t \in [t_0, T], \ \text{for} j = 1, \quad 2, g_j(x(t)) \leq e_j = g_j(x(T)).$$

In order to fix the ideas, we assume that $x(T) \geq x_0$. It is easy to see that one can choose $x(\cdot)$ such that $x'(\cdot) = 0$ on $[t_0, \theta_1)$ and $x'(\cdot) = 2$ on $[\theta_1, T]$, where $\theta_1 = T - (x(T) - x_0)/2$.

Let $x^*(\cdot) \in \mathscr{X}(t_0, x_0, u^*, v^*)$. We know, from Proposition 2.14, that $x^*(\cdot)$ is a Nash trajectory and, from Proposition 2.3, that

$$\forall t \in [t_0, T], \quad J(t, x^*(t), u^*, v^*) = J(t_0, x_0, u^*, v^*) = g(x^*(T)).$$

Hence, if $x^*(T) \geq x(T)$, there exists some $t \in [t_0, T]$ such that $x^*(t) = x(T)$, and thus

$$J(t_0, x_0, u^*, v^*) = J(t, x^*(t), u^*, v^*) \geq g(x^*(t)) = g(x(T)) = (e_1, e_2)$$

because $x^*(\cdot)$ is a Nash trajectory (here and throughout the proof we endow $\mathbb{R}^2$ with the usual partial ordering). Therefore, if $x^*(T) \geq x(T)$, the result is proved.

Let us assume from now on that $x^*(T) < x(T)$. Let us set

$$\theta_2 = \max\{t \in [\theta_1, T] \,|\, x^*(t) = x(t)\},$$

with $\theta_2 = \theta_1$ if there is no $t \in [\theta_1, T]$ with $x^*(t) = x(t)$. Let us point out that $\forall t \in (\theta_2, T]$, $x^*(t) < x(t)$. We claim that

$$\forall t \in (\theta_2, T], \quad \forall y \in [x^*(t), x(t)], \quad J^\sharp(t, y, u^*, v^*) \leq (e_1, e_2). \tag{24}$$

Proof of (24): Let us first prove that

$$\forall t \in (\theta_2, T], \forall y \in (x^*(t), x(t)) \text{ with } y \notin S_t, J(t, y, u^*, v^*) \leq (e_1, e_2). \tag{25}$$

Let t and y be as above, and let $y(\cdot) \in \mathscr{X}(t, y, u^*, v^*)$. Then $y_1(\cdot) = \max\{y(\cdot), x^*(\cdot)\}$ also belongs to $\mathscr{X}(t, y, u^*, v^*)$ and $y_1(T) \in [x^*(T), x(T)]$. Using now the fact that $x(\cdot)$ and $x^*(\cdot)$ are Nash trajectories with $x(t_0) = x^*(t_0) = x_0$, and that $g(x^*(T)) \leq (e_1, e_2)$, it is easy to see that, for any $z \in [x^*(T), x(T)]$, $g(z) \leq (e_1, e_2)$. Therefore

$$J(t, y, u^*, v^*) = g(y_1(T)) \leq (e_1, e_2),$$

so that (25) is proved.

Thanks to Corollary 4.2, inequality (25) implies inequality (24) for $y \in (x^*(t), x(t))$. Inequality (24) also holds for $y = x^*(t)$ since $J(t, x^*(t), u^*, v^*) = J(t_0, x_0, u^*, v^*) \leq (e_1, e_2)$ from the assumption. Let us finally prove the result for $x(t)$. Corollary 4.2 states that there are two sequences $(x_k^j)_k$ (for $j = 1, 2$) converging to $x(t)$, with $x_k^j \notin S_t$, such that

$$\lim_k J_j(t, x_k^j, u^*, v^*) = J_j^\sharp(t, x(t), u^*, v^*).$$

If $x_k^j \leq x(t)$ for infinitely many k, then, since for such a k we already know that $J_j(t, x_k^j, u^*, v^*) \leq e_j$, we have $J_j^\sharp(t, x(t), u^*, v^*) \leq e_j$. If on the contrary, $x_k^j > x(t)$ for any k sufficiently large, let us consider some solutions $x_k^j(\cdot) \in \mathscr{X}(t, x_k^j, u^*, v^*)$. If $x_k^j(T) \leq x(T)$, then we can complete the proof as above. If $x_k^j(T) \geq x(T)$, then $\lim_k x_k^j(T) = x(T)$ because $x(T) = x(t) + 2(T - t)$ and the $x_k^j(\cdot)$ are 2-Lipschitz continuous. Then the result follows from the continuity of g. Therefore, (24) holds in any case.

Let us now set

$$\theta_3 = \inf\{t \in [\theta_2, T] \mid \forall s \in [t, T], J^\sharp(s, x(s), u^*, v^*) = (e_1, e_2)\},$$

with $\theta_3 = T$ if there is no such a t. Our aim is to prove that $\theta_3 = \theta_2$.

For doing so, we argue by contradiction by assuming that $\theta_3 > \theta_2$. Let us prove that in this case, for any $\epsilon > 0$ sufficiently small, (e_1, e_2) is a maximal payoff at the point $(\theta_3 - \epsilon, x(\theta_3 - \epsilon))$ for the game with horizon θ_3 and terminal payoff $J^\sharp(\theta_3, \cdot, u^*, v^*)$ (let us notice that $x(\theta_3 - \epsilon) = x(\theta_3) - 2\epsilon$).

Indeed, we have, thanks to (24) and the fact that the $J_j^\sharp$ are usc, that:

$$\forall z \in [x(\theta_3) - 4\epsilon, x(\theta_3)), J^\sharp(\theta_3, z, u^*, v^*) \leq (e_1, e_2) \text{ and } J^\sharp(\theta_3, x(\theta_3), u^*, v^*) = (e_1, e_2).$$

If $\theta_3 < T$, Lemma 4.3 shows that

$$\forall s \in [\theta_3, T], \quad J^\sharp(\theta_3, x(s), u^*, v^*) \geq J^\sharp(s, x(s), u^*, v^*) = (e_1, e_2),$$

with $x(s) = x(\theta_3) + 2(s - \theta_3)$. Using the fact that the maps $J_j^\sharp(\theta_3, \cdot, u^*, v^*)$ (for $j = 1, 2$) are usc then gives that

$$(e_1, e_2) \in ess - \limsup_{x' \to x(\theta_3)} J^\sharp(\theta_3, x', u^*, v^*).$$

If $\theta_3 = T$, the above formula is also clear because g is continuous.

Hence we have proved that (e_1, e_2) is a maximal payoff at the point $(\theta_3 - \epsilon, x(\theta_3 - \epsilon))$ for the game with horizon θ_3 and terminal payoff $J^\sharp(\theta_3, \cdot, u^*, v^*)$. Since (u^*, v^*) is completely maximal, this implies that

$$J^\sharp(\theta_3, x(\theta_3 - \epsilon), u^*, v^*) = (e_1, e_2)$$

for any $\epsilon > 0$ sufficiently small, which is in contradiction with the definition of θ_3. Therefore we have proved that $\theta_3 = \theta_2$.

Hence

$$J^\sharp(\theta_2, x(\theta_2), u^*, v^*) \geq (e_1, e_2),$$

because the maps $J_j^\sharp$ are usc. So, if $\theta_2 > \theta_1$, we have $x^*(\theta_2) = x(\theta_2)$, and thus

$$J(t_0, x_0, u^*, v^*) = J(\theta_2, x^*(\theta_2), u^*, v^*) \geq (e_1, e_2),$$

and equality $J(t_0, x_0, u^*, v^*) = (e_1, e_2)$ is proved. If, on the contrary, $\theta_2 = \theta_1$, then $x(\theta_2) = x_0$, and Lemma 4.3 implies that

$$J(t_0, x_0, u^*, v^*) \geq J^\sharp(\theta_1, x_0, u^*, v^*) \geq (e_1, e_2).$$

This completes the proof of Theorem 2.15. □

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