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# The assignment game: the $\tau$-value

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**Abstract.** We provide some formulae for the $\tau$-value in the case of the assignment game and prove that it coincides with the midpoint between the buyers-optimal and the sellers-optimal core allocations. As a consequence, the $\tau$-value of an assignment game always lies in the core. Some comparative statics of this solution is analyzed: the pairwise monotonicity and the effect of new entrants.

**Key words:** assignment games, core, $\tau$-value, pairwise monotonicity

## 1. Introduction

Since its definition by Shapley and Shubik (1972), the assignment game has been studied from a number of perspectives, but above all from the viewpoint of the core. The aim of this paper is to analyze a point-solution for this class of games which has interesting properties: the $\tau$-value (Tijs, 1981), a well known solution in the general framework of cooperative TU games. Other well known solution concepts are the Shapley value (Shapley, 1953) and the nucleolus (Schmeidler, 1969). The Shapley value of an assignment game may lie outside the core; this is an important drawback. For its part, the nucleolus of the assignment game, which is always a core allocation although is not easy to compute, has been studied in Solymosi and Raghavan (1994), who give an algorithm to find it.

In section three, an expression for the $\tau$-value of an assignment game is obtained. This expression implies that for the assignment game, unlike games

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in general, the $\tau$-value is a core allocation. In addition to this, for games with many players, the $\tau$-value follows from the solution of a few linear programs.

Finally, in section four, some comparative statics are studied. Although core selection and coalitional monotonicity are not compatible in the general framework of cooperative games (Young, 1985), not even in the subclass of assignment games with at least two agents on each side of the market (Housman and Clark, 1998), there is a more suitable monotonicity property for the assignment game, which we name pairwise monotonicity; we prove the $\tau$-value to be pairwise monotonic. Similarly, although population monotonic allocations schemes do not exist in assignment games with a $2 \times 2$ positive submatrix (Sprumont, 1990), the $\tau$-value is proved to be type-monotonic, a property which better reflects the two-sided structure of the market. Some definitions, notations and remarks are needed first.

## 2. Definitions and preliminaries

Assignment games were introduced by Shapley and Shubik as a model for a two-sided market with transferable utility. The player set consists of the union of two finite disjoint sets $M \cup M'$, where $M$ is the set of buyers and $M'$ is the set of sellers. We denote by $n$ the cardinality of $M \cup M'$, $n = m + m'$, where $m$ and $m'$ are, respectively, the cardinalities of $M$ and $M'$. Given a non negative matrix $A = (a_{ij})_{(i,j) \in M \times M'}$, where $a_{ij}$ is the joint profit that the mixed pair coalition $\{i, j\}$ can obtain if they trade, a cooperative game can be defined where the worth of coalition $\{i, j\}$ is $w(i, j) = a_{ij} \geq 0$. The matrix $A$ determines the worth of any other coalition $S \cup T$, where $S \subseteq M$ and $T \subseteq M'$. A matching (or assignment) between $S$ and $T$ is a subset $\mu$ of $S \times T$ such that each player belongs at most to one pair in $\mu$. Then, $w(S \cup T) = \max\{\sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathscr{M}(S, T)\}$, $\mathscr{M}(S, T)$ being the set of matchings between $S$ and $T$. It is assumed as usual that a coalition formed only by sellers or only by buyers has worth zero. We denote the game as $(M \cup M', w)$. We say a matching $\mu \in \mathscr{M}(M, M')$ is optimal if for all $\mu' \in \mathscr{M}(M, M')$, $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$ and we denote by $\mathscr{M}^*(A)$ the set of optimal matchings for the grand coalition. Moreover, we say that a buyer $i \in M$ is not assigned by $\mu$ if $(i, j) \notin \mu$ for all $j \in M'$ (and similarly for sellers).

Assignment games are examples of cooperative games with transferable utility (TU). A TU game is a pair $(N, v)$, where $N = \{1, 2, \dots, n\}$ is its finite player set and $v : 2^N \to \mathbf{R}$ its characteristic function satisfying $v(\emptyset) = 0$. The set of games with the above player set is denoted by $G^N$. A payoff vector is $x \in \mathbf{R}^n$ and, for every coalition $S \subseteq N$ we write $x(S) := \sum_{i\in S} x_i$ the payoff to coalition $S$ (where $x(\emptyset) = 0$). The core of the game $(N, v)$ consists of those payoff vectors that allocate the worth of the grand coalition (efficient payoffs) in such a way that every other coalition receives at least its worth by the characteristic function: $C(v) = \{x \in \mathbf{R}^n \mid x(N) = v(N)$ and $x(S) \geq v(S)$ for all $S \subset N\}$. An interesting class of games with nonempty core is the class of convex games (Shapley, 1971).

Assignment games are not in general convex. Roughly speaking, an assignment game is convex when it is defined by a diagonal matrix. However, Shapley and Shubik prove that the core of the assignment game $(M \cup M', w)$ is nonempty and can be represented in terms of an optimal matching in

$M \cup M'$. Let $\mu$ be one such optimal matching. Then

$$C(w) = \left\{ (u,v) \in \mathbf{R}^{M \times M'} \;\middle|\; \begin{array}{l} u_i \geq 0, \text{ for all } i \in M; v_j \geq 0, \text{ for all } j \in M' \\ u_i + v_j = a_{ij} \text{ if } (i,j) \in \mu \\ u_i + v_j \geq a_{ij} \text{ if } (i,j) \notin \mu \\ u_i = 0 \text{ if } i \text{ not assigned by } \mu \\ v_j = 0 \text{ if } j \text{ not assigned by } \mu. \end{array} \right\} \quad (1)$$

Moreover, if we denote for all $i \in M$,

$$\bar{u}_i = \max_{(u,v) \in C(w)} u_i \quad \text{and} \quad \underline{u}_i = \min_{(u,v) \in C(w)} u_i,$$

and for all $j \in M'$,

$$\bar{v}_j = \max_{(u,v) \in C(w)} v_j \quad \text{and} \quad \underline{v}_j = \min_{(u,v) \in C(w)} v_j,$$

it happens that all players on the same side of the market achieve their maximum core payoff in the same core allocation. As a consequence, there are two special extreme core allocations: in one of them, $(\bar{u}, \underline{v})$, each buyer achieves his maximum core payoff and in the other, $(\underline{u}, \bar{v})$, each seller does.

Demange (1982) and Leonard (1983) prove that this maximum payoff of a player in the core of the assignment game is his marginal contribution,

$$\bar{u}_i = w(M \cup M') - w(M \cup M' \backslash \{i\}) \quad \text{for all } i \in M \quad (2)$$

and for all $j \in M'$, $\bar{v}_j = w(M \cup M') - w(M \cup M' \backslash \{j\})$. The minimum core payoff of a player also follows easily. If $i \in M$ is matched with $j \in M'$ by an optimal matching $\mu$, then $\underline{u}_i + \bar{v}_j = a_{ij} = w(M \cup M') - w(M \cup M' \backslash \{i,j\})$ and, by replacing $\bar{v}_j$ by the marginal contribution of player $j$, we get

$$\underline{u}_i = w(M \cup M' \backslash \{j\}) - w(M \cup M' \backslash \{i,j\}). \quad (3)$$

Similarly, $\underline{v}_j = w(M \cup M' \backslash \{i\}) - w(M \cup M' \backslash \{i,j\})$.

A point-solution concept for TU games selects for any game $(N,v)$ an efficient payoff $\alpha(v) \in \mathbf{R}^n$. In the next section we study the $\tau$-value as a point-solution for the assignment game.

## 3. The $\tau$-value of an assignment game

The $\tau$-value was introduced by Tijs (1981) and it is essentially a compromise value between an upper bound payoff vector and a lower bound payoff vector for the game. If $(N,v)$ is a cooperative TU game, let $M(v) \in \mathbf{R}^n$ be the vector whose coordinates are the marginal contribution of each player to the grand coalition,

$$M_i(v) = v(N) - v(N \backslash \{i\}), \quad \text{for all } i \in N.$$

The vector $M(v)$ is called the utopia vector and each $M_i(v)$ can be regarded as a maximal payoff player $i$ can expect to obtain, as $M_i(v)$ is an upper bound (not always attainable) for player $i$'s payoff in the core of the cooperative game.

By using the utopia vector, we can now compute what remains for player $i \in N$ when coalition $S$ forms, $i \in S$, and all other players in $S$ are paid their utopia payoff. The remainder for player $i$, $R^v(S, i)$ is defined by

$$R^v(S, i) = v(S) - \sum_{j \in S \setminus \{i\}} M_j(v).$$

The vector $m(v) \in \mathbf{R}^n$, defined by

$$m_i(v) = \max_{\{S \mid i \in S\}} R^v(S, i), \quad \text{for all } i \in N$$

is the minimal rights vector. Notice that player $i \in N$ can guarantee himself the payoff $m_i(v)$ by offering the members of a suitable coalition (the one where the above maximum is achieved) their utopia payoffs.

When the game is balanced, as in the case of the assignment game, it is straightforward to see that vectors $m(v)$ and $M(v)$ satisfy the following inequalities

$$m_i(v) \leq M_i(v), \quad \text{for all } i \in N \tag{4}$$

$$\sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v). \tag{5}$$

The $\tau$-value is then defined as the **unique** efficient payoff vector on the line segment between $m(v)$ and $M(v)$. Formally,

$$\tau(v) = \lambda m(v) + (1 - \lambda) M(v),$$

where $\lambda \in [0, 1]$ is unique satisfying $\sum_{i \in N} \tau_i(v) = v(N)$.

In some games the $\tau$-value does not lie in the core. This is not the case for the assignment game.

Notice first that, in the case of the assignment game, the utopia payoffs, $M_i(w) = w(M \cup M') - w(M \cup M' \setminus \{i\})$, are reasonable upper bounds, as each player attains his utopia payoff at least in one core allocation, either $(\bar{u}, \underline{v})$ or $(\underline{u}, \bar{v})$. As a consequence, the minimal rights vector also has full meaning in this case.

We first prove that the minimal rights payoff for each optimally matched player is his marginal contribution to the coalition that remains once his partner has left.

**Proposition 1.** *If* $(i_*, j_*) \in \mu$, *where* $\mu$ *is an optimal assignment of* $(M \cup M', w)$, *then*

$$m_{i_*}(w) = w(M \cup M' \setminus \{j_*\}) - w(M \cup M' \setminus \{i_*, j_*\}) = \underline{u}_{i_*} \tag{6}$$

$$m_{j_*}(w) = w(M \cup M' \setminus \{i_*\}) - w(M \cup M' \setminus \{i_*, j_*\}) = \underline{v}_{j_*} \tag{7}$$

*Proof:* From its definition, $m_{i_*}(w) = \max_{\{S \mid i_* \in S\}} w(S) - \sum_{i \in S \setminus \{i_*\}} M_i(w)$ and, taking coalition $S = \{i_*, j_*\}$, we obtain

$$\begin{aligned} m_{i_*}(w) &\geq w(i_*, j_*) - M_{j_*}(w) = a_{i_* j_*} - (w(M \cup M') - w(M \cup M' \setminus \{j_*\})) \\ &= w(M \cup M' \setminus \{j_*\}) - \sum_{\substack{(i,j) \in \mu \\ (i,j) \neq (i_*, j_*)}} a_{ij} \\ &= w(M \cup M' \setminus \{j_*\}) - w(M \cup M' \setminus \{i_*, j_*\}). \end{aligned}$$

Moreover, for all $S \subseteq M \cup M'$ containing $i_*$, and taking into account (2) and (3), we obtain

$$\begin{aligned} & w(M \cup M' \setminus \{j_*\}) - w(M \cup M' \setminus \{i_*, j_*\}) + \sum_{j \in S \setminus \{i_*\}} M_j(w) \\ &\quad = \underline{u}_{i_*} + \sum_{i \in (S \setminus \{i_*\}) \cap M} M_i(w) + \sum_{j \in S \cap M'} M_j(w) \geq \\ &\quad = \underline{u}_{i_*} + \sum_{i \in (S \setminus \{i_*\}) \cap M} \underline{u}_i + \sum_{j \in S \cap M'} \bar{v}_j = (\underline{u}, \bar{v})(S) \geq w(S). \end{aligned}$$

Then, $w(M \cup M' \setminus \{j_*\}) - w(M \cup M' \setminus \{i_*, j_*\}) \geq w(S) - \sum_{j \in S \setminus \{i_*\}} M_j(w)$ and hence $w(M \cup M' \setminus \{j_*\}) - w(M \cup M' \setminus \{i_*, j_*\}) = m_{i_*}(w)$, while a similar argument proves that $w(M \cup M' \setminus \{i_*\}) - w(M \cup M' \setminus \{i_*, j_*\}) = m_{j_*}(w)$.

Now the $\tau$-value can be easily computed.

**Theorem 1.** *Let $(M \cup M', w)$ be an assignment game and $\mu$ an optimal matching.*

1. *If a player $k \in M \cup M'$ is not matched by $\mu$, then $\tau_k(w) = 0$.*
2. *If players $i_*$ and $j_*$ are matched by $\mu$, $(i_*, j_*) \in \mu$, then*

$$\tau_{i_*}(w) = \frac{w(M \cup M') - w(M \cup M' \setminus \{i_*\}) + w(M \cup M' \setminus \{j_*\}) - w(M \cup M' \setminus \{i_*, j_*\})}{2} \tag{8}$$

$$\tau_{j_*}(w) = \frac{w(M \cup M') - w(M \cup M' \setminus \{j_*\}) + w(M \cup M' \setminus \{i_*\}) - w(M \cup M' \setminus \{i_*, j_*\})}{2} \tag{9}$$

*Proof:* 1) If $k \in M \cup M'$ is not assigned by $\mu$, then $M_k(w) = w(M \cup M') - w(M \cup M' \setminus \{k\}) = 0$ and $m_k(w) = \max_{k \in S} R^w(S, k) \geq w(k) = 0$. From inequality (4), $\tau_k(w) = 0$.

2) Assume $(i_*, j_*) \in M \times M'$ are matched by the optimal assignment $\mu$ of $(M \cup M', w)$. To see that the midpoint of the line segment between the utopia vector $M(w)$ and the minimal rights vector $m(w)$ is the $\tau$-value of the assignment game $(M \cup M', w)$, we have to prove only that $\frac{m(w)+M(w)}{2}$ is an efficient allocation, which is a direct consequence of proposition 1, since

$$\begin{aligned}(m(w) + M(w))(M \cup M') &= \sum_{(i,j) \in \mu} [(m_i(w) + M_i(w)) + (m_j(w) + M_j(w))] \\ &= \sum_{(i,j) \in \mu} 2(w(M \cup M') - w(M \cup M' \backslash \{i, j\}) \\ &= 2 \sum_{(i,j) \in \mu} a_{ij} = 2w(M \cup M').\end{aligned}$$

The expression of the $\tau$-value given in the above theorem can be rewritten taking into account that for any pair $(i_*, j_*) \in \mu$, it holds $a_{i_* j_*} = w(M \cup M') - w(M \cup M' \backslash \{i_*, j_*\})$, and we then obtain:

$$\tau_{i_*}(w) = \frac{a_{i_* j_*} - w(M \cup M' \backslash \{i_*\}) + w(M \cup M' \backslash \{j_*\})}{2} \quad (10)$$

$$\tau_{j_*}(w) = \frac{a_{i_* j_*} - w(M \cup M' \backslash \{j_*\}) + w(M \cup M' \backslash \{i_*\})}{2} \quad (11)$$

Moreover, by Demange (1982), $M(w) = (\bar{u}, \bar{v})$ and, by proposition 1 above, $m(w) = (\underline{u}, \underline{v})$, where $(\bar{u}, \underline{v})$ and $(\underline{u}, \bar{v})$ are, respectively, the buyers-optimal and the sellers-optimal core allocations. In other words, the utopia payoff vector is defined by the maximum amount each player can obtain in the core of the assignment game, and the minimum rights vector is based on the minimum amount each player can get in the core. An interesting expression of the $\tau$-value follows from this remark.

**Corollary 1.** *For any assignment game $(M \cup M', w)$, the $\tau$-value is the midpoint between the buyers-optimal and the sellers-optimal core allocation,*

$$\tau(w) = \frac{(\underline{u}, \bar{v}) + (\bar{u}, \underline{v})}{2}, \quad (12)$$

*and hence it is a core allocation.*

From the above corollary, the $\tau$-value of an assignment game coincides with the *fair solution* or *fair division point*, the solution concept defined by Thompson (1981) for this class of games as the midpoint of the segment determined by the buyers-optimal and the sellers-optimal core allocations.

To compute the $\tau$ value of an assignment game $(M \cup M', w)$ we need only $w(M \cup M')$ and the marginal contributions of all players on one side of the market. It is well known that $w(M \cup M')$ follows from the solution of a linear program (Shapley and Shubik, 1972). We can also obtain all the marginal contributions of players on one side of the market, for instance

$w(M \cup M') - w(M \cup M' \setminus \{i\})$ for all $i \in M$, as the first $m$ variables in the solution of the linear program

$$\max \quad \sum_{i \in M} x_i$$

$$\text{subject to } x_i + x_j \geq a_{ij}, \quad \text{for all } i \in M \text{ and } j \in M'$$

$$\sum_{i \in M} x_i + \sum_{j \in M'} x_j = w(M \cup M')$$

$$x_i \geq 0,\ x_j \geq 0, \quad \text{for all } i \in M \text{ and } j \in M'.$$

This last fact can be easily justified as all players on the same side of the market attain their maximum core payoffs, which are their marginal contributions, in the same core allocation, which is $(\bar{u}, \underline{v})$. Therefore, to compute the $\tau$-value of an assignment game, apart from efficiency, we only need to solve two linear programs, one for each side of the market.

## 4. Comparative statics

In the general framework of cooperative TU games, a monotonicity property called *coalitional monotonicity* is proved to be incompatible with core selection for games with at least five players (Young, 1985). The above impossibility result is extended by Housman and Clark (1998) for games with at least four players, by means of a $2 \times 2$-assignment game for which no point-solution can be defined that is a core selection and preserves coalitional monotonicity.

A point-solution $\alpha : G^N \to \mathbf{R}^n$, is coalitionally monotonic if for any two games $v, v' \in G^N$ such that there exists $\emptyset \neq T \subseteq N$ with $v(T) \leq v'(T)$ and $v(S) = v'(S)$ for all $S \neq T$, we have $\alpha_i(v) \leq \alpha_i(v')$ for all $i \in T$. That is to say, when one coalition increases its worth while the other coalitions remain the same, the players in this coalition cannot be paid less than in the original game.

Nevertheless coalitional monotonicity does not seem a sensible requirement for assignment games because when the worth of only one coalition is increased, the resulting game is no longer an assignment game. What is of interest in an assignment game is to study the behaviour of a solution when one entry in the assignment matrix increases.

Let $\mathscr{A}_{M \cup M'}$ be the set of assignment games with set of buyers $M$ and set of sellers $M'$. Given an assignment game $w \in \mathscr{A}_{M \cup M'}$ defined by matrix $A = (a_{ij})_{(i,j) \in M \times M'}$, consider the assignment game $w'$ defined by matrix $A' = (a'_{ij})_{(i,j) \in M \times M'}$ where there exists a pair $(i^*, j^*) \in M \times M'$ such that

$$a_{i^*j^*} \leq a'_{i^*j^*} \quad \text{and}$$

$$a_{ij} = a'_{ij} \quad \text{for all } (i,j) \neq (i^*, j^*).$$

We denote this fact by $A \leq_{(i^*, j^*)} A'$.

The above movement in the matrix entries has the following effect in the characteristic function: for all $S \subseteq N = M \cup M'$, if $\{i^*, j^*\} \nsubseteq S$, then $w(S) = w'(S)$, while if $\{i^*, j^*\} \subseteq S$, then $w(S) \leq w'(S)$. Moreover, the optimal matchings may also change.

**Definition 1.** *Let $w, w' \in \mathscr{A}_{M \cup M'}$ be defined by matrices $A$ and $A'$.*

*A point-solution $\alpha$ in $\mathscr{A}_{M \cup M'}$ is pairwise monotonic*[1] *if and only if whenever $A \leq_{(i,j)} A'$ we have $\alpha_i(w) \leq \alpha_i(w')$ and $\alpha_j(w) \leq \alpha_j(w')$.*

If the entry $a_{ij}$ increases, ceteris paribus for the rest of entries, then, by the above remark, the worth of all coalitions containing players $i$ and $j$ may also increase and therefore it seems natural to require the solution to be monotonic in the sense that these two players should not obtain a smaller payoff. Let us now see that this monotonicity property holds for the buyers-optimal core allocation and the sellers-optimal core allocation, and thus also for any fixed convex combination of both.

In theorem 2 which follows, when we prove that the buyers-optimal core allocation is pairwise monotonic, we use the fact that all marginal contributions of a player $i$ to a coalition $M \cup M' \backslash \{j\}$, where $j$ belongs to the opposite side of the market, are attainable in the core of the assignment game. This proposition is a consequence of the fact that for each ordering $\theta$ in the player set $M \cup M'$, the reduced marginal worth vector $rm_\theta^w$ is an extreme core allocation of the assignment game $(M \cup M', w)$ (Núñez and Rafels, 2003). For the sake of completeness we give the definition of the reduced marginal worth vectors.

The reduced marginal worth vectors are inspired by the marginal worth vectors. For each ordering $\theta = (k_1, k_2, \ldots, k_{n-1}, k_n)$, the reduced marginal worth vector $rm_\theta^w$ is a vector in $\mathbf{R}^{m+m'}$ where each player receives his marginal contribution to his set of predecessors, and a reduction of the game is performed in each step: $(rm_\theta^w)_{k_n} = w(k_1, k_2, \ldots, k_n) - w(k_1, \ldots, k_{n-1})$ and, for all $1 \leq r < n$,

$$(rm_\theta^w)_{k_r} = w^{k_n k_{n-1} \ldots k_{r-1}}(k_1, k_2, \ldots, k_r) - w^{k_n k_{n-1} \ldots k_{r-1}}(k_1, k_2, \ldots, k_{r-1}).$$

To complete the definition of these vectors, as in each step only one player leaves the game, it only rests to say that the game $w^{k_n}$, which we call $k_n$-marginal game, is no more that the reduced game *à la* Davis and Maschler on coalition $M \cup M' \backslash \{k_n\}$ and at the payoff $w(M \cup M') - w(M \cup M' \backslash \{k_n\})$. This means that, for each non empty coalition $S \subseteq M \cup M' \backslash \{k_n\}$,

$$w^{k_n}(S) = \max\{w(S), w(S \cup \{k_n\}) - (w(M \cup M') - w(M \cup M' \backslash \{k_n\}))\}.$$

**Proposition 2.** *Let $(M \cup M', w)$ be an assignment game. For all $i \in M$ and all $j \in M'$, the marginal contribution $w(M \cup M' \backslash \{j\}) - w(M \cup M' \backslash \{i, j\})$ is attained by player $i$ in the core of $w$, i.e. there exists $(u, v) \in C(w)$ such that $u_i = w(M \cup M' \backslash \{j\}) - w(M \cup M' \backslash \{i, j\})$.*

[1] Pairwise monotonicity was already defined in Sasaki (1995), although in his paper attention is focused on a weaker form of this property. In the present framework of point-solutions, Sasaki's weak pairwise monotonicity would demand $\alpha_i(w) + \alpha_j(w) \leq \alpha_i(w') + \alpha_j(w')$ whenever $A \leq_{(i,j)} A'$.

*Proof:* Take $\theta = (k_1, k_2, \ldots, k_{n-1}, k_n)$ an ordering in $M \cup M'$ such that $k_n = j$ and $k_{n-1} = i$. Let us consider the reduced marginal worth vector $rm_\theta^w$ related to this ordering. From Núñez and Rafels (2003), $rm_\theta^w \in C(w)$. We now prove that $(rm_\theta^w)_i = w(M \cup M' \backslash \{j\}) - w(M \cup M' \backslash \{i, j\})$.

By the definition of the reduced marginal worth vectors, $(rm_\theta^w)_j = w(M \cup M') - w(M \cup M' \backslash \{j\})$ and $(rm_\theta^w)_i = w^j(M \cup M' \backslash \{j\}) - w^j(M \cup M' \backslash \{i, j\})$, where $(M \cup M' \backslash \{j\}, w^j)$ is the marginal game corresponding to player $j$ and is defined by $w^j(\varnothing) = 0$ and, for all $\varnothing \neq S \subseteq M \cup M' \backslash \{j\}$,

$$w^j(S) = \max\{w(S), w(S \cup \{j\}) - (w(M \cup M') - w(M \cup M' \backslash \{j\}))\}.$$

From the definition, $w^j(M \cup M' \backslash \{j\}) = w(M \cup M' \backslash \{j\})$ and

$$w^j(M \cup M' \backslash \{i, j\}) = \max\{w(M \cup M' \backslash \{i, j\}, w(M \cup M' \backslash \{i\}) - w(M \cup M') + w(M \cup M' \backslash \{j\})\}.$$

But, since players $i$ and $j$ are from different sides of the market, we know from Shapley (1962) that

$$w(M \cup M' \backslash \{i\}) - w(M \cup M' \backslash \{i, j\}) \leq w(M \cup M') - w(M \cup M' \backslash \{j\})$$

and thus $w^j(M \cup M' \backslash \{i, j\}) = w(M \cup M' \backslash \{i, j\})$.

Once proved that $(rm_\theta^w)_i = w(M \cup M' \backslash \{j\}) - w(M \cup M' \backslash \{i, j\})$, this marginal contribution is attained in $C(w)$.

**Theorem 2.** *The buyers-optimal core allocation and the sellers-optimal core allocation are pairwise monotonic.*

*Proof:* We prove pairwise monotonicity only for the buyers-optimal core allocation, as the proof for the sellers-optimal core allocation is analogous and is left to the reader.

Take $(M \cup M', w)$ and $(M \cup M', w')$ two assignment games respectively defined by matrices $A$ and $A'$. Assume there exists one pair $(i, j) \in M \times M'$ such that $a_{ij} < a'_{ij}$ while the rest of entries coincide for both matrices. Then,

$$w(M \cup M' \backslash \{i\}) = w'(M \cup M' \backslash \{i\}) \tag{13}$$

$$w(M \cup M' \backslash \{j\}) = w'(M \cup M' \backslash \{j\}) \tag{14}$$

$$w(M \cup M' \backslash \{i, j\}) = w'(M \cup M' \backslash \{i, j\}) \tag{15}$$

$$w(M \cup M' \backslash \{k\}) \leq w'(M \cup M' \backslash \{k\}) \quad \text{for all } k \neq i, j \tag{16}$$

Let $(\bar{u}^w, \underline{v}^w)$ and $(\bar{u}^{w'}, \underline{v}^{w'})$ be the buyers-optimal core allocation of the games $w$ and $w'$. When proving that $\bar{u}_i^w \leq \bar{u}_i^{w'}$ and $\underline{v}_j^w \leq \underline{v}_j^{w'}$, we consider three cases.

*Case 1:* Assume $(i, j) \in \mu$, where $\mu$ is an optimal assignment for $A$. Then $\mu$ is an optimal assignment for $A'$ and $w(M \cup M') \leq w'(M \cup M')$. From equations (13) to (16) above,

$$\bar{u}_i^w = w(M \cup M') - w(M \cup M' \backslash \{i\}) \leq w'(M \cup M') - w'(M \cup M' \backslash \{i\})$$
$$= \bar{u}_i^{w'}$$
$$\underline{v}_j^w = w(M \cup M' \backslash \{j\}) - w(M \cup M' \backslash \{i, j\})$$
$$= w'(M \cup M' \backslash \{j\}) - w'(M \cup M' \backslash \{i, j\}) = \underline{v}_j^{w'}.$$

*Case 2:* Assume $(i, j) \notin \mu$, for any optimal assignment of $A$, but $(i, j) \in \mu'$ for some optimal assignment $\mu'$ of $A'$. Then, on one side, from $w(M \cup M') \leq w'(M \cup M')$, and taking into account equation (13),

$$\bar{u}_i^w = w(M \cup M') - w(M \cup M' \backslash \{i\}) \leq w'(M \cup M') - w'(M \cup M' \backslash \{i\})$$
$$= \bar{u}_i^{w'}.$$

Moreover, if $j$ was not assigned by some optimal matching of $A$, then it follows that $\underline{v}_j^w = 0 \leq \underline{v}_j^{w'}$. On the other hand, if $j$ is assigned to $i_j$ for some optimal matching $\mu$ of $A$, to prove $\underline{v}_j^w \leq \underline{v}_j^{w'}$ it is enough to prove that

$$w(M \cup M' \backslash \{i_j\}) - w(M \cup M' \backslash \{i_j, j\})$$
$$\leq w(M \cup M' \backslash \{i\}) - w(M \cup M' \backslash \{i, j\}), \tag{17}$$

since, from (13) and (15),

$$w'(M \cup M' \backslash \{i\}) - w'(M \cup M' \backslash \{i, j\})$$
$$= w(M \cup M' \backslash \{i\}) - w(M \cup M' \backslash \{i, j\}).$$

Now, (17) follows because $w(M \cup M' \backslash \{i_j\}) - w(M \cup M' \backslash \{i_j, j\})$ is the minimal payoff to player $j$ in $C(w)$ and $w(M \cup M' \backslash \{i\}) - w(M \cup M' \backslash \{i, j\})$, by proposition 2, is attained by player $j$ in $C(w)$. Then, also in this second case $\underline{v}_j^w \leq \underline{v}_j^{w'}$.

*Case 3:* If $(i, j)$ does not belong to an optimal matching in $A$ nor in $A'$, then $w(M \cup M') = w'(M \cup M')$ and an optimal matching $\mu$ of $A$ is also optimal for $A'$. Then

$$\bar{u}_i^w = w(M \cup M') - w(M \cup M' \backslash \{i\}) = w'(M \cup M') - w'(M \cup M' \backslash \{i\})$$
$$= \bar{u}_i^{w'}.$$

As in the above case, if $j$ is not assigned by $\mu$, then $\underline{v}_j^w = 0$ and the claim follows. Otherwise, let $(i_j, j) \in \mu$. It follows that $\underline{v}_j^w \leq \underline{v}_j^{w'}$ if and only if

$$w(M \cup M' \backslash \{i_j\}) - w(M \cup M' \backslash \{i_j, j\})$$
$$\leq w'(M \cup M' \backslash \{i_j\}) - w'(M \cup M' \backslash \{i_j, j\}),$$

since $\mu$ is an optimal matching in both $A$ and $A'$. Last inequality holds as it is straightforward to see that $w(M \cup M' \backslash \{i_j, j\}) = w'(M \cup M' \backslash \{i_j, j\})$, while $w(M \cup M' \backslash \{i_j\}) \leq w'(M \cup M' \backslash \{i_j\})$ from inequality (16).

We have thus proved that there exist point-solutions in the class of assignment games that are core selection and pairwise monotonic. In fact there is an infinity of them. As a result of the above theorem, all point-solutions defined as a fixed convex combination of the buyers-optimal and the sellers-optimal core allocations are pairwise monotonic and thus the $\tau$-value is pairwise monotonic.

**Corollary 2.** *Let $\lambda \in [0,1]$ and $\alpha_\lambda : \mathscr{A}_{M \cup M'} \to \mathbf{R}^{m+m'}$ be the point-solution in $\mathscr{A}_{M \cup M'}$ defined by*

$$\alpha_\lambda(w) = \lambda(\bar{u}^w, \underline{v}^w) + (1-\lambda)(\underline{u}^w, \bar{v}^w) \text{ for all } w \in \mathscr{A}_{M \cup M'}.$$

*Then $\alpha_\lambda$ is a pairwise monotonic core selection.*

However, not all solutions that are core selections are also pairwise monotonic.

Another monotonicity property is incompatible with core selections for the assignment game. Sprumont (1990) defines population monotonic allocation schemes and proves that no assignment game with at least a $2 \times 2$ submatrix with positive entries has a population monotonic allocation scheme.

An allocation scheme for a game $(N, v)$ is a payoff vector $(x_{i,S})_{i \in S}$ for each nonempty coalition $S \subseteq N$. It is said to be population monotonic if $\sum_{i \in S} x_{i,S} = v(S)$ for all $S \subseteq N$ and, whenever $S \subseteq T$, it holds $x_{i,S} \leq x_{i,T}$ for all $i \in S$. That is to say, in a population monotonic allocation scheme, whenever there is a new entrant, all existing agents are better off.

However, in an assignment game there are two types of agents and the possibilities of trading of an agent can decrease when there is a new entrant on his side of the market.

Let us now consider the class $\mathscr{A}$ of assignment games with any (finite) set of buyers and sellers. Take the assignment game $(M \cup M', w)$ defined by a matrix $A = (a_{ij})_{(i,j) \in M \times M'}$ and assume, without loss of generality, that a new buyer, $i^* \in M$ joins the market. Let $((M \cup \{i^*\}) \cup M', w')$ be a new assignment game defined by $A' = (a'_{ij})_{(i,j) \in (M \cup \{i^*\}) \times M'}$ where $a'_{ij} = a_{ij}$ if $(i,j) \in M \times M'$. From Mo (1988), if $(\bar{u}^w, \underline{v}^w)$ and $(\bar{u}^{w'}, \underline{v}^{w'})$ are, respectively, the buyers-optimal core allocation of the games $w$ and $w'$, then $\bar{u}_i^{w'} \leq \bar{u}_i^w$, for all $i \in M$, while $\underline{v}_j^{w'} \geq \underline{v}_j^w$ for all $j \in M'$. Similarly, if $(\underline{u}^w, \bar{v}^w)$ and $(\underline{u}^{w'}, \bar{v}^{w'})$ are the sellers-optimal core allocation of the games $w$ and $w'$, then $\underline{u}_i^{w'} \leq \underline{u}_i^w$ while $\bar{v}_j^{w'} \geq \bar{v}_j^w$.

We then say that such a solution defined in $\mathscr{A}$ is type monotonic (see Brânzei et al, 2001).

**Definition 2.** *A point-solution $\alpha$ in $\mathscr{A}$ is type monotonic if whenever there is a new entrant $k$:*

1. *If $k$ is a buyer then $\alpha_i(w) \geq \alpha_i(w')$ for all $i \in M$ and $\alpha_j(w) \leq \alpha_j(w')$ for all $j \in M'$.*
2. *If $k$ is a seller, $\alpha_i(w) \leq \alpha_i(w')$ for all $i \in M$ and $\alpha_j(w) \geq \alpha_j(w')$ for all $j \in M'$.*

The same qualitative effects are found for all point-solutions defined as a fixed convex combination of the buyers-optimal and the sellers-optimal core allocations: for any agent $i \in M \cup M'$, when an agent of the same side enters the market, player $i$'s payoff does not increase and if an agent of the opposite side enters the market, player $i$'s payoff does not decrease.

**Proposition 3.** *Let $\lambda \in [0, 1]$ and $\alpha_\lambda$ be the point-solution in $\mathscr{A}$ defined by*

$$\alpha_\lambda(w) = \lambda(\bar{u}^w, \underline{v}^w) + (1 - \lambda)(\underline{u}^w, \bar{v}^w) \quad \textit{for all } w \in \mathscr{A}.$$

*Then $\alpha_\lambda$ is type monotonic.*

The proof is straightforward from Mo (1988) and is left to the reader. Notice that as an immediate consequence the $\tau$-value is type monotonic.

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