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Assignment games with stable core*

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Abstract. We prove that the core of an assignment game (a two-sided matching game with transferable utility as introduced by Shapley and Shubik, 1972) is stable (i.e., it is the unique von Neumann-Morgenstern solution) if and only if there is a matching between the two types of players such that the corresponding entries in the underlying matrix are all row and column maximums. We identify other easily verifiable matrix properties and show their equivalence to various known sufficient conditions for core-stability. By these matrix characterizations we found that on the class of assignment games, largeness of the core, extendability and exactness of the game are all equivalent conditions, and strictly imply the stability of the core. In turn, convexity and subconvexity are equivalent, and strictly imply all aformentioned conditions.

Key words: assignment game, stable core, large core, exact game

1. Introduction

Assignment games (Shapley and Shubik, 1972) are models of two-sided matching markets with transferable utilities where the aim of each player on one side is to form a profitable coalition with a player on the other side. Since only such bilateral cooperations are worthy, these games are completely defined by the matrix containing the cooperative worths of all possible pairings of players from the two sides.

Shapley and Shubik (1972) showed that the core of an assignment game is precisely the set of dual optimal solutions to the assignment optimization problem on the underlying matrix of mixed-pair profits. This result not only implies

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that all assigment games have non-empty core but also that the core can be determined without explicitly generating the entire coalitional function of the game. The algorithm of Solymosi and Raghavan (1994) demonstrates that for assignment games the nucleolus can also be directly computed from the data that induces the game. The bargaining set $\mathscr{M}_1^{(i)}$ (Aumann and Maschler, 1964; Davis and Maschler, 1967) is another solution which is completely known and easily computable for assignment games, since for these games it coincides with the core (Solymosi, 1999).

The stable set, the classical solution suggested by von Neumann and Morgenstern (1944), need not exist for all TU-games (Lucas, 1968). Besides, it need not be unique. Indeed, if a game has a stable set then it typically has a multitude of stable sets. On the other hand, there are classes of special games on which the stable set exists and is unique. Perhaps the best known such class is that of convex games for which Shapley (1971) proved that the core is the unique stable set.

The main purpose of this paper is to identify those assignment games whose core is stable, hence it is the unique stable set. Since assignment games induced by diagonal matrices are convex (cf. Theorem 3), there are such games. A secondary goal is to give the characterization in terms of properties of the underlying matrix so that its verification does not require the explicit knowledge of the entire coalitional function. We achieve both goals by proving that the core of an assignment game is stable if and only if the entries in an optimal assignment for the grand coalition are row and column maximums in the underlying matrix (Theorem 1). In other words, to have core-stability it is necessary and sufficient that there is a matching between the two types of players in which each player is paired with whom his/her profitability is the highest. This implies that assignment games with a different number of players on the two sides cannot have a stable core. It remains an open problem whether or not all assignment games have a stable set.

Several sufficient conditions for stability of the core have been discussed in the literature. Convexity of the game (Shapley, 1971) is a well-known one. Subconvexity of the game and largeness of the core were introduced by Sharkey (1982) who showed that convexity implies subconvexity; subconvexity implies largeness of the core; which in turn implies stableness of the core. In an unpublished paper Kikuta and Shapley (1986) investigated another condition, baptized to extendability of the game in (Gellekom et al., 1999), and proved that it is necessary for core-largeness and still sufficient for core-stability. A unified proof of all these relations was given by Gellekom et al. (1999). Their valid counter-examples demonstrate that these conditions are indeed all different.

Shapley (1971) proved that in a convex game, for each coalition there is a core allocation which gives exactly its worth to the coalition. This condition, baptized to exactness of the game in (Schmeidler, 1972), typically neither implies nor is implied by core-stability (Biswas et al., 1999; Gellekom et al., 1999). On the other hand, for totally balanced games, exactness is implied by largeness of the core (Sharkey, 1982) and even by extendability (Biswas et al., 1999). It follows from the results of Biswas et al. (1999 and 2000) that for totally balanced symmetric games and for totally balanced games with no more than four players, core-largeness, extendability, exactness, and core-stability are all equivalent conditions.

We provide matrix characterizations also for these sufficient conditions (Theorem 2 and Theorem 3). It turns out that on the class of assignment games, largeness of the core, extendability and exactness of the game are all equivalent conditions, but are strictly stronger than stability of the core. Convexity and subconvexity are also equivalent, and are strictly stronger than all the other conditions.

2. Definitions and preliminaries

A transferable utility cooperative game on the nonempty finite *set P of players* is defined by a *coalitional function* $V : 2^P \to \mathbf{R}$ satisfying $V(\emptyset) = 0$. The function V specifies the worth of every *coalition* $S \subseteq P$.

Given a game (P, V), a *payoff allocation* $x \in \mathbf{R}^P$ is called *efficient*, if $x(P) = V(P)$; *individually rational*, if $x_i = x(\{i\}) \geq V(\{i\})$ for all $i \in P$; *coalitionally rational*, if $x(S) \geq V(S)$ for all $S \subseteq P$; where, by the standard notation, $x(S) = \sum_{i \in S} x_i$ if $S \neq \emptyset$, and $x(\emptyset) = 0$. We denote by $\mathscr{I}(P, V)$ the *imputation set* (i.e., the set of efficient and individually rational payoffs), and by $\mathscr{C}(P, V)$ the *core* (i.e., the set of efficient and coalitionally rational payoffs) of the game (P, V).

The game (P, V) is called *superadditive*, if $S \cap T = \emptyset$ implies $V(S \cup T) \geq V(S) + V(T)$ for all $S, T \subseteq P$; *balanced*, if its core $\mathscr{C}(P, V)$ is not empty; and *totally balanced*, if every *subgame* (i.e., the game obtained by restricting the player set to a coalition and the coalitional function to the power set of that coalition) is balanced. Note that totally balanced games are superadditive.

Given a game (P, V), the *excess* $e(S, x) := V(S) - x(S)$ is the usual measure of gain (or loss if negative) to coalition $S \subseteq P$ if its members depart from allocation $x \in \mathbf{R}^P$ in order to form their own coalition. Note that $e(\emptyset, x) = 0$ for all $x \in \mathbf{R}^P$, and

$$\mathscr{C}(P, V) = \{x \in \mathbf{R}^P : e(P, x) = 0, e(S, x) \leq 0 \ \forall S \subset P\},$$

i.e., the core is the set of allocations which yield nonpositive excess for all coalitions.

We say that *allocation y dominates allocation x via coalition S* if $y(S) \leq V(S)$ and $y_k > x_k \ \forall k \in S$. Note that an allocation can be dominated only via coalitions having positive excess at that allocation. The (nonempty) core $\mathscr{C}$ of a game is called *stable* if for every imputation $x \in \mathscr{I} \backslash \mathscr{C}$ there exists a core allocation $y \in \mathscr{C}$ and a coalition S such that y dominates x via S.

Given two finite sets S and T, we call $\mu \subseteq S \times T$ an (S, T)-*assignment*, if it is a bijection from some $S' \subseteq S$ to some $T' \subseteq T$ such that $|S'| = |T'| = \min(|S|, |T|)$. Trivially, $\mu = \emptyset$ if $S = \emptyset$ or $T = \emptyset$. We shall write $(i, j) \in \mu$ as well as $\mu(i) = j$. We denote by $\Pi_{(S,T)}$ the set of all (S, T)-assignments. Obviously, $\Pi_{(S,T)} = \{\emptyset\}$ if $S = \emptyset$ or $T = \emptyset$.

A game (P, V) is called an *assignment game* if there exists a partition $P = I \cup J$, $I \cap J = \emptyset$, of the player set and a nonnegative matrix $A = [a_{ij}]_{i \in I, j \in J}$ such that

$$V(S) = V_A(S) := \max_{\mu \in \Pi_{(S \cap I, S \cap J)}} \sum_{(i,j) \in \mu} a_{ij} \quad \forall S \subseteq P.$$

By adding dummy player(s) (i.e. zero rows/columns to the matrix), we can assume without loss of generality that there are the same number of players of both types (i.e. the underlying matrix is square). It will be convenient to identify the players with the row/column indices. So we shall use $N = \{1, 2, \ldots, n\}$ for the set of indices, while $I = \{1, 2, \ldots, n\}$ and $J = \{1', 2', \ldots, n'\}$ for the set of row and column players, respectively. In other words, we put a prime ($'$) on the index j to distinguish the j-th column player $j' \in J$ from the j-th row player $j \in I$. Coalitions of the type $\{i, j'\}$ are called mixed-pair coalitions.

Throughout the paper we assume that the rows and columns of the underlying matrix A are arranged such that the diagonal assignment $\{(i, i) : i \in N\}$ is maximal in A, i.e., $V_A(I \cup J) = \sum_{i=1}^{n} a_{ii}$.

To emphasize the bipartite nature of assignment games, we shall write the payoff allocations as $(u, v) \in \mathbf{R}^n \times \mathbf{R}^n$. Let us introduce the notation

$$e_{ij}(u, v) := a_{ij} - u_i - v_j \quad i, j \in N$$

for the excess of coalition $\{i, j'\}$ at allocation (u, v). We associate with $(u, v) \in \mathbf{R}^n \times \mathbf{R}^n$ the $n \times n$ *excess matrix* $E(u, v) = [e_{ij}(u, v)]_{i,j \in N}$.

The (total) balancedness of assignment games was proved by Shapley and Shubik (1972). One key point in their characterization of the core of assignment games is that, besides efficiency, it suffices to require rationality only for single-player and mixed-pair coalitions. Formally,

$$\mathscr{C}(V_A) = \{(u, v) \in \mathscr{I}(V_A) : e_{ii}(u, v) = 0, e_{ij}(u, v) \leq 0 \ \forall i, j \in N\},$$

where for the imputation set we clearly have

$$\mathscr{I}(V_A) = \left\{(u, v) \in \mathbf{R}^n \times \mathbf{R}^n : \sum_{k=1}^{n} e_{kk}(u, v) = 0; u_i \geq 0, v_i \geq 0 \ \forall i \in N\right\}.$$

3. Assignment games with stable core

We say that a matrix A has *dominant diagonal* if all of its diagonal entries are row and column maximums, i.e., $a_{ii} \geq a_{ij}$ and $a_{ii} \geq a_{ji}$ for all $i, j \in N$. Note that A has dominant diagonal if and only if in the assignment game V_A the special imputations

$$(\underline{u}, \bar{v}) := (\underline{u}_i = 0, \bar{v}_i = a_{ii} : i \in N) \quad (\bar{u}, \underline{v}) := (\bar{u}_i = a_{ii}, \underline{v}_i = 0 : i \in N) \tag{1}$$

are both core allocations.

Now we are ready to prove our main result.

Theorem 1. *Let A be a nonnegative $n \times n$-matrix such that its main diagonal is an optimal assignment, and let V_A be the $(n+n)$-player assignment game induced by A. Then the following are equivalent:*

(i) *$\mathscr{C}(V_A)$ is stable;*
(ii) *A has dominant diagonal.*

Proof: (i) $\Rightarrow$ (ii) Consider the imputation $(\underline{u}, \bar{v})$ defined in (1). If $(\underline{u}, \bar{v}) \notin \mathscr{C}(V_A)$ then, by the stability of the core, there exists an allocation $(u', v') \in \mathscr{C}(V_A)$ such that (u', v') dominates $(\underline{u}, \bar{v})$ via some coalition S. Since then $V_A(S)$ must be positive, S contains at least one column player, say the j-th one. Then $v'_j > \bar{v}_j = a_{jj}$, a contradiction to $v'_j \le a_{jj}$ that holds in any core allocation (u', v'). We get that $(\underline{u}, \bar{v}) \in \mathscr{C}(V_A)$. Interchanging the roles of rows and columns and repeating the above argument with the other special imputation $(\bar{u}, \underline{v})$, we similarly get that $(\bar{u}, \underline{v}) \in \mathscr{C}(V_A)$. It follows that A has dominant diagonal.

(ii) $\Rightarrow$ (i) Let $(u, v) \in \mathscr{I}(V_A) \backslash \mathscr{C}(V_A)$. We show that if A has dominant diagonal then there exists $(u', v') \in \mathscr{C}(V_A)$ such that (u', v') dominates (u, v) via some coalition S. Actually, we show domination via a mixed-pair coalition.

Case 1: $e_{ii}(u, v) \neq 0$ for some $i \in N$.

Since the sum of the diagonal entries in $E(u, v)$ is 0, there exists an $i \in N$ such that $e_{ii}(u, v) > 0$, i.e., $u_i + v_i < a_{ii}$. Since (u, v) is an imputation, $a_{ii} > 0$. Then clearly there is a $\lambda \in (0, 1)$ such that $u_i < (1 - \lambda) a_{ii}$ and $v_i < \lambda a_{ii}$. (Take, e.g., $\lambda = \dfrac{a_{ii} - u_i + v_i}{2a_{ii}}$.) This implies that the allocation $(u', v') := \lambda(\underline{u}, \bar{v}) + (1 - \lambda) \cdot (\bar{u}, \underline{v})$ dominates (u, v) via the diagonal mixed-pair coalition $\{i, i'\}$. Since obviously $(u', v') \in \mathscr{C}(V_A)$, the claim follows.

Case 2: $e_{ii}(u, v) = 0$ for all $i \in N$.

With such an imputation (u, v) we associate a directed graph $G(u, v)$ with node set N such that there is an arc (i, j) of "length" $e_{ij} := e_{ij}(u, v)$ from any node i to any node $j \neq i$. Note that the arc lengths can be positive, negative or zero.

Let us recall some graph terminology. In a directed graph a *path* from node s to node t, or an (s, t)-path for short, is a sequence $(s, i_1), (i_1, i_2), \ldots, (i_k, t)$ of arcs. The path is called *simple* if the visited nodes $s, i_1, \ldots, i_k, t$ are all distinct. A *cycle* is a set of arcs such that exactly one arc goes in and exactly one arc goes out from each visited node.

Since the diagonal is a maximal assignment in A, the graph $G(u, v)$ does not contain a cycle of positive length. Suppose not, and the cycle $(i_1, i_2), \ldots, (i_k, i_1)$ is of positive length. Then, with $i_{k+1} = i_1$, we get $\sum_{h=1}^{k} e_{i_h i_{h+1}} = \sum_{h=1}^{k} a_{i_h i_{h+1}} - \sum_{h=1}^{k} u_{i_h} - \sum_{h=1}^{k} v_{i_{h+1}} = \sum_{h=1}^{k} a_{i_h i_{h+1}} - \sum_{h=1}^{k} a_{i_h i_h} > 0$, a contradiction to the optimality of the diagonal assignment. (Notice that the graph $G(u, v)$ contains a cycle of zero length if and only if the diagonal is not the only maximal assignment in A.) It follows that for any node $k \in N$, the numbers

$$\ell_k := \text{the length of the longest path ending in } k, \quad d_k := \max\{0, \ell_k\}$$

are both well defined. Moreover, d_k is nonnegative.

Since A has dominant diagonal, from

$$u_i + v_i = a_{ii} \ge a_{ij} = u_i + v_j + e_{ij}; \quad u_j + v_j = a_{jj} \ge a_{ij} = u_i + v_j + e_{ij}$$

we get for any $i \neq j$,

$$v_i - v_j \ge e_{ij}; \quad u_j - u_i \ge e_{ij}. \tag{2}$$

It follows from the telescopic nature of these inequalities that for any two nodes s and t in $G(u, v)$,

$$\min\{u_t - u_s, v_s - v_t\} \geq \text{the length of the longest } (s,t)\text{-path}. \tag{3}$$

Let us define the allocation (u', v') by $u'_i = u_i - d_i$ and $v'_i = v_i + d_i$ for all $i \in N$. We claim that (u', v') is in the core. Obviously, $u'_i + v'_i = a_{ii}$ and $v'_i \geq 0$ for all $i \in N$. To see $u'_i \geq 0$ in case $d_i > 0$, let k be a node such that there is a (k, i)-path of length d_i. Then from (3) we get $u_i - u_k \geq d_i$, hence $u'_i = u_i - d_i \geq u_k \geq 0$. Obviously, $u'_i \geq 0$ in case $d_i = 0$. It remains to see $u'_i + v'_j \geq a_{ij}$ for all $i \neq j \in N$. By definitions, $u'_i + v'_j = u_i + v_j + d_j - d_i$ and $a_{ij} = u_i + v_j + e_{ij}$, so we only need to show $d_j \geq d_i + e_{ij}$. If $d_i = 0$, we are done by noting that arc (i, j) is itself a path ending in node j, so $d_j \geq \ell_j \geq e_{ij}$. If $d_i > 0$, take a longest path ending in node i. Its length is $d_i = \ell_i$. Adding to this path arc (i, j) gives a (not necessarily simple) path ending in j of length $d_i + e_{ij}$, so $d_j \geq \ell_j \geq d_i + e_{ij}$. Therefore, (u', v') is indeed a core allocation.

Since (u, v) is not in the core, there is at least one positive arc in the graph $G(u, v)$. Thus, $d_i > 0$ for at least one node i. We claim that there exists an arc (p, q) of positive length such that $d_q = e_{pq}$. Let r be a node for which $d_r > 0$, and let us consider a path of length d_r that ends in r. Suppose this path starts from node p. We can assume without loss of generality that the path is simple (just leave out the cycles of zero length, if any) and minimal with respect to inclusion (i.e., there are no paths of zero length ending in p). Clearly, $d_p = 0$ for the starting node p, and our path is a longest (p, r)-path. It follows that for any node t on this path the length of the subpath from p to t is positive. In particular, the first arc in this path, ending in node q, has positive length. Therefore, we indeed have an arc (p, q) such that $d_p = 0$ and $d_q = e_{pq} > 0$.

We finish the proof by constructing a core allocation (u'', v'') which dominates (u, v) via this particular mixed-pair coalition $\{p, q'\}$. To this end, let

$$\varepsilon := \frac{1}{2}\min\{e_{pq}, \min\{v'_j : v'_j > 0\}\}.$$

Clearly, $\varepsilon > 0$. Define (u'', v'') by

$$u''_i := u'_i + \varepsilon \quad \text{and} \quad v''_i := v'_i - \varepsilon \quad \text{if } v'_i > 0$$

$$u''_i := u'_i \; (= a_{ii}) \quad \text{and} \quad v''_i := v'_i \quad \text{if } v'_i = 0$$

for all $i \in N$.

We claim that (u'', v'') is in the core. Since $(u', v') \in \mathscr{C}$, we obviously have $u''_i \geq 0$, $v''_i \geq 0$ and $u''_i + v''_i = a_{ii}$ for all $i \in N$. It remains to check that $u''_i + v''_j \geq a_{ij}$ for all $i \neq j$. When both v'_i and v'_j are positive or when both are zero, we have $u''_i + v''_j = u'_i + v'_j \geq a_{ij}$. If $v'_i > 0$ and $v'_j = 0$ then $u''_i + v''_j = u'_i + \varepsilon + v'_j \geq a_{ij} + \varepsilon > a_{ij}$. On the other hand, if $v'_i = 0$ and $v'_j > 0$ then $u''_i := u'_i = a_{ii}$ and $u''_i + v''_j = u'_i + v'_j - \varepsilon > a_{ii} \geq a_{ij}$. Thus, (u'', v'') is indeed a core allocation.

Finally, we show that (u'', v'') dominates (u, v) via the mixed-pair coalition $\{p, q'\}$. Firstly, since $e_{pq} > 0$, (2) implies $v_p > v_q \geq 0$, so $v'_p \; (= v_p) > 0$. Then $d_p = 0$ gives $u''_p = u'_p + \varepsilon = u_p - d_p + \varepsilon > u_p$. Secondly, it follows from $d_q = e_{pq}$ that $v'_q = v_q + d_q > 0$. Then $\varepsilon \leq \frac{1}{2}e_{pq}$ gives $v''_q = v'_q - \varepsilon \geq v_q + \frac{1}{2}e_{pq} > v_q$. Thirdly, by definitions, $u''_p + v''_q = u'_p + v'_q = u_p + v_q + e_{pq} = a_{pq}$. Thus, $(u'', v'') \in \mathscr{C}$ indeed dominates (u, v) via the mixed-pair coalition $\{p, q'\}$.

4. Assignment games with large core

A (balanced) game (P, V) is said to have a *large core* if for every coalitionally rational allocation y there is a core allocation $x \in \mathscr{C}(P, V)$ such that $x_k \leq y_k$ for all $k \in P$. Sharkey (1982) proved that largeness implies stableness of the core. He also proved that if a game has a large core then the totally balanced cover of the game is exact. For assignment games this means that largeness of the core implies exactness of the game. This latter property is defined as follows. A (balanced) game (P, V) is said to be *exact* if for every coalition S there is a core allocation $x \in \mathscr{C}(P, V)$ such that $x(S) = V(S)$. A related property, weaker than largeness of the core but (for totally balanced games) stronger than exactness, was investigated by Kikuta and Shapley (1986). They showed that this property is also sufficient for stability of the core. A (balanced) game (P, V) is said to be *extendable* if any core allocation of any subgame can be extended to a core element of (P, V).

We say that matrix A has *doubly dominant diagonal* if $a_{ii} + a_{jk} \geq a_{ik} + a_{ji}$ for all $i, j, k \in N$. Notice that this property is restrictive only if i is distinct from j and k. Also note that having a dominant diagonal and having a doubly dominant diagonal are independent properties, i.e., a matrix can have a dominant but not a doubly dominant diagonal and vice versa.

Now we are ready to prove the main result in this section.

Theorem 2. *Let A be a nonnegative $n \times n$-matrix such that its main diagonal is an optimal assignment, and let V_A be the $(n+n)$-player assignment game induced by A. Then the following are equivalent:*

(i) *$\mathscr{C}(V_A)$ is large;*
(ii) *V_A is extendable;*
(iii) *V_A is exact;*
(iv) *A has dominant and doubly dominant diagonal.*

Proof: (i) $\Rightarrow$ (ii) It holds for any TU-game, see (Kikuta and Shapley, 1986) or the proof of Proposition 1 in (Biswas et al., 1999).

(ii) $\Rightarrow$ (iii) It is straightforward to establish for totally balanced games, see the proof of Proposition 2 in (Biswas et al., 1999).

(iii) $\Rightarrow$ (iv) Let V_A be exact.

Take the coalition J of all column players. Then there is a $(u, v) \in \mathscr{C}(V_A)$ such that $(u, v)(J) = v(J) = V_A(J) = 0$, thus $v_j = 0$ for all $j \in J$. It follows that $u_i = a_{ii}$ for every row player i, i.e., $(u, v) = (\bar{u}, \underline{v})$. Thus, the core contains one of the special imputations in (1). The mirror argument gives that the core contains the other special imputation in (1). Therefore, the underlying matrix A has dominant diagonal.

As we remarked above, when we check that our matrix A has a doubly dominant diagonal we can assume without loss of generality that $i \neq j$ and $i \neq k$. Take coalition $\{j, k'\}$. Then there is a $(u, v) \in \mathscr{C}(V_A)$ such that $u_j + v_k = a_{jk}$. Adding it to $u_i + v_i = a_{ii}$ and using coalitional rationality of (u, v) give $a_{ii} + a_{jk} = u_i + v_k + u_j + v_i \geq a_{ik} + a_{ji}$. Thus, A has doubly dominant diagonal.

(iv) $\Rightarrow$ (i) Let A have dominant and doubly dominant diagonal.

Take an allocation (u'', v'') that is coalitionally rational in V_A. Then $u_i'' \geq 0$, $v_j'' \geq 0$ and $e_{ij}'' := e_{ij}(u'', v'') \leq 0$ for all $i, j \in N$. Let us define (u', v') by $u_i' := u_i'' - \min\{u_i'', \min_j(-e_{ij}'')\}$ and $v_i' := v_i''$ for all $i \in N$. Clearly, (u', v') is coalitionally rational, and $u_i' \leq u_i''$ for all $i \in N$. Moreover, for each row player i, either $u_i' = 0$ or $\min_j(-e_{ij}') = 0$, where $e_{ij}' := e_{ij}(u', v')$. Now, let us decrease the payoff of each column player as much as possible without loosing coalitional rationality, i.e., let us define allocation (u, v) by $u_i := u_i'$ and $v_i := v_i' - \min\{v_i', \min_j(-e_{ij}')\}$ for all $i \in N$. As before, (u, v) is coalitionally rational, $v_i \leq v_i'$ for all $i \in N$, moreover, for each column player i', either $v_i = 0$ or $\min_j(-e_{ij}) = 0$.

We claim that this (u, v) is a core allocation. Since it is coalitionally rational, we are done in case $e_{ii} = 0$ for all $i \in N$. Suppose $e_{ii} < 0$ for some $i \in N$. Then at least one of u_i and v_i is positive, because A is nonnegative. Let us suppose that $u_i = 0$. Then for column player i', $v_i > 0$ and $e_{ii} < 0$, so there is a row player $j \neq i$ such that $e_{ji} = 0$, i.e., $u_j + v_i = a_{ji}$. Subtracting this from $u_i + v_i > a_{ii}$ gives $0 \geq -u_j > a_{ii} - a_{ji}$, a contradiction to A having dominant diagonal. We obtain that if $e_{ii} < 0$ for some $i \in N$ then both u_i and v_i are positive. Then however, there must be indices $j \neq i$ and $k \neq i$ such that $e_{ji} = 0$ and $e_{ik} = 0$. Since (u, v) is coalitionally rational, $e_{jk} \leq 0$. It follows that $e_{ii} + e_{jk} < 0 = e_{ik} + e_{ji}$, a contradiction to A having doubly dominant diagonal. Therefore, we must have $e_{ii} = 0$ for all $i \in N$, so (u, v) is indeed a core allocation that is componentwise less than or equal to the arbitrarily chosen coalitionally rational allocation (u'', v''). Hence, $\mathscr{C}(V_A)$ is large.

5. Convex and subconvex assignment games

A game (P, V) is called *convex* if for all $S, T \subseteq P$ the coalitional function satisfies $V(S) + V(T) \leq V(S \cup T) + V(S \cap T)$. Sharkey (1982) proved that subconvexity, a weaker version of convexity, also implies largeness and hence stability of the core. Here we use the following alternative definition of subconvexity that was given by Gellekom et al. (1999).

Given a game (P, V), a bijective map $\pi : P \to \{1, 2, \ldots, |P|\}$ will be called an *enumeration* of P. The set of predecessors of $i \in P$ is defined by $\text{Pred}_\pi(i) = \{j \in P : \pi(j) < \pi(i)\}$. Given an enumeration π of P, let the payoff vector y^π be given by

$$y_i^\pi := \max\{V(Q \cup i) - V(Q) : Q \subseteq \text{Pred}_\pi(i), Q \cup i \neq P\}.$$

Namely, y_i^π is the maximal marginal contribution of player i with respect to a subset of his predecessors. The game (P, V) is called *subconvex* if $y^\pi(P) \leq V(P)$ for all enumeration π of the player set P.

Now we are ready to prove the main result in this section.

Theorem 3. *Let A be a nonnegative $n \times n$-matrix such that its main diagonal is an optimal assignment, and let V_A be the $(n + n)$-player assignment game induced by A. Then the following are equivalent:*

(i) *V_A is convex;*
(ii) *V_A is subconvex;*
(iii) *A is a diagonal matrix (i.e., $a_{ij} = 0$ if $i \neq j$).*

Proof: (i) $\Rightarrow$ (ii) Straightforward (cf. Sharkey, 1982).

(ii) $\Rightarrow$ (iii) Suppose A is not diagonal and $a_{12} > 0$. Consider an enumeration

π which specifies the following order of the players: $(1, 2', 2, 1', \ldots)$. Clearly, $y_1^\pi = 0$ and $y_{2'}^\pi = a_{12}$. Since $2'$ is a predecessor of his optimally matched partner 2, we get $y_2^\pi \geq a_{22}$. Similarly, column player $1'$ can secure for himself at least a_{11} by joining his optimally matched partner 1 who is his predecessor. The same reasoning and the nonnegativity of y^π give that for any pair $\{j, j'\}$ of players on the diagonal, $y_j^\pi + y_{j'}^\pi \geq a_{jj}$ holds. It follows that $y^\pi(P) \geq a_{12} + V_A(P) > V_A(P)$, a contradiction to subconvexity.

(iii) $\Rightarrow$ (i) If A is a diagonal matrix, the value of a coalition S is simply the sum of the diagonal entries in S, i.e., $V_A(S) = \sum_{\{i,i'\} \subseteq S} a_{ii}$. Therefore, V_A is an additive set function and so the game is convex.

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