Int J Game Theory (2001) 30:167–176

International Journal of Game Theory

An axiomatization of the modified Banzhaf Coleman index

M. J. Albizuri*

Department of Applied Economics IV, Facultad de Ciencias Económicas y Empresariales, University of the Basque Country, Avenida de Lehendakari Agirre, 83, 48015 Bilbao, Spain (e-mail: elpalirm@bs.ehu.es)

Final version: June 2001

Abstract. In this paper we provide a characterization for the modified Banzhaf-Coleman indexes by employing some amalgamation axioms as Lehrer (1988) did for the Banzhaf-Coleman indexes.

Key words: modified Banzhaf-Coleman index, axiomatization

1. Introduction

In (1981) Owen defined the modified Banzhaf-Coleman (B-C) index, which is a modification of the (normalized) B-C index (1965, 1971) when a priori unions represented by coalitional structures are considered. He defined this modified index in a similar way as he defined the so called Owen value (1977), which is a modification of the Shapley value (1953) to the coalitional framework.

In the case of the Owen value (1977), an axiomatic characterization was given when it was defined. However, for the modified B-C index (1981), the own author writes that the characterization he provides 'does seem to beg the question', since he fixes for the unanimity games the own indexes to be characterized. Given a game and a coalitional structure, he remarks that the point is that the modified B-C index of a coalition belonging to the coalitional structure does not depend only on the game between the coalitions associated with the coalitional structure. It can be said equivalently that the modified B-C index does not verify symmetry between the coalitions of the coalitional structure, axiom that the Owen value verifies.

When Owen defined the modified B-C index (1981), Dubey and Shapley

* This research has been partially supported by the University of the Basque Country (project UPV 031.321-HA042/99) and DGES Ministerio de Educación y Ciencia (project PB96-0247).

(1979) had characterized the B-C index. Several years later, Lehrer (1988) provided a new characterization of the the B-C index by employing several common axioms and a reduction or amalgamation axiom. In the paper we present, we will also employ some amalgamation axioms in order to obtain a characterization of the modified B-C indexes. In this way, we can overcome the trouble caused by the modified B-C index for not verifying the above axioms.

2. Preliminaries

Let N be a finite set of players. A transferable utility (TU) game on N is a function $v : 2^N \to \Re$ such that $v(\emptyset) = 0$, where 2^N denotes the family of all subsets or coalitions of N. A game v on N is said to be simple if for each $S \subseteq N$, we have either $v(S) = 0$ or $v(S) = 1$. A simple game on N is monotonic if $v(S) \geq v(T)$ for each $S, T \subseteq N$ such that $T \subseteq S$. We denote by SG the set of all monotonic simple games on any finite player set N and by SG_N the subset of SG formed by those games on the player set N.

Let v and $w \in SG_N$. We define the games $v \vee w$ and $v \wedge w$ as follows:

$$(v \vee w)(S) = \max(v(S), w(S)), \quad (v \wedge w)(S) = \min(v(S), w(S)) \quad (\forall S \subseteq N).$$

Let $v \in SG_N$, $A = \{i, j\} \subseteq N$ and $\bar{A}$ be a player who does not belong to N. We denote by v_A the TU game with player set $(N \backslash A) \cup \{\bar{A}\}$ defined by

$v_A(S) = v(S)$ if $S \subseteq N \backslash A$

$v_A(S) = v((S \backslash \bar{A}) \cup \{i, j\})$ if $S \ni \bar{A}$.

Thus, v represents the TU game that results from v after players i and j have been amalgamated into one (who is $\bar{A}$).

Along this paper an index is a mapping $\psi : SG_N \to \Re^N$ for some finite set of players N.

Lehrer (1988) considered an index as a mapping from SG into $\bigcup_N \Re^N$ and he characterized the B-C index β. This index associates with each $v \in SG_N$ and $i \in N$ the real number

$$\beta_i(v) = 2^{1-|N|} \sum_{\substack{S \subseteq N \\ i \notin S}} [v(S \cup \{i\}) - v(S)]. \tag{1}$$

Lehrer characterized β by using the following axioms.

$\psi(v \vee w) + \psi(v \wedge w) = \psi(v) + \psi(w)$ $(\forall v, w \in SG)$, that is, ψ is a valuation.

Dummy player axiom: let $v \in SG_N$. If $i \in N$ is a dummy player, i.e., if it verifies $v(S \cup \{i\}) = v(S) + v(i)$ $(\forall S \subseteq N \backslash \{i\})$, then $\psi_i(v) = v(i)$.

Equal treatement: let $v \in SG_N$ and $i, j \in N$. If

$$v(S \cup \{i\}) = v(S \cup \{j\}) \quad (\forall S \subseteq N \backslash \{i, j\}),$$

then $\psi_i(v) = \psi_j(v)$.

Amalgamation axiom: let $v \in SG_N$. Then, $\psi_i(v) + \psi_j(v) \leq \psi_{\bar{A}}(v_A)$ $(\forall i, j \in N)$.

So, the Banzhaf semivalue is the only index that is a valuation and verifies the dummy player axiom, equal treatment and the amalgamation axiom.

The first three axioms are well known axioms, the first one in the context of simple games while the others for any TU game. With respect to the last axiom, it was introduced by Lehrer (1988) and according to it, the value does not decrease when two players are amalgamated.

On the other hand, we denote by $\mathbf{B}$ a finite partition or coalitional structure $\{B_p\}_{p\in\mathbf{N}}$ of N. Owen (1981) defined the modified Banzhaf-Coleman index in the following way. Let v be a TU game on N and $\mathbf{B}$ a coalitional structure of N. For a fixed $q \in \mathbf{N}$ and any $S \subseteq B_q$, he considered $\mathbf{B}_S = \{B_p\}_{\substack{p\in\mathbf{N}\\ p\neq q}} \cup \{S\}$ and the TU game $v/\mathbf{B}_S$ whose players are the coalitions that form $\mathbf{B}_S$ and whose worths are

$$v/\mathbf{B}_S(\mathbf{T}) = v\Big(\bigcup_{T\in\mathbf{T}} T\Big) \quad (\forall \mathbf{T} \subseteq \mathbf{B}_S),$$

that is, $v/\mathbf{B}_S$ represents the game associated with v whose worths are the worths of the coalitions that form the players (coalitions) in $\mathbf{B}_S$.

For each $S \subseteq B_q$, Owen considered the Banzhaf-Coleman index β of the player S in the game $v/\mathbf{B}_S$, that is $\beta_S(v/\mathbf{B}_S)$. In this way, he obtained a TU game w_q whose player set is B_q and is defined by $w_q(S) = \beta_S(v/\mathbf{B}_S)$ for all $S \subseteq B_q$. He considered again the Banzhaf-Coleman index of w_q, obtaining the modified Banzhaf-Coleman index $\beta^{\mathbf{B}}$ of the game v, that is

$$\beta_i^{\mathbf{B}}(v) = \beta_i(w_q) \quad (\forall i \in B_q). \tag{2}$$

The above equality and (1) give rise to the following explicit formula for the modified B-C index: let v be a game with player set N, $\mathbf{B}$ a coalitional structure of N and $i \in B_q$,

$$\beta_i^{\mathbf{B}}(v) = 2^{2-|\mathbf{B}|-|B_q|} \sum_{\substack{\mathbf{T}\subseteq\mathbf{B}\\ B_q\notin\mathbf{T}}} \sum_{\substack{S\subseteq B_q\\ i\notin S}} [v(L\cup S\cup\{i\}) - v(L\cup S)], \tag{3}$$

where $L = \bigcup_{T\in\mathbf{T}} T$.

3. A characterization on the Class of Monotonic Simple Games

In this section we are going to characterize the modified Banzhaf-Coleman indexes on the Class of Monotonic Simple Games. As a first approach we can say that the axioms of the characterization we provide are adaptations to the coalitional framework of the ones that characterize the Banzhaf-Coleman index (1988) to which we have added a stability axiom.

Indeed, we will require the solutions to be valuations and verify the dummy player axiom, as Lerher asked for the Banzhaf index. Moreover, we will require an anonymity axiom, two amalgamation axioms similar to Lerher's amalgamation one and an additional axiom which will be called amalgamation stability. We will write the last four axioms. To write them we will associate an index $\psi^{\mathbf{B}}$ with each coalitional structure $\mathbf{B}$. Let $\mathbf{B}$ be a coalitional structure of N. If $B_p = \{i\}$ and $B_q = \{j\}$ for some $i, j \in N$ and p, q, or $i, j \in B_q$ for some $i, j \in N$ and q, we will denote by $\hat{\mathbf{B}}$ the resultant coalitional structure on $(N - \{i,j\}) \cup \bar{A}$ when i, j are amalgamated in $\mathbf{B}$. The axioms are the following ones.

B-anonymity: let $v \in SG_N$, π be a permutation of N and πv the game defined by $(\pi v)(S) = v(\pi S)$ $(\forall S \subseteq N)$. Then,

$$\psi_i^{\mathbf{B}}(\pi v) = \psi_{\pi i}^{\pi\mathbf{B}}(v) \quad (\forall i \in N).$$

Inside **B**-amalgamation: let $v \in SG_N$ and $B_q \in \mathbf{B}$. Then,

$$\psi_i^{\mathbf{B}}(v) + \psi_j^{\mathbf{B}}(v) \leq \psi_{\hat{A}}^{\hat{\mathbf{B}}}(v_{\{i,j\}}) \quad (\forall i, j \in B_q).$$

Outside **B**-amalgamation: let $v \in SG_N$, **B** be a coalitional structure and $i, j \in N$. If there exist p, q such that $B_p = \{i\}$ and $B_q = \{j\}$, it holds,

$$\psi_i^{\mathbf{B}}(v) + \psi_j^{\mathbf{B}}(v) \leq \psi_{\hat{A}}^{\hat{\mathbf{B}}}(v_{\{i,j\}}).$$

B-amalgamation stability: let $v \in SG_N$ and $i, j \in B_q$ for some q. Then,

$$\psi_k^{\mathbf{B}}(v) = \psi_k^{\hat{\mathbf{B}}}(v_{\{i,j\}}) \quad (\forall k \in N \backslash B_q).$$

Observe that by **B**-anonymity the indexes are independent of the players' names. This axiom does not require symmetry between the coalitions of the coalitional structure since, as we have mentioned in the introduction, the Banzhaf-Coleman index, in contrast to the Owen value, does not verify this property.

Notice also that inside **B**-amalgamation and outside **B**-amalgamation are natural requirements in the coalitional context if one tries to ask for a two players' amalgamation axiom. Indeed, the inequalities are valid when two players are amalgamated inside one coalition of the coalitional structure or when both players are themselves coalitions of such a coalitional structure. These amalgamations yield naturally a coalitional structure. However, if two players belong to different coalitions of the coalitional structure and they do not form themselves those coalitions, there is no direct natural coalitional structure and the amalgamation property does not need to hold any more.

Finally, **B**-amalgamation stability says that the index of a player $i \in B_p$ is independent of the amalgamation of any two players that belong to any coalition B_q different from B_p. That is, the index of a player $i \in B_p$ is not affected by the players j that belong to any coalition B_q different from B_p when these players j are considered alone, but the index of i could be affected by the whole coalitions B_q he does not belong to. This property can be seen as a coalitional stability property. Notice that the Owen value verifies this axiom.

We now prove that the above six axioms characterize the family of all the B-C indexes, that is, the indexes that are associated with any coalitional structure **B** with any player set N. First, we stablish the following proposition.

Proposition 1. *Let* **B** *be a coalitional structure of N. The index* $\beta^{\mathbf{B}}$ *is a valuation and verifies the dummy axiom,* **B**-*anonymity, inside* **B**-*amalgamation, outside* **B**-*amalgamation and* **B**-*amalgamation stability.*

Proof: By formula (3), $\beta^{\mathbf{B}}$ is a valuation and verifies the dummy axiom and **B**-anonymity. To prove that it verifies inside **B**-amalgamation let $v \in SG_N$ and $i, j \in B_q$. By (3),

$$
\begin{aligned}
&\beta_i^{\mathbf{B}}(v) + \beta_j^{\mathbf{B}}(v) \\
&= 2^{2-|\mathbf{B}|-|B_q|} \sum_{\substack{\mathbf{T}\subseteq\mathbf{B} \\ B_q \notin \mathbf{T}}} \sum_{\substack{S\subseteq B_q \\ i\notin S}} [v(L\cup S\cup\{i\}) - v(L\cup S)] \\
&\quad + 2^{2-|\mathbf{B}|-|B_q|} \sum_{\substack{\mathbf{T}\subseteq\mathbf{B} \\ B_q \notin \mathbf{T}}} \sum_{\substack{S\subseteq B_q \\ j\notin S}} [v(L\cup S\cup\{j\}) - v(L\cup S)] \\
&= 2^{2-|\mathbf{B}|-|B_q|} \sum_{\substack{\mathbf{T}\subseteq\mathbf{B} \\ B_q \notin \mathbf{T}}} \sum_{\substack{S\subseteq B_q \\ i,j\notin S}} \begin{bmatrix} v(L\cup S\cup\{i\}) - v(L\cup S) + v(L\cup S\cup\{i,j\}) \\ -v(L\cup S\cup\{j\}) \end{bmatrix} \\
&\quad + 2^{2-|\mathbf{B}|-|B_q|} \sum_{\substack{\mathbf{T}\subseteq\mathbf{B} \\ B_q \notin \mathbf{T}}} \sum_{\substack{S\subseteq B_q \\ i,j\notin S}} \begin{bmatrix} v(L\cup S\cup\{j\}) - v(L\cup S) + v(L\cup S\cup\{i,j\}) \\ -v(L\cup S\cup\{i\}) \end{bmatrix} \\
&= 2^{2-|\mathbf{B}|-|B_q|} 2 \sum_{\substack{\mathbf{T}\subseteq\mathbf{B} \\ B_q \notin \mathbf{T}}} \sum_{\substack{S\subseteq B_q \\ i,j\notin S}} [v(L\cup S\cup\{i,j\}) - v(L\cup S)] \\
&= 2^{2-|\mathbf{B}|-(|B_q|-1)} \sum_{\substack{\mathbf{T}\subseteq\mathbf{B} \\ B_q \notin \mathbf{T}}} \sum_{\substack{S\subseteq B_q \\ i,j\notin S}} [v(L\cup S\cup\{i,j\}) - v(L\cup S)] = \beta_{\bar{A}}^{\hat{\mathbf{B}}}(v_{\{i,j\}}).
\end{aligned}
$$

Let us prove now outside **B**-amalgamation. Let $v \in SG_N$, $i, j \in N$ and p, q such that $B_p = \{i\}$ and $B_q = \{j\}$. Applying (2)

$$\beta_i^{\mathbf{B}}(v) + \beta_j^{\mathbf{B}}(v) = \beta_i(w_p) + \beta_j(w_q) = w_p(i) + w_q(j),$$

where the last equality is obtained by taking into account that the B-C index of a player when there are no more players is equal to his worth. Moreover, by definition, $w_p(i) = \beta_i(v/\mathbf{B}_{\{i\}}) = \beta_i(v/\mathbf{B})$ and similarly, $w_q(j) = \beta_j(v/\mathbf{B})$. Thus,

$$\beta_i^{\mathbf{B}}(v) + \beta_j^{\mathbf{B}}(v) = \beta_i(v/\mathbf{B}) + \beta_j(v/\mathbf{B}).$$

As β verifies amalgamation axiom $\beta_i(v/\mathbf{B}) + \beta_j(v/\mathbf{B}) \le \beta_{\bar{A}}((v/\mathbf{B})_{\{B_p, B_q\}})$ and taking into account that

$$\beta_{\bar{A}}((v/\mathbf{B})_{\{B_p, B_q\}}) = \beta_{\bar{A}}(v_{\{i,j\}}/\hat{\mathbf{B}}) = \beta_{\bar{A}}^{\hat{\mathbf{B}}}(v_{\{i,j\}}),$$

then,

$$\beta_i^{\mathbf{B}}(v) + \beta_j^{\mathbf{B}}(v) \le \beta_{\bar{A}}^{\hat{\mathbf{B}}}(v_{\{i,j\}}),$$

that is, $\beta^{\mathbf{B}}$ verifies outside **B**-amalgamation.

Finally, let us prove that $\beta^{\mathbf{B}}$ verifies **B**-amalgamation stability. Let $v \in SG$ and $i, j \in B_q$ for some q. It can be clearly observed in (3) that $\beta_k^{\mathbf{B}}(v)$ and $\beta_k^{\hat{\mathbf{B}}}(v_{\{i,j\}})$ are equal for all $k \in N \backslash B_q$, and hence **B**-amalgamation stability is verified. ■

We stablish in the following theorem that the B-C indexes are the only one which satisfy the above six axioms. We employ the unanimity games u_T $(T \subseteq N)$ defined by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise.

Theorem 2. *There is only one family of indexes* $\{\psi^{\mathbf{B}}\}_{\mathbf{B}}$ *such that every* $\psi^{\mathbf{B}}$ *is a valuation and verifies the dummy axiom,* **B**-*anonymity, inside* **B**-*amalgamation, outside* **B**-*amalgamation and* **B**-*amalgamation stability. Furthermore,* $\psi^{\mathbf{B}} = \beta^{\mathbf{B}}$ *for all coalitional structure* **B**.

Proof: Proposition 1 guarantees existence, so let us prove uniqueness. Let $\{\psi^{\mathbf{B}}\}_{\mathbf{B}}$ be a family where each $\psi^{\mathbf{B}}$ verifies the six axioms.

First of all we will see that it is sufficient to prove uniqueness for games u_S, that is, for games with one minimal winning coalition (note also that if $v = 0$, the dummy axiom implies $\psi_i^{\mathbf{B}}(0) = 0$ for all $i \in N$). We will prove it by induction on the minimal winning coalitions. Let $v \in SG$ and N the player set of v. Since N is finite, v has a finite number of minimal winning coalitions $S_1, \ldots, S_m$. By monotonicity of v, we have $v = u_{S_1} \vee \cdots \vee u_{S_m}$.

Since $\psi^{\mathbf{B}}$ is a valuation

$$\begin{aligned}\psi^{\mathbf{B}}(u_{S_1} \vee \cdots \vee u_{S_m}) &= \psi^{\mathbf{B}}(u_{S_1} \vee (u_{S_2} \vee \cdots \vee u_{S_m})) \\ &= \psi^{\mathbf{B}}(u_{S_1}) + \psi^{\mathbf{B}}(u_{S_2} \vee \cdots \vee u_{S_m}) \\ &\quad - \psi^{\mathbf{B}}(u_{S_1} \wedge (u_{S_2} \vee \cdots \vee u_{S_m})),\end{aligned}$$

and taking into account that

$$\begin{aligned}u_{S_1} \wedge (u_{S_2} \vee \cdots \vee u_{S_m}) &= (u_{S_1} \wedge u_{S_2}) \vee \cdots \vee (u_{S_1} \wedge u_{S_m}) \\ &= (u_{S_1 \cup S_2}) \vee \cdots \vee (u_{S_1 \cup S_m}),\end{aligned}$$

the three games in the summation above have respectively 1, $m-1$ and $\leq m-1$ minimal winning coalitions, and thus by induction $\psi^{\mathbf{B}}(v)$ will be determined.

To determine $\psi^{\mathbf{B}}(u_S)$ we will apply induction on the player set N of u_S.

If $|N| = 1$, then $N = \{i\}$, $\mathbf{B} = \{\{i\}\}$ and $u_S = u_{\{i\}}$. Therefore, by the dummy player axiom $\psi^{\mathbf{B}}(u_{\{i\}})$ is determined.

Suppose that if $|N| < n$, then $\psi^{\mathbf{B}}(u_S)$ is determined for every coalitional structure **B** of N.

Suppose $|N| = n$. If $i \in N \backslash S$, then i is a dummy player in u_S and the dummy player axiom implies $\psi_i^{\mathbf{B}}(u_S) = 0$. To see that $\psi_i^{\mathbf{B}}(u_S)$ is determined when $i \in S$ we will distinguish three possible situations for **B** that cover all the possible ones.

i) **B** contains B_p satisfying $|B_p| \geq 2$ and $B_p \cap S = \emptyset$.

If two players $i, j \in B_p$ are amalgamated, then by **B**-amalgamation stability $\psi_k^{\mathbf{B}}(u_S) = \psi_k^{\hat{\mathbf{B}}}(u_S)$ for all $k \in S$. Since by induction $\psi_k^{\hat{\mathbf{B}}}(u_S)$ is determined, so is $\psi_k^{\mathbf{B}}(u_S)$.

ii) **B** contains two coalitions B_p, B_q such that $|B_p| \geq 2$, $|B_q| \geq 2$, $|S \cap B_p| \geq 1$ and $|S \cap B_q| \geq 1$.

Now, we can reason in a similar way as in i). Amalgamating a player $i \in S \cap B_p$ and $j \in B_p$ such that $i \neq j$, $\psi_k^{\mathbf{B}}(u_S)$ is determined for every $k \in S \backslash B_p$. Amalgamating a player $i \in S \cap B_q$ and $j \in B_q$ such that $i \neq j$, $\psi_k^{\mathbf{B}}(u_S)$ is determined for every $k \in S \backslash B_q$. Hence, $\psi_k^{\mathbf{B}}(u_S)$ is determined for all $k \in S$.

iii) There is at most one $B_r \in \mathbf{B}$ such that $|B_r| \geq 2$. Furthermore, if such a B_r exists, $S \cap B_r \neq \emptyset$.

In this case we will apply induction on $|S|$. From now on we will apply this second induction.

If $|S| = 1$, by the dummy player axiom $\psi_i^{\mathbf{B}}(u_S)$ is determined for all $i \in S$.

Suppose that for all $\mathbf{B}'$ and i, $\psi_i^{\mathbf{B}'}(u_T)$ is determined if $|T| < |S|$.

Let us prove now that $\psi_i^{\mathbf{B}}(u_S)$ is determined for every $i \in S$.

Let $S' = S \backslash \{i\}$. Since $\psi^{\mathbf{B}}$ is a valuation,

$$\psi^{\mathbf{B}}(u_{S'} \vee u_{\{i\}}) + \psi^{\mathbf{B}}(u_{S'} \wedge u_{\{i\}}) = \psi^{\mathbf{B}}(u_{S'}) + \psi^{\mathbf{B}}(u_{\{i\}}),$$

that is,

$$\psi^{\mathbf{B}}(u_{S'} \vee u_{\{i\}}) + \psi^{\mathbf{B}}(u_S) = \psi^{\mathbf{B}}(u_{S'}) + \psi^{\mathbf{B}}(u_{\{i\}}). \tag{4}$$

On the other hand, applying the **B**-anonymity axiom, for each $B_q \in \mathbf{B}$ such that $B_q \cap S \neq \emptyset$ there exists $a_q \in \Re$ such that $\psi_k^{\mathbf{B}}(u_S) = a_q$ for all $k \in B_q \cap S$. Similarly, for the same $B_q \in \mathbf{B}$ there exist $b_q \in \Re$ such that $\psi_k^{\mathbf{B}}(u_{S'} \vee u_{\{i\}}) = b_q$ for all $k \in B_q \cap S \backslash \{i\}$. And finally, there exists $c \in \Re$ such that $\psi_i^{\mathbf{B}}(u_{S'} \vee u_{\{i\}}) = c$.

Hence, by (4) and applying induction, for every q, k such that $k \in B_q \cap S \backslash \{i\}$,

$$a_q + b_q = \beta_k^{\mathbf{B}}(u_{S'}) = 2^{2-|\mathbf{B}|+|U(S')|-|B_q \cap S'|}, \tag{5}$$

where $U(S') = \{B_q \in \mathbf{B} / B_q \cap S' = \emptyset\}$. Moreover, the dummy axiom implies

$$a_p + c = 1 \quad \text{for } p \text{ such that } i \in B_p. \tag{6}$$

Now we distinguish two cases:

1) Let us suppose $|B_p \cap S| \geq 2$ (notice that in this case $p = r$). If i and some $k \in B_p$ are amalgamated in u_S, induction and inside **B**-amalgamation yield

$$\begin{aligned} 2 \cdot a_p &\leq \psi_{\bar{A}}^{\hat{\mathbf{B}}}((u_S)_{\{i,k\}}) = \psi_{\bar{A}}^{\hat{\mathbf{B}}}(u_{(S-\{i,k\}) \cup \bar{A}}) \\ &= \beta_{\bar{A}}^{\hat{\mathbf{B}}}(u_{(S-\{i,k\}) \cup \bar{A}}) = 2^{2-|\mathbf{B}|+|U(S')|-|B_p \cap S'|}. \end{aligned}$$

Thus,

$$2 \cdot a_p \leq 2^{2-|\mathbf{B}|+|U(S')|-|B_p \cap S'|}. \tag{7}$$

Moreover, if we amalgamate the same players in $u_{S'} \vee u_{\{i\}}$ we obtain a dictatorial game and by the dummy axiom and inside **B**-amalgamation

$$b_p + c \leq 1. \tag{8}$$

Then, adding (8) and (7), it holds $2 \cdot a_p + b_p + c \leq 1 + 2^{2-|\mathbf{B}|+|U(S')|-|B_p \cap S'|}$, and taking into account (5), with $q = p$, and (6) this is an equality. Therefore, so are (8) and (7). Hence, rewriting (7) we have

$$a_p = \frac{2^{2-|\mathbf{B}|+|U(S')|-|B_p \cap S'|}}{2},$$

and $\psi_i^{\mathbf{B}}(u_S) = a_p$ is determined.

2) Let us suppose $|B_p \cap S| = 1$, that is, $B_p \cap S = \{i\}$.

Since $|S| \neq 1$ there is $q \neq p$ such that $B_q \cap S \neq \emptyset$.

Let us distinguish now two cases.

First, suppose there exists $q \neq p$ such that $|B_q \cap S| > 1$ (note that $q = r$). If we amalgamate two players in $B_q \cap S$ we obtain u_T with $|T| < |S|$. By **B**-amalgamation stability $\psi_i^{\mathbf{B}}(u_S) = \psi_i^{\hat{\mathbf{B}}}(u_T)$ and applying induction $\psi_i^{\hat{\mathbf{B}}}(u_T)$ is determined. Therefore, $\psi_i^{\mathbf{B}}(u_S)$ is determined.

Second, suppose there is $q \neq p$ such that $|B_q \cap S| = 1$, that is, $B_q \cap S = \{k\}$. Then, if there are players in $B_q \backslash S$ we amalgamate two by two all the players in B_q. By **B**-amalgamation stability

$$\psi_i^{\mathbf{B}}(u_S) = \psi_i^{\hat{\mathbf{B}}}(u_S) \tag{9}$$

where we have identified $\bar{A}$ with k without misunderstanding.We can obtain again (5) and (6) for u_S and $\hat{\mathbf{B}}$, equalities which we will write as follows,

$$\hat{a}_q + \hat{b}_q = \beta_k^{\hat{\mathbf{B}}}(u_{S'}) = 2^{2-|\mathbf{B}|+|U(S')|-|B_q \cap S'|} = 2^{2-|\mathbf{B}|+|U(S')|-1} \tag{10}$$

$$\hat{a}_p + \hat{c} = 1. \tag{11}$$

We will distinguish two subcases.

2.a) $B_p \subseteq S$. That is, $B_p = \{i\}$.

If we amalgamate i and k in u_S, by outside $\hat{\mathbf{B}}$-amalgamation and induction, if $\mathbf{B}^*$ denotes the resultant coalitional structure

$$\begin{aligned} \hat{a}_p + \hat{a}_q \leq \psi_{\bar{A}}^{\mathbf{B}^*}((u_S)_{\{i,k\}}) &= \psi_{\bar{A}}^{\mathbf{B}^*}(u_{(S-\{i,k\}) \cup \bar{A}}) \\ &= \beta_{\bar{A}}^{\mathbf{B}^*}(u_{(S-\{i,k\}) \cup \bar{A}}) \\ &= 2^{2-|\mathbf{B}^*|+|U((S-\{i,k\}) \cup \bar{A})|-|\bar{A}|} \\ &= 2^{2-(|\mathbf{B}|-1)+(|U(S')|-1)-1}, \end{aligned} \tag{12}$$

where the last equality is true because $|\mathbf{B}^*| = |\mathbf{B}| - 1$, $i \in U(S')$ and $i \notin U((S - \{i,k\}) \cup \bar{A})$.

Furthermore, applying outside $\hat{\mathbf{B}}$-amalgamation, reasoning as in 1) it holds

$$\hat{b}_q + \hat{c} \leq 1, \tag{13}$$

and adding (12) and (13) and taking into account (10) and (11) we obtain the equality

$$\hat{a}_p + \hat{a}_q = 2^{2-|\mathbf{B}|+|U(S')|-1}.$$

Finally, $\hat{\mathbf{B}}$-anonymity implies $\hat{a}_p = \hat{a}_q$, and therefore

$$\psi_i^{\hat{\mathbf{B}}}(u_S) = \hat{a}_p = \frac{2^{2-|\mathbf{B}|+|U(S')|-1}}{2}.$$

So, by (9), $\psi_i^{\mathbf{B}}(u_S)$ is determined.

2.b) $B_p \cap (N \backslash S) \neq \varnothing$ (observe that $p = r$).

Now, since $B_p \neq \{i\}$, the direct application of outside $\hat{\mathbf{B}}$-amalgamation is impossible, but if i is amalgamated with the other players in $\hat{B}_p \backslash S$ we have that $\hat{a}_q$ does not change due to $\hat{\mathbf{B}}$-amalgamation stability. Moreover, inside $\hat{\mathbf{B}}$-amalgamation and the dummy player axiom yield

$$\hat{a}_p \leq a_p^*, \tag{14}$$

where a_p^* is the index of the amalgamated player. As in 2.a) we obtain

$$a_p^* + \hat{a}_q \leq 2^{2-|\mathbf{B}|+|U(S')|-1},$$

and thus, by (14),

$$\hat{a}_p + \hat{a}_q \leq 2^{2-|\mathbf{B}|+|U(S')|-1}.$$

Similarly, $\hat{b}_q + \hat{c} \leq 1$. Therefore, we have again the equality

$$\hat{a}_p + \hat{a}_q = 2^{2-|\mathbf{B}|+|U(S')|-1}. \tag{15}$$

Notice that in this equality, $\hat{a}_q = \psi_k^{\hat{\mathbf{B}}}(u_S)$ and $\hat{B}_q = \{k\} \subseteq S$. Then, case 2) with only 2.a) implies $\hat{a}_q$ is determined and by (15) so is $\hat{a}_p$. Hence, $\psi_i^{\mathbf{B}}(u_S)$ is determined. ■

Remark 1: Note that **B**-amalgamation stability does not depend on the other axioms of the characterization. For example, if we consider the indexes $\xi^{\mathbf{B}} = \beta$ for all coalitional structure **B**, these indexes verify all the axioms in theorem 2 except **B**-amalgamation stability.

Example 1. Let us consider the following easy example to see the way in which inside **B**-amalgamation and **B**-amalgamation stability work. Let $N = \{1, 2, 3, 4, 5\}$, $\mathbf{B} = \{\{1, 2, 3\}, \{4, 5\}\}$, and $v \in SG_N$ such that $v(i) = 0$ for all $i \in N$ and $v(ij) = 1$ if and only if $\{i, j\} \in \{\{4, 5\}, \{1, 2\}, \{1, 3\}\}$. The modified B-C index $\beta^{\mathbf{B}}$ gives the following results: $\beta_1^{\mathbf{B}}(v) = \frac{3}{8}$, $\beta_2^{\mathbf{B}}(v) = \frac{1}{8}$, $\beta_3^{\mathbf{B}}(v) = \frac{1}{8}$, $\beta_4^{\mathbf{B}}(v) = \frac{1}{4}$ and $\beta_5^{\mathbf{B}}(v) = \frac{1}{4}$. If players 1 and 3 are amalgamated into one (denoted by $\bar{A}$), then we have the coalitional structure $\hat{\mathbf{B}} = \{\{\bar{A}, 2\}, \{4, 5\}\}$ and the game $v_{\{1,3\}}$. According to **B**-amalgamation stability the indexes of 3 and 4 do not change, that is, $\beta_4^{\hat{\mathbf{B}}}(v_{\{1,3\}}) = \frac{1}{4}$ and $\beta_5^{\hat{\mathbf{B}}}(v_{\{1,3\}}) = \frac{1}{4}$. Moreover, inside **B**-amalgamation requires $\beta_{\bar{A}}^{\hat{\mathbf{B}}}(v_{\{1,3\}}) \geq \frac{3}{8} + \frac{1}{8}$. In fact, the reader can check that $\beta_{\bar{A}}^{\hat{\mathbf{B}}}(v_{\{1,3\}}) = \frac{1}{2}$.

4. References

Dubey P, Shapley LS (1979) Mathematical properties of the Banzhaf power index. Math Oper Res 4:99–131

Lehrer E (1988) An axiomatization of the Banzhaf value. International Journal of Game Theory 17:89–99

Owen G (1977) Values of games with a priori unions. In: Henn R and Moschlin O (eds.) Lecture notes in Economics and Mathematical systems. Essays in Honor of Oskar Morgenstern. Springer Verlag, Nueva York, pp 76–88

Owen G (1981) Modification of the Banzhaf-Coleman index for games with a priori unions. In: Manfred J Holler (ed.) Power, voting and voting power. Physica-Verlag, Wurzburg, pp 232–238

Shapley LS (1953) A value for *n*-person games. Contributions to the theory of games (Annals of Mathematics Studies 28). Princeton Univ. Press, Princeton, pp 307–317