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An axiomatic approach to the concept of interaction among players in cooperative games

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Abstract. An axiomatization of the interaction between the players of any coalition is given. It is based on three axioms: linearity, dummy and symmetry. These interaction indices extend the Banzhaf and Shapley values when using in addition two equivalent recursive axioms. Lastly, we give an expression of the Banzhaf and Shapley interaction indices in terms of pseudo-Boolean functions.

Key words: Cooperative games, interaction among players, Shapley values, Banzhaf values, coalition

1. Introduction

Since the work of Shapley [18], the concept of value of a game has been widely used in cooperative game theory, and many other researchers have proposed their own approach of values (see e.g. the Banzhaf value [1], the weighted Shapley value [10], probabilistic values [19], random order values and sharing values [19], etc.).

Roughly speaking, a value of a game v is a real function ϕ^v which assigns to every player i his prospect $\phi^v(i)$ from playing the game[1]. For simple games, the value is more often called a power index, relating the number of swings occuring when a player i joins coalitions. For general games, the notion of swing is replaced by what could be called the *added worth* – i.e. the difference of worth $v(S \cup i) - v(S)$ when player i joins coalition S –, and a value is then more or less an average of added worths.

Taking the Shapley value as a typical example, the fact that in general

[1] This non-standard notation will be justified later.

$\phi^v(i)$ is different from $v(i)$, except for additive games, shows that players have some interest to make coalitions: it may happen that $v(i)$ and $v(j)$ are small and at the same time $v(\{i,j\})$ is large. The converse could have happened as well, and in this last case, players i and j have better not to join together. Clearly, the value $\phi^v(i)$ merely measures the *average* added worth that player i brings to all possible coalitions, but it does not explain why player i may have a large value. In other words, it gives no information on the phenomena of *interaction* or *cooperation* existing among players. Taking again the above example of players i and j, we could say:

- players i and j have interest to cooperate, or have, exhibit a *positive interaction* when the worth of coalition $\{i,j\}$ is *more* than the sum of individual worths,
- players i and j have no interest to cooperate, or have, exhibit a *negative interaction* when the worth of coalition $\{i,j\}$ is *less* than the sum of individual worths,
- players i and j can act independently in case of equality.

Of course, this intuitive concept requires a more elaborated definition than the one above. Obviously, one should not only compare $v(\{i,j\})$ with $v(i)+v(j)$, but consider also what happens when i,j or $\{i,j\}$ join coalitions, so that the sign of the quantity $v(S\cup\{i,j\})-v(S\cup i)-v(S\cup j)+v(S)$, or its average over all coalitions, should play a central role for explaining the interaction between players i and j. Following the same reasoning, it would be interesting to define interaction among more than two players.

Strangely enough, as far as the authors know, the notion of interaction has never been considered in cooperative game theory, and indices to measure it have never been proposed. However, we believe that this concept is very useful for the description of a game, of which it provides a new viewpoint, exactly as dividends [9, 14] do. The purpose of the paper is to bring an axiomatic foundation *à la Shapley* of the interaction index, in order to formalize and justify the intuitive above presentation of the concept. We will see that interaction indices appear to be an extension of the notion of value, where a value is considered as a function over the set of players, while the extension goes over all subsets of players (or coalitions), hence the notation $\phi^v(i)$. However in this paper, we will restrict ourself to the axiomatization of the interaction indices extending the Shapley and the Banzhaf values, proposed by the authors [2, 6, 16]. Axiomatization of probabilistic interaction indices, extending probabilistic values, will be presented in a forthcoming companion paper.

In the last but one section, we give an expression of the interaction indices extending the Shapley and the Banzhaf values in terms of pseudo-Boolean functions.

We begin by recalling some results on the axiomatic of the Shapley and the Banzhaf values. This will permit us to introduce notations and definitions.

2. The Shapley and Banzhaf values

2.1. Notations and definitions

Let U be an infinite set, the universe of players. We consider $\mathcal{G}^N$ the set of all games on a finite support $N\subset U$, that is, set functions v from 2^N to $\mathbb{R}$ such that $v(\emptyset) = 0$. We denote by $\mathscr{G}$ the space of all games which have a finite support. When necessary, the set of players will be indicated by a superscript: v^N.

In order to avoid heavy notations, we will whenever possible omit braces for singletons, e.g. writing $v(i), S \cup i$ instead of $v(\{i\}), S \cup \{i\}$. Also, for pairs, triples, we will write ij, ijk instead of $\{i, j\}, \{i, j, k\}$, as for example $S \cup ijk$. Set difference of S and T is denoted by $S \backslash T$. Cardinality of sets $S, T, \ldots$ will be denoted whenever possible by corresponding lower cases $s, t, \ldots$, otherwise by the standard notation $|S|, |T|, \ldots$.

An element $i \in N$ is said to be a *dummy* if $v(S \cup i) = v(S) + v(i)$ for any $S \subset N \backslash i$. A game is *monotonic* if $v(S) \leq v(T)$ whenever $S \subset T$. It is *anti-monotonic* if the inequalities are reversed.

Let us consider a permutation π on N. The game πv is defined by $\pi v(\pi S) = v(S)$, where $\pi S = \{\pi(i), i \in S\}$. The *unanimity game* for $T \subset N$, denoted v_T, is such that $v_T(S) = 1$ iff $S \supset T$, and is 0 otherwise. A slightly different type of game is denoted $\hat{v}_T$, and is defined by $\hat{v}_T(S) = 1$ iff $S \supsetneqq T$, and 0 otherwise.

Let v^N be a game on N, and T a non empty subset of N. The *reduced game with respect to* T [12] is a game denoted $v_{[T]}^{(N \backslash T) \cup [T]}$ defined on the set $(N \backslash T) \cup [T]$ of $n - t + 1$ players, where $[T]$ indicates a single hypothetical player, which is the union (or representative) of the players in T. When there is no fear of ambiguity, the superscript will be omitted. The reduced game $v_{[T]}$ is defined as follows for any $S \subset N \backslash T$:

$$v_{[T]}(S) = v(S)$$

$$v_{[T]}(S \cup \{[T]\}) = v(S \cup T).$$

Let v^N be a game on N, and $i \in N$. The *game on* $N \backslash i$ *in the presence of* i, denoted $v_{\cup i}^{N \backslash i}$ is defined by

$$v_{\cup i}^{N \backslash i}(S) = v^N(S \cup i) - v^N(i), \quad \forall S \subset N \backslash i. \tag{1}$$

In fact, this is equivalent to consider only coalitions containing i. Substraction of $v^N(i)$ is introduced to satisfy the constraint $v_{\cup i}^{N \backslash i}(\emptyset) = 0$. Again, the superscript will be omitted if there is no fear of ambiguity.

The two preceding notions can be merged. Let us consider a game v^N on N, a subset $T \subset N$, and a player $i \in N \backslash T$. The *reduced game on* $N \backslash i$ *with respect to* T *in the presence of* i, denoted $v_{[T], \cup i}^{(N \backslash (T \cup i)) \cup [T]}$, is defined by:

$$v_{[T], \cup i}^{(N \backslash (T \cup i)) \cup [T]}(S) = v^N(S \cup i) - v^N(i)$$

$$v_{[T], \cup i}^{(N \backslash (T \cup i)) \cup [T]}(S \cup \{[T]\}) = v^N(S \cup T \cup i) - v^N(i)$$

for any $S \subset N \backslash T$.

A *value* or *power index* ϕ^v of the game $v \in \mathscr{G}^N$ is a real valued function on N. According to our notations, we will write ϕ^{v^N} for indicating the underlying set of players.

A value ϕ^v is said to be *efficient* if $\sum_{i \in N} \phi^v(i) = v(N)$. The property of *2-efficiency* [12] is satisfied for a value ϕ related to a game v^N if

$$\phi^v(i) + \phi^v(j) = \phi^{v_{[ij]}}([ij]) \quad \text{for every pair } \{i, j\} \subset N$$

where $\phi^{v_{[ij]}}([ij])$ is the payoff to player $[ij]$ in the reduced game $v_{[ij]}$.

A weaker concept was introduced by Lehrer [11], called *super-additivity*, where the equalities are replaced by inequalities $(\leq)$ in the above equations.

The function v^N can be derivated to give

$$\delta_{\varnothing} v^N(T) = v^N(T) \quad \text{for all } T \subset N$$

$$\delta_i v^N(T) = v^N(T) - v^N(T \backslash i) \quad \text{for all } T \text{ such that } \{i\} \subset T \subset N$$

$$\delta_{ij} v^N(T) = \delta_i[\delta_j v^N(T)] = v^N(T) - v^N(T \backslash i) - v^N(T \backslash j) + v^N(T \backslash \{ij\})$$

$$\text{for all } T \text{ and } i \text{ such that } \{i\} \subset T \subset N$$

$$\delta_S v^N(T) = \delta_{S \backslash i}[\delta_i v^N(T)] \quad \text{for all } T \text{ and } i \text{ such that } \{i\} \subset S \subset T \subset N.$$

Note that $\delta_S v^N(T \cup S) = \sum_{L \subset S} (-1)^{\ell - s} v^N(T \cup L)$.

2.2. Characterization of the Shapley and Banzhaf values

We follow Weber [19] in our presentation. Let us introduce the following axioms.

- *linearity axiom (L)*: ϕ^v is a linear function on $\mathcal{G}^N$, that is $\phi^{(v+w)} = \phi^v + \phi^w$, and $\phi^{c \cdot v} = c \cdot \phi^v$, for any $v, w \in \mathcal{G}^N$ and any $c \in \mathbb{R}$.
- *dummy axiom (D)*: if $i \in N$ is a dummy player, then $\phi^v(i) = v(i)$.
- *monotonicity axiom (M)*: if v is monotonic, then $\phi^v(i) \geq 0$, for all $i \in N$.
- *symmetry axiom (S)*: for all $v \in \mathcal{G}^N$, for all permutation π on N,

$$\phi^v(i) = \phi^{\pi v}(\pi(i)), \quad \forall i \in N. \tag{2}$$

- *efficiency axiom (E)*: for any $v \in \mathcal{G}^N$, ϕ^v is efficient.
- *2-efficiency axiom (2-E)*: for any $v \in \mathcal{G}^N$, the property of 2-efficiency is satisfied for every two-players coalition in N.

Weber has shown the following result.

Theorem 1. *Let ϕ^v a value defined for any $v \in \mathcal{G}^N$.*

(i) *if ϕ^v satisfies the linearity axiom (L), then there exists a family of constants a_T^i, $T \subset N$, such that*

$$\phi^v(i) = \sum_{T \subset N} a_T^i v(T). \tag{3}$$

(ii) *if ϕ^v satisfies axioms L, D, then there exists a family of constants p_T^i, $T \subset N\backslash i$, such that*

$$\phi^v(i) = \sum_{T \subset N\backslash i} p_T^i(v(T \cup i) - v(T)). \tag{4}$$

and $\sum_{T \subset N\backslash i} p_T^i = 1$.

(iii) *if ϕ^v satisfies axioms L, D, and M, then in addition $p_T^i \geq 0$, $\forall i \in N$, $\forall T \subset N\backslash i$.*

(iv) *let ϕ^v be a value of the form $\phi^v(i) = \sum_{T \subset N\backslash i} p_T^i(v(T \cup i) - v(T))$, for all $i \in N$, $v \in \mathcal{G}^N$. If ϕ^v satisfies the symmetry axiom, then there exists a family of constants $p_0, \ldots, p_{n-1}$ such that $p_T^i = p_{|T|}$, $\forall T \subset N\backslash i$, $\forall i \in N$.*

(v) *if ϕ^v satisfies axioms L, D, S, and E, then it is the Shapley value.*

The Shapley value, which is the only value satisfying axioms L, D, S, and E (and also M, but not 2-E), is given by

$$\phi_{\mathrm{S}}^v(i) = \sum_{T \subset N\backslash i} \frac{(n-t-1)!t!}{n!}[v(T \cup i) - v(T)]. \tag{5}$$

Concerning the Banzhaf value, the following can be shown.

Theorem 2. *Let ϕ^v a value defined for any $v \in \mathcal{G}$, and satisfying axioms L, D, S and 2-E. Then ϕ^v is the Banzhaf value, defined by*

$$\phi_{\mathrm{B}}^v(i) = \frac{1}{2^{n-1}} \sum_{T \subset N\backslash i} [v(T \cup i) - v(T)]. \tag{6}$$

Proof: from axioms L, D and S and Theorem 1, we can deduce that there exist real constants $p_t(n)$, $t = 0, \ldots, n-1$, such that $\sum_{T \subset N\backslash i} p_t(n) = 1$, and

$$\phi^v(i) = \sum_{T \subset N\backslash i} p_t(n)(v(T \cup i) - v(T)).$$

Taking $n = 1$ leads to $p_0(1) = 1$. We consider $n > 1$. Rewriting the above equation we have:

$$\begin{aligned}
\phi^v(i) = & \sum_{T \subset N\backslash ij} p_t(n)[v(T \cup i) - v(T)] \\
& + \sum_{T \subset N\backslash ij} p_{t+1}(n)[v(T \cup ij) - v(T \cup j)] \\
\phi^v(j) = & \sum_{T \subset N\backslash ij} p_t(n)[v(T \cup j) - v(T)] \\
& + \sum_{T \subset N\backslash ij} p_{t+1}(n)[v(T \cup ij) - v(T \cup i)].
\end{aligned}$$

Considering the reduced game $v_{[ij]}$, we have

$$\phi^{v_{[ij]}}([ij]) = \sum_{T \subset N \setminus ij} p_t(n-1)[v(T \cup ij) - v(T)].$$

Using 2-efficiency, by identification we get:

$$p_t(n) = p_{t+1}(n)$$

$$2p_t(n) = p_t(n-1)$$

$$2p_{t+1}(n) = p_t(n-1).$$

The first equation tells us that $p_t(n)$ is independent of t, while the second one, together with the initial condition $p_0(1) = 1$ leads to $p_t(n) = \frac{1}{2^{n-1}}$, the desired result. The third equation is redundant. □

Nowak has shown a similar result [12], but without using linearity. Lehrer [11] has also axiomatized the Banzhaf value using the super-additivity axiom, thus showing a stronger result than ours. However our proof is much simpler, and will be useful in the sequel (see section 4).

3. Axiomatization of the interaction index

This paragraph presents a complete axiomatization of interaction indices based on Shapley and Banzhaf values. We will denote by $I^v(S)$ the interaction index of players among coalition $S \subset N$ for the game $v \in \mathscr{G}^N$. The rationale behind $I^v(ij)$, interaction index for the pair $\{i, j\}$, has been explained in the introduction, and will serve as a basis for finding reasonable axioms.

Clearly, the notion of interaction among players should have a meaning only for at least two players. The interaction of a single player, or of the empty set, has no meaning with regard to the intuitive idea of interaction. Thus, if I^v is considered as a set function, i.e. defined for *any* subset, we must find a natural extension, or a natural explanation, for singletons and the empty set.

We adopt an approach similar to the one of Weber [19], trying to introduce axioms step by step.

An *interaction index* I^v of the game $v \in \mathscr{G}^N$ is a real valued function on 2^N.

A first reasonable axiom is to say that $I^v(S)$ should be linear on $\mathscr{G}^N$ for every $S \subset N$. We call this axiom LI.

Proposition 1. *If I^v satisfies the linearity axiom LI, then for every $S \subset N$, there exists a family of real constants $\{\alpha_T^S\}_{T \subset N}$ such that*

$$I^v(S) = \sum_{T \subset N} \alpha_T^S v(T). \tag{7}$$

Proof: Simply remark that this result is contained in the proof of Theorem 1 (i). □

A dummy player i should have no interaction with any coalition, since its contribution to a coalition is always $v(i)$, so there is no interest to cooperate or not to cooperate with player i. We propose the following axiom.

Dummy axiom (D'): If i is a dummy player for $v \in \mathcal{G}^N$, then for every $S \subset N\backslash i, S \neq \varnothing, I^v(S \cup i) = 0$.

Proposition 2. *If I^v satisfies LI and D', then for every $S \subset N$, $|S| \geq 2$, there exists a family of constants $\{p_T^S\}_{T \subset N\backslash S}$, such that*

$$I^v(S) = \sum_{T \subset N\backslash S} p_T^S \delta_S v(S \cup T), \tag{8}$$

Proof: (1.) We consider $S = \{i, j\}$ fixed. By axiom LI, we have $I^v(ij) = \sum_{T \subset N} \alpha_T^{ij} v(T)$. Let us choose $T \subset N\backslash i$, and consider the unanimity game v_T. Clearly, i is a dummy player for v_T since $v_T(S \cup i) = v_T(S) + v_T(i) = 1$ iff $S \supset T$, and 0 otherwise. Thus, by axiom D$'$, $I^{v_T}(ij) = 0$, for every $T \subset N\backslash i$. Now for the particular case $T = N\backslash i$, we have by axiom LI

$$I^{v_{N\backslash i}}(ij) = \alpha_{N\backslash i}^{ij} + \alpha_N^{ij} = 0,$$

so that $\alpha_{N\backslash i}^{ij} = -\alpha_N^{ij}$. Similarly for $T = N\backslash ik$, $k \neq i$, we have

$$I^{v_{N\backslash ik}}(ij) = \alpha_{N\backslash ik}^{ij} + \alpha_{N\backslash i}^{ij} + \alpha_{N\backslash k}^{ij} + \alpha_N^{ij} = 0,$$

implying $\alpha_{N\backslash k}^{ij} = -\alpha_{N\backslash ik}^{ij}$, for every $k \neq i$. By continuing the same process, we conclude that

$$\alpha_{N\backslash T}^{ij} = -\alpha_{N\backslash (T \cup i)}^{ij}, \quad \forall T \subset N\backslash i.$$

We could have done the same with j, so we can write

$$\alpha_{N\backslash T}^{ij} = -\alpha_{N\backslash (T \cup j)}^{ij}, \quad \forall T \subset N\backslash j.$$

Let us substitute this in the general expression of $I^v(ij)$. For any $v \in \mathcal{G}^N$,

$$\begin{aligned} I^v(ij) &= \sum_{T \subset N\backslash ij} (\alpha_{T \cup ij}^{ij} v(T \cup ij) + \alpha_{T \cup i}^{ij} v(T \cup i) + \alpha_{T \cup j}^{ij} v(T \cup j) + \alpha_T^{ij} v(T)) \\ &= \sum_{T \subset N\backslash ij} ([v(T \cup ij) - v(T \cup j)]\alpha_{T \cup ij}^{ij} + [v(T \cup i) - v(T)]\alpha_{T \cup i}^{ij}) \\ &= \sum_{T \subset N\backslash ij} \alpha_{T \cup ij}^{ij} [v(T \cup ij) - v(T \cup j) - v(T \cup i) + v(T)]. \end{aligned}$$

Letting $p_T^{ij} := \alpha_{T \cup ij}^{ij}$, we get the desired result.

(2.) We consider the general case, $S \subset N$, $|S| > 2$. We consider some $i \in S$ and the unanimity game $v_{N\backslash i}$. Since i is a dummy for this game, clearly

$I^{v_{N\setminus i}}(S) = 0$, which implies $\alpha^S_{N\setminus i} = -\alpha^S_N$. Continuing as in (1.) leads to

$$\alpha^S_{N\setminus T} = -\alpha^S_{N\setminus(T\cup i)}, \quad \forall i \in S, \forall T \subset N\setminus i.$$

It follows that for any $T \subset N\setminus S$, for any $i \in S$, $\alpha^S_{T\cup i} = -\alpha^S_T$, which in turn implies

$$\alpha^S_{T\cup L} = (-1)^l \alpha^S_T, \quad \forall L \subset S, \forall T \subset N\setminus S.$$

Substituting in $I^v(S)$ for any $v \in \mathcal{G}^N$, we get

$$\begin{aligned} I^v(S) &= \sum_{T\subset N\setminus S} \sum_{L\subset S} \alpha^S_{T\cup L} v(L\cup T) \\ &= \sum_{T\subset N\setminus S} \alpha^S_{T\cup S} \sum_{L\subset S} (-1)^{s-l} v(L\cup T) \\ &= \sum_{T\subset N\setminus S} \alpha^S_{T\cup S} \delta_S v(S\cup T). \end{aligned}$$

The desired result holds, with $p^S_T := \alpha^S_{T\cup S}$. $\square$

Remark that equation (8) is defined even for S being singletons or the empty set, so that this could be a natural generalization in order to get a set function. Remark that for $S = \{i\}$, we get $I^v(\{i\}) = \sum_{T\subset N\setminus i} p^i_T (v(S\cup i) - v(S))$, so that I^v coincides on singletons with a value ϕ^v satisfying axioms L and D. This can be interpreted, saying that the interaction of a singleton is nothing else than its value. This shows also that any kind of value (e.g. Shapley, Banzhaf) should lead to a different kind of interaction, since the latter can be viewed as an extension of the former. Since $\delta_\emptyset v(S) = v(S)$, we obtain $I^v(\emptyset) = \sum_{T\subset N} p^\emptyset_T v(T)$, which can be deduced solely from axiom LI.

From now on, due to the above explanation, we reformulate as follows the dummy axiom in order to take into account both the value and the interaction for coalitions of more than two players:

Dummy axiom (*DI*): If i is a dummy player for $v \in \mathcal{G}^N$, then

(i) $I^v(i) = v(i)$
(ii) for every $S \subset N\setminus i$, $S \neq \emptyset$, $I^v(S\cup i) = 0$.

We now rewrite Proposition 2 in the following form:

Proposition 3. *If I^v satisfies LI and DI, then for every $S \subset N$ there exists a family of constants $\{p^S_T\}_{T\subset N\setminus S}$ such that*

$$I^v(S) = \sum_{T\subset N\setminus S} p^S_T \delta_S v(S\cup T)$$

An *additive* game is such that $v(S\cup T) = v(S) + v(T)$ whenever $S\cap T = \emptyset$. Using Proposition 2, the following holds trivially.

Proposition 4. *Let I^v be an interaction index satisfying axioms LI and DI. If v is additive, then $I^v(S) = 0$ for every S such that $|S| > 1$, and $I(i) = v(i)$.*

Thus, there is no interaction among players in an additive game, as expected.

We introduce now the following axiom.

Symmetry axiom (SI): for all $v \in \mathscr{G}^N$, for all permutation π on N,

$$I^v(S) = I^{\pi v}(\pi S), \quad \forall S \subset N.$$

As this axiom contains the preceding symmetry axiom introduced by Weber, we keep its name. The following can be shown.

Proposition 5. *Let I^v be an interaction index satisfying axioms LI, DI and SI for every $v \in \mathscr{G}$. Then there exist real constants $p_t^s(n)$, $s = 0, \dots, n$, $t = 0, \dots, n-s$, such that*

$$I^{v^N}(S) = \sum_{T \subset N \setminus S} p_t^s(n) \delta_S v(S \cup T).$$

Proof: The proof is similar to the one of Weber for values [19]. Let us first consider a given set of players N. Since I^v satisfies axioms LI and DI, it can be put under the form (8) by proposition 2. Consider the game $\hat{v}_T$, for any $T \subset N$. From the definition of $\hat{v}_T$, we have for any $S \subset N \setminus T$, $S \neq \varnothing$

$$\begin{aligned} I^{\hat{v}_T}(S) &= \sum_{K \subset N \setminus S} p_K^S \sum_{L \subset S} (-1)^{s-l} \hat{v}_T(L \cup K) \\ &= (-1)^{s+1} p_T^S, \end{aligned} \tag{9}$$

since every sum $\sum_{L \subset S}$ is zero except when $K = T$. Now observe that $p_T^{\varnothing} = (-1)^l p_{T \cup L}^{\varnothing}$ (see proof of proposition 2, part 2), so that from the definition of $\hat{v}$,

$$I^{\hat{v}_T}(\varnothing) = \sum_{L \subset N \setminus T, L \neq \varnothing} p_{T \cup L}^{\varnothing} = \sum_{L \subset N \setminus T, L \neq \varnothing} p_T^{\varnothing} (-1)^l = -p_T^{\varnothing}.$$

Thus, equation (9) holds also for $S = \varnothing$.

Let $S \subset N, |S| \leq n-2$ be fixed, and consider two different non empty sets $T_1, T_2 \subsetneq N \setminus S$, with $|T_1| = |T_2|$. Let us consider a permutation π, which transforms T_1 into T_2, leaving S invariant. Then by axiom SI we have $I^{\hat{v}_{T_1}}(S) = I^{\pi \hat{v}_{T_1}}(\pi S) = I^{\hat{v}_{T_2}}(S)$. Using (9), we conclude that $p_{T_1}^S = p_{T_2}^S$, for any S, and any $\varnothing \neq T_1, T_2 \subsetneq N \setminus S$ of same cardinality.

Now let us consider two different non empty lets $S_1, S_2 \subsetneq N$, with $|S_1| = |S_2|$, and $T \subset N \setminus (S_1 \cup S_2)$. Let us consider a permutation π which transforms S_1 in S_2 and leaves other elements invariant. Then clearly $\pi \hat{v}_T = \hat{v}_T$, and $I^{\hat{v}_T}(S_1) = I^{\hat{v}_T}(S_2)$ by axiom SI. From (9) we conclude that $p_T^{S_1} = p_T^{S_2}$, for any distinct $S_1, S_2 \subsetneq N$ of same cardinality, and any $T \subset N \setminus (S_1 \cup S_2)$. The above two results show that there exist constants p_t^s, $s = 1, \dots, n-1$, $t = 1, \dots, n-s-1$, such that $p_{|T|}^{|S|} = p_t^s$. Remark that for $S = \varnothing$ the result also holds, so that s may be 0.

Next we consider two distinct non empty sets S_1, S_2 of same cardinality, and a permutation π transforming S_1 in S_2. Then by axiom SI, $I^{v_N}(S_1) = I^{v_N}(S_2)$. It is not difficult to see that $I^{v_N}(S) = p^S_{N\backslash S}$, since the only non zero term in the double sum happens when $K = N\backslash S$ and $L = S$. This implies $p^{S_1}_{N\backslash S_1} = p^{S_2}_{N\backslash S_2}$, so that p^s_t is defined for $t = n - s$. Finally, we have

$$p^{S_1}_{\varnothing} = I^{v_{S_1}}(S_1) - \sum_{\varnothing \neq K \subset N\backslash S_1} p^{S_1}_K = I^{v_{S_1}}(S_1) - \sum_{k=1}^{n-|S_1|} p^{|S_1|}_k \binom{|N| - |S_1|}{k}.$$

By the symmetry axiom SI, $I^{v_{S_1}}(S_1) = I^{v_{S_2}}(S_2)$, which implies $p^{S_1}_{\varnothing} = p^{S_2}_{\varnothing}$ and p^s_t is also defined for $t = 0$.

All the proof is based on a given finite set N but the symmetry axiom SI implies that p^s_t is independent from the particular choice of N but depends on its cardinality and the definition makes sense. $\square$

The next step is to propose some recursive axiom in order to link somehow value and interaction. Let us first limit ourself to a pair of players i, j, and consider the reduced game $v_{[ij]}$. What could be the value of $[ij]$ for this reduced game? We guess that this value should depend on the values of i when j is absent, and j when i is absent, and somehow their interaction should be also taken into account. Clearly, if the interaction is positive (profitable cooperation), then the value of $[ij]$ should be greater than simply the sum of individual values. If on the contrary the interaction is negative (harmful cooperation), the value of $[ij]$ should be less than the sum. In summary, the following formula is natural:

$$\phi^{v_{[ij]}}([ij]) = \phi^{v^{N\backslash j}}(i) + \phi^{v^{N\backslash i}}(j) + I^v(ij),$$

where, according to our notations, $v^{N\backslash j}$ is the game v of which domain is restricted to $N\backslash j$. Put in another form, we have an expression of $I^v(ij)$ solely in terms of values (or interaction for singletons):

$$I^v(ij) = I^{v_{[ij]}}([ij]) - I^{v^{N\backslash j}}(i) - I^{v^{N\backslash i}}(j), \tag{10}$$

which is clearly recursive. The problem is now to extend (10) to any coalition. We propose two generalizations of this formula, which will be called "recursive axioms".

Recursive axiom 1 (R1): I^v obeys the following recurrence formula for every $S \subset N, |S| > 1$ and for any $v \in \mathscr{G}$:

$$I^v(S) = I^{v_{[S]}}([S]) - \sum_{K \subsetneq S, K \neq \varnothing} I^{v^{N\backslash K}}(S\backslash K).$$

This is a straightforward generalization, expressing interaction of S in terms of all successive interactions of subsets.

An equivalent form can be found for this axiom.

Proposition 6. *An interaction index I^v satisfies axiom R1 if and only if*

$$I^v(S) = \sum_{L\subset S, L\neq\emptyset} (-1)^{s-l} I^{v_{[L]}^{(N\setminus S)\cup[L]}}([L]). \tag{11}$$

for any $v \in \mathcal{G}^N$.

Proof: we consider the "only if" part, and show the property by recurrence. Let us show it for $S = \{i, j\}$. In fact, in this case equations (10) and (11) are identical since $v_{[i]}^{(N\setminus ij)\cup[i]} = v^{N\setminus j}$. We suppose now that equation (11) is true under axiom R1 for any subset up to $(s-1)$ elements, and try to prove it for s elements. We have:

$$\begin{aligned}
I^v(S) &= I^{v_{[S]}}([S]) - \sum_{K\subsetneq S, K\neq\emptyset} I^{v^{N\setminus K}}(S\setminus K)\\
&= I^{v_{[S]}}([S]) - \sum_{K\subsetneq S, K\neq\emptyset}\ \sum_{L\subset S\setminus K, L\neq\emptyset} (-1)^{s-k-l} I^{v_{[L]}^{(N\setminus S)\cup[L]}}([L])\\
&= I^{v_{[S]}}([S]) - \sum_{L\subsetneq S, L\neq\emptyset} I^{v_{[L]}^{(N\setminus S)\cup[L]}}([L]) \sum_{K\subset S\setminus L, K\neq\emptyset} (-1)^{s-k-l}\\
&= I^{v_{[S]}}([S]) - \sum_{L\subsetneq S, L\neq\emptyset} I^{v_{[L]}^{(N\setminus S)\cup[L]}}([L])(-1)^{s-l+1}\\
&= \sum_{L\subset S, L\neq\emptyset} (-1)^{s-l} I^{v_{[L]}^{(N\setminus S)\cup[L]}}([L]).
\end{aligned}$$

The "if" part follows when using the same derivation but starting from the end. □

For the second axiom, we define

Recursive axiom 2 (R2): I^v obeys the following recurrence formula for every $S \subset N, |S| > 1$ and for any $v \in \mathcal{G}^N$:

$$I^v(S) = I^{v_{\cup j}^{N\setminus j}}(S\setminus j) - I^{v^{N\setminus j}}(S\setminus j), \quad \forall j \in S.$$

To see if this is effectively a generalization, the above equation for $S = \{i, j\}$ becomes $I^{v^N}(ij) = I^{v_{\cup j}^{N\setminus j}}(i) - I^{v^{N\setminus j}}(i)$. Now, under the assumption of axioms LI, DI and SI, we have (the absence of superscript on v always indicates v^N):

$$\begin{aligned}
I^{v_{[ij]}}([ij]) - I^{v^{N\setminus i}}(j) = &\sum_{T\subset N\setminus ij} p_t^1(n-1)[v(T\cup ij) - v(T)]\\
&- \sum_{T\subset N\setminus ij} p_t^1(n-1)[v(T\cup j) - v(T)]
\end{aligned}$$

$$= \sum_{T \subset N \setminus ij} p_t^1 (n-1)[v(T \cup ij) - v(T \cup j)]$$

$$= I^{v_{\cup j}^{N \setminus j}}(i).$$

The axiom R2 has an interesting interpretation. It says that the interaction of the players in S is equal to the interaction between the players in $S \setminus j$ in the omnipresence of j, minus the interaction between the players of $S \setminus j$ (in the absence of j).

The next proposition shows that R2 too is equivalent to (11).

Proposition 7. *Under axioms LI, DI and SI, an interaction I^v satisfies axiom R2 if and only if*

$$I^v(S) = \sum_{L \subset S, L \neq \emptyset} (-1)^{s-l} I^{v_{[L]}^{(N \setminus S) \cup [L]}}([L]) \tag{12}$$

for any $v \in \mathcal{G}$.

Proof: we consider the "only if" part, and show the property by recurrence. It is already shown for the case $S = \{i, j\}$, since R2 and R1 coincides in this case. We suppose (12) is true for any subset of at most $(s-1)$ elements, and try to show it for s elements, supposing LI, DI, SI and R2 are satisfied. We have for any $j \in S$:

$$I^v(S) = I^{v_{\cup j}^{N \setminus j}}(S \setminus j) - I^{v^{N \setminus j}}(S \setminus j).$$

Since the two terms in righthand part are for $(s-1)$ elements, we can apply the hypothesis:

$$I^{v^{N \setminus j}}(S \setminus j) = \sum_{L \subset S \setminus j, L \neq \emptyset} (-1)^{s-l-1} I^{v_{[L]}^{(N \setminus S) \cup [L]}}([L])$$

$$I^{v_{\cup j}^{N \setminus j}}(S \setminus j) = \sum_{L \subset S \setminus j, L \neq \emptyset} (-1)^{s-l-1} I^{v_{[L], \cup j}^{(N \setminus S) \cup [L]}}([L]).$$

Now, using axioms LI, DI and SI, we have

$$I^{v_{[L \cup j]}^{(N \setminus S) \cup [L \cup j]}}([L \cup j]) = \sum_{T \subset N \setminus S} p_t^1 [v_{[L \cup j]}^{(N \setminus S) \cup [L \cup j]}(T \cup [L \cup j]) - v_{[L \cup j]}^{(N \setminus S) \cup [L \cup j]}(T)]$$

$$= \sum_{T \subset N \setminus S} p_t^1 [v(T \cup L \cup j) - v(T)],$$

$$I^{v^{(N \setminus S) \cup j}}(j) = \sum_{T \subset N \setminus S} p_t^1 [v(T \cup j) - v(T)],$$

$$I^{v^{(N\setminus S)\cup[L]}_{[L],\cup j}}([L]) = \sum_{T\subset N\setminus S} p^1_t[v^{(N\setminus S)\cup[L]}_{[L],\cup j}(T\cup[L]) - v^{(N\setminus S)\cup[L]}_{[L],\cup j}(T)]$$

$$= \sum_{T\subset N\setminus S} p^1_t[v(T\cup L\cup j) - v(T\cup j)],$$

from which we deduce that

$$I^{v^{(N\setminus S)\cup[L]}_{[L],\,\cup j}}([L]) = I^{v^{(N\setminus S)\cup[L\cup j]}_{[L\cup j]}}([L\cup j]) - I^{v^{(N\setminus S)\cup j}}(j).$$

Substituting in $I^v(S)$ gives

$$I^v(S) = \sum_{L\subset S\setminus j, L\neq\varnothing} (-1)^{s-l-1}[I^{v^{(N\setminus S)\cup[L\cup j]}_{[L\cup j]}}([L\cup j]) - I^{v^{(N\setminus S)\cup j}}(j)]$$

$$+ \sum_{L\subset S\setminus j, L\neq\varnothing} (-1)^{s-l} I^{v^{(N\setminus S)\cup[L]}_{[L]}}([L])$$

$$= \sum_{L\subset S\setminus j, L\neq\varnothing} (-1)^{s-l-1} I^{v^{(N\setminus S)\cup[L\cup j]}_{[L\cup j]}}([L\cup j]) + (-1)^{s-1} I^{v^{(N\setminus S)\cup j}}(j)$$

$$+ \sum_{L\subset S\setminus j, L\neq\varnothing} (-1)^{s-l} I^{v^{(N\setminus S)\cup[L]}_{[L]}}([L])$$

$$= \sum_{L\subset S, L\neq\varnothing} (-1)^{s-l} I^{v^{(N\setminus S)\cup[L]}_{[L]}}([L]).$$

The "if" part can be shown taking the same derivation but starting from the end. $\square$

A corollary of Propositions 6 and 7 is the following.

Corollary 1. *Under axioms LI, DI and SI, axioms R1 and R2 are equivalent.*

The recursive axioms entail a particular property of the coefficients p^s_t.

Proposition 8. *If I^v satisfies axioms LI, DI, SI and (R1 or R2), then*

$$p^s_t(n) = p^1_t(n-s+1).$$

for any $v\in\mathcal{G}$.

In other words, the coefficients depend only on t and $n-s$.

Proof: by axioms LI, DI and SI we can write for any $L\subset S$:

$$I^{v^N}(S) = \sum_{T\subset N\setminus S} p^s_t(n)\delta_S v(T\cup S)$$

$$= \sum_{T\subset N\setminus S} p^s_t(n)\sum_{L\subset S}(-1)^{s-l} v(T\cup L) \tag{13}$$

$$I^{v^{(N\setminus S)\cup[L]}_{[L]}}([L]) = \sum_{T\subset N\setminus S} p^1_t(n-s+1)[v(T\cup L) - v(T)],$$

and from proposition 6,

$$I^{v^N}(S) = \sum_{L \subset S, L \neq \varnothing} (-1)^{s-l} I^{v_{[L]}^{(N \setminus S) \cup [L]}}([L])$$
$$= \sum_{L \subset S, L \neq \varnothing} (-1)^{s-l} \sum_{T \subset N \setminus S} p_t^1(n-s+1)[v(T \cup L) - v(T)]$$
$$= \sum_{T \subset N \setminus S} p_t^1(n-s+1) \left[\sum_{L \subset S} (-1)^{s-l} [v(T \cup L) - v(T)] \right].$$

Since $\sum_{L \subset S} (-1)^{s-l} = 0$, we obtain

$$I^{v^N}(S) = \sum_{T \subset N \setminus S} p_t^1(n-s+1) \sum_{L \subset S} (-1)^{s-l} v(T \cup L). \tag{14}$$

Comparing (13) and (14), one obtains

$$p_t^s(n) = p_t^1(n-s+1). \qquad \square$$

The recursive axiom permits to link interaction indices to values in a unique way. That is, if for example the Shapley value is chosen, the interaction index based on the Shapley value is uniquely determined, and the coefficients $p_t^s(n)$ are known. The same will be true for any value, *provided it satisfies the linearity, dummy and symmetry axioms.*

We are now ready to state the main result of the paper.

Theorem 3. *Let I^v be an interaction index defined for any game v in $\mathcal{G}$.*

(i) *the Shapley interaction index, defined by*

$$I_{\mathrm{S}}^v(S) = \sum_{T \subset N \setminus S} \frac{(n-t-s)!t!}{(n-s+1)!} \sum_{L \subset S} (-1)^{s-l} v(L \cup T), \quad \forall S \subset N \tag{15}$$

is the only interaction index satisfying axioms LI, DI, SI, (R1 or R2), which restriction to singletons corresponds to the Shapley value.

(ii) *the Banzhaf interaction index, defined by*

$$I_{\mathrm{B}}^v(S) = \frac{1}{2^{n-s}} \sum_{T \subset N \setminus S} \sum_{L \subset S} (-1)^{s-l} v(L \cup T), \quad \forall S \subset N \tag{16}$$

is the only interaction index satisfying axioms LI, DI, SI, (R1 or R2), which restriction to singletons corresponds to the Banzhaf value.

Proof: clear from Theorems 1, 2, and Propositions 1, 2, 5, and 8. Also, the Shapley and Banzhaf interaction indices clearly satisfy the corresponding axioms. $\square$

4. Alternative axiomatization of the Banzhaf interaction index

It is possible to find another set of axioms for the Banzhaf interaction, which do not use the recursive axiom. We introduce the following new axiom, which is a generalization of the 2-efficiency.

Generalized 2-efficiency axiom (*G2-E*)*:* for any pair $i, j \in N$, for any $S \subset N \backslash ij$,

$$I^{v_{[ij]}^{(N\backslash ij)\cup[ij]}}(S \cup [ij]) = I^{v^N}(S \cup i) + I^{v^N}(S \cup j).$$

The following limit condition will be also useful.

Limit condition (*LIM*): for all $S \subset N$,

$$I^{v^S}(S) = \delta_S v^S(S).$$

We can show the following.

Theorem 4. *The Banzhaf interaction index is the only interaction index satisfying axioms LI, DI, SI, G2-E, and LIM.*

Proof: (similar to the proof of Theorem 2) using LI, DI, and SI, we have for any game v on N, any $i, j \in N$, and any $S \subset N \backslash ij$:

$$\begin{aligned}
I^v(S \cup i) &= \sum_{T \subset N\backslash(S\cup i)} p_t^{s+1}(n)\delta_{S\cup i}v(T \cup S \cup i) \\
&= \sum_{T \subset N\backslash(S\cup i)} p_t^{s+1}(n)[\delta_S v(T \cup S \cup i) - \delta_S v(T \cup S)] \\
&= \sum_{T \subset N\backslash(S\cup ij)} p_t^{s+1}(n)[\delta_S v(T \cup S \cup i) - \delta_S v(T \cup S)] \\
&\quad + \sum_{T \subset N\backslash(S\cup ij)} p_{t+1}^{s+1}(n)[\delta_S v(T \cup S \cup ij) - \delta_S v(T \cup S \cup j)]. \qquad (17)
\end{aligned}$$

Similarly,

$$\begin{aligned}
I^v(S \cup j) &= \sum_{T \subset N\backslash(S\cup ij)} p_t^{s+1}(n)[\delta_S v(T \cup S \cup j) - \delta_S v(T \cup S)] \\
&\quad + \sum_{T \subset N\backslash(S\cup ij)} p_{t+1}^{s+1}(n)[\delta_S v(T \cup S \cup ij) - \delta_S v(T \cup S \cup i)], \qquad (18)
\end{aligned}$$

and

$$\begin{aligned}
&I^{v_{[ij]}^{(N\backslash ij)\cup[ij]}}(S \cup [ij]) \\
&\quad = \sum_{T \subset N\backslash(S\cup ij)} p_t^{s+1}(n-1)[\delta_S v(T \cup S \cup ij) - \delta_S v(T \cup S)] \qquad (19)
\end{aligned}$$

Comparing (17), (18), and (19) wih the use of axiom G2-E, one obtains:

$$p_t^{s+1}(n) = p_{t+1}^{s+1}(n)$$

$$2p_t^{s+1}(n) = p_t^{s+1}(n-1)$$

The first equation tells us that $p_t^{s+1}(n)$ is independent of t, while the second one, together with LIM which gives the initial condition $p_0^{s+1}(s+1) = 1$, leads to

$$p_t^s(n) = p_0^s(n) = \frac{1}{2^{n-s}},$$

which is the desired result. □

5. Expressions of I_B and I_S in terms of multilinear extension of v

Following Owen [13, 14], Hammer and Rudeanu [8], we define the multilinear extension of $v \in \mathcal{G}^N$ (MLE of game v):

$$\begin{aligned} g(x_1, \ldots, x_n) &:= \sum_{S \subset N} a(S) \left[\prod_{i \in S} x_i \right] \\ &= \sum_{S \subset N} v(S) \left[\prod_{i \in S} x_i \cdot \prod_{i \in S^c} (1 - x_i) \right] \end{aligned}$$

where $x_i \in [0, 1]$, $i \in N$. The real coefficients $a(S)$ are called the dividends [9, 14]. In combinatorics, a viewed as a set function on N is called the Möbius transform of v (see e.g. Rota [15]), which is given by

$$a(S) = \sum_{T \subset S} (-1)^{s-t} v(T), \quad \forall S \subset N.$$

Reciprocally, the dividend being given, one can recover the game v by

$$v(S) = \sum_{T \subset S} a(T), \quad \forall S \subset N. \tag{20}$$

Using equation (20), it is easy to see that for any $S \subset N$,

$$g(\alpha_S) = v(S),$$

where $\alpha_S \in \{0, 1\}^n$ is a Boolean vector $[\alpha_S^1 \cdots \alpha_S^n]$ corresponding to S by

$$\alpha_S^j = \begin{cases} 1, & \text{if } j \in S \\ 0, & \text{otherwise.} \end{cases}$$

This shows clearly that g coincides with v on the vertices of the hypercube $[0,1]^n$.

Let us introduce the notation $\underline{c} := (c, c, \dots, c)$ for any constant c belonging to $[0,1]$. If we define the T-derivative of g as

$$\Delta_T g(x_1, \dots, x_n) = \frac{\partial^t g(x_1, \dots, x_n)}{\partial x_{i_1} \cdots \partial x_{i_t}} \quad \text{where } T = \{i_1, \dots, i_t\}$$

we easily obtain:

$$g(x_1, \dots, x_n) = \sum_{T \subset N} \Delta_T g\left(\underline{\frac{1}{2}}\right) \left[\prod_{i \in T} \left(x_i - \frac{1}{2}\right)\right] \tag{21}$$

and

$$\Delta_T g(x_1, \dots, x_n) = \sum_{S \subset T^c} a(S \cup T) \left[\prod_{i \in S} x_i\right] \tag{22}$$

Let us now express the Banzhaf and Shapley interaction index using g. First, we express these indices in terms of the dividends $a(S)$. Grabisch has shown that [6]:

$$I_{\mathrm{S}}(S) = \sum_{T \supset S} \frac{1}{t - s + 1} a(T) \tag{23}$$

while Roubens has shown that for the Banzhaf interaction index [16]:

$$I_{\mathrm{B}}(S) = \sum_{T \supset S} \frac{1}{2^{t-s}} a(T). \tag{24}$$

Using these two expressions we finally obtain

$$I_{\mathrm{B}}(T) = \Delta_T g\left(\underline{\frac{1}{2}}\right), \quad T \neq \varnothing \tag{25}$$

$$I_{\mathrm{S}}(T) = \int_0^1 \Delta_T g(\underline{\lambda})\, d\lambda, \quad T \neq \varnothing \tag{26}$$

We see that the Banzhaf interaction index related to coalition T is the value of the T-gradient of the MLE of game v on the center of the unit cube, while the Shapley interaction index related to T is obtained by integrating the t-th gradient of the MLE of game v along the main diagonal of the cube.

6. Related topics

The notion of interaction as presented here has strong links in the field of multicriteria decision making (MCDM), where criteria stand for players, and gives the theoretical basis for a new class of methods in MCDM around non-additive measures and the Choquet integral. The Choquet integral is in fact a generalization of the Lebesgue integral when the measure is non additive (e.g. a game). The reader is referred to [4] for a general survey of MCDM with Choquet integral, and to [6] for a use of the interaction index in multicriteria decision making.

On a more abstract point of view, it can be said that the interaction index, viewed as a set function, is another representation of a game, exactly as dividends (or Möbius transform) are. More formally, a set function $\omega : 2^N \to \mathbb{R}$ is a representation of v if there exists an invertible transform $\mathscr{T}$ such that

$$\omega = \mathscr{T}(v) \quad \text{and} \quad v = \mathscr{T}^{-1}(\omega).$$

The transformations $\mathscr{T}$ related to I_S and I_B have been studied by Denneberg and Grabisch [2] and by Grabisch, Marichal and Roubens [7]. To obtain these operators, two types of transforms have been pointed out: fractal and cardinality linear transforms, the names of which coming from the particular structure of the matrices.

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