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Characterizations of a multi-choice value*

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Abstract. A multi-choice game is a generalization of a cooperative game in which each player has several activity levels. We study the extended Shapley value as proposed by Derks and Peters (1993). Van den Nouweland (1993) provided a characterization that is an extension of Young's (1985) characterization of the Shapley value. Here we provide several other characterizations, one of which is the analogue of Shapley's (1953) original characterization. The three other characterizations are inspired by Myerson's (1980) characterization of the Shapley value using balanced contributions.

Key words: Multi-choice games, Shapley value, characterizations, balanced contributions

1. Introduction

Multi-choice games were introduced by Hsiao and Raghavan (1993). A multi-choice game is a cooperative game in which each player has a certain number of activity levels at which he can choose to play. The reward that a group of players can obtain depends on the efforts of the cooperating players.

Hsiao and Raghavan (1993) considered games in which all players have the same number of activity levels. We allow for different numbers of activity levels for different players. Several concepts from TU-games can be extended to the setting of multi-choice games in a straightforward way. For instance, straightforward extensions of convexity and the core solution have been studied by van den Nouweland *et al.* (1995). For the Shapley value (see

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Shapley (1953)), however, there exist several more or less natural extensions to the setting of multi-choice games. Here we study the extended Shapley value as proposed by Derks and Peters (1993) and give several characterizations of it.

The work is organized as follows. Section 2 deals with notation, definitions, and the formal description of our model. In section 3 we discuss several extensions of the Shapley value to multi-choice games. In section 4 we present the characterizations of the extended Shapley value as proposed by Derks and Peters (1993).

2. The model

Let $N = \{1, \dots, n\}$ be a set of players. Suppose each player $i \in N$ has m_i levels at which he can actively participate. Let $m = (m_1, \dots, m_n)$ be the vector that describes the number of activity levels for every player, at which he can actively participate. We set $M_i := \{0, \dots, m_i\}$ as the action space of player $i \in N$, where the action 0 means not participating. Let $M := \prod_{i \in N} M_i$ be the product set of the action spaces. A characteristic function is a function $v : M \to \mathbb{R}$ which assigns to each coalition $s = (s_1, \dots, s_n)$ the worth that the players can obtain when each player i plays at activity level $s_i \in M_i$ with $v(0) = 0$. A multi-choice game is given by a triple (N, m, v). If no confusion can arise a game (N, m, v) will sometimes be denoted by its characteristic function v. Let us denote the class of multi-choice games with player set N and activity level vector m by $MC^{N,m}$, and the class of all multi-choice games by MC. Clearly, the class of ordinary TU-games is a subclass of the class of multi-choice games, because a TU-game can be viewed as a multi-choice game in which every player has two activity levels, participate and not participate.

3. Multi-choice values

We will now discuss several solutions on MC that are extensions of the Shapley value. For $i \in N$, let $M_i^+ := M_i \backslash \{0\}$. Further, let $M^+ := \bigcup_{i \in N}(\{i\} \times M_i^+)$. A solution on MC is a map Ψ assigning to each multi-choice game $(N, m, v) \in MC$ an element $\Psi(N, m, v) \in \mathbb{R}^{M^+}$. As is pointed out in van den Nouweland (1993) there exists more than one reasonable extension of the definition of the Shapley value for TU-games to multi-choice games. The first extension of the Shapley value was introduced by Hsiao and Raghavan (1993). They restricted themselves to multi-choice games where all players have the same number of activity levels and defined Shapley values using weights on the actions, thereby extending ideas of weighted Shapley values (cf. Kalai and Samet (1988)). Another extension of the Shapley value was introduced by van den Nouweland *et al.* (1995). They define the extended Shapley value as the average of all marginal vectors that correspond to admissible orders for the multi-choice game. Calvo and Santos (1997) study this value and focus on total payoff instead of payoff per level. Here we will consider a third extension, the value as proposed by Derks and Peters (1993). For this, let us start with some additional notation.

The analogue of unanimity games for multi-choice games are minimal effort games $(N,m,u_s) \in MC^{N,m}$, where $s \in \prod_{i\in N} M_i$, defined by

$$u_s(t) := \begin{cases} 1 & \text{if } t_i \geq s_i \text{ for all } i \in N; \\ 0 & \text{otherwise} \end{cases}$$

for all $t \in \prod_{i\in N} M_i$. One can prove that the minimal effort games form a basis of the space $MC^{N,m}$, and that for a multi-choice game (N,m,v) it holds that

$$v = \sum_{s\in M} \Delta_v(s) u_s,$$

where the $\Delta_v(s)$ are the extended dividends defined by

$$\Delta_v(0) := 0 \quad \text{and}$$

$$\Delta_v(s) := v(s) - \sum_{t\leq s, t\neq s} \Delta_v(t) \quad \text{for } s \neq 0.$$

Now we can go on to the extension of the Shapley value of Derks and Peters (1993).

For a multi-choice game $(N,m,v) \in MC^{N,m}$ the value $\Theta(N,m,v)$ of Derks and Peters (1993) is given by

$$\Theta_{ij}(N,m,v) := \sum_{s\in M: s_i \geq j} \frac{\Delta_v(s)}{\sum_{k\in N} s_k} \tag{1}$$

for all $(i,j) \in M^+$. So, the dividend $\Delta_v(s)$ is divided equally among the necessary levels.

In fact, this value can be seen as the vector of average marginal contributions of the pairs $(i,j) \in M^+$. Let us point this out formally. For this, we may suppose that $M^+ \neq \emptyset$. An order for a multi-choice game (N,m,v) is a bijection $\sigma : M^+ \to \{1,\ldots,\sum_{i\in N} m_i\}$. The subset $\sigma^{-1}(\{1,\ldots,k\})$ of M^+, which is present after k steps according to σ, is denoted by $S^{\sigma,k}$. The marginal vector $w^\sigma \in \mathbb{R}^{M^+}$ corresponding to σ is defined by

$$w^\sigma_{ij} := v(\rho(S^{\sigma,\sigma(i,j)})) - v(\rho(S^{\sigma,\sigma(i,j)-1})) \tag{2}$$

for all $(i,j) \in M^+$. Here ρ is the map that assigns to every subset $S \subseteq M^+$ the maximal feasible coalition $\rho(S)$ that is a 'subset' of S. Formally, for $S \subseteq M^+$,

$$\rho(S) := (t_1,\ldots,t_n),$$

where

$$t_i = \begin{cases} \max\{k \in M_i^+ : (i,1),\ldots,(i,k) \in S\} & \text{if } (i,1) \in S; \\ 0 & \text{otherwise.} \end{cases}$$

Now, define

$$\Lambda_{ij}(N,m,v) := \frac{1}{(\sum_{k\in N} m_k)!}\sum_{\sigma} w_{ij}^{\sigma} \tag{3}$$

for all $(i,j) \in M^+$. The number $\Lambda_{ij}(N,m,v)$ is the average marginal contribution of the pair $(i,j) \in M^+$ to the maximal feasible coalition. In fact, the number $\Lambda_{ij}(N,m,v)$ is equal to the Shapley payoff of player (i,j) in the ordinary TU-game $(M^+,\bar{v})$, where the characteristic function $\bar{v}$ is defined by

$$\bar{v}(T) := v(\rho(T)) \quad \text{for all } T \subseteq M^+.$$

One can prove that a multi-choice game (N,m,v) is convex[1] if and only if the TU-game $(M^+,\bar{v})$ is convex.

It is not difficult to see that for a minimal effort game (N,m,u_s) we have

$$\Theta_{ij}(N,m,u_s) = \Lambda_{ij}(N,m,u_s) = \begin{cases} \dfrac{1}{\sum_{k\in N} s_k} & \text{if } j \le s_i; \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

for all $(i,j) \in M^+$. Derivation of formula (4) is straightforward for $\Theta_{ij}(N,m,u_s)$ by using formula (1). To see the equality for $\Lambda_{ij}(N,m,u_s)$ first note that for all σ and all $(i,j) \in M^+$ we have $w_{ij}^{\sigma} \in \{0,1\}$. Now, if $j > s_i$, then $w_{ij}^{\sigma} = 0$. If $j \le s_i$, then note that the number of σ for which $w_{ij}^{\sigma} = 1$, or equivalently

$$S^{\sigma,\sigma(i,j)} \supseteq S := \{(1,1),\ldots,(1,s_1),\ldots,(n,1),\ldots,(n,s_n)\} \tag{5}$$

does not depend upon $(i,j) \in S$; the number of permutations of M^+ with (i,j) last element of S is the same for every $(i,j) \in S$. Hence, the number of permutations for which (5) holds true is the same for all $(i,j) \in S$ and is therefore equal to $(\sum_{k\in N} m_k)!/(\sum_{k\in N} s_k)$. This implies that indeed formula (4) holds true for $\Lambda_{ij}(N,m,u_s)$.

From formula (4) and the linearity of both Λ and Θ it follows that $\Lambda = \Theta$.

The following example shows that in some situations the extension of the Shapley value by Derks and Peters (1993) seems to be more appropriate than the extension of the Shapley value by van den Nouweland *et al.* (1995). Further, it illustrates why the players may be interested in the payoff for each level, not solely the sum of their levels, which is the case in Calvo and Santos (1997).

Example 3.1: Consider the following cost allocation problem related to airlines. Suppose there is an airline with several divisions, where each division has available a finite number of sizes of planes. Suppose further that each division has to perform a flight schedule, and therefore has to decide which

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[1] A multi-choice game (N,m,v) is said to be convex if $v(s \vee t) + v(s \wedge t) \ge v(s) + v(t)$ for all $s,t \in \prod_{k\in N} M_k$, where $(s \wedge t)_i := \min\{s_i,t_i\}$ and $(s \vee t)_i := \max\{s_i,t_i\}$ for all $i \in N$. For ordinary TU-games this definition is equivalent to the usual one.

sizes of planes it will use. Then the airline builds the smallest runway that suffices for the largest planes chosen by the divisions. The costs depend on the length of the runway. The question now arises how to allocate the forthcoming costs among the divisions.

For example, consider the situation of an airline with two divisions, a passenger division (division 1) and a cargo division (division 2). Suppose further that the company possesses small planes and large planes. The small planes need a runway of length 1 and are suitable for passengers as well as for cargo. The large planes need a runway of length 2 and can only carry cargo. Suppose also that the costs of a runway of length l ($l = 1, 2$) are l.

We model this situation as a multi-choice game as follows. Let $N = \{1, 2\}$ be the set of players, i.e. the divisions. Let $m = (1, 2)$ be the activity levels from which the players can choose, i.e. the sizes of the available planes. Now, the game (N, m, c), where c is the cost function defined by $c := u_{(0,1)} + u_{(1,0)} - u_{(1,1)} + u_{(0,2)}$, models the situation above.

The value of Derks and Peters (1993) gives $\Theta_{1,1}(N, m, c) = \frac{1}{2}$, $\Theta_{2,1}(N, m, c) = 1$, and $\Theta_{2,2}(N, m, u_s) = \frac{1}{2}$, while the value Γ of van den Nouweland *et al.* (1995) gives $\Gamma_{1,1}(N, m, c) = \frac{1}{3}$, $\Gamma_{2,1}(N, m, c) = \frac{2}{3}$, and $\Gamma_{2,2}(N, m, c) = 1$.

Now suppose that instead of modeling that division 1 has no possibility to use larger planes, we model the situation by allowing it to use 0 large planes. So, if they use all their large planes there will be no effect on the costs. Formally, the cost function c remains unchanged, but the vector of activity levels changes to $m' = (2, 2)$. Some calculations yield $\Theta_{1,1}(N, m', c) = \Gamma_{1,1}(N, m', c) = \frac{1}{2}$, $\Theta_{1,2}(N, m', c) = \Gamma_{1,2}(N, m', c) = 0$, $\Theta_{2,1}(N, m', c) = 1$, $\Gamma_{2,1}(N, m', c) = \frac{1}{2}$, $\Theta_{2,2}(N, m', c) = \frac{1}{2}$, and $\Gamma_{2,2}(N, m', c) = 1$. We see that the value of van den Nouweland *et al.* (1995) has a serious drawback in this example, since division 1 has to pay for being allowed to choose larger planes, although it does not use these planes.

Finally, note that the determination of costs per plane size can be an aid in cost allocation within the divisions. $\diamond$

4. Characterizations

In this section we recall one characterization of the extended Shapley value by Derks and Peters (1993), and provide five other characterizations. Therefore, consider the following properties of solutions on MC. A solution Ψ on MC satisfies

- *efficiency* (EFF) if for all games $(N, m, v) \in MC$:

$$\sum_{i \in N} \sum_{j=1}^{m_i} \Psi_{ij}(N, m, v) = v(m).$$

- *strong monotonicity* (SMON) if for all games (N, m, v) and $(N, m, w) \in MC$, whenever $(i, j) \in M^+$ is such that for all $s \in \prod_{k \in N} M_k$ with $s_i = j$

$$v(s) - v(t) \geq w(s) - w(t),$$

where $t \in \prod_{k \in N} M_k$ is such that $t_k = s_k$ if $k \neq i$ and $t_i = s_i - 1$, then

$$\Psi_{ij}(N, m, v) \geq \Psi_{ij}(N, m, w).$$

- *the veto property* (VETO) if for all games $(N, m, v) \in MC$, and all $i_1, i_2 \in N$, whenever $j_1 \in M_{i_1}^+$, and $j_2 \in M_{i_2}^+$ are veto levels, then

$$\Psi_{i_1 j_1}(N, m, v) = \Psi_{i_2 j_2}(N, m, v).$$

Here, $j \in M_i^+$ is a veto level if $v(s) = 0$ for all $s \in \prod_{k \in N} M_k$ with $s_i < j$.

Property SMON says that if for two games (N, m, v) and $(N, m, w) \in MC$ and a player $i \in N$ it holds that the marginal contribution of level $j \in M_i^+$ in the game (N, m, v) is not smaller than the marginal contribution in the game (N, m, w), then the payoff to level $j \in M_i^+$ in the game (N, m, v) is not smaller than the payoff in the game (N, m, w). Property VETO says that for a game $(N, m, v) \in MC$ the payoffs to all players $i \in N$ and levels $j \in M_i^+$ that have veto power (i.e. a level of player i less than j yields worth 0, independent of the levels of the other players) should be equal. The properties EFF and SMON reduce to the properties with same names given in Young's (1985) characterization of the Shapley value for TU-games. Furthermore, VETO restricted to TU-games is implied be the symmetry property that Young uses, but would be sufficient to replace symmetry in Young's characterization. The following theorem can be found in van den Nouweland (1993) and is an extension of Young's theorem to the multi-choice framework.

Theorem 4.1. *A solution Ψ satisfies EFF, SMON, and VETO if and only if $\Psi = \Theta$.*

Inspired by Theorem 4.1 we will provide a characterization of Θ using the following properties. A solution Ψ on MC satisfies

- *additivity* (ADD) if for all games (N, m, v) and $(N, m, v) \in MC$:

$$\Psi(N, m, v + w) = \Psi(N, m, v) + \Psi(N, m, w).$$

- *the dummy property* (DUM) if for all games $(N, m, v) \in MC$, and all $i \in N$, whenever $j \in M_i^+$ is a dummy level, then

$$\Psi_{ij}(N, m, v) = 0.$$

Here, $j \in M_i^+$ is a dummy level if $v(s_{-i}, j - 1) = v(s_{-i}, l)$ for all $s_{-i} \in \prod_{k \in N \setminus \{i\}} M_k$ and all $j \leq l \leq m_i$.

Next, we prove that by replacing the property SMON in Theorem 4.1 with ADD and DUM we get another characterization. It is readily verified that SMON does not imply ADD nor DUM, and that ADD and DUM do not imply SMON. Theorem 4.2 is the analogue of Shapley's (1953) original characterization, since VETO restricted to TU-games is implied by the symmetry property that Shapley uses, but would be sufficient to replace symmetry in Shapley's characterization. Furthermore, EFF and DUM restricted to TU-games is equivalent with the carrier axiom of Shapley. Finally, ADD restricted to TU-games coincides with Shapley's 'law of aggregation'.

Theorem 4.2. *A solution Ψ satisfies EFF, ADD, VETO, and DUM if and only if $\Psi = \Theta$.*

Proof. First we prove that Θ satisfies the properties. Note that EFF and VETO follow from Theorem 4.1. Property ADD follows readily from (1). Finally, Θ satisfies DUM as is easily seen with formulas (2) and (3).

To prove uniqueness, we note that, by additivity, it is sufficient to show that Ψ and Θ coincide on the class of minimal effort games. Let (N, m, u_s) be a minimal effort game. Let $i \in N$. Every level $k_i \in M_i^+$ with $k_i > s_i$ is a dummy level, and therefore, by DUM, we have $\Psi_{ik_i}(N, m, u_s) = 0$. All other levels $k_i \in M_i^+$ are veto levels. Then, by VETO, we have

$$\Psi_{ik_i}(N, m, u_s) = c \quad \forall (i, k_i) \in M^+, \quad k_i \leq s_i$$

for some constant $c \in \mathbb{R}$. By EFF, $c = 1/(\sum_{k \in N} s_k)$. Now formula (4) gives $\Psi_{ij}(N, m, u_s) = \Theta_{ij}(N, m, u_s)$ for all $(i, j) \in M^+$, which proves the theorem. $\square$

In the next theorem we present the first of our series of three characterizations of the extended Shapley value based on balanced contributions properties. For $i \in N$, let e^i be the i-th unit vector in $\mathbb{R}^n$. A solution Ψ on MC satisfies[2]

- *the equal loss property* (EL) if for all games $(N, m, v) \in MC$, all $(i, k) \in M^+$, $k \neq m_i$:

$$\Psi_{ik}(N, m, v) - \Psi_{ik}(N, m - e^i, v) = \Psi_{im_i}(N, m, v).$$

- *the upper balanced contributions property* (UBC) if for all games $(N, m, v) \in MC$, and all $(i, m_i), (j, m_j) \in M^+$, $i \neq j$:

$$\Psi_{im_i}(N, m, v) - \Psi_{im_i}(N, m - e^j, v) = \Psi_{jm_j}(N, m, v) - \Psi_{jm_j}(N, m - e^i, v).$$

The equal loss property and the upper balanced contributions property are inspired by the balanced contributions property of Myerson (1980). Property EL says that whenever a player gets available a higher activity level the payoff for all original levels changes with an amount equal to the payoff for the highest level in the new situation. Property UBC says that for every pair i, j of different players the change in payoff for the highest level of player i when player j gets available a higher activity level is equal to the change in payoff for the highest level of player j when player i gets available a higher activity level. Note that for TU-games EL is a vacuous property and that the following characterization boils down to Myerson's (1980) balanced contributions

[2] With a slight abuse of notation we write (N, m', v) for the restriction of the game (N, m, v) to the activity levels $m' \in M$.

characterization of the Shapley value, as will also be the case for the characterizations in Theorem 4.4 and Theorem 4.5.

Theorem 4.3. *A solution Ψ satisfies EFF, EL, and UBC if and only if $\Psi = \Theta$.*

Proof. First we prove that Θ satisfies the properties. By linearity of Θ and Theorem 4.1 it is sufficient to prove that Θ satisfies EL and UBC on all minimal effort games. Let (N, m, u_s) be a minimal effort game. Let $(i,k) \in M^+$. Then

$$\Theta_{ik}(N,m,u_s) = \begin{cases} \dfrac{1}{\sum_{l\in N} s_l} & \text{if } k \leq s_i; \\ 0 & \text{if } k > s_i, \text{ and} \end{cases}$$

$$\Theta_{ik}(N,m-e^i,u_s) = \begin{cases} \dfrac{1}{\sum_{l\in N} s_l} & \text{if } k \leq s_i < m_i; \\ 0 & \text{if } m_i = s_i \text{ or } k > s_i. \end{cases}$$

Now one easily verifies that Θ indeed satisfies the equalities of EL. Let $(i,m_i),(j,m_j) \in M^+,\ i \neq j$. Then

$$\Theta_{im_i}(N,m,u_s) = \begin{cases} \dfrac{1}{\sum_{l\in N} s_l} & \text{if } m_i = s_i; \\ 0 & \text{if } m_i > s_i, \text{ and} \end{cases}$$

$$\Theta_{im_i}(N,m-e^j,u_s) = \begin{cases} \dfrac{1}{\sum_{l\in N} s_l} & \text{if } m_j > s_j \text{ and } m_i = s_i; \\ 0 & \text{otherwise} \end{cases}$$

Similar expressions hold when we interchange i and j. Again, one can check that Θ satisfies the equalities of UBC.

To prove uniqueness, suppose there are two solutions, denoted Φ and Ψ, that satisfy EFF, EL, and UBC. We will prove that $\Phi = \Psi$. The proof is with induction on the total number of levels $\sum_{k\in N} m_k$. It is clear that for all multi-choice games (N,m,v) with $\sum_{k\in N} m_k = 0$ we have $\Phi(N,m,v) = \Psi(N,m,v)$. Assume that for some $p \geq 1$ and for all multi-choice games (N,m,v) with $\sum_{k\in N} m_k = p-1$ it holds that $\Phi(N,m,v) = \Psi(N,m,v)$. We will prove that Φ and Ψ coincide on the class of multi-choice games (N,m,v) with $\sum_{k\in N} m_k = p$. Let (N,m,v) be a multi-choice game with $\sum_{k\in N} m_k = p$. Then, by EL and the induction hypothesis, we have for all $(i,k) \in M^+$, $k \neq m_i$ that

$$\begin{aligned} \Phi_{ik}(N,m,v) - \Phi_{im_i}(N,m,v) &= \Phi_{ik}(N,m-e^i,v) \\ &= \Psi_{ik}(N,m-e^i,v) \\ &= \Psi_{ik}(N,m,v) - \Psi_{im_i}(N,m,v). \end{aligned}$$

So,

$$\Phi_{ik}(N,m,v) - \Psi_{ik}(N,m,v)$$
$$= \Phi_{im_i}(N,m,v) - \Psi_{im_i}(N,m,v) \quad \forall (i,k) \in M^+. \tag{6}$$

Furthermore, by UBC and the induction hypothesis, we have for all (i,m_i), $(j,m_j) \in M^+,\ i \neq j$ that

$$\begin{aligned}\Phi_{im_i}(N,m,v) - \Phi_{jm_j}(N,m,v) &= \Phi_{im_i}(N,m-e^j,v) - \Phi_{jm_j}(N,m-e^i,v)\\ &= \Psi_{im_i}(N,m-e^j,v) - \Psi_{jm_j}(N,m-e^i,v)\\ &= \Psi_{im_i}(N,m,v) - \Psi_{jm_j}(N,m,v).\end{aligned}$$

So,

$$\Phi_{im_i}(N,m,v) - \Psi_{im_i}(N,m,v)$$
$$= \Phi_{jm_j}(N,m,v) - \Psi_{jm_j}(N,m,v) \quad \forall (i,m_i),(j,m_j) \in M^+. \tag{7}$$

Combining (6) and (7) yields

$$\Phi_{ik}(N,m,v) - \Psi_{ik}(N,m,v) = c \quad \forall (i,k) \in M^+,$$

for some constant $c \in \mathbb{R}$. Finally, EFF gives $c=0$, implying that $\Phi(N,m,v) = \Psi(N,m,v)$. $\square$

We say that a solution Ψ on MC satisfies

- *the lower balanced contributions property* (LBC) if for all games $(N,m,v) \in MC$, and all $(i,1),(j,1) \in M^+,\ i \neq j$:

$$\Psi_{i1}(N,m,v) - \Psi_{i1}(N,m-m_je^j,v) = \Psi_{j1}(N,m,v) - \Psi_{j1}(N,m-m_ie^i,v).$$

One can characterize the Shapley value by replacing property UBC with LBC in Theorem 4.3. The proof of the characterization using LBC is similar to that of the characterization using UBC, and is therefore omitted.

Theorem 4.4. *A solution Ψ satisfies EFF, EL, and LBC if and only if $\Psi = \Theta$.*

Consider the following property for a solution Ψ on MC.

- *the strong balanced contributions property* (SBC): for all games $(N,m,v) \in MC$, and all $(i,k_i),(j,k_j) \in M^+, i \neq j$:

$$\Psi_{ik_i}(N,m,v) - \Psi_{ik_i}(N,m-(m_j-k_j+1)e^j,v)$$
$$= \Psi_{jk_j}(N,m,v) - \Psi_{jk_j}(N,m-(m_i-k_i+1)e^i,v).$$

Property SBC is stronger than UBC and LBC: if we take $k_i = m_i$ and $k_j = m_j$ in SBC we get UBC, if we take $k_i = k_j = 1$ in SBC we get LBC. One can verify similarly as for UBC in the proof of Theorem 4.3 that Θ satisfies SBC.

Since SBC is stronger than both UBC and LBC, one might want to characterize Θ using only EFF and SBC. This, however, is not possible, as the following example shows.

Example 4.1: We define the solution Υ on MC as follows. Let (N, m, u_s) be a minimal effort game. If there are $i, j \in N$ with $i \neq j$ and $s_i, s_j \geq 1$ then we define

$$\Upsilon(N, m, u_s) := \Theta(N, m, u_s).$$

If there is a $i \in N$ with $s_i \geq 1$ and $s_j = 0$ for all $j \neq i$, then we define

$$\Upsilon_{ik}(N, m, u_s) := \begin{cases} \dfrac{1}{m_i} & \text{if } s_i \geq 1 \text{ and } k \in M_i^+; \\ 0 & \text{if } s_i = 0 \text{ and } k \in M_i^+. \end{cases}$$

Now extend Υ linearly to the class of multi-choice games. Obviously, $\Upsilon \neq \Theta$. One can verify that Υ satisfies EFF and SBC. Hence, EFF and SBC are not sufficient to characterize Θ. $\diamond$

Example 4.1 shows that besides EFF and SBC we need a third property, weaker than EL, to characterize Θ. This property is needed to show that a solution satisfying these three properties coincides with Θ on multi-choice games (N, m, v) with the following property: there exists an $i \in N$ such that $m_i \geq 2$ and $m_j = 0$ for all $j \neq i$. Note that this is accomplished in Theorem 4.3 and Theorem 4.4 using EL. For a characterization with SBC we can accomplish this by taking the restriction of EL to the class of multi-choice games of the form $(N, m_i e^i, v)$. Formally, a solution Ψ on MC satisfies

- *the weak equal loss property* (WEL) if for all games $(N, m, v) \in MC$ with $m = m_i e^i$ for some $i \in N$ and all $(i, k) \in M^+, k \neq m_i$:

$$\Psi_{ik}(N, m, v) - \Psi_{ik}(N, m - e^i, v) = \Psi_{im_i}(N, m, v).$$

Theorem 4.5. *A solution Ψ on MC satisfies EFF, WEL, and SBC if and only if $\Psi = \Theta$.*

Proof. From Theorem 4.1 it follows that Θ satisfies EFF. Since Θ satisfies EL, it also satisfies WEL. Furthermore, we already noticed that Θ satisfies SBC. Hence, Θ satisfies the properties.

To prove uniqueness, suppose that there are two solutions, denoted Φ and Ψ, that satisfy EFF, WEL, and SBC. We will prove that $\Phi = \Psi$. The proof is with induction on the total number of levels $\sum_{k \in N} m_k$. It is clear that for all multi-choice games $(N, m, v) \in MC$ with $\sum_{k \in N} m_k = 0$ we have $\Phi(N, m, v) = \Psi(N, m, v)$. Assume that for some $p \geq 1$ and all multi-choice games $(N, m, v) \in MC$ with $\sum_{k \in N} m_k \geq p - 1$ it holds that $\Phi(N, m, v) = \Psi(N, m, v)$. We will prove that Φ and Ψ also coincide on the class of multi-choice games

$(N, m, v) \in MC$ with $\sum_{k \in N} m_k = p$. Let $(N, m, v) \in MC$ be a multi-choice game with $\sum_{k \in N} m_k = p$. By SBC and the induction hypothesis, we have for all $(i, k_i), (j, k_j) \in M^+$, $i \neq j$ that

$$\begin{aligned}
&\Phi_{ik_i}(N, m, v) - \Phi_{jk_j}(N, m, v) \\
&\quad = \Phi_{ik_i}(N, m - (m_j - k_j + 1)e^j, v) - \Phi_{jk_j}(N, m - (m_i - k_i + 1)e^i, v) \\
&\quad = \Psi_{ik_i}(N, m - (m_j - k_j + 1)e^j, v) - \Psi_{jk_j}(N, m - (m_i - k_i + 1)e^i, v) \\
&\quad = \Psi_{ik_i}(N, m, v) - \Psi_{jk_j}(N, m, v).
\end{aligned}$$

So,

$$\Phi_{ik_i}(N, m, v) - \Psi_{ik_i}(N, m, v) = \Phi_{jk_j}(N, m, v) - \Psi_{jk_j}(N, m, v) \quad \forall (i, k_i), (j, k_j) \in M^+, \quad i \neq j. \tag{8}$$

Let $(i, m_i) \in M^+$ (note that this is possible since $\sum_{k \in N} m_k = p \geq 1$). If there is an agent $j \neq i$ with $m_j > 0$, then it follows from (8) that for all $k, l \in M_i^+$

$$\begin{aligned}
\Phi_{ik}(N, m, v) - \Psi_{ik}(N, m, v) &= \Phi_{j1}(N, m, v) - \Psi_{j1}(N, m, v) \\
&= \Phi_{il}(N, m, v) - \Psi_{il}(N, m, v).
\end{aligned}$$

If there is *not* an agent $j \neq i$ with $(j, m_j) \in M^+$, then it follows from WEL and the induction hypothesis, that for all $(i, k) \in M_i^+$, $k \neq m_i$ we have that

$$\begin{aligned}
\Phi_{ik}(N, m, v) - \Phi_{im_i}(N, m, v) &= \Phi_{ik}(N, m - e^i, v) \\
&= \Psi_{ik}(N, m - e^i, v) \\
&= \Psi_{ik}(N, m, v) - \Psi_{im_i}(N, m, v).
\end{aligned}$$

So,

$$\Phi_{ik}(N, m, v) - \Psi_{ik}(N, m, v) = \Phi_{im_i}(N, m, v) - \Psi_{im_i}(N, m, v) \quad \forall (i, k) \in M^+.$$

Hence, in both cases we have that for all $k, l \in M_i^+$

$$\Phi_{ik}(N, m, v) - \Psi_{ik}(N, m, v) = \Phi_{il}(N, m, v) - \Psi_{il}(N, m, v).$$

Together with (8) this gives

$$\Phi_{ik}(N, m, v) - \Psi_{ik}(N, m, v) = c \quad \forall (i, k) \in M^+,$$

for some constant $c \in \mathbb{R}$. Finally, EFF gives $c = 0$, implying that $\Phi(N, m, v) = \Psi(N, m, v)$. $\square$

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