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The existence of TU α-core in normal form games

Jingang Zhao*

Department of Economics, Ohio State University, 1945 North High Street, Columbus, OH 43210-1172, USA (e-mail: Zhao.18@Osu.Edu)

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Abstract. This paper provides a TU α-core existence result in a large class of normal form games. In the oligopoly markets of a homogeneous good, the TU α-core is non-empty if all profit functions are continuous and concave. In a general game, the existence of TU α-core follows from the weak separability, the compactness and convexity of choice sets, and the concavity and continuity of payoff functions.

Key words: Core existence, α-core, hybrid solution, transferable utility

1. Introduction

This paper studies the α-core for transferable utility (TU) games in normal form. Such study might provide insights into the question of "how to split joint profits among firms" because oligopoly markets are a special class of TU games in normal from. The TU α-core is obtained by using side payments or transfer payments, where the players in the grand coalition first maximize their joint payoffs and then decide how to split their maximal joint payoffs. This contrasts sharply with the NTU (non-transferable utility) α-core, obtained without using any side payment, where players in the grand coalition can only coordinate their strategies and they do not maximize nor split their joint payoffs.

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In oligopoly markets, cooperation with side payments can be interpreted as overt collusion, and cooperation without side payments as covert collusion. Thus, TU α-core describes the cooperative outcomes (i.e., allocations of the monopoly profits) in a monopoly merger, and NTU α-core describes the collusive outcomes for the grand cartel (or the cartel of all firms, usually illegal)[1]. Consequently, the existence of TU α-core (NTU α-core) can be understood as a necessary condition for monopoly merger (formation of the grand cartel). In other words, monopoly merger (formation of the grand cartel) can only take place if the TU α-core (NTU α-core) is non-empty[2].

The general existence of NTU α-core is established by Scarf (1971): A game in normal form has a non-empty NTU α-core if (a) all strategy sets are compact and convex, (b) all payoff functions are continuous and quasi-concave. This paper extends such NTU α-core theorem to a TU α-core theorem as follows: A game in normal form has a non-empty TU α-core if (a) all strategy sets are compact and convex, (b) all payoff functions are continuous and *concave*, and (c) the game satisfies weak separability. Since a coalition with side payments has larger blocking power, it is harder for a payoff vector to be unblocked with side payment than without side payments. This is the reason why stronger conditions are required for the existence of TU α-core.

The rest of the paper is organized as follows. The next section describes the problem and defines the α-core concept. Section 3 provides sufficient conditions for the existence of TU α-core and then extends the result to hybrid solutions. Section 4 concludes, and the appendix provides all proofs.

2. Description of the problem

Throughout the paper superscripts in small letters denote individual players, and subscripts in capital letters denote coalitions. An n-person game in normal form is given by

$$\Gamma = \{N, X^i, u^i\}, \tag{1}$$

where $N = \{1, 2, \ldots, n\}$ is the set of players. For each $i \in N$, X^i (a non-empty subset in $\mathbf{R}^{m_i}$) is i's strategy set, $u^i : X \to \mathbf{R}$, is i's payoff function, where $X = \Pi_{i=1}^n X^i$.

We shall focus on a large class of these games, called TU games[3] in normal form, in which the payoffs can be transferred among players. These become the oligopoly markets when $X^i = [0, \bar{y}^i]$ is i's production set ($\bar{y}^i > 0$ is i's capacity), $u^i = p(\Sigma x^j)x^i - c^i(x^i)$ is i's profit, where $p(\Sigma x^j)$ is the inverse demand function, and $c^i(x^i)$ is i's cost function.

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[1] Cartel members can get away with "colluding on choices" because regulators can not prove collusion without evidence like a contract or side payments.

[2] Otherwise, any split of monopoly profits (efficient choice of the grand cartel) has some blocking coalitions. Since firms in these blocking coalitions can do better themselves, they will object to the merger contract (cartel agreement), so the monopoly merger (the grand cartel) will not be formed.

[3] Note that TU games are usually given as $\Gamma^{CF} = \{N, V(\cdot)\}$, where the superscript CF refers to either characteristic form or coalition function form. This game is a function from the subsets of N to $\mathbf{R}$, which specifies a joint payoff $V(S)$ for each $S \subseteq N$. Because there are no strategies in this game form, it can not be directly applied to economic problems like oligopoly markets. However, TU games in normal form can be readily applied in these situations.

The following notations are useful in defining the core concept. Let $\mathcal{N}$ denote the set of all coalitions (the set of all nonempty subsets of N). For each $S \in \mathcal{N}$, let $|S|$ be the number of players in S, and $\mathbf{R}^S$ denote the $|S|$-dimensional Euclidean space whose coordinates have as superscripts the members in S. For any $x = \{x^1, \ldots, x^n\} \in X$, where $x^i = \{x^i_1, \ldots, x^i_{m_i}\} \in X^i$ $(i = 1, \ldots, n)$, let $x_S = \{x^i \mid i \in S\} \in X_S = \Pi_{i \in S} X^i$ be the strategies of a coalition S, and $x_{-S} = \{x^i \mid i \notin S\} \in X_{-S} = \Pi_{i \notin S} X^i$ be the strategies of the players in the complementary coalition $N \backslash S$. Similar notation applies to any $u = \{u^1, \ldots, u^n\} \in \mathbf{R}^n$.

We shall write $x_N = x$ and $u_N = u$ for simplicity. For any two vectors u, v, $u \geq v \Leftrightarrow u^i \geq v^i$, all i; $u > v \Leftrightarrow u \geq v$ and $u \neq v$; and $u \gg v \Leftrightarrow u^i > v^i$, all i.

For each S, the outsiders' *punishment function*, $\bar{y}_{-S}(x_S)$, is the *minimal solution* in[4]

$$\bar{u}_S(x_S) = \operatorname*{Min}_{y_{-S} \in X_{-S}} \Sigma_{i \in S}\, u^i(x_S, y_{-S}) = \Sigma_{i \in S}\, u^i(x_S, \bar{y}_{-S}(x_S)), \tag{2}$$

where $\bar{u}_S(x_S)$ is called the *guaranteed payoff function* of S. In the definition, (x_S, y_{-S}) denotes a vector $z \in X$ such that $z^i = x^i$ if $i \in S$ and $z^i = y^i$ if $i \notin S$. Now, a coalition's payoff in *the α-core fashion* is given by

$$\begin{aligned} v_\alpha(S) &= \operatorname*{Max}_{x_S \in X_S} \bar{u}_S(x_S) = \bar{u}_S(\tilde{x}_S) = \Sigma_{i \in S}\, u^i(\tilde{x}_S, \bar{y}_{-S}(\tilde{x}_S)) \\ &= \operatorname*{Max}_{x_S \in X_S} \operatorname*{Min}_{y_{-S} \in X_{-S}} \Sigma_{i \in S}\, u^i(x_S, y_{-S}), \end{aligned} \tag{3}$$

where $\tilde{x}_S$ is the maximal solution. This defines a *coalition function form game* derived in *the α-core fashion* from the original game (1) as follows:

$$\Gamma_\alpha^{CF} = \{N, v_\alpha(\cdot)\}, \tag{4}$$

and any core vector of (4) is equivalent to the α-core (Aumann, 1959):

Definition 1: *A TU α-core solution of the game (1) is any pair of joint strategy and payoff allocation* $(\bar{x}, \sigma)$ *such that (a)* $\Sigma_{i \in N} u^i(x) \leq \Sigma_{i \in N}\, u^i(\bar{x}) = \Sigma_{i \in N}\, \sigma^i$ *for all* $x \in X$; *and (b) for each* S, $\Sigma_{i \in S}\, \sigma^i \geq v_\alpha(S)$.

In other words, $(\bar{x}, \sigma)$ is in the α-core if (a) $\bar{x}$ maximizes the grand coalition's joint payoff, and (b) there is no S having $x_S \in X_S$ such that $\Sigma_{i \in S}\, u^i(x_S, z_{-S}) > \Sigma_{i \in S} \sigma^i$ for all $z_{-S} \in X_{-S}$. In an oligopoly market, the above definition incorporates two cooperative actions observed in a horizontal merger: (i) Reorganizing production to maximize joint profit (represented by the optimal choice $\bar{x}$), and (ii) Agreeing on a take-over price (i.e., the split of merger benefits, as represented by the profit allocation σ).

On the other hand, if the TU α-core is empty in game (1), then for any payoff vector σ satisfying $\Sigma_{i \in N}\, \sigma^i = \Sigma_{i \in N}\, u^i(\bar{x})$, there is always a coalition S with some $x_S \in X_S$ such that $\Sigma_{i \in S}\, u^i(x_S, z_{-S}) > \Sigma_{i \in S}\, \sigma^i$ for all $z_{-S} \in X_{-S}$.

[4] We assume that all Min/Max problems have optimal solutions. This holds under the usual conditions that all u^i are continuous and all X^i are compact. Otherwise, we should replace the Max by Sup and Min by Inf.

Remark 1: The above TU α-core is defined both by a joint strategy and by a feasible payoff vector, while the core for games in coalition function form is defined only by a payoff vector. However, the NTU α-core can be defined either by a joint strategy or by a feasible payoff vector.

Remark 2: A merited criticism for the above TU α-core is that a coalition's value $v_\alpha(S)$ is pessimistically small, because players in the coalition worry about the worst outcome when attempting to deviate (or to improve for themselves). One consequence of such pessimistic assumption is the largeness of the α-core. However, such criticism (i.e., "α-core is too large") reinforces, rather than harms, the claim that the existence of TU α-core (NTU α-core) is a necessary condition for horizontal merger (formation of grand cartel). Should the pessimistic $v_\alpha(S)$ be replaced by larger values, the coalition is more likely to block a payoff vector, and the necessary condition would become "more necessary."

3. The existence of TU α-core

This section first introduces the concept of weak separability, which is a key element in our sufficient conditions. Then it provides the main existence result and extensions to oligopoly markets.

Definition 2: *(i) For each coalition $T \in \mathcal{N}$, its guaranteed payoff function $\bar{u}_T(x_T)$ defined in (2) is* **weakly separable** *at a point x_T if for all $i \in T$,*

$$u^i(x_T, \bar{y}_{-T}(x_T)) = \operatorname*{Min}_{y_{-T} \in X_{-T}} u^i(x_T, y_T).$$

(ii) Its payoff in the α-core fashion (i.e., $v_\alpha(T)$ given by (3)) is **weakly separable** *if its guaranteed payoff function $\bar{u}_T$ is weakly separable at the maximal solution $\tilde{x}_T$.*

(iii) The game (1) satisfies **weak separability**[5] *if for each $T \in \mathcal{N}$ and $T \neq N$, $v_\alpha(T)$ given by (3) is weakly separable.*

The above weak separability assumes that for each $T \neq N$, all the following $|T| + 1$ functions of $z_{-T} : \Sigma_{i \in T} u^i(\tilde{x}_T, y_{-T})$, and $u^i(\tilde{x}_T, y_{-T})$, all $i \in T$, reach their minimum at the same point $\bar{z}_{-T} = \bar{y}_T(\tilde{x}_T)$, where x_{-T} is fixed at $\tilde{x}_T$. In other words, the outside choice $\bar{z}_{-T}$ that best punishes the coalition as whole (i.e., minimizing $\Sigma_{i \in T} u^i(\tilde{x}_T, y_{-T})$) also best punishes each individual player in the coalition (i.e., minimizing $u^i(\tilde{x}_T, y_{-T})$ for each $i \in T$).

Theorem 1: *The game (1) has non-empty TU α-core if the following three conditions hold: (i) It satisfies the weak separability; (ii) For each $i \in N$, X^i is compact and convex, and $u^i(x)$ is continuous; and (iii) For each i, $u^i(x)$ is concave in x.*

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[5] Recently, the weak separability has been extended to the strong separability in Zhao (1996a), and Theorem 1 becomes a TU β-core theorem when weak separability is replaced by strong separability and all other conditions are kept unchanged. Note that T (instead of S) is used here in order to reduce confusion in proofs, where a balanced collection is given by $\mathcal{B} = \{T_1, \ldots, T_k\}$, and coalition structure by $\Delta = \{S_1, \ldots, S_k\}$.

This theorem and the weak separability are illustrated in the following Example 1.

Example 1: Let $N = \{1,2,3\}, X^i = [0,1]$, all i, $u^1(x^1,x^2,x^3) = 2 - |x^1 - x^2|$, and $u^2(x) = u^3(x) \equiv 1$. The payoffs in the α-core fashion are: $v(1) = 1.5$, $v(2) = v(3) = 1$, $v(12) = 3$, $v(13) = 2.5$, $v(23) = 2$, and $v(123) = 4$ (For $S = 2,3,12,23$, and 123, the computation is straight forward. For $S = 1$, and 13, the computation is more involved). Therefore, the set of α-core allocations is

$$\mathrm{Co}(\Gamma_\alpha^{CF}) = \{(1.5 + t^1, 1 + t^2, 1) \mid t^1 + t^2 = 0.5, t \in \mathbf{R}_+^2\}.$$

The next task is to check the conditions of Theorem 1. Part (ii) is obvious. Part (iii) holds because $u^1(x^1,x^2,x^3) \equiv 2 - |x^1 - x^2|$ is a $\wedge$-shaped linear function. The weak separability for $S = 1,2,3,12,23$, and 123 are obvious. So we only need to check the weak separability for $S = 13$. By

$$\underset{y_{-S}\in X_{-S}}{\mathrm{Min}}\ \Sigma_{i\in S}\, u^i(x_S, y_{-S}) = \underset{y^2}{\mathrm{Min}}\{3 - |x^1 - y^2|\} = \begin{cases} 3-(1-x^1) \text{ if } i \le x^1 \le 1/2 \\ 3 - x^1 \text{ if } 1/2 < x^1 \le 1 \end{cases},$$

we have

$$v_\alpha(13) = \underset{x_{13}}{\mathrm{Max}}\ \underset{y^2}{\mathrm{Min}}\{3 - |x^1 - y^2|\} = \underset{x_1}{\mathrm{Max}}\begin{cases} 3-(1-x^1) \text{ if } 0 \le x^1 \le 1/2 \\ 3 - x^1 \text{ if } 1/2 < x^1 \le 1 \end{cases} = 2.5,$$

where $\tilde{x}_S = (\tilde{x}^1, \tilde{x}^3) = (0.5, x^3)$ for any x^3 $(0 \le x^3 \le 1)$, and $\bar{y}^2(\tilde{x}_S) = 0$ or 1. It follows from

$$u^1(\tilde{x}_S, \bar{y}^2(\tilde{x}_S)) = u^1(0.5, 0, x^3) = 1.5 = \underset{y^2}{\mathrm{Min}}\{2 - |0.5 - y^2|\})$$

$$= \underset{y^2}{\mathrm{Min}}\ u^1(\tilde{x}_S, y^2), \quad \text{and}$$

$$u^3(\tilde{x}_S, \bar{y}^2(\tilde{x}_S)) \equiv 1 = \underset{y^2}{\mathrm{Min}}\ u^3(\tilde{x}_S, y^2)$$

that the weak separability is satisfied.

As can be seen in the next theorem, one can remove the weak separability assumption in an oligopoly market without harming the existence of TU α-core.

Theorem 2: *Consider an oligopoly market $\Gamma = \{N, X^i, \pi^i\}$ defined by $X^i = [0, \bar{y}^i]$ and $\pi^i(x) = p(\Sigma x^j)x^i - c^i(x^i)$. Its TU α-core is non-empty if (i) The inverse demand function $p(\Sigma x^i)$ is decreasing; and (ii) For each $i \in N$, its profit function $\pi^i(x)$ is continuous and concave in x.*

Remark 3: As discussed earlier, it is harder for a payoff vector to be unblocked with side payments than without side payments, and this is the reason why the above TU α-core conditions are stronger than the known NTU α-core conditions (i.e., Scarf, 1971). However, this does not imply that the set of TU α-core payoffs is included in that for NTU α-core. In fact, as illustrated in the next example and in Figure 1, there is no general inclusion relation between TU α-core and NTU α-core.

Example 2. Consider a duopoly industry: $P = 6 - x^1 - x^2$, $\bar{y}^1 = 3$, $\bar{y}^2 = 4$, $c^1(x^1) = x^1$, $c^2(x^2) = 2x^2$. The two guaranteed profit functions are $\bar{\pi}^1(x^1) = [1 - x^1]x^1$, and $\bar{\pi}^2(x^2) = [1 - x^2]x^2$, both attaining a maximum value of 1/4 at $\bar{x}^1 = \bar{x}^2 = 1/2$. Let

$$E = \{(\pi^1, \pi^2) \mid (\pi^1, \pi^2) \text{ is an NTU efficient payoff vector}\}, \quad \text{and}$$

$$F = \{(\pi^1, \pi^2) \in \mathbf{R}^2_+ \mid \pi^1 + \pi^2 = 6.25\}$$

denote respectively the NTU and TU efficient profit frontiers, then

$$\alpha - \text{core}_{\text{NTU}} = \{(\pi^1, \pi^2) \in E \mid \pi^1 \geq \tfrac{1}{4}, \pi^2 \geq \tfrac{1}{4}\}, \quad \text{and}$$

$$\alpha - \text{core}_{\text{TU}} = \{(\pi^1, \pi^2) \in E \mid \pi^1 \geq \tfrac{1}{4}, \pi^2 \geq \tfrac{1}{4}\},$$

which are represented in Figure 1. As shown in the figure, the NTU and TU α-cores do not include one another.

Next, we extend Theorems 1 and 2 to hybrid solutions. A given coalition structure (i.e., a partition of N) $\Delta = \{S_1, S_2, \ldots, S_k\}$ induces k parametric normal form TU games:

$$\Gamma_S(x_{-S}) = \{S, X^i, u^i(\cdot, x_{-S})\} \tag{5}$$

for each $S = S_1, \ldots, S_k$. For each $S \in \Delta$, the players in S will first maximize their joint payoff for each fixed x_{-S}, and then decide how to distribute the payoffs among themselves.

Such environment contains both the element of cooperative behavior and the element of non-cooperative behavior: Different coalitions behave non-cooperatively across the coalitions, but within each coalition the players co-

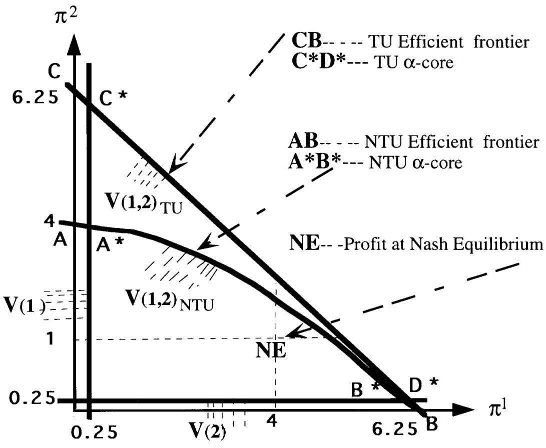

Fig. 1. The α-core payoffs in 2-person TU and NTU games

operate among themselves. The equilibria in these situations are called the hybrid equilibria (Zhao, 1992 and 1996b). A hybrid equilibrium becomes the Nash equilibrium if the partition is the finest, and a cooperative solution if the partition is the coarsest.

Because there exists no best spilt for a coalition, one needs to specify how a coalition S is going to distribute the joint payoffs as defined in (5). Suppose a coalition distributes its maximal joint payoff by using either the NTU or the TU α-core[6], then a distribution rule (DR) for a coalition structure $\Delta = \{S_1, S_2, \ldots, S_k\}$ specifies that a coalition $S \in \Delta$ distributes its joint payoffs according to $\mathrm{DR}(S) =$ TU α-core or = NTU α-core (i.e., DR is a single valued map from the set of all partitions to {TU α-core, NTU α-core}).

Now, the concept of a *hybrid solution with a distribution rule* (or HSDR for short) can be given as:

Definition 3: *Given a coalition structure $\Delta = \{S_1, S_2, \ldots, S_k\}$ and a DR for Δ in the game (1). The* **HSDR** *for Δ is a pair of strategy profile and payoff vector $(\bar{x}, \sigma)$ such that for each $S \in \Delta$, (i) $\bar{x}_S$ is a solution of* $\mathrm{Max}\{\Sigma_{i \in S} u^i(x_S, \bar{x}_{-S}) \mid x_S \in X_S\}$; *and (ii) σ_S splits the joint payoff $\Sigma_{i \in S} u^i(\bar{x}) = \Sigma_{i \in S} \sigma^i$ according to DR(S).*

In other words, a HSDR is reached if each S maximizes its payoff given the outside choice and distributes its payoff according to $\mathrm{DR}(S)$. Clearly, HSDR becomes the Nash equilibrium if Δ is the finest partition (i.e., $\Delta = \{(1), (2), \ldots, (n)\}$), the TU α-core if Δ is the coarsest partition (i.e., $\Delta = \{(1, \ldots, n)\}$) and $\mathrm{DR}(N) =$ TU α-core, and the NTU α-core if Δ is the coarsest partition and $\mathrm{DR}(N) =$ NTU α-core. Thus, the concept of HSDR serves as a connection between the non-cooperative and the cooperative approaches.

Theorem 3: *Given a coalition structure $\Delta = \{S_1, S_2, \ldots, S_k\}$ and a DR for Δ in the game (1). There is at least one HSDR for Δ if (i) X^i is compact and convex, and $u^i(x)$ is continuous for all i; (ii) For each $S \in \Delta$ with $DR(S) =$ NTU α-core, all $u^i(x_S, x_{-S}), i \in S$, are quasi-concave in x_S; and (iii) For each $S \in \Delta$ with $DR(S) =$ TU α-core, all $u^i(x_S, x_{-S}), i \in S$, are concave in x_S, and the game $\Gamma_S(x_{-S})$ defined by (5) satisfies weak separability for all x_{-S}.*

Corollary 1: *Given a coalition structure $\Delta = \{S_1, S_2, \ldots, S_k\}$ and a DR in an oligopoly game $\Gamma = \{N, X^i, \pi^i\}$. There is at least one HSDR for Δ if (i) $\pi^i(x)$ is continuous for all i; (ii) For each $S \in \Delta$ with $DR(S) =$ NTU α-core, all $\pi^i(x_S, x_{-S})$, $i \in S$, are quasi-concave in x_S; and (iii) For each $S \in \Delta$ with $DR(S) =$ TU α-core, the inverse demand function is decreasing, and all $\pi^i(x_S, x_{-S})$, $i \in S$, are concave in x_S.*

Remark 4: Theorem 3 becomes Scarf's theorem on NTU α-core if Δ is the coarsest partition and $\mathrm{DR}(N) =$ NTU α-core, the earlier Theorem 1 on TU α-core if Δ is the coarsest partition and $\mathrm{DR}(N) =$ TU α-core, and the existence theorem of Nash equilibrium if Δ is the finest partition.

[6] This assumption is rather restrictive, because a coalition could use other cooperative solutions (like TU β-core, bargaining solutions or values). By allowing a coalition to use different distribution rules and by considering the stability of partitions, it is possible to predict which cooperative solution is likely to be adopted. More discussions can be found in Zhao (1996b).

4. Concluding remarks

We have established the general existence of TU α-core: It only requires the compactness and convexity of choice sets, the concavity and continuity of payoff functions, and the weak separability of the game. When applied to an oligopoly market, our result updates Scarf's NTU α-core theorem to a TU α-core theorem at the modest cost of changing quasi-concave functions to concave ones. The TU α-core conditions are stronger than those for NTU α-core because a coalition with side payments has larger blocking power and thus makes it more difficult for a payoff vector to be unblocked.

We also extended the TU α-core result to hybrid solutions. One future topic is to study the stability of coalition structure and to predict which partition is more likely to exist: The finest partition (i.e., the Nash equilibrium)? The coarsest partition (i.e., the core)? Or a general partition (i.e., a non-trivial hybrid solution)?

Appendix

A necessary and sufficient condition for core existence in the game (4) is its balancedness. Let $\mathscr{B} = \{T_1, \ldots, T_k\}$ be a collection of coalitions. For each $i \in N$, let $\mathscr{B}(i) = \{T \in B \mid i \in T\}$ denote the set of coalitions of which i is a member, $\mathscr{B}$ is a balanced collection if there are $w_T = 0$ for each $T \in \mathscr{B}$ such that $\Sigma_{T \in \mathscr{B}(i)} w_T = 1$ for all i. A game $\Gamma = \{N, V(\cdot)\}$ is balanced if $\Sigma_{S \in \mathscr{B}} w_S V(S) \leq V(N)$ for any balanced collection $\mathscr{B}$, with w_S for each $S \in \mathscr{B}$.

Proof of Theorem 1: It follows from the assumptions that the product of all players' strategy sets is convex and compact, and that the grand coalition's joint payoff is continuous. Thus there exists an $\bar{x}$ that maximizes the grand coalition's joint payoff.

The existence of a TU α-core is equivalent to the core existence in the TU game $\Gamma_\alpha^{CF} = \{N, v_\alpha(S)\}$ defined in (4). Thus, we only need to show $\Sigma_{S \in \mathscr{B}} w_S V(S) \leq V(N)$ for any balanced collection $\mathscr{B} = \{T_1, T_2, \ldots, T_k\}$ with weight w_T for each $T \in \mathscr{B}$ (note that $V(S) = v_\alpha(S)$). It follows from (2) and (3) that $V(N) = \operatorname*{Max}_{x \in X} \Sigma_{i \in N} u^i(x) = \Sigma_{i \in N} u^i(\bar{x})$, and

$$\begin{aligned} \Sigma_{S \in \mathscr{B}} w_S V(S) &= \Sigma_{S \in \mathscr{B}} w_S \Sigma_{i \in S} u^i(x_S^*, \bar{y}_{-S}(x_S^*)) \\ &= \Sigma_{i=1}^n \Sigma_{S \in \mathscr{B}(i)} w_S u^i(x_S^*, \bar{y}_{-S}(x_S^*)), \end{aligned} \tag{P1}$$

where x_S^* is the solution to the maximization problem (3), and $\bar{y}_{-S}(x_S)$ is the solution to the minimization problem (2). For each $i \in N$, let

$$x^i = \Sigma_{S \in \mathscr{B}(i)} w_S x_S^*(i) \in X^i, \quad \text{and}$$

$$x = \{x^1, \ldots, x^n\} \in X = \prod_{i \in N} X^i \tag{P2}$$

where $x_S^*(i)$ is the i-th component of x_S^*. We shall show that

$$\Sigma_{S \in \mathscr{B}(i)} w_S u^i(x_S^*, \bar{y}_{-S}(x_S^*)) \leq u^i(x) \tag{P3}$$

for each player i. Without loss of generality, we only need to show the above inequality for player 1, as the arguments for other players are essentially the same. In order to show

$$\Sigma_{S \in \mathscr{B}(1)} w_S u^1(x_S^*, \bar{y}_{-S}(x_S^*)) \leq u^1(x),$$

we shall define, for each $S \in \mathscr{B}(1) = \{T \in \mathscr{B} \mid 1 \in T\}$, $y(S) = (x_S^*, z_{-S}) \in X = \prod_{i \in N} X^i$ as follows. For the fixed $S \in \mathscr{B}(1)$ and each $i \in N \backslash S$, let

$$z^i = \Sigma_{T \in \mathscr{B}(i) \backslash \mathscr{B}(1)} \hat{w}_T x_T^*(i) \in X^i,$$

and for each T in the above summation, $\hat{w}_T = w_T / (\Sigma_{E \in \mathscr{B}(i) \backslash \mathscr{B}(1)} w_E)$. It follows from weak separability, the concavity of u^1 and the following equality

$$x = \Sigma_{S \in \mathscr{B}(1)} w_s y(S) \quad \text{(see Scarf, 1971)}$$

that

$$\Sigma_{S \in \mathscr{B}(1)} w_S u^1(x_S^*, \bar{y}_{-S}(x_S^*)) \leq \Sigma_{S \in \mathscr{B}(1)} w_S u^1(y(S)) \leq u^1(x).$$

By (P1) and (P3),

$$\begin{aligned}\Sigma_{S \in \mathscr{B}} w_S V(S) &= \Sigma_{i=1}^n \Sigma_{S \in \mathscr{B}(i)} w_S u^i(x_S^*, \bar{y}_{-S}(x_S^*)) \\ &\leq \Sigma_{i=1}^n u^i(x) \leq \Sigma_{i=1}^n u^i(\bar{x}) = V(N).\end{aligned}$$

Thus the game is balanced. **Q.E.D.**

Proof of Theorem 2: Since the inverse demand function $p(\Sigma x^j)$ is decreasing (see the paragraph after (1)), the outsiders' capacities $\bar{y}_{-T}$ minimizes both the joint profit $\pi_T = \Sigma_{i \in T}[P(\Sigma x^j)x^i - c^i(x^i)]$ and each i's profit $\pi^i = p(\Sigma x^j)x^i - c^i(x^i)$ $(i \in T)$ for all x_T. This observation leads directly to the weak separability. Thus, all conditions of Theorem 1 are satisfied, and the TU α-core is non-empty. **Q.E.D.**

Proof of Theorem 3: For each coalition $S \in \Delta$, let $\delta_S(x_{-S})$ denote its set of optimal responses given the complementary choice x_{-S}, that is, the set of solutions to the optimization problem

$$\underset{x_S \in X_S}{\text{Max}} \; \Sigma_{i \in S} u^i(x_S, x_{-S})$$

for a fixed x_{-S}. By the three conditions, the correspondence $\delta : X \to 2^X$, defined by

$$\delta(x) = \prod_{S \in \Delta} \delta_S(x_{-S}) = \{\delta_{S_1}(x_{-S_1}) \times \cdots \times \delta_{S_k}(x_{-S_k})\}$$

for each $x = \{x^1, \ldots, x^n\} = \{x_{S_1}, \ldots, x_{S_k}\} \in X = \prod_{i \in N} X^i$, satisfies the conditions of Kakutani's fixed point theorem, thus it has a fixed point $\bar{x}$ such that for each coalition $S \in \Delta$, $\bar{x}_S$ maximizes its joint payoff given $\bar{x}_{-S}$. If $\mathrm{DR}(S) =$ TU α-core, the parametric game $\Gamma_S(\bar{x}_{-S})$ of (5) satisfies the conditions of Theorem 1, so it has a TU α-core payoff vector σ_S. If $\mathrm{DR}(S) =$ NTU α-core, the parametric game $\Gamma_S(\bar{x}_{-S})$ satisfies the conditions of Scarf's theorem (1971), so it has an NTU α-core payoff vector σ_S. Let

$$\sigma = \{\sigma^1, \ldots, \sigma^n\} = \{\sigma_S \mid S \in \Delta\} \in \mathbf{R}^n,$$

then $(\bar{x}, \sigma)$ is a HSDR for the coalition structure Δ. **Q.E.D.**

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