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Characterization sets for the nucleolus*

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Abstract. We introduce the concept of a characterization set for the nucleolus of a cooperative game and develop sufficient conditions for a collection of coalitions to form a characterization set thereof. Further, we formalize Kopelowitz's method for computing the nucleolus through the notion of a sequential LP process, and derive a general relationship between the size of a characterization set and the complexity of computing the nucleolus.

Key words: Cooperative game, nucleolus, strongly polynomial algorithms, minimum cost spanning tree games

1. Introduction

The definition of the nucleolus of a cooperative game in characteristic function form entails comparisons between vectors of exponential length. Thus, if one attempts to compute the nucleolus by simply following its definition, it would take an exponential time. However, it is well known that one can compute the nucleolus of many classes of games in polynomial time. Indeed, let n denote the total number of players in a game. Littlechild [1974] has developed an $O(n^2)$ algorithm for computing the nucleolus of an airport game, Granot and Granot [1992] have developed a strongly polynomial algorithm for computing the nucleolus of a fixed cost spanning forest game, Solymosi and Raghavan [1994] constructed an $O(n^4)$ algorithm for computing the nucleolus of an assignment game, Derks and Kuipers [1992] and Granot, Granot and Zhu [1994] have developed $O(n^5)$ and $O(n^3)$ algorithms, respectively, for

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computing the nucleolus of certain routing games, and Megiddo [1978] and Granot, Maschler, Owen and Zhu [1996] have developed strongly polynomial algorithms for computing the nucleolus of tree games. Most of the above efficient algorithms for computing the nucleolus are based on the observation that the information needed to completely characterize the nucleolus for some classes of games is much less than that dictated by its definition.

In this paper, we formalize the above approach for general cooperative games. We provide a uniform scheme, or a general approach, for seeking efficient algorithms for computing the nucleolus. The concept of a *characterization set* (see Section 2 for definition) embodies the notion of 'minimum' relevant information needed to characterize the nucleolus of a class of games. The notion of a *sequential linear programming (LP) process* (see Section 4 for definition), which is a formalization of Kopelowitz's method (Kopelowitz [1967], Maschler, Peleg and Shapley [1979]), provides a generic algorithm for computing the nucleolus. The input to the sequential LP process is a characterization set and the values of the cost function for coalitions contained therein. Its output is the prenucleolus/nucleolus of the corresponding game. The size of the first LP problem in the sequential LP process is proportional to the size of the characterization set. Therefore, to develop an efficient algorithm for computing the nucleolus of a game, one attempts to find the smallest characterization set for the nucleolus. Additional efficiency can sometimes be attained by making use of the structure of the characterization set and/or the characteristic function. In fact, most of the papers cited above follow this line of development.

The plan of this paper is as follows. In Section 2 we introduce the concept of a characterization set for the nucleolus of a game. Therein we provide sufficient conditions for a family of coalitions to form a characterization set. In Section 3 we prove that the class of *irreducible saturated coalitions* (see definition therein) forms a characterization set for the nucleolus of a monotone game having a nonempty core.

In Section 4 we show that if the nucleolus of a game has a characterization set of size polynomially bounded in the total number of players, then the nucleolus of this game can be computed in strongly polynomial time (see definition of *strong polynomiality* therein). Characterization sets for some classes of games, previously studied in the literature, are described in Section 5.

2. Characterization sets for the nucleolus

Let $\Gamma = (N; c)$ be a cooperative game in characteristic function form, where N is a finite set of players and c is a real valued function defined on 2^N, with $c(\emptyset) = 0$. Subsets of N are called *coalitions* and the characteristic function could be interpreted as either the cost, or the revenue of forming the various coalitions. In this paper, we interpret c as a cost function. Thus, in the sequel we often refer to a cooperative game in characteristic function form as a *cost game*. We emphasize, however, that this assumption about the interpretation of the characteristic function does not induce any loss of generality, since analogous results are valid also for *revenue games*.

Let us briefly review a few notation. The set of all pre-imputations (resp., imputations) of Γ is denoted by $X^*(\Gamma)$ (resp., $X(\Gamma)$). Thus, $X^*(\Gamma) = \{x : \sum_{i=1}^n x_i = c(N)\}$ and $X(\Gamma) = \{x : \sum_{i=1}^n x_i = c(N), x_i \le c(\{i\}), i = 1, \dots, n\}$.

The *excess* of coalition S at $x \in \boldsymbol{R}^N$ is defined by $e(S,x) = c(S) - x(S)$, where $x(S) = \sum_{k \in S} x_k$. For a game $\Gamma = (N;c)$ and $x \in \boldsymbol{R}^N$, let $\theta(x;\Gamma)$ be the $|2^N|$-dimensional vector whose components are the values of the excess function $e(S,x)$, for $S \in 2^N$, arranged in a nondecreasing order. Let $\geq_l$ be the "lexicographically greater than" relationship between vectors of the same dimension, and let $X_0 \subseteq \boldsymbol{R}^N$. The *nucleolus of Γ with respect to X_0* is given by,

$$\mathcal{N}(\Gamma, X_0) \overset{\text{def}}{=} \{x \in X_0 \colon \theta(x;\Gamma) \geq_l \theta(y;\Gamma), \forall y \in X_0\}.$$

If $X_0 = X^*(\Gamma)$, $\mathcal{N}(\Gamma, X_0)$ is called the *prenucleolus* of Γ and is denoted by $\mathcal{PN}(\Gamma)$, and if $X_0 = X(\Gamma)$, $\mathcal{N}(\Gamma, X_0)$ is called the *nucleolus* of Γ and is denoted by $\mathcal{N}(\Gamma)$.

It is well known that if X_0 is nonempty and compact then $\mathcal{N}(\Gamma, X_0) \neq \emptyset$, and if, furthermore, X_0 is convex, then $\mathcal{N}(\Gamma, X_0)$ consists of a single point (for proofs see, for example, Schmeidler [1969]). Similarly, if X_0 is nonempty, compact and convex then the prenucleolus also consists of a unique point. In this paper, we are mainly concerned with the prenucleolus and the nucleolus of a cost game Γ. Therefore, unless otherwise specified, X_0 is taken to be either $X^*(\Gamma)$ or $X(\Gamma)$.

Define *a cost game with coalition formation restrictions* to be a triple $\Gamma^{\mathcal{T}} \overset{\text{def}}{=} (N, \mathcal{T}, c)$, where N is the set of players, $\mathcal{T}$ is a family of subsets of N which consists of all 'permissible coalitions' and c is the characteristic function of $\Gamma^{\mathcal{T}}$, which is a real-valued function defined on $\mathcal{T} \cup \{\{i\}, i \in N\}$. We assume in the sequel that $\mathcal{T}$ contains the grand coalition N. Let $X \subseteq \boldsymbol{R}^N$. Define the nucleolus of $\Gamma^{\mathcal{T}}$ with respect to X by,

$$\mathcal{N}(\Gamma^{\mathcal{T}}, X) = \{x \in X \colon \theta(x, c, \mathcal{T}) \geq_l \theta(y, c, \mathcal{T}), \forall y \in X\},$$

where $\theta(y, c, \mathcal{T})$ is the $|\mathcal{T}|$-dimensional vector whose components are the excesses $e(S, y)$, $S \in \mathcal{T}$, arranged in a nondecreasing order.

Using the notion of a cost game with coalition formation restrictions, we introduce the concept of a *characterization set* for the nucleolus of a cooperative game.

Definition 2.1. *A subset $\mathcal{T}$ of 2^N is called a* ***characterization set****, or a* ***c-set*** *for short, for the nucleolus of a game $\Gamma = (N;c)$ with respect to X_0, if $\mathcal{N}(\Gamma^{\mathcal{T}}, X_0) = \mathcal{N}(\Gamma, X_0)$.*

Two questions regarding c-sets naturally arise. The first one is how to verify whether a family of coalitions forms a *c*-set for the nucleolus of a game (or for a class of games). The second question is whether a smaller *c*-set would lead to a more efficient algorithm for computing the nucleolus. For the remainder of this section and in the next section, we address the first question. We provide some sufficient conditions for a collection of coalitions to form a *c*-set for the nucleolus. The second question is considered in Section 4.

Games with coalition formation restrictions were studied in Maschler, Potters and Tijs [1992]. The following theorem is an important special case of one of their results. It also generalizes Kohlberg's [1971] characterization of the nucleolus, and Sobolev's [1975] characterization of the prenucleolus. First, we need to introduce some notation and terminology.

The *characteristic vector*, e_R, of coalition R is the $|N|$-tuple vector whose i-th component is equal to 1 if $i \in R$ and is equal to 0, otherwise. A collection of coalitions, $\mathscr{S}$, is said to be a *balanced collection*, or, for short, *balanced*, if there exist positive coefficients α_S, $S \in \mathscr{S}$, such that $\sum \alpha_S \cdot e_S = e_N$. Finally, $e_T \in \mathrm{Span}\{e_S : S \in \mathscr{S}\}$ denotes that e_T can be expressed as a linear combination of e_S, $S \in \mathscr{S}$. Let $x \in \boldsymbol{R}^N$ be an imputation for Γ, or $\Gamma^{\mathscr{T}}$. Denote by $b_0(x)$ the set of singletons:

$$b_0(x) = \{i \in N : x_i = c(\{i\})\}.$$

Theorem 2.2. (Maschler et al. [1992]) *Let $\Gamma^{\mathscr{T}} = (N, \mathscr{T}, c)$ be a game with coalition formation restrictions. An imputation x belongs to $\mathscr{N}(\Gamma^{\mathscr{T}})$, if for every real number α, for which*

$$\{S \in \mathscr{T} : e(S, x) \le \alpha\} \ne \emptyset,$$

there exists a subset $b_0^\alpha(x)$ of $b_0(x)$ such that

$$b_0^\alpha(x) \cup \{S \in \mathscr{T} : e(S, x) \le \alpha\}$$

is a balanced collection.[1]

A preimputation x is in $\mathscr{PN}(\Gamma^{\mathscr{T}})$ if, for every real number α, the collection $\{S \in \mathscr{T} : e(S, x) \le \alpha\}$ is balanced whenver it is not empty.[2]

From Theorem 2.2 we can infer some collections of coalitions that constitute c-sets:

Theorem 2.3. *Let $\Gamma = (N; c)$ be a cost game and let $\mathscr{T}$ be a subset of 2^N. Let x be contained*[3] *in the nucleolus (resp., prenucleolus) of $\Gamma^{\mathscr{T}}$. The collection $\mathscr{T}$ is a c-set for the nucleolus (resp., prenucleolus) of Γ if for every S in $2^N \backslash \mathscr{T}$ there exists a nonempty subcollection $\mathscr{T}_S$ of $\mathscr{T}$, such that*

(i) $e(S, x) \ge e(T, x)$, whenever $T \in \mathscr{T}_S$,

(ii) $e_S \in \mathrm{Span}\{e_T : T \in \mathscr{T}_S\}$.

The proof requires the following lemma:

Lemma 2.4. *Let $\mathscr{A}$, $\mathscr{A} \subseteq 2^N$, be a balanced collection and let S, $S \notin \mathscr{A}$, satisfy $e_S \in \mathrm{Span}\{e_T : T \in \mathscr{A}\}$. Then $\mathscr{A} \cup \{S\}$ is also balanced.*

Proof. There exist postive constants α_T such that

$$\sum_{T \in \mathscr{A}} \alpha_T e_T = e_N.$$

There exist constants k_T such that

[1] In this case we say that *Kohlberg's condition is satisfied for $\mathscr{T}$, at x.*

[2] In this case we say that *Sobolev's condition is satisfied for $\mathscr{T}$, at x.*

[3] In general, the nucleolus (resp., prenucleolus) of a game with coalition formation restrictions need not consist of a unique point.

$$\sum_{T \in \mathscr{A}} k_T e_T = e_S.$$

Let ε be a small enough positive number such that $\alpha_T - \varepsilon k_T > 0$ for all $T \in \mathscr{A}$. Then

$$\begin{aligned} e_N &= \sum_{T \in \mathscr{A}} \alpha_T e_T + \varepsilon e_S - \varepsilon e_S \\ &= \sum_{T \in \mathscr{A}} (\alpha_T - \varepsilon k_T) e_T + \varepsilon e_S, \end{aligned}$$

which proves that $\mathscr{A} \cup \{S\}$ is balanced. ■

Proof of Theorem 2.3. By condition (ii), $\mathrm{Span}\{e_T : T \in \mathscr{T}\} = \mathbf{R}^N$. Therefore, $\mathscr{N}(\Gamma^{\mathscr{T}}, X_0)$ (resp., $\mathscr{PN}(\Gamma^{\mathscr{T}}, X_0)$) is the singleton $\{x\}$, if $X_0 = X(\Gamma)$ (resp., $X_0 = X^*(\Gamma)$). It remains to show that x is also the nucleolus (resp., prenucleolus) of Γ. Let α be any real number for which the collection $\mathscr{B} \equiv \{S \subseteq N : e(S,x) \leq \alpha\}$ is not empty. If the coalitions in $\mathscr{B}$ which are not in $\mathscr{T}$, are removed, the resulting set is still not empty. Indeed, by (i), there exist coalitions in $\mathscr{T}$ whose excess is smaller than or equal to α. Since x is the nucleolus (resp., prenucleolus) of $\Gamma^{\mathscr{T}}$, the collection $b_0^{\alpha}(x) \cup \{S \subseteq \mathscr{T} : e(S,x) \leq \alpha\}$ (resp., $\{S \subseteq \mathscr{T} : e(S,x) \leq \alpha\}$) is balanced. By (ii), and Lemma 2.4 it will remain balanced if we add to this collection all the coalitions in $2^N \setminus \mathscr{T}$, whose excess is not greater than α. By Kohlberg's (resp., Sobolev's) theorem, x is the nucleolus (resp., prenucleolus) point of Γ. ■

The core of a game $\Gamma = (N; c)$ is defined as $C(\Gamma) = \{x \in R^N : x(S) \leq c(S), \forall S \subset N, x(N) = c(N)\}$.

Remark 2.5. *A priori, it may appear that Theorem 2.3 does not provide us with a convenient tool to get c-sets, because we first have to find the nucleolus/prenucleolus of $\Gamma^{\mathscr{T}}$. This need not be the case. Suppose, for example, we know that the game $\Gamma^{\mathscr{T}}$ has a nonempty core and we can show that $e(S,x) \geq e(T,x)$ whenever $x \in C(\Gamma^{\mathscr{T}})$ and $T \in \mathscr{T}_S$. Then (i) is satisfied automatically for the nucleolus, because the core contains the nucleolus. We shall subsequently provide examples where this reasoning can be employed.*

Remark 2.6. *A related result to Theorem 2.3 was independently derived by Reijnierse [1995]. Therein, he characterized coalitions whose removal will not alter the nucleolus of the original game Γ. By contrast, Theorem 2.3 above characterizes coalitions whose addition to the game with coalition restrictions $\Gamma^{\mathscr{T}}$ will not alter the nucleolus of $\Gamma^{\mathscr{T}}$.*

Let $\Gamma = (N; c)$ be a cost game with a nonempty core. A family of coalitions, $\mathscr{T}$, is said to induce a representation of the core if

$$C(\Gamma) = \{x \in \boldsymbol{R}^N : x(S) \leq c(S), \forall S \in \mathscr{T}; x(N) = c(N)\}.$$

In general, a family of coalitions which induces a representation of the core does not always form a c-set for the nucleolus as demonstrated by the following example, which is essentially due to Maschler et al. [1979].

Example 2.7. Let Γ be the game $(N;c)$ with $N = \{1,2,3,4\}$ and c defined by,

$$c(N) = c(\{1,2,3\}) = c(\{1,2,4\}) = c(\{1,3,4\}) = c(\{2,3,4\}) = 2,$$

$$c(\{1,2\}) = c(\{3,4\}) = c(\{1,4\}) = c(\{2,3\}) = 1,$$

$$c(\{1,3\}) = 3/2, \quad c(\{2,4\}) = 2,$$

$$c(\{i\}) = 1 \quad \text{for all } i \in N,$$

$$c(\emptyset) = 0.$$

Let $\Gamma' = (N;c')$ be the same as Γ, except that $c'(\{1,2,3\}) = 15/8$. It can be shown that $X(\Gamma) = X(\Gamma')$ and that $C(\Gamma) = c(\Gamma') = \{\lambda \cdot (\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{1}{4}) + (1-\lambda) \cdot (0,1,0,1)\colon 0 \leq \lambda \leq 1\}$. Observe that Γ and Γ' only differ by the cost of coalition $\{1,2,3\}$, and both hyperplanes $x(\{1,2,3\}) = c(\{1,2,3\})$ and $x(\{1,2,3\}) = c'(\{1,2,3\})$ do not intersect $C(\Gamma) = C(\Gamma')$. Thus, one could easily verify that coalition $\{1,2,3\}$ is not contained in any efficient representation for $C(\Gamma)$ or $C(\Gamma')$, and that any efficient representation for $C(\Gamma)$ is also an efficient representation for $C(\Gamma')$. Therefore, if every family of coalitions which induces a representation of the core forms a c-set for the nucleolus, we should have that $\mathscr{N}(\Gamma) = \mathscr{N}(\Gamma')$. But, it can be verified e.g., by using Kohlberg's [1971] criterion, that

$$\mathscr{N}(\Gamma) = \left\{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\right\} \quad \text{and} \quad \mathscr{N}(\Gamma') = \left\{\left(\frac{7}{16},\frac{9}{16},\frac{7}{16},\frac{9}{16}\right)\right\}.$$

However, by Theorem 2.3 and Remark 2.5, we can claim:

Corollary 2.8. *Let $(N;c)$ be a cost game with a nonempty core and suppose that $\mathscr{T}$, a proper subset of 2^N, induces a representation of the core. Then, $\mathscr{T}$ is a c-set for $\mathscr{N}(\Gamma, X_0)$ if for every $S \in 2^N \backslash \mathscr{T}$ there exists a $\mathscr{T}_S$, $\mathscr{T}_S \subseteq \mathscr{T}$, such that,*
(i) $e(S,x) \geq e(T,x)$, $\forall T \in \mathscr{T}_S$, $\forall x \in C(\Gamma^{\mathscr{T}})$;[4]
(ii) $e_S \in \mathrm{Span}\{e_T\colon T \in \mathscr{T}_S\}$.

Two games $\Gamma_1 = (N;c_1)$ and $\Gamma_2 = (N;c_2)$ are said to be *strategically equivalent* if there exist a positive real number α and a constant vector $a \in \boldsymbol{R}^N$ such that $c_2(S) = \alpha c_1(S) + a(S)$ for all $S \subseteq N$, where $a(S) = \sum_{k \in S} a_k$. It follows that the concept of a characterization set is covariant under strategic equivalence, because passing from one game to a strategically equivalent one causes all excesses to be multiplied by a positive constant.

3. Saturated coalitions for monotone games

In this section, which is restricted to monotone cost games, we introduce the notion of saturated coalitions, and prove that the class of all irreducible saturated coalitions forms a c-set for the nucleolus of a monotone game having a non-empty core.

[4] Since $\mathscr{T}$ induces a representation of the core of $(N;c)$, $C(\Gamma^{\mathscr{T}})$ can be equivalently replaced by $C(\Gamma)$.

A cost game $(N;c)$ is said to be *monotone*, if $c(S) \le c(T)$, whenever $S \subseteq T$. Further,

Definition 3.1.
(1) *A coalition S of $\Gamma = (N;c)$ is said to be* ***saturated*** *if $i \in S$ whenever*[5] $c(S \cup i) = c(S)$.
(2) *A closure of S is a* ***saturated coalition*** $\bar{S}$ *such that* $S \subseteq \bar{S}$ *and* $c(S) = c(\bar{S})$.

Lemma 3.2. *Any non-saturated coalition has at least one closure.*

Proof. This is demonstrated by adding players to a non-saturated coalition S until any further addition will have to increase $c(S)$. Since $|N|$ is finite, the process of adding players will stop after at most $|N|$ additions. By definition, the resulting coalition is saturated, hence, is a closure of S. ■

Definition 3.3. *A saturated coalition S is* ***irreducible*** *if there is no partition $\{S_1, \dots, S_p\}$ of S such that S_i are saturated and $c(S) \ge c(S_1) + \cdots + c(S_p)$.*

For a cost game Γ, let $\mathscr{IS}$ denote the class of all irreducible saturated coalitions, and define $\mathscr{IS}' \stackrel{\text{def}}{=} \mathscr{IS} \cup \{N \setminus i \colon i \in N\} \cup \{N\}$.

Lemma 3.4. *A saturated coalition S is irreducible if and only if S has no partition $P = \{S_1, \dots, S_k\}$ such that $S_i \in \mathscr{IS}$ and $\sum_{i=1}^{k} c(S_i) \le c(S)$.*

Proof. The sufficiency follows from the definition. To show the necessity, suppose a saturated coalition S is not irreducible. Then, from the definition of irreducibility, S has a partition $P_1 = \{S_1, \dots, S_p\}$ such that S_i's are saturated and $\sum_{i=1}^{p} c(S_i) \le c(S)$. Since any refinement of P_1 is still a partition of S, the proof follows by applying inductive arguments to those subsets S_i which are not irreducible. ■

Theorem 3.5. *Let $\Gamma = (N;c)$ be a monotone cost game with a nonempty core. Then $\mathscr{IS}'$ induces a representation of $C(\Gamma)$.*

Proof. Clearly,

$$C(\Gamma) \subseteq C' \stackrel{\text{def}}{=} \{x \in \boldsymbol{R}^N : x(S) \le c(S), S \in \mathscr{IS}'; x(N) = c(N)\}.$$

To prove the other direction of the inclusion, we need to show that for all $x \in C'$, $x(S) \le c(S)$ for any $S \in 2^N \setminus \mathscr{IS}'$. Let $S \in 2^N \setminus \mathscr{IS}'$. Clearly, S is either unsaturated, or it is saturated but not irreducible. If S is saturated but not irreducible, then, by Lemma 3.4, S has a partition $P = \{S_1, \dots, S_k\}$ such that $S_i \in \mathscr{IS}$ and $\sum_{i=1}^{k} c(S_i) \le c(S)$. Thus, $x(S) = \sum_{i=1}^{k} x(S_i) \le \sum_{i=1}^{k} c(S_i) \le c(S)$ for $x \in C'$. If S is unsaturated, let $\bar{S}$ be a closure of S. Then, $S \subseteq \bar{S}$, $c(S) = c(\bar{S})$ and $x(\bar{S}) \le c(\bar{S})$. Now, observe that $x \ge 0$ for all $x \in C'$.

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[5] For simplicity of notation, we omit sometimes curly brackets. Thus, we write $c(S \cup i)$, $c(i)$, and $N \setminus i$ instead of $c(S \cup \{i\})$, $c(\{i\})$ and $N \setminus \{i\}$, respectively.

Indeed, for any $i \in N$ and $x \in C'$, $x_i = x(N) - x(N\backslash i) \geq c(N) - c(N\backslash i) \geq 0$. Thus, $x(S) \leq x(\bar{S}) \leq c(\bar{S}) = c(S)$. ■

Theorem 3.6. *Suppose Γ is a monotone cost game and $C(\Gamma) \neq \emptyset$. Then, $\mathscr{IS}'$ is a c-set for $\mathscr{N}(\Gamma)$.*

Proof. By Theorem 3.5, $\mathscr{IS}'$ induces a representation of $C(\Gamma)$. Let $S \in 2^N \backslash \mathscr{IS}'$.

Case 1. If S is saturated, then S is not irreducible and it follows from Lemma 3.4 that S has a partition $P = \{S_1, \ldots, S_k\}$ such that $S_i \in \mathscr{IS}$ and $\sum_{i=1}^{k} c(S_i) \leq c(S)$. Let $\mathscr{T}_S = P$. Clearly, $\mathscr{T}_S$ defined this way satisfies condition (ii) of Corollary 2.8. Further, for $x \in C(\Gamma^{\mathscr{IS}'})$, $e(S_i, x) \geq 0$ for all $1 \leq i \leq k$. Thus, for $1 \leq j \leq k$, $e(S_j, x) \leq \sum_{i=1}^{k} e(S_i, x) \leq e(S, x)$, where the last inequality follows from $\sum_{i=1}^{k} c(S_i) \leq c(S)$. Hence, $\mathscr{T}_S$ also satisfies condition (i) of Corollary 2.8.

Case 2. If S is not saturated, let $\bar{S}$ be a closure of S. Define $\mathscr{T}_S = \{N\backslash i : i \in \bar{S}\backslash S\} \cup \{N\} \cup \mathscr{T}_{\bar{S}}$, where $\mathscr{T}_{\bar{S}} = \{\bar{S}\}$ if $\bar{S} \in \mathscr{IS}$ and otherwise $\mathscr{T}_{\bar{S}}$ is defined as in Case 1. $\mathscr{T}_S$ defined this way satisfies condition (ii) of Corollary 2.8. To show that $\mathscr{T}_S$ also satisfies condition (i) of Corollary 2.8, let $x \in C(\Gamma^{\mathscr{IS}'})$. If $\mathscr{T}_{\bar{S}} = \{\bar{S}\}$, then $e(\bar{S}, x) \leq e(S, x)$ is implied by $c(\bar{S}) = c(S)$ and $x \geq 0$. Else, for $T \in \mathscr{T}_{\bar{S}}$, $e(T, x) \leq e(S, x)$ follows as shown in Case 1. For $i \in \bar{S}\backslash S$,

$$\begin{aligned} e(N\backslash i, x) &= c(N\backslash i) - x(N\backslash i) = c(N\backslash i) - c(N) + x_i \\ &\leq x_i, \quad \text{since } \Gamma \text{ is monotone,} \\ &= (x(S \cup i) - x(S)) + (c(S) - c(S \cup i)), \quad \text{since } c(S \cup i) = c(S), \\ &\leq e(S, x), \quad \text{since } e(S \cup i, x) \geq 0. \end{aligned}$$

Finally, $0 = e(N, x) \leq e(S, x)$ since $x \in C(\Gamma)$. Hence, $\mathscr{T}_S$ satisfies both conditions of Corollary 2.8, and $\mathscr{IS}'$ is a c-set for $\mathscr{N}(\Gamma)$ as claimed. ■

A coalition S is said to be *essential* for a game $\Gamma = (N; c)$ if for every proper partition, $P = \{S_1, \ldots, S_p\}$, of S, $c(S) < \sum_{i=1}^{p} c(S_i)$. By convention, single member coalitions are essential. Let $\mathscr{E}$ denote the class of all essential coalitions of Γ. The following result is essentially proved in Huberman [1980].

Theorem 3.7. *If $C(\Gamma) \neq \emptyset$, then $\mathscr{E}$ is a c-set for $\mathscr{N}(\Gamma)$.*

Theorem 3.7 follows trivially from Corollary 2.8. Indeed, $\mathscr{E}$ induces a representation of $C(\Gamma)$, and for $S \in 2^N \backslash \mathscr{E}$, $\mathscr{T}_S$ can be chosen to be an arbitrary partition $P = \{S_1, \ldots, S_p\}$ of S, such that $c(S) \geq \sum_{i=1}^{p} c(S_i)$. For such a partition, Condition (ii) of Corollary 2.8 is trivially satisfied. Condition (i) therein is satisfied by such a partition since, by assumption, $C(\Gamma) \neq \emptyset$.

Remark 3.8 below relates Huberman's essential coalitions to the collection of irreducible saturated coalitions introduced above.

Remark 3.8.

(1) *It follows from the definition of irreducibility that a saturated coalition is irreducible if and only if it is essential among the class of all saturated coalitions.*

(2) *Let $\Gamma = (N; c)$ be an arbitrary cost game. Then, for an additive game $(N; d)$, where $d(i)$ is sufficiently large for all $i \in N$, $\Gamma' = (N; c + d)$ is a monotone game with every subset of N therein being saturated. For such a game Γ', the class of all irreducible saturated coalitions coincides with the class of all essential coalitions.*

(3) *In general, for an arbitrary game Γ, it may happen that neither $\mathscr{E} \subseteq \mathscr{IS}$ nor $\mathscr{IS} \subseteq \mathscr{T}$.*

Finally, we comment that as (2) in Remark 3.8 suggests, monotonicity, as well as other concepts introduced in this section, are not covariant under the addition of additive games.

4. Sequential LP processes and polynomial computability of the nucleolus

In this section, we formalize Kopelowitz's method (Kopelowitz [1967], Maschler, Peleg and Shapley [1979]) for computing the nucleolus of a cooperative game through the notion of a sequential LP process. We further study the relationship between the size of a *c*-set and the complexity of computing the nucleolus.

Notation 4.1. Let c be a function defined on $\mathscr{T}$, $\mathscr{T} \subseteq 2^N$, and let X be a polyhedron in $\boldsymbol{R}^N$. Denote by $P(\mathscr{T}, c, X)$ the following LP problem:

$$P(\mathscr{T}, c, X)\colon \operatorname{Max}\{r\colon r + x(S) \le c(S), S \in \mathscr{T}; x \in X\}.$$

The sequential LP process for $P(\mathscr{T}, c, X)$, denoted by $\mathrm{SLP}(P(\mathscr{T}, c, X))$, is the process of solving a sequence of LP problems, $\{P_k\colon k \ge 1\}$, led by $P_1 = P(\mathscr{T}, c, X)$ and terminated with P_κ ($\kappa \ge 1$) which has a unique optimal solution or has only equality constraints. For the kth LP problem P_k, let r_k be its optimal value, X_k be the projection of its optimal solution set on $\boldsymbol{R}^N$ and $\Sigma_k = \{S \subseteq N\colon e(S, x) = \text{constant} \ge r_k \text{ on } X_k\}$. Then P_{k+1} is derived from P_k by converting all constraints induced by subsets $S \in \Sigma_k$ into equalities of the form, $x(S) = c(S) - e(S, y)$ where y is an arbitrary vector in X_k. We will refer to X_κ, the projection of the optimal solution for the last LP in $\mathrm{SLP}(P(\mathscr{T}, c, X))$ on $\boldsymbol{R}^N$, as the *outcome* of $\mathrm{SLP}(P(\mathscr{T}, c, X))$, and to $\rho \overset{\text{def}}{=} \{X, X_1, \dots, X_\kappa\}$ as the *trajectory* of $\mathrm{SLP}(P(\mathscr{T}, c, X))$. We also let Σ^k denote $\Sigma_1 \cup \cdots \cup \Sigma_k$.

Remark 4.2. *Equivalently, r_k, X_k, Σ^k and P_{k+1} for $k \ge 1$ can be defined as,*

(1) $r_1 = \max_{x \in X} \min_{S \in \mathscr{T}} e(S, x)$ *and* $r_k = \max_{x \in X_{k-1}} \min_{S \in \mathscr{T} \setminus \Sigma^{k-1}} e(S, x)$ *for* $k > 1$.

(2) $X_k = \{x \in X_{k-1}\colon e(S, x) \ge r_k, \forall S \in \mathscr{T} \setminus \Sigma^{k-1}\} = \{x \in X_{k-1}\colon \min_{S \in \mathscr{T} \setminus \Sigma^{k-1}} e(S, x) = r_k\}$.

(3) *Σ^k is the set of all coalitions S in $\mathscr{T}$ such that $e(S, x)$ is constant for all $x \in X_k$.*

(4) *P_{k+1} is the LP problem derived from P_1 by setting all constraints induced by subsets in Σ^k into equalities of the form, $x(S) = c(S) - e(S, y)$, where y is an arbitrary vector in X_k.*

The sequential LP process depicted in Notation 4.1 is slightly different from the process described in Macshler et al. [1979]. Indeed, therein they derive P_{k+1} from P_k by converting into equalities all constraints induced by coalitions in $\{S \subseteq N : e(S,x) = r_k \text{ for all } x \in X_k\}$, which is a subset of Σ_k defined in Notation 4.1. But, as mentioned at Footnote 37 therein, these two processes have the same outcome. Actually, we can prove the following:

Lemma 4.3. *The outcome of the process described in Notation 4.1 will not change if in the derivation of P_{k+1} from P_k at least one but not necessarily all constraints induced by coalitions in Σ_k are converted into equalities.*

Proof. It is sufficient to prove that any such modified process and $\mathrm{SLP}(P(\mathscr{T}, c, X))$ have the same outcome. Such a proof follows the same line as the proof of Theorem 6.6 in Maschler et al. [1979]. After proving analogous results as Lemmas 6.3 and 6.5 therein, we can conclude that the modified process (in particular, $\mathrm{SLP}(P(\mathscr{T}, c, X))$) terminates after a finite number of steps with the following outcome,

$$\{x \in X : \theta(x, c, \mathscr{T}) \geq_l \theta(y, c, \mathscr{T}) \text{ for all } y \in X\},$$

where $\theta(x, c, \mathscr{T})$ is the $|\mathscr{T}|$-dimensional vector consisting of the components $e(S, x)$, $S \in \mathscr{T}$, arranged in a non-decreasing order. (See also Remark 4.4(2) below.) For brevity, we omit this part of the proof. ■

For a cost game $\Gamma = (N; c)$, denote by $P(\mathscr{T}, \Gamma, X_0)$ the LP problem $P(\mathscr{T}, c, X_0)$, where c is taken to be the characteristic function of Γ and $X_0 = X^*(\Gamma)$ or $X(\Gamma)$. With the notion of a sequential LP process, Kopelowitz's procedure for computing the nucleolus of Γ with respect to X_0 can be described as $\mathrm{SLP}(P(2^N, \Gamma, X_0))$, whose outcome is $\mathscr{N}(\Gamma, X_0)$.

Remark 4.4.

(1) *For a game with a nonempty core, the prenucleolus coincides with the nucleolus and is contained in its core. For such a game Γ, the optimal value r_1 of $P(\mathscr{T}, \Gamma, X^*(\Gamma))$ is nonnegative. So the projection of the optimal solution set of $P(\mathscr{T}, \Gamma, X^*(\Gamma))$ on $\boldsymbol{R}^N$ is contained in the core of Γ, which is contained in $X(\Gamma)$. Therefore, if Γ has a nonempty core, the constraint $x \in X_0$ can be replaced by $x(N) = c(N)$, or by $x \in C(\Gamma)$ in all LP problems encountered in Kopelowitz's procedure for computing the prenucleolus/nucleolus of Γ.*

(2) *The outcome of $\mathrm{SLP}(P(\mathscr{T}, \Gamma, X_0))$ is $\mathscr{N}(\Gamma^{\mathscr{T}}, X_0)$, where $\Gamma^{\mathscr{T}}$ is the game derived from Γ by imposing coalition formation restrictions $\mathscr{T}$.*

A geometrical interpretation for $\mathrm{SLP}(P(2^N, \Gamma, X(\Gamma)))$, which dubs its outcome, $\mathscr{N}(\Gamma)$, as the lexicographical center of the imputation set $X(\Gamma)$, was given in Maschler, Peleg and Shapley [1979]. Analogously, a similar geometrical explanation can be produced for $\mathrm{SLP}(P(\mathscr{T}, \Gamma, X_0))$ for any $\mathscr{T} \subseteq 2^N$. For this reason, we refer to the outcome of $\mathrm{SLP}(P(\mathscr{T}, \Gamma, X_0))$, which is a nonempty compact convex subset of X_0, *the lexicographical center of X_0 with respect to $\mathscr{T}$.*

In general, the outcome of $\mathrm{SLP}(P(\mathscr{T}, \Gamma, X_0))$ might not be a singleton set. However, when $\mathscr{T} = 2^N$ the outcome of $\mathrm{SLP}(P(\mathscr{T}, \Gamma, X_0))$ is the prenucleolus or nucleolus of Γ, and hence is a singleton set. Further:

Proposition 4.5. *The outcome of* $\mathrm{SLP}(P(\mathscr{T}, \Gamma, X_0))$ *is a singleton set if and only if the set of incidence vectors of subsets in* $\mathscr{T}$ *spans* $\boldsymbol{R}^N$.

The proof is immediate, and thus omitted.

An algorithm is said to be *strongly polynomial* if: (1) it consists of the elementary operations: additions, comparisons, multiplications and divisions, (2) the elementary operations are carried out on rationals of size polynomially bounded in the dimension of the input, and (3) the number of such operations is polynomially bounded in the dimension of the input. A problem is said to be *solvable in strongly polynomial time* if there is a strongly polynomial algorithm for solving the problem. The *dimension of a cooperative game* is defined to be the total number of players in the game. We prove in this section that if the nucleolus of a cost game Γ has a c-set whose size is polynomially bounded in the number of players, then $\mathscr{N}(\Gamma, X_0)$ can be computed in strongly polynomial time.

For a cost game $\Gamma = (N; c)$, let $\mathscr{T}$ be a c-set for $\mathscr{N}(\Gamma, X_0)$. Recall that in this case, $\mathscr{N}(\Gamma, X_0) = \mathscr{N}(\Gamma^{\mathscr{T}}, X_0)$ is the outcome of $\mathrm{SLP}(P(\mathscr{T}, \Gamma, X_0))$ (Remark 4.4(2)), where

$$P(\mathscr{T}, \Gamma, X_0): \quad \mathrm{Max}\{r: r + x(S) \leq c(S), S \in \mathscr{T}; x \in X_0\}.$$

We develop next an algorithm for computing the outcome of $\mathrm{SLP}(P(\mathscr{T}, \Gamma, X_0))$ and show that the proposed algorithm is strongly polynomial if $\mathscr{T}$ has size polynomially bounded in $|N|$. For brevity, we assume that $X_0 = X(\Gamma)$ in the sequel. The case where $X_0 = X^*(\Gamma)$ is similar. Recall that when $X_0 = X(\Gamma)$, the constraint $x \in X_0$ in $P(\mathscr{T}, \Gamma, X_0)$ can be replaced by: $x_i \leq c(i)$ for $i \in N$, $x(N) = c(N)$.

Let $\{\mathscr{T}_1, \mathscr{T}_2\}$ be a partition of $\mathscr{T} \cup \{N\}$ with $N \in \mathscr{T}_2$. Denote by $P_{\mathscr{T}_1,\mathscr{T}_2}$ the LP problem,

$$\begin{aligned} P_{\mathscr{T}_1,\mathscr{T}_2}: \quad & \mathrm{Max} \quad r \\ & \text{s.t.} \quad r + x(S) \leq c(S), \quad S \in \mathscr{T}_1; \\ & \qquad x(S) = c(S) - r_S, \quad S \in \mathscr{T}_2; \\ & \qquad x_i \leq c(i), \quad i \in N, \end{aligned}$$

where r_S, for $S \in \mathscr{T}_2$, are constants depending on S, and in particular, $r_N = 0$. The dual LP problem of $P_{\mathscr{T}_1,\mathscr{T}_2}$ is

$$\begin{aligned} D_{\mathscr{T}_1,\mathscr{T}_2}: \quad & \mathrm{Min} \quad \sum_{S \in \mathscr{T}_1} c(S)\pi_S + \sum_{S \in \mathscr{T}_2} (c(S) - r_S)\pi_S + \sum_{i \in N} c(i)\lambda_i \\ & \text{s.t.} \quad \sum_{S \in \mathscr{T}_1} \pi_S = 1; \\ & \qquad \sum_{S: i \in S} \pi_S + \lambda_i = 0, \quad i \in N; \\ & \qquad \pi_S \geq 0, \ \forall S \in \mathscr{T}_1, \ \lambda_i \geq 0, \ \forall i \in N. \end{aligned}$$

Observe that all coefficients in the constraints of $P_{\mathscr{T}_1,\mathscr{T}_2}$ and $D_{\mathscr{T}_1,\mathscr{T}_2}$ are zero or one. Thus, both problems are combinatorial linear programs[6] and by applying Tardos' [1986] celebrated algorithm for solving combinatorial LP problems, we conclude that:

Lemma 4.6. *If $\mathscr{T}$ has size polynomially bounded in $|N|$, then there is a strongly polynomial algorithm which computes the optimal value and an optimal solution for $P_{\mathscr{T}_1,\mathscr{T}_2}$ or $D_{\mathscr{T}_1,\mathscr{T}_2}$.*

Since a superset of a c-set for the nucleolus of a game is still a c-set for the nucleolus, without loss of generality, we can assume in the sequel that $\mathscr{T} \supseteq \{N \backslash i : i \in N\}$. Algorithm 4.7 for computing the outcome of $\mathrm{SLP}(P(\mathscr{T}, \Gamma, X_0))$ can be stated as follows.

Algorithm 4.7.
Input: *The c-set $\mathscr{T}$ for the nucleolus of a cost game Γ. Assume $\mathscr{T} \supseteq \{N \backslash i : i \in N\}$.*
Output: $\{y\} = \mathscr{N}(\Gamma)$.
Step 0. *Set $P_1 = P(\mathscr{T}, \Gamma, X_0)$, $\mathscr{T}_1 = \mathscr{T} \backslash \{N\}$, $\mathscr{T}_2 = \{N\}$, $k = 1$ and $\xi = \{N \backslash i : i \in N\}$.*
Step 1. *Repeat the following until $\xi = \emptyset$:*

- *Solve $D_{\mathscr{T}_1,\mathscr{T}_2}$ by applying Tardos' algorithm. Let r_k be its optimal value, π^* be an optimal solution, and let $\Sigma_k \stackrel{\text{def}}{=} \{S \in \mathscr{T}_1 : \pi_S^* \neq 0\}$. Set $\mathscr{T}_1 = \mathscr{T}_1 \backslash \Sigma_k$, $\mathscr{T}_2 = \mathscr{T}_2 \cup \Sigma_k$, $\xi = \xi \backslash \Sigma_k$ and $k = k + 1$.*
- *For each i such that $N \backslash i \in \Sigma_k$, set $y_i = r_k + c(N) - c(N \backslash i)$.*

Step 2. *Output*[7] $\{y\}$.

Theorem 4.8. *Let G be a class of games and suppose that for every game in this class one can find in polynomial time a c-set of size polynomially bounded in the number of players. Then, the nucleolus of each game in this class can be calculated in strongly polynomial time.*

Proof. Let $\Gamma = (N; c) \in G$ and $\mathscr{T}$ be the c-set for Γ, which can be found in polynomial time. We start by showing that Algorithm 4.7 computes the outcome of $\mathrm{SLP}(P(\mathscr{T}, \Gamma, X_0))$. Indeed, for each P_k $(k \geq 1)$, Algorithm 4.7 computes the optimal value and identifies some inequality constraints which are binding at all of its optimal solutions by finding the optimal value and an optimal solution for the dual LP problem of P_k. Explicitly, in the first iteration of Step 1, Algorithm 4.7 computes the optimal value and an optimal solution of the LP problem $D_{\mathscr{T} \backslash \{N\},\{N\}}$, which is the dual LP problem of P_1. Let r_1 be its optimal value and π^* be an optimal solution. Then, by the duality

[6] A linear program is said to be combinatorial if the coefficients in the constraint matrix are polynomially bounded in the dimension of the problem. Thus, if, in particular, the coefficients are zero or one, as is the case with $P_{\mathscr{T}_1,\mathscr{T}_2}$ and $D_{\mathscr{T}_1,\mathscr{T}_2}$, the associated linear program is combinatorial.
[7] In Algorithm 4.7 we solve a sequence of dual problems of the general form $D_{\mathscr{T}_1,\mathscr{T}_2}$. Equivalently, the algorithm could have been described in terms of solving a sequence of primal problems of the form $P_{\mathscr{T}_1,\mathscr{T}_2}$. However, in the latter case one would need to resort to an optimal dual solution at each stage in order to characterize the collection Σ_k (see Step 1). In order to simplify the presentation, Algorithm 4.7 is described in terms of solving a sequence of dual problems $D_{\mathscr{T}_1,\mathscr{T}_2}$.

theorem of linear programming, r_1 is the optimal value of P_1 and subsets in $\Sigma_1 \stackrel{\text{def}}{=} \{S \in \mathscr{T}: \pi^*_S \neq 0\}$ induce constraints which are binding at all optimal solutions of P_1. Let P_2 be the LP problem derived from P_1 by converting some constraints induced by subsets in Σ_1 into equalities as done in Notation 4.1. The dual of P_2 is $D_{\mathscr{T}_1,\mathscr{T}_2}$, where $\mathscr{T}_1 = \mathscr{T} \setminus (\{N\} \cup \Sigma_1)$ and $\mathscr{T}_2 = \{N\} \cup \Sigma_1$. In general, in the kth iteration of Step 1, Algorithm 4.7 computes the optimal value and an optimal solution of the dual LP problem of P_k, which has the form $D_{\mathscr{T}_1,\mathscr{T}_2}$ for some partition $\{\mathscr{T}_1, \mathscr{T}_2\}$ of $\mathscr{T} \cup \{N\}$. Let r_k be its optimal value and π^* be an optimal solution. Then r_k is the optimal value of P_k and subsets in $\Sigma_k \stackrel{\text{def}}{=} \{S \in \mathscr{T}_1 : \pi^*_S \neq 0\}$ induce constraints which are binding at all optimal solutions of P_k. Derive P_{k+1} from P_k by converting constraints induced by subsets in Σ_k into equalities, and note that the dual of P_{k+1} is $D_{\mathscr{T}_1 \setminus \Sigma_k, \mathscr{T}_2 \cup \Sigma_k}$.

By the time constraints induced by all coalitions $N \setminus i$, $i \in N$, have been converted into equalities, $\text{SLP}(P(\mathscr{T}, \Gamma, X_0))$ has come to its end. Indeed, when the latter has occurred the values of all variables x_i, $i \in N$, are fixed and thus the last LP problem solved must have a unique solution. Algorithm 4.7 tests this stopping criterion by checking if $\{N \setminus i: i \in N\} \subseteq \mathscr{T}_2$ at the end of every iteration of Step 1. Whenever $N \setminus i$, for some $i \in N$, is moved from $\mathscr{T}_1$ to $\mathscr{T}_2$, Algorithm 4.7 sets $y_i = r_k + c(N) - c(N \setminus i)$, where k is chosen such that $N \setminus i \in \Sigma_k$. Clearly, y defined this way is the outcome of $\text{SLP}(P(\mathscr{T}, \Gamma, X_0))$.

Algorithm 4.7 stops after finitely many iterations since, at each iteration of Step 1, the optimal solution for $D_{\mathscr{T}_1,\mathscr{T}_2}$ must satisfy $\sum_{S \in \mathscr{T}_1} \pi^*_S = 1$, so Σ_k is nonempty. Hence after every iteration at least one inequality will be converted into an equality. Therefore, the algorithm will terminate after at most $|\mathscr{T}|$ iterations.

To show that Algorithm 4.7 is strongly polynomial if $\mathscr{T}$ has size polynomially bounded in $|N|$, observe that, at each iteration, computations are dominated by those needed for solving $D_{\mathscr{T}_1,\mathscr{T}_2}$. From Lemma 4.6, Tardos' algorithm for solving $D_{\mathscr{T}_1,\mathscr{T}_2}$ is strongly polynomial if $|\mathscr{T}|$ is polynomial in $|N|$. Since Algorithm 4.7 terminates after at most $|\mathscr{T}|$ iterations, which is polynomial in $|N|$, it follows that the nucleolus is computed in strongly polynomial time, completing the proof. ■

5. Examples

To illustrate their usefulness and prevalence, we briefly describe in this section c-sets for the classes of minimum cost spanning tree games and assignment games

5.1. Minimum cost spanning tree games

A minimum cost spanning tree (MCST) game, Γ_G, is defined on a simple complete graph $G = (V, E)$ with node set $V = N \cup \{0\}$. Node 0 is the supplier node and $N = \{1, 2, \ldots, n\}$ is the set of customer nodes. Thus, customer i resides at node i. A nonnegative real number is associated with each edge of G and is called the edge cost. We use c_{ij} to denote the cost of the edge joining nodes i and j. For $S \subseteq N$, let $T_S \stackrel{\text{def}}{=} (V_S, E_S)$ represent a MCST for the induced subgraph of G with node set $V_S = S \cup \{0\}$. In the MCST game, the player set is identified with the set of customers N and the cost function is defined as $c(\emptyset) = 0$ and $c(S) = \sum\{c_{ij}\colon (i,j) \in E_S\}$ for $S \subseteq N$ and $S \neq \emptyset$. The class of MCST games has been studied, for example, by Bird [1976], Granot and Huberman [1981, 1984] and Megiddo [1978], and is known to have a nonempty core (see, e.g. Bird [1976] or Granot and Huberman [1981]).

To describe a *c*-set for the nucleolus of a MCST game Γ_G, we need some more notation. Let $T_N = (V_N, E_N)$ be a MCST for Γ. A subset $S \subseteq N$ will be called T_N-*connected* if it induces a connected subgraph on T_N. That is, if i and j are in S, then so are all other nodes on the unique path between i and j in T_N. Let $\mathscr{L}'$ denote the class of coalitions S whose complements, $N \backslash S$, are T_N-connected, and let $\mathscr{L} = \mathscr{L}' \cup \{N\}$.

Theorem 5.1. (Granot and Huberman [1984]) *$\mathscr{L}$ induces a representation of the core of a MCST game.*

For $Q \subset N$, let $P(Q) \overset{\text{def}}{=} \{\bar{Q}_1, \bar{Q}_2, \ldots, \bar{Q}_{r(Q)}\}$ be the collection of all maximal connected components of $N \backslash Q$ in T_N. Thus, if $|P(Q)| = 1$, then $Q \in \mathscr{L}'$. Further, for $Q \notin \mathscr{L}'$, let $\mathscr{T}_Q = \{N \backslash T\colon T \in P(Q)\}$. Then, for each $Q \notin \mathscr{L}'$, $e_Q = \sum(e_{N\backslash T}\colon T \in P(Q)) - (|P(Q)| - 1) \cdot e_N$, and condition (ii) of Corollary 2.8 is satisfied. Moreover, Granot and Huberman's proof of Theorem 5.1, in fact, establishes that for each $Q \notin \mathscr{L}'$, $e(Q, x) \geq e(T, x)$, $\forall T \in \mathscr{T}_Q$, for each x in the core of a MCST game. Thus, by Corollary 2.8, we conclude:

Corollary 5.2. *$\mathscr{L}$ is a c-set for $\mathscr{N}(\Gamma_G)$.*

Corollary 5.2 was explicitly proved by Granot and Huberman [Theorem 5, 1984]. However, as it was shown above, it follows immediately from Corollary 2.8, once conditions (i) and (ii) therein are satisfied for Γ_G.

For an arbitrary graph G, $\mathscr{L}$ consists of an exponential number (in terms of $|N|$) of coalitions. Therefore, for general MCST games, the computation of the nucleolus may still require exponential time. For the special case when T_N is a chain, $\mathscr{L}$ consists of $n(n+1)/2$ coalitions. Thus, from Theorem 4.8, the nucleolus of this special class of MCST games can be computed in strongly polynomial time.

An even more special case is obtained when G itself is restricted to be a chain. Games associated with such chain graphs were first studied by Littlechild [1974], Littlechild and Owen [1977] and Littlechild and Thompson [1977]. In Littlechild [1974], the author has identified a class of $O(|N|)$ coalitions which were shown to be the only relevant coalitions for calculating the nucleolus, and has essentially developed therein an $O(|N|^2)$ algorithm for computing the nucleolus. Galil [1980] and lately, Granot, Maschler, Owen and Zhu [1996] have derived a linear time algorithm for computing the nucleolus of this class of games. For this special class of games, it was established by Littlechild [1974] that the characterization set, $\mathscr{T}$, is given by $\mathscr{T} = \{\{1\}, \{1,2\}, \ldots, \{1,2,\ldots,n-1\}\} \cup \{\{N \backslash \{i\}, i \in N\} \cup \{N\}$.

5.2. Assignment games

Assignment games were introduced by Shapley and Shubik [1972] as a model for a simple two-sided market. In such a market:

(1) There are two disjoint sets of agents, say "producers" and "consumers", denoted by N_1 and N_2, respectively.

(2) The agents are constrained to conduct their business under exclusive bilateral contracts, and for that purpose partnerships are formed.

(3) All transactions are limited to exchanges between partners. For $i \in N_1$ and $j \in N_2$, there is associated a number $a_{ij} \geq 0$, representing the potential profit of the partnership between i and j if it is formed.

We denote a simple market by $M = (N_1, N_2, A)$, where $A = (a_{ij}) \in \boldsymbol{R}^{|N_1| \times |N_2|}$. The assignment game associated with M is the game $\Gamma_M = (N; v)$, where $N = N_1 \cup N_2$, and for $S \subseteq N$, $v(S)$ is the maximum profit that can be attained by matching producers in $S \cap N_1$ with consumers in $S \cap N_2$.

For an assignment game $\Gamma_M = (N; v)$, Shapley and Shubik [1972] proved that:

Proposition 5.3. *$C(\Gamma_M) \neq \emptyset$, and*[8]

$$C(\Gamma_M) = \{x \in \boldsymbol{R}^N : x_i + x_j \geq a_{ij} \textit{ for all } i \in N_1, j \in N_2; x \geq 0; x(N) = v(N)\}.$$

Let $\mathscr{T} \stackrel{\text{def}}{=} \{S : S = \{i,j\}$ for some $i \in N_1$, $j \in N_2$ or $S = \{k\}$ for some $k \in N\} \cup \{N\}$. From Proposition 5.3, it follows that $\mathscr{T}$ induces a representation of the core of Γ_M. Now, for each $S \notin \mathscr{T}$, let $P(S) = \{\{i_1,j_1\}, \{i_2,j_2\}, \ldots, \{i_{k(S)}, j_{k(s)}\} : i_l \in N_1,\ j_l \in N_2,\ l = 1, \ldots, k(S);\ Q_S, Q_S \subset N_1 \text{ or } Q_S \subset N_2\}$ be a partition of S satisfying $v(S) = \sum(a_{i_l,j_l} : i_l \in N_1, j_l \in N_2, l = 1, \ldots, k(S))$, and let $J_S = \{\{i_1,j_1\}, \{i_2,j_2\}, \ldots, \{i_{k(S)}, j_{k(S)}\} \cup \{\{j\} : j \in Q_S\}$. Then e_S is simply the sum of the characteristic vectors of subsets in J_S, and condition (ii) in Corollary 2.8 is satisfied. Further, for each $x \in C(\Gamma_M)$, $x(S) \geq v(S)$ and therefore,[9]

$$\begin{aligned} x(S) - v(S) &= \sum_{l=1,\ldots,k(S)} (x_{i_l} + x_{j_l} - v(\{i_l, j_l\})) + \sum_{j \in Q_S} x_j \\ &\geq x_{i_l} + x_{j_l} - v(\{i_l, j_l\}), \quad l = 1, \ldots, k(S), \end{aligned}$$

and

$$x(S) - v(S) \geq x_j = x_j - v(\{j\}), \quad j \in Q_S.$$

Thus, condition (i) in Corollary 2.8 is also satisfied, and we conclude:

Proposition 5.4. *$\mathscr{T}$ is a c-set for $\mathscr{N}(\Gamma_M)$.*

Proposition 5.4 was proved explicitly in Solymosi and Raghavan [1994]. However, again, as it was shown above, it follows immediately from Corollary 2.8, once conditions (i) and (ii) therein are established for Γ_M.

Since there are only $|N_1| \cdot |N_2| + |N_1| + |N_2|$ coalitions in $\mathscr{T}$, it follows from Theorem 4.8 that $\mathscr{N}(\Gamma_M)$ can be computed in strongly polynomial time.

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[8] Since assignment games are *revenue games*, we need to reverse the direction of inequalities in the definition of all solution concepts.

[9] Since assignment games are revenue games, we need to reverse the direction of the inequalities in condition (i) in Theorem 2.3 and Corollary 2.8.

In fact, an $O(|N|^4)$ algorithm to compute $\mathcal{N}(\Gamma_M)$ is developed in Solymosi and Raghavan [1994].

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