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The selectope for cooperative games*

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Abstract. The selectope of a cooperative transferable utility game is the convex hull of the payoff vectors obtained by assigning the Harsanyi dividends of the coalitions to members determined by so-called selectors. The selectope is studied from a set-theoretic point of view, as superset of the core and of the Weber set; and from a value-theoretic point of view, as containing weighted Shapley values, random order values, and sharing values.

Key words: Cooperative game, selectope core, Weber set, sharing value

1. Introduction

A (*transferable utility*) *game* is a pair (N, v) (often simply denoted by v) where $N = \{1, 2, \dots, n\}$ is the set of *players* and $v : 2^N \to \mathbb{R}$ is the *characteristic function* assigning to each *coalition* S the *worth* $v(S)$, with the convention that $v(\emptyset) = 0$. A central question concerning such a game v is: Assuming that the *grand coalition* N forms, how to distribute its worth $v(N)$ among the players? One way to answer this question is to specify a *value*, i.e., a map $\mathcal{G}^N \to \mathbb{R}^N$, where $\mathcal{G}^N$ denotes the set of all games with player set N.

For a game v and coalitions S the *dividends* (Harsanyi, 1963) are defined, recursively, by

$$\Delta_v(S) := \begin{cases} 0 & \text{if } S = \emptyset \\ v(S) - \sum_{T \subseteq S, T \neq S} \Delta_v(T) & \text{otherwise.} \end{cases}$$

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Note that for every $v \in \mathscr{G}^N$ and $S \subseteq N$:

$$v(S) = \sum_{T \subseteq S} \Delta_v(T). \tag{1}$$

A *selector* is a function $\alpha : 2^N \backslash \{\emptyset\} \to N$ with $\alpha(S) \in S$ for every nonempty coalition S. The set of all selectors on $2^N \backslash \{\emptyset\}$ is denoted by A^N. The *selector value corresponding to* α is the value m^α defined by

$$m_i^\alpha(v) := \sum_{S: i = \alpha(S)} \Delta_v(S)$$

for every $i \in N$ and $v \in \mathscr{G}^N$. In Hammer *et al.* (1977) the *selectope* for a game v was introduced, which is defined by

$$\mathcal{S}(v) := \text{conv}\{m^\alpha(v) \in \mathbb{R}^N : \alpha \in A^N\}.$$

The selectope can be viewed as containing all possible reasonable ways to distribute the dividends of a game among the players. Therefore it is interesting to investigate its relation to other solution concepts that were proposed in the literature. Specifically, the purpose of this paper is to study the selectope from two points of view.

In Section 2 the selectope is considered in relation to other set-theoretic solution concepts, in particular the imputation set, Weber set, and core. The main results are as follows. The Weber set (the convex hull of the so-called marginal values) is a subset of the selectope and, consequently, the core (which is a subset of the Weber set) is a subset of the selectope. The core and the selectope coincide if, and only if, the selectope is a subset of the imputation set, and this holds for the class of almost positive games, i.e., games where the dividends of all non-singleton coalitions are non-negative. Coincidence of the Weber set and the selectope is also characterized, by considering so-called greedy allocations. Moreover, it is shown that the selectope of a game is equal to the core of a corresponding convex game of which the core and the Weber set are determined by these greedy allocations. A consequence is a new characterization of the Shapley value in terms of these greedy allocations.

In Section 3 the selectope is considered from a value-theoretic, axiomatic point of view; in particular, it contains sharing values, random order values, and weighted Shapley values. We present a coherent system of axioms characterizing these classes of values and also give the relation with Hart-Mas-Colell consistency.

Section 4 contains a few concluding remarks and further research questions.

2. Set-theoretic approach

The *imputation set* of a game v is the set of all individually rational and efficient payoff vectors:

$$I(v) := \{x \in \mathbb{R}^N : x(N) = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N\}.$$

Here $x(S) := \sum_{i \in S} x_i$ for every $S \subseteq N$ and $x \in \mathbb{R}^N$. The *core* is the set of all imputations that are also coalitionally rational:

$$C(v) := \{x \in \mathbb{R}^N : x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \subseteq N\}.$$

For a permutation π of N and a player i, let $P_\pi(i)$ denote the set of predecessors of i according to π, that is:

$$P_\pi(i) := \{j \in N : \pi^{-1}(j) < \pi^{-1}(i)\}.$$

Then the *marginal value* m^π is defined by

$$m_i^\pi(v) := v(P_\pi(i) \cup \{i\}) - v(P_\pi(i)) \text{ for every } v \in \mathcal{G}^N \text{ and } i \in N.$$

Observe that, by (1):

$$m_i^\pi(v) = \sum_{T \subseteq P_\pi(i) \cup \{i\}, T \ni i} \Delta_v(T) \text{ for every } v \in \mathcal{G}^N \text{ and } i \in N. \quad (2)$$

In words this says that at a marginal value m^π a player obtains the dividends of every coalition in which his rank according to π is maximal.

The *Weber set* is the convex hull of the marginal values:

$$W(v) := \text{conv}\{m^\pi(v) : \pi \text{ a permutation of } N\}.$$

It is well known that the core is always contained in the Weber set (Weber, 1988; Derks, 1992). Furthermore, the Weber set is a subset of the selectope. This is a consequence of Lemma 1 below.

In order to obtain some feeling for these definitions we include an example of a three-person game in Figure 1.

Lemma 1. *Let π be a permutation of N and let $\alpha : 2^N \backslash \{\emptyset\} \to N$ be defined by*

$$\alpha(S) := \pi(\max\{j \in N : \pi(j) \in S\}) \text{ for every nonempty coalition } S.$$

Then α is a selector and $m^\alpha(v) = m^\pi(v)$ for every $v \in \mathcal{G}^N$.

Proof: Obviously α is a selector. For $i \in N$ and $v \in \mathcal{G}^N$:

$$m_i^\pi(v) = \sum_{T \subseteq P_\pi(i) \cup \{i\}, T \ni i} \Delta_v(T) = m_i^\alpha(v)$$

where the first equality follows by (2) and the second equality follows because $i = \alpha(T)$ for every $T \subseteq P_\pi(i) \cup \{i\}$ with $T \ni i$. □

For a converse of Lemma 1 the following condition on a selector is required. A selector α is *consistent* if $\alpha(S) = \alpha(T)$ whenever $S \subseteq T$ and $\alpha(T) \in S$.[1]

[1] This condition has the same mathematical form as Nash's (1950) condition of Independence of Irrelevant Alternatives for bargaining solutions.

S	1	2	3	12	13	23	123	
$v(S)$	0	0	0	1	1	-1	3	
$\Delta_v(S)$	0	0	0	1	1	-1	2	$m^\alpha(v)$
α^1	1	2	3	1	1	2	1	$4, -1, 0$
α^2	1	2	3	1	1	2	2	$2, 1, 0$
α^3	1	2	3	1	1	2	3	$2, -1, 2$
α^4	1	2	3	1	1	3	1	$4, 0, -1$
α^5	1	2	3	1	1	3	2	$2, 2, -1$
α^6	1	2	3	1	1	3	3	$2, 0, 1$
α^7	1	2	3	1	3	2	1	$3, -1, 1$
α^8	1	2	3	1	3	2	2	$1, 1, 1$
α^9	1	2	3	1	3	2	3	$1, -1, 3$
α^{10}	1	2	3	1	3	3	1	$3, 0, 0$
α^{11}	1	2	3	1	3	3	2	$1, 2, 0$
α^{12}	1	2	3	1	3	3	3	$1, 0, 2$
α^{13}	1	2	3	2	1	2	1	$3, 0, 0$
α^{14}	1	2	3	2	1	2	2	$1, 2, 0$
α^{15}	1	2	3	2	1	2	3	$1, 0, 2$
α^{16}	1	2	3	2	1	3	1	$3, 1, -1$
α^{17}	1	2	3	2	1	3	2	$1, 3, -1$
α^{18}	1	2	3	2	1	3	3	$1, 1, 1$
α^{19}	1	2	3	2	3	2	1	$2, 0, 1$
α^{20}	1	2	3	2	3	2	2	$0, 2, 1$
α^{21}	1	2	3	2	3	2	3	$0, 0, 3$
α^{22}	1	2	3	2	3	3	1	$2, 1, 0$
α^{23}	1	2	3	2	3	3	2	$0, 3, 0$
α^{24}	1	2	3	2	3	3	3	$0, 1, 2$

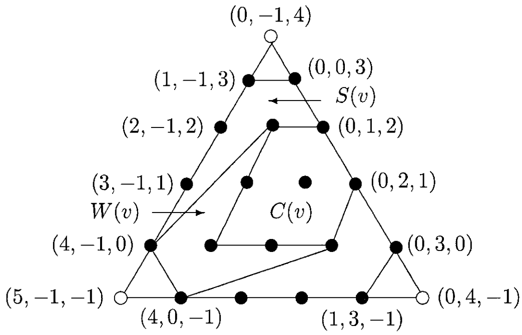

Fig. 1. Dividends, core, Weber set, and selectope in a three-person game

Lemma 2. *Let α be a selector. Then there is a permutation π of N with $m^\pi(v) = m^\alpha(v)$ for every game $v \in \mathcal{G}^N$ if and only if α is consistent. In that case, π is unique and described, recursively, by $\pi(n) = \alpha(N)$ and*

$$\pi(i) = \alpha(N \backslash \{\pi(n), \ldots, \pi(i+1)\}) \text{ for every } i \in N \backslash \{n\}.$$

Proof: Let π be a permutation of N with $m^{\pi}(v) = m^{\alpha}(v)$ for every $v \in \mathscr{G}^N$. Let T and S be coalitions with $S \subseteq T$ and $\alpha(T) \in S$. Say $i = \alpha(T)$. Define $w \in \mathscr{G}^N$ by $\Delta_w(T) := 1$ and $\Delta_w(M) := 0$ for all $M \neq T$. Then $m_i^{\alpha}(w) = 1$, so $m_i^{\pi}(w) = 1$ and by (2) $\pi^{-1}(i) = \max\{\pi^{-1}(j) : j \in T\}$. Hence also $\pi^{-1}(i) = \max\{\pi^{-1}(j) : j \in S\}$. Define $\tilde{w}$ by $\Delta_{\tilde{w}}(S) := 1$ and $\Delta_{\tilde{w}}(M) := 0$ for all $M \neq S$. Then $m_i^{\alpha}(\tilde{w}) = m_i^{\pi}(\tilde{w}) = \Delta_{\tilde{w}}(S) = 1$ and therefore $\alpha(S) = i$. It follows that α is consistent.

Next let α be a consistent selector. Define $v^n \in \mathscr{G}^N$ by $\Delta_{v^n}(N) := 1$ and $\Delta_{v^n}(M) := 0$ otherwise. Then by (2), for every permutation π, $m^{\pi}(v^n) = m^{\alpha}(v^n)$ if and only if $\alpha(N) = \pi(n)$. Define $v^{n-1} \in \mathscr{G}^N$ by $\Delta_{v^{n-1}}(N\backslash\{\pi(n)\}) := 1$ and $\Delta_{v^{n-1}}(M) := 0$ otherwise. By (2) again, for every permutation π, $m^{\pi}(v^{n-1}) = m^{\alpha}(v^{n-1})$ if and only if $\alpha(N\backslash\{\pi(n)\}) = \pi(n-1)$. By repeating this argument the unique permutation π as in the statement of the lemma is obtained. □

Summarizing, there is a one-to-one correspondence between permutations and consistent selectors, and between marginal values and selector values corresponding to consistent selectors. In Figure 1, for example, the consistent selectors are α^1, α^4, α^{12}, α^{14}, α^{20}, and α^{24}.

The relations between core and selectope, already proved by Hammer *et al.* (1977) with the aid of flows, can also be proved in a direct and elementary way, without duality-based arguments. See Corollary 1 and the equivalence between statements (i) and (iii) in Theorem 2. In the proofs of these results a lemma (Lemma 3 below) will be used which has some significance on its own.

From equation (1) one immediately derives the well known fact that each game $v \in \mathscr{G}^N$ can uniquely be represented by:

$$v = \sum_{\emptyset \neq S \subseteq N} \Delta_v(S) u_S$$

where for every $S \neq \emptyset$ the *unanimity game* u_S is defined by $u_S(T) := 1$ if $S \subseteq T$ and $u_S(T) := 0$ otherwise. These unanimity games form a basis of the linear space $\mathscr{G}^N$. Given this representation one defines

$$v^+ = \sum_{S:\Delta_v(S)>0} \Delta_v(S) u_S \quad \text{and} \quad v^- = \sum_{S:\Delta_v(S)<0} -\Delta_v(S) u_S.$$

Then $v = v^+ - v^-$.

Lemma 3. *Let $v \in \mathscr{G}^N$. Then:*

(a) $S(v^+) = C(v^+)$ *and* $S(v^-) = C(v^-)$.
(b) $S(v) = C(v^+) - C(v^-)$.[2]

Proof: (a) We only prove the first equality, the second is similar. Because $C(v^+) \subseteq S(v^+)$ it is sufficient to prove that $m^{\alpha}(v^+) \in C(v^+)$ for any selector α. This, however, follows because

[2] Sums and differences of sets are defined vector-wise.

$$\begin{aligned} m^{\alpha}(v^{+})(S) &= \sum_{i\in S}\sum_{T:i=\alpha(T)} \Delta_{v^{+}}(T) \\ &\geq \sum_{T\subseteq S} \Delta_{v^{+}}(T) \\ &= v^{+}(S) \end{aligned}$$

for every coalition $S \neq \emptyset$, where the inequality follows because all dividends in v^{+} are nonnegative and the last equality follows from (1).
(b) For any selector α, $m^{\alpha}(v) = m^{\alpha}(v^{+}) - m^{\alpha}(v^{-})$, hence $m^{\alpha}(v) \in C(v^{+}) - C(v^{-})$ in view of part (a). This proves $S(v) \subseteq C(v^{+}) - C(v^{-})$. For the converse it is, in view again of part (a), sufficient to prove that for selectors $\alpha, \beta \in A^{N}$ it holds that $m^{\alpha}(v^{+}) - m^{\beta}(v^{-}) \in S(v)$. Define the selector γ by $\gamma(S) := \alpha(S)$ if $\Delta_v(S) \geq 0$ and $\gamma(S) := \beta(S)$ if $\Delta_v(S) < 0$. Then $S(v) \ni m^{\gamma}(v) = m^{\alpha}(v^{+}) - m^{\beta}(v^{-})$. □

Lemma 3 and in particular part (b) has some important consequences. The first one is that the selectope has a core-like structure. To see this we need to recall some well known facts concerning dual and convex games.

The *dual* of a game v, denoted by v^{*}, is defined as $v^{*}(S) = v(N) - v(N\backslash S)$ for each $S \subseteq N$. It can easily be checked that, in general, $-C(v) = C((-v)^{*})$ so that in particular $-C(v^{-}) = C((-v^{-})^{*})$. Further, observe that v^{+} is a convex game and $-v^{-}$ a concave game, where a game w is *convex* if $w(S\cup T) + w(S\cap T) \geq w(S) + w(T)$ holds for all $S, T \subseteq N$, and w is *concave* if $-w$ is convex. The dual of a concave game is generally known and can easily be shown to be convex. Therefore, Lemma 3 implies that the selectope of a game is a sum of the cores of two convex games.

It is well known that the core of a game is equal to the Weber set if, and only if, the game is convex (Shapley, 1971; Ichiishi, 1981). The Weber set has the subadditivity property $W(v) + W(w) \supseteq W(v+w)$ and the core has the superadditivity property $C(v) + C(w) \subseteq C(v+w)$ for all games v, w. This implies that the core is additive on the cone of convex games, so that we have the following result.

Theorem 1. *The selectope of a game v equals the core of the convex game $\tilde{v} = v^{+} + (-v^{-})^{*}$.*

From this theorem we conclude that the geometric structure of the selectope is of a much simpler nature than that of the Weber set, since we now know that the facets of the selectope correspond to normals with $(0,1)$-coefficients, like the core, and the number of extreme elements is majorized by the maximal number of extreme elements of the Weber set. Further, the candidates for these elements are as easy to compute as the marginal values, the candidates for the extreme elements of the Weber set. This feature will be elaborated in the subsection on the Shapley value below.

An interpretation of the worth $\tilde{v}(S)$ of coalition S (see Theorem 1) is obtained by rewriting this worth in terms of dividends:

$$\begin{aligned} \tilde{v}(S) &= v^{+}(S) - v^{-}(N) + v^{-}(N\backslash S) \\ &= \sum_{T:T\subseteq S, \Delta_v(T)>0} \Delta_v(T) + \sum_{T:T\cap S\neq\emptyset, \Delta_v(T)<0} \Delta_v(T). \end{aligned} \tag{3}$$

Thus, if the worth $v(N) = \sum_T \Delta_v(T)$ is distributed among the players, so that each portion $\Delta_v(T)$ is allocated to only players in T, then the value $\tilde{v}(S)$ is a sharp lower bound for the amount the players can achieve.

Corollary 1. *The core is a subset of the selectope.*

Proof: Observe that by (3) the game v majorizes the corresponding game $\tilde{v}$. Together with $v(N) = \tilde{v}(N)$, this implies that $C(v) \subseteq C(\tilde{v})$. So by Theorem 1 we have $C(v) \subseteq S(v)$. □

Call a game v *almost positive* if the dividends of all non-singleton coalitions are non-negative. One easily shows that the set of almost positive games is a subset of the class of convex games. Theorem 2 shows that the core and selectope coincide for almost positive games, which was already noticed by Hammer *et al.* (1977).

Call v *additive* if the worth of the union of any two disjoint coalitions equals the sum of the separate worths. An additive game may be identified with the vector of the individual worths, and this vector is the unique core element of the game. Also, $C(v + w) = C(v) + C(w)$ if w is an additive game.

Theorem 2 presents another consequence of Lemma 3(b).

Theorem 2. *Let $v \in \mathcal{G}^N$. The following three statements are equivalent.*

(i) $C(v) = S(v)$,
(ii) $S(v) \subseteq I(v)$,
(iii) *v is almost positive.*

Furthermore, $S(v) = I(v)$ if and only if v is additive.

Proof: (i) implies (ii) because $C(v) \subseteq I(v)$ by definition. Assume (ii) and suppose there is a coalition S with $|S| \geq 2$ and $\Delta_v(S) < 0$. Take $i \in S$ and let α be a selector with the property that $i = \alpha(S)$ and $i \neq \alpha(T)$ for every $T \neq S, \{i\}$. Then $m_i^\alpha(v) = v(\{i\}) + \Delta_v(S) < v(\{i\})$, contradicting $S(v) \subseteq I(v)$. This proves that (ii) implies (iii). Next, assume that v is almost positive. Then $C(v) = C(v^+ - v^-) = C(v^+) - C(v^-) = S(v)$, where the second equality follows because $-v^-$ is an additive game and the third equality follows by Lemma 3.

For the final statement of the theorem, observe that v is additive if, and only if, $\Delta_v(S) = v(\{i\})$ for $S = \{i\}$ and $\Delta_v(S) = 0$ otherwise, which is true if, and only if, $S(v) = \{(v\{1\}), \ldots, v(\{n\})\} = I(v)$. □

See Crama *et al.* (1989) for a characterization of the class of almost positive games in terms of inequalities on the coalitional worths.

The remainder of this section is organized in two subsections. The first one deals with the relationship between the selectope and the Shapley value. Although values are the topic of the next section we discuss the Shapley value at this place because of the close relationship with the second subsection, on the coincidence of the Weber set and the selectope.

2.1. Selector values and the Shapley value

The Shapley value (Shapley, 1953) is the best known solution concept within the theory of cooperative games. There are many different ways to introduce this value and here we will describe a new one. Well known is the description of the Shapley value as the average of the marginal values:

$$\varphi(v) = \frac{1}{n!} \sum_{\pi} m^{\pi}(v),$$

where the summation is taken over all permutations of the player set N. Alternatively therefore, φ is the average of the consistent selector values (see Lemma 1 and Lemma 2). Now consider the average of all selector values, not just the consistent ones, say ϕ. This value is related to the selectope in the same way as the Shapley value is related to the Weber set. In the sum of all selector values the dividend $\Delta_v(T)$ has to be distributed equally often to all players in T, implying that in the average the dividend of T is equally shared between the players of T. Therefore,

$$\phi_i(v) = \sum_{T:T \subseteq N, T \ni i} \frac{\Delta_v(T)}{|T|},$$

and this expression is again a standard description of the Shapley value (hence $\phi = \varphi$).

Next, consider the marginal values $m^{\pi}(\tilde{v})$ of the game $\tilde{v}$. The convex hull of these values is the Weber set and hence the core of the convex game $\tilde{v}$ and by Theorem 1 this set is equal to the selectope of v. Then,

$$\begin{aligned} m_i^{\pi}(\tilde{v}) &= \tilde{v}(P_{\pi}(i) \cup \{i\}) - \tilde{v}(P_{\pi}(i)) \\ &= v^{+}(P_{\pi}(i) \cup \{i\}) - v^{+}(P_{\pi}(i)) \\ &\quad + v^{-}(N \backslash (P_{\pi}(i) \cup \{i\})) - v^{-}(N \backslash P_{\pi}(i)) \\ &= \sum_{T:i=\max_{\pi}(T), \Delta_v(T)>0} \Delta_v(T) + \sum_{T:i=\min_{\pi}(T), \Delta_v(T)<0} \Delta_v(T), \end{aligned} \tag{4}$$

where $\max_{\pi}(T)$ denotes the player in T which is last in the queue $\pi(1), \ldots, \pi(n)$, and similarly, $\min_{\pi}(T)$ is the player in T who is the first player of T in this queue.

From the point of view of the first players one can argue that this allocation is far from profitable: the first player $\pi(1)$ in the queue receives all negative dividends of the coalitions he is a member of, and his own dividend $\Delta_v(\{\pi(1)\})$. The players thereafter receive, one by one, the positive dividends of only those coalitions where they are the last player, and the negative dividends of those coalitions where they are the first player. From this point of view, the last player is best off. Nevertheless, we will call these allocations *greedy allocations* because looking in the reverse direction of the 'queue' a player may grab all positive dividends yet to come while leaving the negative dividends for the players lower in the queue. These greedy allocations are to be considered as allocations in the game v. In the example in Figure 1 the greedy allocations are exactly the extreme points of the selectope $S(v)$, i.e., of $C(\tilde{v}) = W(\tilde{v})$ (cf. Theorem 1).

The greedy allocations are the marginal values in the game $\tilde{v}$ so that the average of these allocations is the Shapley value of $\tilde{v}$. Moreover, the Shapley value is a linear function on the game space, with the property that the values of any game and its dual coincide. Therefore,

$$\varphi(\tilde{v}) = \varphi(v^+) + \varphi((-v^-)^*) = \varphi(v^+) - \varphi(v^-) = \varphi(v),$$

and we conclude that the Shapley value equals the average of the greedy allocations as well.

2.2. The selectope and the Weber set

In this subsection we characterize those games for which the selectope and the Weber set coincide. The question is tackled by examining the greedy allocations: When is the allocation $m^\pi(\tilde{v})$ an element of the Weber set for a game v and permutation π? Observe that for any selector α, with

$$\alpha(T) = \max_\pi(T) \quad \text{for all } T \text{ with } \Delta_v(T) > 0, \quad \text{and}$$

$$\alpha(T) = \min_\pi(T) \quad \text{for all } T \text{ with } \Delta_v(T) < 0,$$

we have $m^\pi(\tilde{v}) = m^\alpha(v)$ in view of (4). If it is possible to choose α to be consistent, then we call permutation π *consistent in the game* v (the consistency property here is indeed dependent on data of the game; see also later). So, if π is consistent in v then the greedy allocation $m^\pi(\tilde{v})$ equals a consistent selector value, which by Lemma 2 is equal to a marginal value. Therefore, if all permutations are consistent in the game then the Weber set must contain the selectope, implying equality. The converse also holds:

Theorem 3. *The selectope and the Weber set of a game v coincide if and only if all permutations are consistent in v.*

Proof: The if-part has been addressed above. For the only-if part suppose that for a game v there is a permutation π that is not consistent in v. Without loss of generality assume that π is the identity permutation. Take $p \in \mathbb{R}^N$ with $p_1 < p_2 < \cdots < p_n$. Let α be a consistent selector such that $p \cdot m^\alpha(v)$ is maximal subject to all consistent selector values. Because π is not consistent in v we can take a coalition S with either $\alpha(S) \neq \max_\pi(S)$ and $\Delta_v(S) > 0$, or $\alpha(S) \neq \min_\pi(S)$ and $\Delta_v(S) < 0$. Let the selector α' be equal to α for all coalitions $T \neq S$, and $\alpha'(S) = \max_\pi(S)$ if $\Delta_v(S) > 0$ or $\alpha'(S) = \min_\pi(S)$ if $\Delta_v(S) < 0$. If $\Delta_v(S) > 0$ then

$$p \cdot m^{\alpha'}(v) = p \cdot m^\alpha(v) + (p_{\max_\pi(S)} - p_{\alpha(S)})\Delta_v(S) > p \cdot m^\alpha(v),$$

and if $\Delta_v(S) < 0$ then

$$p \cdot m^{\alpha'}(v) = p \cdot m^{\alpha}(v) + (p_{\min_\pi(S)} - p_{\alpha(S)})\Delta_v(S) > p \cdot m^{\alpha}(v).$$

In both cases we have $p \cdot m^{\alpha'}(v) > p \cdot m^{\alpha}(v)$ which proves that $S(v) \neq W(v)$. □

The consistency property of a permutation in a game v is dependent on the signs of the dividends and not on their actual values. It is therefore possible to generalize the consistency definition as follows. Let $\mathscr{U}$ and $\mathscr{W}$ be two disjoint collections of coalitions. A permutation π is called *$(\mathscr{U}, \mathscr{W})$-consistent* if a consistent selector α exists such that $\alpha(S) = \max_\pi(S)$ if $S \in \mathscr{U}$, and $\alpha(S) = \min_\pi(S)$ if $S \in \mathscr{W}$.

In Derks and Peters (1998) it is proved that the existence of a permutation that is not $(\mathscr{U}, \mathscr{W})$-consistent is equivalent to the existence of a sequence $S_1, S_2, \ldots, S_k$ of coalitions, at least one of which is in $\mathscr{U}$ and at least one of which is in $\mathscr{W}$, and k different players $i_1, \ldots, i_k$, such that $i_j \in S_j \cap S_{j+1}$, with $j = 1, \ldots k$ and S_{k+1} equal to S_1. Applying this result one easily proves the following corollary.

Let $\mathscr{D}^+(v)$ denote the set of players who are members of non-singleton coalitions with positive dividends in the game v, and similarly, $\mathscr{D}^-(v) = \bigcup\{S : |S| > 1, \Delta_v(S) < 0\}$.

Corollary 2. *In a game v the Weber set and the selectope coincide if*

$$|\mathscr{D}^+(v) \cap \mathscr{D}^-(v)| \leq 1. \tag{5}$$

Almost positive games obviously satisfy (5). The same is true for *almost negative* games (which have all non-singleton dividends nonpositive). Every two-person game trivially is almost negative or almost positive. In a game satisfying (5) non-singleton subcoalitions of coalitions with positive [resp. negative] dividend have nonnegative [resp. nonpositive] dividend. In general, there are games satisfying (5) other than the almost positive or almost negative games. Moreover, not every convex game satisfies (5), as the following example shows.

Example 1. Let $v \in \mathscr{G}^{\{1,2,3\}}$ be defined by $v(\{1,2\}) = v(\{1,3\}) = 1$, $v(\{2,3\}) = 9$, $v(\{1,2,3\}) = 10$, and $v(\{i\}) = 0$ for every $i \in \{1,2,3\}$. It is easy to check that this game is convex. The dividend of the grand coalition is equal to -1 and the dividends of the two-person coalitions $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$ are equal to 1, 1, and 9, respectively, so the game does not satisfy (5). Also note that the maximum that player 1 can obtain in any marginal value is equal to 1, whereas $m_1^\alpha(v) = 2$ for any selector α with $\alpha(\{1,2\}) = \alpha(\{1,3\}) = 1$ and $\alpha(\{1,2,3\}) \neq 1$. This implies that for this game the Weber set is strictly smaller than the selectope.

3. Value-theoretic approach

In this section the selectope and its subsets will be considered from a value-theoretic point of view.

A *sharing system* is a collection $q = (q_{S,\cdot})_{S \in 2^N \setminus \{\emptyset\}}$ with $q_S \in \mathbb{R}^S$, $q_{S,i} \geq 0$ for all $i \in S$, and $\sum_{i \in S} q_{S,i} = 1$, for every nonempty coalition S. Let Q^N

denote the set of all sharing systems on N. With every sharing system $q \in Q^N$ a *sharing value* m^q is associated, defined by:

$$m_i^q(v) := \sum_{S:i\in S} q_{S,i}\Delta_v(S) \text{ for every } v \in \mathcal{G}^N \text{ and every } i \in N.$$

Such a sharing value distributes the dividends of a game between the players according to the weights given by the sharing system q. The following lemma relates sharing values with the selectope. First note that all values m^q are linear; in general, a value $\psi : \mathcal{G}^N \to \mathbb{R}^N$ is called *linear* if $\psi(\beta v + \gamma w) = \beta\psi(v) + \gamma\psi(w)$ for all $v, w \in \mathcal{G}^N$ and $\beta, \gamma \in \mathbb{R}$.

Lemma 4. *Let v be a game.*

(i) *For any sharing system q,*

$$m^q(v) = \sum_{\alpha:\alpha\in A^N} q_\alpha m^\alpha(v) \tag{6}$$

where $q_\alpha := \prod_{S\neq\emptyset} q_{S,\alpha(S)}$.

(ii) *For any collection $(p_\alpha)_{\alpha\in A^N}$ with $p_\alpha \geq 0$ for all $\alpha \in A^N$ and $\sum_{\alpha\in A^N} p_\alpha = 1$,*

$$m^q(v) = \sum_{\alpha:\alpha\in A^N} p_\alpha m^\alpha(v) \tag{7}$$

where $q_{S,i} := \sum_{\alpha:\alpha(S)=i} p_\alpha$.

(iii) $S(v) = \{m^q(v) : q \in Q^N\}$.

Proof: (iii) follows from (i) and (ii). By linearity of the values m^q (and m^α) it is sufficient to show (6) and (7) for a unanimity game u_S. For a player $j \notin S$ equation (7) has zero on both sides because in u_S the coalition S has dividend equal to 1 and all other coalitions have zero dividend. For a player $i \in S$ equation (7) reduces to $q_{S,i} = \sum_{\alpha:\alpha(S)=i} p_\alpha$, which is true by definition. Also equation (6) has zero on both sides for players outside S. For a player $i \in S$ it reduces to $q_{S,i} = \sum_{\alpha:i=\alpha(S)} \prod_{T\neq\emptyset} q_{T,\alpha(T)}$. This equality is a standard property that can be proved in an elementary way. □

Thus, sharing values are convex combinations of selector values and fill up the selectope. If in equation (6) or (7) the summation is taken only over consistent selectors, then the corresponding sharing value is called a *random order value*, cf. Weber (1988) and Lemma 2 above. Thus, random order values are convex combinations of marginal values and fill up the Weber set.

A sharing system q is called *consistent* if the following holds for all non-empty coalitions S and T:

$$\text{If } i \in S \subseteq T \text{ and } \sum_{j\in S} q_{T,j} > 0, \text{then } q_{S,i} = \frac{q_{T,i}}{\sum_{j\in S} q_{T,j}}.$$

Sharing values corresponding to consistent sharing systems are *weighted*

Shapley values: see Shapley (1953), Kalai and Samet (1987), and Monderer *et al.* (1992). In the last paper it is shown that weighted Shapley values cover the core. They are always contained in the Weber set, and marginal values are weighted Shapley values. The Shapley value (introduced in the preceding section) is the unique symmetrically weighted Shapley value, with weights $q_{S,i} = 1/|S|$ for every nonempty coalition S and every $i \in S$. Here $|S|$ denotes the cardinality of the set S.

Sharing values, random order values, weighted Shapley values, and the Shapley value can be characterized by coherent sets of axioms. The axioms under consideration are the following, defined for a value $\psi : \mathcal{G}^N \to \mathbb{R}^N$.

(i) *Additivity*: $\psi(v + w) = \psi(v) + \psi(w)$ for all $v, w \in \mathcal{G}^N$.
(ii) *Null-Player*: $\psi_i(v) = 0$ whenever i is a null player in v, i.e., $v(S \cup \{i\}) = v(S)$ for every $S \in 2^N$.
(iii) *Efficiency*: $\psi(v)(N) = v(N)$ for every game v.
(iv) *Positivity*: $\psi(v) \geq 0$ whenever v is almost positive and $v(\{i\}) \geq 0$ for every $i \in N$.
(v) *Monotonicity*: $\psi(v) \geq 0$ whenever v is monotonic, i.e., $S \subseteq T$ implies $v(S) \leq v(T)$ for all $S, T \in 2^N$.
(vi) *Partnership*: $\psi_i(\psi(u_T)(S)u_S) = \psi_i(u_T)$ for all $S \subseteq T \subseteq N$ and all $i \in S$.
(vii) *Symmetry*: $\psi_i(v) = \psi_j(v)$ whenever i and j are symmetric in v, i.e., $v(S \cup \{i\}) = v(S \cup \{j\})$ for every $S \in 2^{N \setminus \{i,j\}}$.

Most axioms can be found in the literature. The partnership axiom is slightly weaker than the corresponding axiom in Kalai and Samet (1987). The positivity axiom is new. Table 1 gives an overview over values and axioms. A "+" means that the (all) corresponding value(s) satisfies or satisfy the corresponding axiom. A "−" means the opposite. Characterizing systems are found by collecting axioms denoted by "⊕". The results summarized in Table 1 are collected in the following theorem.

Theorem 4. *Let $\psi : \mathcal{G}^N \to \mathbb{R}^N$ be a value satisfying Additivity, Null-Player, and Efficiency. Then:*

(a) ψ satisfies Positivity if and only if ψ is a sharing value.
(b) ψ satisfies Monotonicity if and only if ψ is a random order value.

Table 1. Axiomatics of sharing values

	Shapley value	weighted Shapley values	random order values	sharing values
Additivity	⊕	⊕	⊕	⊕
Null-Player	⊕	⊕	⊕	⊕
Efficiency	⊕	⊕	⊕	⊕
Positivity	+	⊕	+	⊕
Monotonicity	+	+	⊕	−
Partnership	+	⊕	−	−
Symmetry	⊕	−	−	−

(c) ψ satisfies Positivity and Partnership if and only if ψ is a weighted Shapley value.
(d) ψ satisfies Symmetry if and only if ψ is the Shapley value.

Proof: To prove that all values satisfy the desired axioms is left to the reader. Attention will now be restricted to the opposite directions of the characterizations.
(a) Suppose ψ satisfies Positivity. For every $\emptyset \neq S \subseteq N$ and $i \in S$ define $q_{S,i} := \psi_i(u_S)$. By Efficiency, Null-Player and Positivity it follows that $(q_{S,\cdot})_{S \in 2^N \backslash \{\emptyset\}}$ is a sharing system. In order to prove that $\psi = m^q$ it is, in view of Additivity of ψ and m^q, sufficient to prove that for a unanimity game u_S and $\alpha \in \mathbb{R}$, $\psi(\alpha u_S) = m^q(\alpha u_S)$. Because $m^q(\alpha u_S) = \alpha m^q(u_S) = \alpha\psi(u_S)$ by definition of q and m^q, it is sufficient to prove $\psi(\alpha u_S) = \alpha\psi(u_S)$. By Additivity this is true for all $\alpha \in \mathbb{N}$, and by Efficiency and Positivity also for $\alpha = 0$. Furthermore, for any $\alpha \in \mathbb{R}$ note that $0 = \psi((\alpha - \alpha)u_S) = \psi(\alpha u_S) + \psi(-\alpha u_S)$ so that $\psi(-\alpha u_S) = -\psi(\alpha u_S)$. For α rational, say $\alpha = \frac{k}{\ell}$ with k, ℓ integer, observe that by Additivity $\ell\psi\left(\frac{k}{\ell}u_S\right) = k\psi(u_S)$ so that $\psi(\alpha u_S) = \alpha\psi(u_S)$ follows after dividing by ℓ. Finally, for $\alpha \in \mathbb{R}$, let γ, δ be rational numbers with $\gamma < \alpha < \delta$. By Positivity, $\psi((\alpha - \gamma)u_S) \geq 0$ and $\psi((\delta - \alpha)u_S) \geq 0$, hence

$$\gamma\psi(u_S) = \psi(\gamma u_S) \leq \psi(\alpha u_S) \leq \psi(\delta u_S) = \delta\psi(u_S).$$

By letting $\gamma \uparrow \alpha$ and $\delta \downarrow \alpha$ it follows that $\psi(\alpha u_S) = \alpha\psi(u_S)$.
(b) Suppose ψ satisfies Monotonicity. Then it also satisfies Positivity and hence, by (a), Linearity. The desired result now follows from Theorems 4 and 13 in Weber (1988).
(c) This is straightforward from (a) and the definition of a consistent sharing system.
(d) Well known and straightforward. □

It can be shown that all characterizations in Theorem 4 are tight: The axioms are logically independent. The characterization of the sharing values is new. The characterization of the random order values of Weber (1988) is slightly strenghtened here by the use of Additivity instead of Linearity. There does not seem to be an elementary and simple characterizing description of the sharing systems in random order values. A property satisfied by such sharing systems is monotonicity, where a sharing system $(q_{S,\cdot})_{S \in 2^N \backslash \{\emptyset\}}$ is called *monotonic* if $q_{S,i} \geq q_{T,i}$ for all $i \in N$ and $S, T \subseteq N$ with $i \in S \subseteq T$. In other words, a player's weight cannot increase as the coalition becomes larger. In order to prove that the sharing system q in a random order value m^q is monotonic, let $S \subseteq T$ and consider the monotonic game $w = u_S + u_{T \backslash S} - u_T$. Then $\Delta_w(S) = \Delta_w(T \backslash S) = 1$, $\Delta_w(T) = -1$, and all other dividends in w are equal to 0. For a player $i \in S$ one has $m_i^q(w) = q_{S,i} - q_{T,i}$ by definition of m^q. Because m^q is monotonic, $m_i^q(w) \geq 0$ implying $q_{S,i} \geq q_{T,i}$ (which was to be proved). However, monotonic weights not necessarily imply Monotonicity of the value, as the following example shows.

Example 2. Consider the four-person game (N, v) with $N = \{1, 2, 3, 4\}$ and with $v(S) = 0$ for $S \in \{\{2, 4\}, \{3, 4\}\}$ and for S with $|S| = 1$, and with $v(S) =$

1 otherwise. This is clearly a monotonic game. Consider the sharing system q defined by $q_N = (0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $q_S = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ for all S with $|S| = 3$, $q_S = (\frac{1}{3}, \frac{2}{3})$ for all S with $1 \in S$ and $|S| = 2$, and with $q_S = (\frac{1}{2}, \frac{1}{2})$ for all two-person coalitions not containing player 1. Clearly, q is a monotonic system. It is straightforward to check, however, that $m_1^q(v) = -\frac{1}{2}$. So the sharing value m^q is not monotonic.

The characterization of the weighted Shapley values in Theorem 4 is a variation on the characterization by Kalai and Samet (1987). The partnership axiom is basically a translation of the idea of consistency of a sharing system to consistency in a game-theoretic sense. The latter property is usually related to the idea of a reduced game, and the partnership axiom indeed has the flavour of a reduced game property. The last result of this section is explicitly concerned with reduced games. To this end the definition of a value ψ is extended to $\mathcal{G} := \bigcup_{N \subseteq \mathbb{N}, |N| < \infty} \mathcal{G}^N$, such that $\psi(N, v) \in \mathbb{R}^N$ for every $(N, v) \in \mathcal{G}$. In Hart and Mas-Colell (1989) the following *reduced game* is introduced. Let $(N, v) \in \mathcal{G}^N$ and $U \subseteq N$, then $(N \backslash U, v_{U,\psi}) \in \mathcal{G}^{N \backslash U}$ is defined by $v_{U,\psi}(\emptyset) := 0$ and:

$$v_{U,\psi}(S) := v(S \cup U) - \sum_{i \in U} \psi_i(S \cup U, v) \quad \text{for all } S \subseteq N \backslash U, S \neq \emptyset.$$

A value ψ is called *consistent* if $\psi_i(N, v) = \psi_i(N \backslash U, v_{U,\psi})$ for all $(N, v) \in \mathcal{G}$, $U \subseteq N$, and $i \in N \backslash U$.

Also the concept of a sharing system can be extended to a system $q = (q_{S,\cdot})_{S \subseteq \mathbb{N}, |S| < \infty}$ in the obvious way. Consistency of such an extended system q is defined in the same way as for the finite case.

Theorem 5. *Let q be a (an extended) sharing system. The following two statements are equivalent.*

(i) q is consistent,
(ii) m^q is consistent.

Proof: For (ii) $\Rightarrow$ (i) assume that m^q is a consistent sharing value. In order to prove that q is consistent, let $i \in S \subseteq T$ and assume that $\sum_{j \in S} q_{T,j} > 0$. It is straightforward to calculate that the reduced game $(S, (u_T)_{N \backslash S, m^q})$, that is, the unanimity game u_T reduced to the coalition S, assigns $\sum_{j \in S} q_{T,j}$ to the grand coalition S and 0 to all other coalitions. So $m_i^q(S, (u_T)_{N \backslash S, m^q}) = q_{S,i} \sum_{j \in S} \cdot q_{T,j}$. By consistency of m^q one has $m_i^q(S, (u_T)_{N \backslash S, m^q}) = m_i^q(u_T)$, hence $q_{S,i} \sum_{j \in S} q_{T,j} = m_i^q(u_T)$. This proves consistency of q.

For (i) $\Rightarrow$ (ii) first note that for a linear value ψ and games v and w one has $(v + w)_{U,\psi} = v_{U,\psi} + w_{U,\psi}$, and $(\alpha v)_{U,\psi} = \alpha v_{U,\psi}$ for any scalar α. Hence, to deduce consistency of m^q from consistency of the sharing system q it suffices to consider unanimity games and their reduced games. The proof is then completed by reversing the above argument. □

Thus, within the class of sharing values the weighted Shapley values are exactly the values that are consistent. Alternatively, weighted Shapley values can be characterized by consistency together with "consistency across two-person games". This is proved in Hart and Mas-Colell (1989) for the case of strictly positive weights.

4. Concluding remarks

One of the remaining questions on this research topic is which other solution concepts – besides the core and the sharing values are caught by the selectope.

For the *nucleolus* (Schmeidler, 1969) the answer is negative. The nucleolus, an imputation by definition, need not belong to the Weber set. In particular, the set of imputations may have an empty intersection with the Weber set (see Martinez-de-Albinez and Rafels, 1996) and also with the selectope, as the following example shows.

Example 3. Let the 4-person game v be defined by $v(S) = 0$ for $|S| \in \{0, 1, 4\}$, $v(3,4) = -10$, $v(S) = 1$ for all other 2-person coalitions, $v(1,2,3) = v(1,2,4) = 4$, and $v(1,3,4) = v(2,3,4) = -7$. The dividends in this game are equal to 0 for the 1-person coalitions, -10 for coalition $\{3,4\}$, and 1 for all the other coalitions. In any point of the selectope players 1 and 2 together obtain at least 1, whereas the unique imputation and thus the nucleolus is the zero vector. So the latter is not in the selectope.

Also the *prenucleolus* (see Sobolev, 1975) – which is not necessarily an imputation – need not belong to the Weber set, as the following example shows.

Example 4. Let the 4-person game v be defined as in the following table.

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$v(S)$	2	0	6	3	2	0	5	0	1	0	1	3	0	6	6

The prenucleolus of this game is the vector $(1.75, -1.5, 4.25, 1.5)$. It can be shown that this vector is not in the Weber set, by checking that it is separated from the Weber set by a hyperplane with normal vector equal to $(3, 0, 1, 0)$.

It is an open problem whether the prenucleolus has to be in the selectope.

Another interesting question is whether the selectope can be characterized by a reduced game property. Since the selectope is always equal to the core of an associated convex game, a candidate would be the Davis-Maschler reduced game property (cf. Peleg (1992)). This topic is left for further research.

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