Int J Game Theory (2000) 29:413–432

International Journal of Game Theory

First price auctions: Monotonicity and uniqueness*

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Received: August 1997/Last revision: March 2000

Abstract. I study monotonicity and uniqueness of the equilibrium strategies in a two-person first price auction with affiliated signals. Existing results establish uniqueness within the class of non-decreasing bidding strategies. I show that there is an *effectively* unique Nash equilibrium within the class of *piecewise monotone* strategies. The main result is that in equilibrium, the strategies must be strictly increasing within the support of *winning bids*. This result provides the missing link for the analysis of uniqueness in two-person first price auctions. The analysis applies to asymmetric environments as well and does not require risk neutrality.

Journal of Economic Literature Classification Number: D44.

Key words: auctions, monotonicity, uniqueness.

1. Introduction

Most of the literature on auctions focuses on symmetric equilibria in which the buyers' bids increase monotonically in the signals they observe. For symmetric sealed bid first price auctions there is only one symmetric equilibrium in monotone strategies, but it is not yet fully understood whether there may be other equilibria that do not satisfy symmetry and monotonicity (Wilson, 1988).

* This paper is based on the second chapter of my Ph.D thesis.

† I am especially grateful to Charles Wilson for innumerable discussions on this topic. I also appreciate the very useful comments of two referees.

There are a few authors who address the question of uniqueness but only in very specific environments. For instance, Griesmer, Levitan and Shubik (1967) provide a complete characterization of the equilibrium set of a two-person first price auction in which the bidders' private valuations are independently and uniformly distributed. More recently, Lebrun (1996) and Maskin and Riley (1996a and 1996b) establish existence and uniqueness for the independent private values model with symmetric bidders and provide partial extensions for asymmetric versions of that model.

However, a general treatment of environments that involve common values and correlated signals is still missing. Maskin and Riley (1986) constitutes a first step in tackling this problem. For the case of a two-person symmetric auction with affiliated signals, that paper shows that the symmetric equilibrium is the only equilibrium in which the bidders' strategies are monotone non-decreasing in their respective signals. Lizzeri and Persico (1995, 1997) extend this result to general asymmetric two-bidder auctions with a reserve price. Using a representation of the equilibrium conditions as a system of differential equations, their analysis proves the existence of a pure strategy equilibrium in monotone strategies, and also shows that this equilibrium is *effectively unique* within the class of monotone non-decreasing strategies (effectively unique in the sense that uniqueness needs to hold only on the support of bids that can win with positive probability or *winning bids*). Using different techniques, Athey (1997) also establishes the existence of a pure strategy equilibrium in monotone strategies for the two-bidder model. Her argument is based on the fact that the best reply to a monotone strategy must be monotone. However, a best reply to a non-monotone strategy is not necessarily monotone, so it is not immediately clear why there cannot also be equilibria in non-monotone strategies.

Ruling out the possibility that there are other equilibria in non-monotone strategies remains as the main hurdle to solving the problem of uniqueness in the two-bidder case. In this paper, I address this problem within the context of a sealed bid first price auction with affiliated signals. Aside from the restriction to two bidders, the environment that I consider is rather general, encompassing the *general symmetric model* of Milgrom and Weber (1982) as well as the preceding models as particular cases. Importantly, the analysis does not rely on either symmetry or risk neutrality requirements. However, I require that the bidding strategies satisfy a regularity condition: buyers' bids must be *piecewise monotone* in their respective signals. Note that since all strategies are piecewise monotone when the signals are discrete-valued, the condition is not restrictive in this particular case.

The main findings are the following. I show that the equilibrium bidding strategies must actually be strictly increasing in the respective signals, within the support of winning bids (Proposition 1). Combining with the preceding results, it then follows that the equilibrium in non-decreasing strategies is effectively unique (Proposition 2). As an illustration, I also include a complete characterization of the equilibrium strategies for the case of symmetric auctions with discrete-valued signals and no reserve price (Proposition 3).

Most of the paper is devoted to the monotonicity argument, which is rather elaborate and relies heavily on the assumption that the signals are affiliated. The complexity of the argument stems from the fact that, generally, it is not possible to prove directly a global result about best replies being monotone. Instead, the approach used in this paper consists in proving a local result about best responses. Roughly, this local result shows that if the strategy of a bidder never crosses a bid b_0 from above, then for some $\varepsilon > 0$ and every $b \in [b_0 b_0 + \varepsilon)$, the other bidder's best response never crosses b from above. Combined with a statement about the infimum of the support of winning bids $\underline{b}$, which says that the strategy of at least one bidder never crosses $\underline{b}$ from above, the local result can be expanded into a global result about the equilibrium strategies: for any bid b in the support of winning bids, none of the strategies ever crosses b from above. In other words, the strategies must be monotone non-decreasing on the support of winning bids. Given the technical character of the analysis, it is convenient to introduce the model and some definitions before attempting a more detailed discussion. An informal account of the main issues involved is included at the beginning of section 4.

The paper is organized as follows. In section 2 I lay out the model and introduce the basic notation. In section 3 I derive some basic results about the support of the equilibrium distribution of bids. Section 4 is devoted to establishing monotonicity of the equilibrium strategies (Proposition 1). Section 5 applies this result to establish uniqueness (Proposition 2) and characterizes the solution for the discrete, symmetric case as an illustration (Proposition 3). Section 6 is devoted to concluding remarks. Most of the proofs are relegated to the appendix.

2. The model

2.1. Description and basic assumptions

In this section I lay out the model of a sealed bid first price auction. There are two bidders who submit their bids simultaneously. A single object is awarded to the highest bidder as long as his bid is at least as large as a reserve price r. In case of a tie, the winner is selected randomly. Moreover, the price paid by the winner equals his own bid.

The problem is modeled as a two-person Bayesian game. Throughout the paper I refer to the players as bidder i or bidder j, making the convention that $i, j = 1, 2$ and that $i \neq j$. Each player selects his bid after privately observing the value of a signal. Let $\mathbf{s}_i$ denote the signal observed by bidder i (where I adopt the convention of using bold case letters for random variables). This signal takes values in a set $S_i \subset \mathbb{R}$, which I suppose to be either a closed interval or a finite set. Let $F : S_1 \times S_2 \to [0, 1]$ denote the joint distribution of the players' signals. I suppose that F has a density function f with support $S_1 \times S_2$ (in the discrete case, f denotes the density with respect to the counting measure). Moreover, I assume that the signals are *affiliated*. In general, the components of a random vector are affiliated if any two non-decreasing functions of the random vector are positively correlated conditional on every sublattice of the support of the joint distribution. In particular, two random variables whose distribution has a density are affiliated if and only if the probability density function is *Totally Positive of Order 2* (TP_2) (Karlin and Rinott, 1980; Milgrom and Weber, 1982; Whitt, 1982). In the present case, this means that the distribution of signals satisfies the following assumption:

(A_1) $f(s)f(s') \leq f(s \vee s')f(s \wedge s')$ *for every* $s, s' \in S_1 \times S_2$, *where* $s \vee s'$ *and* $s \wedge s'$ *respectively denote the component-wise maximum and minimum of* s *and* s'.

The bidders' preferences are represented by a von Neumann-Morgenstern utility function. For each $(s_1, s_2) \in S_1 \times S_2$, let $u_i(b, s_i, s_j)$ denote bidder i's utility if he makes a bid $b \in \mathbb{R}_+$ and obtains the object. The utility function is normalized so that the utility of a bidder that does not obtain the object is zero. I also assume that u_i satisfies the following conditions:

(A_2) *The function u_i is continuously differentiable, strictly decreasing in b, strictly increasing in s_i and non-decreasing in s_j.*

(A_3) *For $b_1 > b_0$, the difference $u_i(b_1, s_i, s_j) - u_i(b_0, s_i, s_j)$ is non-decreasing in s_i.*[1]

2.2. Strategy sets and additional notation

Let $\mathscr{B}$ denote the Borel σ-algebra associated to $\mathbb{R}_+$ and let $\mathscr{P}(\mathscr{B})$ denote the set of probability measures on $\mathscr{B}$. A behavior strategy for bidder i is a mapping $\beta_i : S_i \to \mathscr{P}(\mathscr{B})$ such that, for each $B \in \mathscr{B}$, the application $\langle \beta_i, B \rangle$ is a measurable function from S_i to the interval $[0, 1]$. The mapping $\tilde{\beta}_i : S_i \to \mathbb{R}_+$ is a *selection* from β_i if $\tilde{\beta}_i(s_i) \in supp\, \beta_i(s_i)$ for every $s_i \in S_i$. I also denote $\beta_i^{-1}(b) = \{s_i \in S_i : b \in supp\, \beta_i(s_i)\}$. The strategy β_i and the marginal distribution of bidder i's signal induce a probability distribution over buyer i's bids. The infimum and the supremum of the support of this distribution will be denoted $\underline{b}_i$ and $\bar{b}_i$, respectively. Given a strategy profile $\beta = (\beta_1, \beta_2)$, I will denote $\underline{b} = \max\{\underline{b}_1, \underline{b}_2, r\}$ and $\bar{b} = \max\{\bar{b}_1, \bar{b}_2\}$.

The following concepts of monotonicity and sign-monotonicity provide a convenient way of describing the model and stating the results.

Definition 1. *Let Y denote a measurable subset of $\mathbb{R}$ and consider a mapping $\varphi : Y \to \mathbb{R}$. The map φ is* **monotone non-decreasing** *(***strictly increasing***) if $\varphi(y) \leq (<)\ \varphi(y')$ when $y < y'$. The map φ is* **monotone non-increasing** *(***strictly decreasing***) if $-\varphi$ is monotone non-decreasing (strictly increasing). The map φ is* **quasi single crossing** *(***QSC***) if $\varphi(y) \geq 0 \Rightarrow \varphi(y') \geq 0$ for every $y' > y$. The map φ is* **single crossing** *(***SC***) if it is QSC and $\varphi(y) > 0 \Rightarrow \varphi(y') > 0$ for every $y' > y$. The map φ is* **weakly QSC** *if $\varphi(y) > 0 \Rightarrow \varphi(y') \geq 0$ for every $y' > y$. Finally, φ is* **strictly SC** *if $\varphi(y) \geq 0 \Rightarrow \varphi(y') > 0$ for every $y' > y$.*

For any of the preceding properties, I will say that this property holds **almost everywhere** as long as it is satisfied in a set $\tilde{S}_i \subset S_i$ whose complement in S_i is a set of measure zero, with respect to the Lebesgue measure in the case in which the strategy sets are closed intervals, and with respect to the counting measure in the case of discrete-valued signals.

Applying the preceding definitions to our strategy space, I define the following concepts.

[1] A map $\phi : Y \to \mathbb{R}$, where $Y \subset \mathbb{R}^n$, is said to be *supermodular* if $\phi(y \vee y') + \phi(y \wedge y') \geq \phi(y) + \phi(y')$ for every $y, y' \in Y$. Using this terminology, A_1 says that $\log f$ is supermodular, and A_3 says that, for every s_j, the function $u_i(b, s_i, s_j)$ is supermodular in (b, s_i). Athey (1997) employs a related but slightly stronger condition by assuming that u_i is supermodular both in (b, s_i) and in (b, s_j).

Definition 2. *A strategy β_i is* **monotone non-decreasing** *(***strictly increasing***) if every selection $\tilde{\beta}_i$ from β_i is monotone non-decreasing (strictly increasing) almost everywhere. Similarly, a strategy β_i is* **monotone non-increasing** *(***strictly decreasing***) if every selection $\tilde{\beta}_i$ from β_i is monotone non-increasing (strictly decreasing) almost everywhere. A strategy β_i is* **piecewise monotone** *if there is a finite increasing sequence $(k_m)_{m=1,\dots,M}$ in S_i such that $k_1 = \inf S_i$, $k_M = \sup S_i$ and, for each $m = 1, \dots, M-1$, the restriction of β_i to $\{s_i \in S_i : k_m \le s_i \le k_{m+1}\}$ is either monotone non-decreasing or monotone non-increasing. A strategy β_i is* **monotone at** *b if $\tilde{\beta}_i - b$ is SC almost everywhere, for every selection $\tilde{\beta}_i$ from β_i. Similarly, we define* **quasi-monotonicity at** *b (***weakly quasi-monotonicity at** *b) by replacing SC by QSC (weakly QSC) in the last definition.*

Now we can state briefly the regularity condition that the strategy sets are required to satisfy:

(A_4) *The bidders are restricted to using piecewise monotone strategies.*

This assumption is not restrictive at all when the strategy sets S_i are discrete (to see this, let $\{k_1, \dots, k_M\} = S_i$ and notice that the restriction of β_i to each subset $\{s_i \in S_i : k_m \le s_i \le k_{m+1}\} = \{k_m, k_{m+1}\}$ clearly is either monotone non-decreasing or monotone non-increasing). However, it imposes significant restrictions in the continuous case. For instance, it requires that the bidders use *essentially pure* strategies in this case.[2] Still, it allows for a meaningful variety of strategic behavior, including all the essentially pure strategies that can be represented almost everywhere by functions that are piecewise polynomial. In fact, if the pure strategy $\beta_i : [a, b] \to \mathbb{R}$ is a differentiable function, it is piecewise monotone if and only if the sign of the derivative changes finitely many times. However, pure strategies that exhibit *wildly* oscillating behavior, like the differentiable function $\varphi(x) = x^2 \sin(1/x)$ defined on $[-1, 1]$, are not piecewise monotone.

The following notation will be used throughout the paper. Suppose that bidder j employs the strategy β_j. Given β_j, let $Q_i(b, s_j)$ denote the probability that bidder i obtains the object provided that he bids b and that bidder j's signal is s_j. Also let $\pi_i(b, s_i)$ denote bidder i's expected payoff when he bids b and his signal is s_i. More explicitly, $\pi_i(b, x) = E(u_i(b, \mathbf{s}_i, \mathbf{s}_j) Q_i(b, \mathbf{s}_j) | \mathbf{s}_i = x)$. Thus, bidder i's optimization problem when his signal is s_i can be concisely stated as selecting a bid b to maximize $\pi_i(b, s_i)$.

[2] A strategy β_i is *essentially pure* if *supp* $\beta_i(x)$ is a singleton for almost every $x \in S_i$. Since in the continuous case a given piecewise monotone strategy β_i must be monotone in each element of a finite partition of S_i in intervals, it should not come as a surprise that like monotone strategies, piecewise monotone strategies rule out randomization in the continuous (atomless) case but not in the discrete case. In fact, showing that every strategy that satisfies assumption A_4 is *essentially pure* in the atomless case reduces to showing that monotone strategies must be essentially pure. The following brief argument is included for completeness. Suppose that β_i is piecewise monotone but not essentially pure. Then we can select a set of positive (Lebesgue) measure $X \subset S_i$ such that the restriction of β_i to X is monotone, and that, for every $x \in X$, there are at least two different bids a_x and b_x in *supp* $\beta_i(x)$, with $a_x < b_x$. Denote $I_x = (a_x, b_x)$. By monotonicity, $I_x \cap I_y = \emptyset$ for every $x, y \in X$. Since each interval I_x contains a rational number, there can be at most countably many such sets, which contradicts the fact that, being a set of positive measure, X is uncountable in the atomless case.

Since $Q_i(b, s_j)$ is monotone non-decreasing in its first argument, the corresponding right and left limits exist everywhere. Moreover, I denote $Q_i(b^+, s_j) = \lim_{t \downarrow b} Q_i(t, s_j)$ and $Q_i(b^-, s_j) = \lim_{t \uparrow b} Q_i(t, s_j)$. Similarly, I denote $\pi_i(b^+, s_i) = \lim_{t \downarrow b} \pi_i(t, s_i)$ and $\pi_i(b^-, s_i) = \lim_{t \uparrow b} \pi_i(t, s_i)$. The following result follows directly from the definitions.

Lemma 1. *Suppose that β_j is monotone at b. Then there is a parameter $\hat{s}_j$ such that $Q_i(b^+, s_j) = 1$ for almost every $s_j < \hat{s}_j$ and $Q_i(b^+, s_j) = 0$ for almost every $s_j > \hat{s}_j$.*

Proof: By definition of monotonicity at b, there exist some signals $\tilde{s}_j$ and $\hat{s}_j$, with $\tilde{s}_j < \hat{s}_j$, such that β_j selects (with probability 1) a bid lower than b at (almost every) $s_j < \tilde{s}_j$, a bid higher than b at (almost every) $s_j > \hat{s}_j$, and selects exactly b at (almost every) $s_j \in [\tilde{s}_j, \hat{s}_j]$. Take a sequence $b_n \downarrow b$, and note that as $n \to \infty$, $Q_i(b_n, s_j) \to 1$ for almost every $s_j < \hat{s}_j$, and $Q_i(b_n, s_j) \to 0$ for almost every $s_j > \hat{s}_j$. □

3. Preliminary results

This section reviews some properties of the support of the equilibrium bidding distributions. In essence, I rule out the possibility of gaps and, in some sense, mass points in the interior of the support of winning bids. Since most of the arguments are already standard, I will only discuss the results briefly. Nonetheless, detailed proofs are supplied in the appendix for completeness.

The first observation is that none of the bidders ever selects a winning bid at which his expected payoff exhibits a jump discontinuity. Since this kind of discontinuity can only occur because the other bidder's strategy has a mass point, the tie-breaker makes the expected payoff of the player in question an average of the right and left limits of his expected payoff at the point of discontinuity. But then he can improve by either bidding a little more or a little less. Thus, we can state the following:

Lemma 2. *Suppose that the bid $\hat{b} > \underline{b}$ is optimal for bidder i when he observes the signal x. Then $\pi_i(b, x)$ is continuous at $\hat{b}$.*

To rule out gaps in the support the intuition is simple: If a bidder's support presents a gap $(b - \varepsilon, b]$, the other bidder never bids an amount close to b, since he can reduce his bid, and expected payment, without affecting his chance of winning. This line of thought leads to contradiction, establishing the following result.

Lemma 3. *The support of the distribution of bids of each bidder contains the interval $[\underline{b}, \bar{b}]$. Moreover, $\bar{b}_1 = \bar{b}_2 = \bar{b}$.*

Finally, note that when a bidder's strategy is quasi-monotone and presents a mass point at a bid $b > b$, the "types" of this player that bid b are larger than the ones that bid less than b. Since the other bidder can increase his payoff by increasing slightly his bid to "capture" these higher types, his payoff jumps discontinuously at b. Thus, he never bids in some interval $(b - \varepsilon, b]$, contradicting Lemma 3. In fact, one can show the following.

Lemma 4. *Suppose that β_j is quasi-monotone at $b \in (\underline{b}, \bar{b}]$. Then β_j is atomless and, consequently, monotone at b.*

4. Monotonicity of the equilibrium strategies

The monotone comparative statics literature identifies an intimate link between the *single crossing property of incremental returns* and monotonicity of the optimal choices with respect to a parameter (Milgrom and Shannon, 1994). In the present context, we can fix the strategy of bidder *j* and apply that methodology to analyze the best reply by bidder *i*. I say that *i*'s incremental returns satisfy the single crossing property if, for every pair of bids b_0 and b_1 such that $b_0 < b_1$, the incremental return $\pi_i(b_1, s_i) - \pi_i(b_0, s_i)$ is strictly SC in s_i. It is clear that when this occurs, bidder *i*'s optimal strategy must be monotone non-decreasing. To verify this fact just note that if a bid b_1 is optimal for *i* when his signal is s_i, it must be strictly preferred to any lower bid b_0 when his signal is larger than s_i, provided that the incremental returns satisfy the single crossing property.

If bidder *i*'s incremental returns satisfied the single crossing property independently of the strategy used by the other bidder, we would be able to conclude directly that, in equilibrium, bidder *i* must employ a non-decreasing strategy. However, the condition is verified only in very special cases (independent private values model). Typically, when the signals are affiliated, we need to restrict the strategy used by bidder *j* to be able to conclude that *i*'s incremental returns satisfy the single crossing property.

Fact 1. *Suppose that the strategy used by bidder j is monotone non-decreasing. Then the incremental returns of bidder i satisfy the single crossing property.*[3]

As a consequence of the preceding result, the best reply to a non-decreasing strategy must be a non-decreasing strategy. This fact plays a central role in the argument to establish existence of equilibrium in non-decreasing strategies used in Athey (1997). However, we cannot build an argument to show that the equilibrium strategies *must* be monotone on the assumption that one of the equilibrium strategies is monotone. Thus, we need a sharper result to address issues of monotonicity and uniqueness.

The single crossing property of incremental returns is a *global property* in the sense that it refers to every pair of bids $b_0, b_1 \in \mathbb{R}_+$ such that $b_0 < b_1$. Since we cannot assume that the strategy of bidder *j* is monotone, we can only hope for a *local version* of that property to be satisfied. Lemma 7 in the next subsection, provides the appropriate generalization of Fact 1. This result relaxes the condition to local monotonicity, but sacrifices the global character of the conclusion. More precisely, it shows that if bidder *j*'s strategy is monotone at a given bid b_0 (i.e.: it never crosses b_0 from above), then, for every $b_1 > b_0$, the incremental return $\pi_i(b_1, s_i) - \pi_i(b_0, s_i)$ is strictly SC in s_i (where the local character of the conclusion stems from the fact that the bid b_0 is given).

Using this result, one can prove a local monotonicity result about the equilibrium strategies: if bidder *j*'s equilibrium strategy is monotone at b_0,

[3] This fact is a consequence of Lemma 7 below. A similar result appears in Athey (1996).

then, for some $\varepsilon > 0$ and every $b \in [b_0, b_0 + \varepsilon)$, bidder i's strategy is monotone at b (Lemma 9). Combining this with a statement saying that the strategy of at least one of the bidders must be monotone at $\underline{b}$ (Lemma 10), it then follows that Lemma 9 can be expanded from a local to a global result: for every b in the support of winning bids, both bidders' strategies must be monotone at b. This clearly means that the equilibrium strategies must be monotone non-decreasing on the support of winning bids. Using the fact that the interior of the equilibrium bid distributions must be atomless, we conclude that the equilibrium strategies must be strictly increasing on the support of winning bids (Proposition 1).

4.1. Single crossing property of incremental returns

The main result of this section is Lemma 7, which provides the conditions that ensure that the incremental returns satisfy the local version of the single crossing property, a result that will play a key role in proving local monotonicity results in the next subsection. Lemmas 5 and 6 contain the steps one must verify to prove Lemma 7. In particular, Lemma 6 contains the main consequence for the present analysis of the assumption that the signals are affiliated. A detailed account of the argument follows.

For the remainder of this section I fix the strategy β_j of bidder j in order to examine the best response by bidder i. I also fix two bids b_0 and b_1, where $b_0 < b_1$. To examine the dependence of $\pi_i(b_1, s_i) - \pi_i(b_0, s_i)$ on the signal s_i, I introduce the following concepts which will help articulate the analysis. I define a mapping $\delta_i : S_i \times S_j \to \mathbb{R}$ such that $\delta_i(s_i, s_j) = u_i(b_1, s_i, s_j)Q_i(b_1, s_j) - u_i(b_0^+, s_i, s_j)Q_i(b_0^+, s_j)$. In words, $\delta_i(s_i, s_j)$ represents bidder i's incremental return (from increasing his bid from a bid just above b_0 to b_1) given that the bidders' signals take the values s_i and s_j. I also define a mapping $h_i : S_i \times S_j \to \mathbb{R}$ such that $h_i(x, y) = E(\delta_i(x, \mathbf{s}_j)|\mathbf{s}_i = y)$. Notice that $\pi_i(b_1, s_i) - \pi_i(b_0^+, s_i) = h_i(s_i, s_i)$. The first argument of h_i reflects the dependence of $\pi_i(b_1, s_i) - \pi_i(b_0^+, s_i)$ on s_i through the direct dependence of δ_i on s_i and the second argument reflects the indirect dependence on s_i through the conditional distribution of $\mathbf{s}_j$. In other words, the first argument reflects direct information about the expected value of the object, and the second argument reflects information about both the expected value of the object and player 2's bidding behavior, which is revealed by s_i indirectly through its statistical linkage with $\mathbf{s}_j$.

Naturally, a first step to understand h_i relies on the analysis of δ_i. Consider first the dependence of δ_i on s_i. Note that

$$\begin{aligned}
&\delta_i(s_i', s_j) - \delta_i(s_i, s_j) \\
&\quad = Q_i(b_1, s_j)[u_i(b_1, s_i', s_j) - u_i(b_1, s_i, s_j)] \\
&\qquad - Q_i(b_0^+, s_j)[u_i(b_0, s_i', s_j) - u_i(b_0, s_i, s_j)] \\
&\quad \geq [Q_i(b_1, s_j) - Q_i(b_0^+, s_j)][u_i(b_0, s_i', s_j) - u_i(b_0, s_i, s_j)] \qquad (1)
\end{aligned}$$

where the inequality is a consequence of assumption A_3. Thus, assumption A_2 and the fact that $Q_i(b_1, s_j) - Q_i(b_0, s_j) \geq 0$ imply that δ_i is non-decreasing in its first argument, and that it is strictly increasing when the preceding inequality is strict. Thus, if bidder i's probability of obtaining the object is strictly larger when he bids b_1 than when he bids b_0, as it will be in the relevant cases, h_i will be strictly increasing in the first argument.

Examination of the dependence of δ_i on s_j is somewhat more intricate. Suppose that β_j is monotone at b_0, Lemma 4 implies that it is also atomless at b_0. Clearly, there must exist a signal $\hat{s}_j$ such that if bidder i bids b_0, he wins if and only if $s_j < \hat{s}_j$. Although bidder i also wins with b_1, he prefers to win with a lower bid. Then, $\delta_i(s_i, s_j) = u_i(b_1, s_i, s_j) - u_i(b_0, s_i, s_j) < 0$ when $s_j < \hat{s}_j$. On the other hand, when $s_j > \hat{s}_j$, bidder i loses when he bids b_0. Thus, in this case $\delta_i(s_i, s_j) = Q_i(b_1, s_j) u_i(b_1, s_i, s_j)$, which can only be positive if $u_i(b_1, s_i, s_j)$ is positive. Since u_i is non-decreasing in s_j by assumption, then $\delta_i(s_i, s_j) > 0$ implies that $\delta_i(s_i, s_j') \geq 0$ for $s_j' > s_j$. We must conclude that δ_i is weakly QSC in s_j. The following lemma summarizes the preceding discussion.

Lemma 5. *The following properties are true for almost every s_j in S_j:*

(i) *δ_i is non-decreasing in s_i. Moreover, $Q_i(b_1, s_j) > Q_i(b_0, s_j)$ implies that $\delta_i(s_i, s_j) < \delta_i(s_i', s_j)$ for $s_i < s_i'$.*
(ii) *Suppose that β_j is monotone at b_0. Then δ_i is weakly QSC in s_j.*

The consequence of Lemma 5(i) is straightforward: h_i is non-decreasing in its first argument, and it is strictly increasing whenever bidder i's probability of obtaining the object is strictly larger when he bids b_1 than when he bids b_0. In fact, it can be shown that h_i is strictly SC in its first argument.

When the signals are independent, h_i is constant in its second argument. However, statistical dependence between signals establishes a probabilistic linkage between the signal observed by a bidder and the bids selected by the other bidder, which also affects the first bidder's expected payoff. In fact, the analysis of the dependence of h_i on its second argument constitutes a more subtle problem whose structure depends crucially on the assumption that the signals are affiliated. To study this effect I rely on the following property of TP_2 transformations (related results appear in Karlin (1968)).

Lemma 6. *Consider two mappings $\varphi : S_j \to \mathbb{R}$ and $\psi : S_i \to \mathbb{R}$ such that $\psi(x) = E(\varphi(\mathbf{s}_j)|\mathbf{s}_i = x)$. Suppose that the joint distribution of $\mathbf{s}_i$ and $\mathbf{s}_j$ satisfies assumption A_1 and that φ is weakly QSC. Then ψ is a QSC mapping.*

I conclude directly from Lemma 5(ii) and Lemma 6 that h_i is QSC in its second argument. Combining this with the fact that h_i is strictly SC in its first argument, I establish the following.

Lemma 7. *Suppose that β_j is monotone at b_0. Then, for every winning bid $b_1 > b_0$, the incremental return $\pi_i(b_1, s_i) - \pi_i(b_0^+, s_i)$ is strictly SC in s_i.*

Proof: Suppose that $h_i(x, x) \geq 0$. Since β_j is monotone at b_0, Lemma 5(ii) implies that δ_i is weakly QSC in s_j. Then Lemma 6 implies h_i is QSC in its second argument. We conclude that $h_i(x, x') \geq 0$ for $x' > x$.

I claim that $Q_i(b_1, \mathbf{s}_j) - Q_i(b_0^+, \mathbf{s}_j) > 0$ with positive probability with respect to the distribution of $\mathbf{s}_j$ conditional on $\mathbf{s}_i = x'$. Arguing by contradiction, suppose $Q_i(b_1, \mathbf{s}_j) - Q_i(b_0^+, \mathbf{s}_j) = 0$ with probability 1. This means that $\delta_i(x, \mathbf{s}_j) = [u_i(b_1, x, \mathbf{s}_j) - u_i(b_0, x, \mathbf{s}_j)] Q_i(b_1, \mathbf{s}_j) \leq 0$ with probability 1. But since

$Q_i(b_1, \mathbf{s}_j) > 0$ with positive probability because b_1 is a winning bid for i, we conclude that $\delta_i(x, \mathbf{s}_j) < 0$ with positive probability. Thus, $h_i(x, x') < 0$, a contradiction that establishes the claim.

Lemma 5(i) and the preceding claim imply that $\delta_i(x, \mathbf{s}_j) \leq \delta_i(x', \mathbf{s}_j)$ with probability 1 and that the inequality is strict with positive probability. Thus, $h_i(x', x') - h_i(x, x') = E(\delta_i(x', \mathbf{s}_j) - \delta_i(x, \mathbf{s}_j) \mid \mathbf{s}_i = x') > 0$. We conclude that $h_i(x', x') > 0$, as desired. □

The following result is a corollary of Lemma 7.

Lemma 8. *Suppose that b_1 is a winning bid for bidder i. Then $\pi_i(b_1, s_i)$ is strictly SC in s_i.*

Proof: Just select $b_0 < \underline{b}$ and apply Lemma 7. □

4.2. Main result

In this section I show that the equilibrium strategies are strictly increasing within the support of winning bids, which is the main result of the paper (Proposition 1). The argument proceeds by first establishing local monotonicity at both the interior and the infimum of the support of winning bids, and then expanding these local results into a global one by an induction sort of logic. I rely on the local single crossing properties derived in the preceding section to obtain the required local monotonicity results, which are contained in the following two lemmas.

The next lemma shows that if a bidder's strategy is monotone at each point of the interval $(\underline{b}, b_0)$, a best response by the other bidder must be monotone at each point of a larger interval $(\underline{b}, b_0 + \varepsilon)$. The following illustration contains the essence of the argument. Suppose that both bidders' strategies are monotone at each bid lower than b_0, where $b_0 > \underline{b}$. Also suppose that there are two signals x_0 and x_1, with $x_0 > x_1$, and a bid $b_1 > b_0$ such that b_1 is optimal for bidder i when his signal is x_1. If, additionally, b_0 is optimal for bidder i when his signal is x_0, bidder i's strategy is not monotone at b_0. However, since b_1 is optimal for i when his signal is x_1, we must have that $\pi_i(b_1, x_1) \geq \pi_i(b_0, x_1)$. Then Lemma 7 implies that $\pi_i(b_1, x_0) > \pi_i(b_0, x_0)$, contradicting the optimality of b_0 when i's signal is x_0.

Lemma 9. *Let β denote an equilibrium. Suppose that β_i is weakly quasi-monotone at b_0 and that β_j is monotone at b_0. Then there is some $\varepsilon > 0$ such that both β_i and β_j are monotone at each bid $b \in (b_0, b_0 + \varepsilon)$.*

Proof: First, I claim that given a selection $\tilde{\beta}_i$ from β_i that is weakly quasi-monotone at b_0, if there is a sequence $(b_n)_{n=1,\dots,\infty}$ such that $b_n \downarrow b_0$ and $\tilde{\beta}_i$ is not monotone at b_n for $n = 1, \dots, \infty$, then the graph of $\tilde{\beta}_i$ must have two cluster points (x_0, b_0) and (x_1, b_1) with $x_0 > x_1$ and $b_0 < b_1$. Denote $\tilde{x} = \operatorname{ess\,inf}\{s_i \in S_i : \tilde{\beta}_i(s_i) > b_0\}$.[4] Since $\tilde{\beta}_i$ is piecewise monotone, there must exist

[4] The essential supremum (infimum) of a set A, denoted ess sup A (ess inf A), is defined as the infimum (supremum) of sup B (inf B) as B ranges over all the sets B such that the symmetric difference between A and B is a set of measure zero.

some $x_1 \geq \tilde{x}$ and some $b_1 > b_0$ such that $\tilde{\beta}_i(x_1) = b_1$ and the restriction of $\tilde{\beta}_i$ to $\{s_i \in S_i : \tilde{x} \leq s_i \leq x_1\}$ is monotone (in particular, in the discrete case, one can always take $x_1 = \tilde{x}$). Using the fact that $\tilde{\beta}_i$ is weakly quasi-monotone at b_0, we conclude that the restriction of $\tilde{\beta}_i$ to $\{s_i \in S_i : s_i \leq x_1\}$ must be monotone at b_n. Let X_n denote the closure of the set $\{s_i \in S_i : b_n \geq \tilde{\beta}_i(s_i)\}$. Also denote $X = \bigcap_{n=1}^{\infty} X_n$ and $x_0 = \operatorname{ess\,sup} X$. Since $\tilde{\beta}_i$ is not monotone at b_n, then the restriction of $\tilde{\beta}_i$ to $\{s_i \in S_i : s_i \leq x_0\}$ cannot be monotone at b_n. We conclude that $x_1 < x_0$. Note that as a consequence of the fact that $\tilde{\beta}_i$ is weakly quasi-monotone at b_0, we have that $\tilde{\beta}_i(s_i) \geq b_0$ for almost every $s_i > \tilde{x}$, so in view of the definition of x_0, the pair (x_0, b_0) must actually be a cluster point of the graph of $\tilde{\beta}_i$. This establishes the claim.

I also claim that if β_i is weakly quasi-monotone at b_0 and, for every winning bid $b > b_0$, the incremental return $\pi_i(b, s_i) - \pi_i(b_0^+, s_i)$ is strictly SC in s_i, then there must be some $\varepsilon > 0$ such that β_i is monotone at each $b \in (b_0, b_0 + \varepsilon)$. I argue by contradiction. Take a selection $\tilde{\beta}_i$ from β_i and suppose that there is a sequence $(b_n)_{n=1,\ldots,\infty}$ such that $b_n \downarrow b_0$ and that $\tilde{\beta}_i$ is not monotone at b_n, for $n = 1, \ldots, \infty$. By the preceding claim, the graph of $\tilde{\beta}_i$ must have two cluster points (x_0, b_0) and (x_1, b_1) with $x_0 > x_1$ and $b_0 < b_1$. By optimality we have that $\pi_i(b_1, x_1) \geq \pi_i(b_0^+, x_1)$. Since the incremental return is strictly SC, we must then have that $\pi_i(b_1, x_0) > \pi_i(b_0^+, x_0)$. Hence, for n sufficiently large and x close to x_0, we have that $\pi_i(b_1, x) > \pi_i(b^n, x)$, a contradiction that establishes the claim.

Since β_j is monotone at b_0, Lemma 7 implies that, for $b > b_0$, the incremental return $\pi_i(b, s_i) - \pi_i(b_0^+, s_i)$ is strictly SC in s_i. Hence, the last claim implies that β_i is monotone at each $b \in (b_0, b_0 + \varepsilon')$, for some $\varepsilon' > 0$. Using again Lemma 7, this implies that, for $b > b_0$, also $\pi_j(b, s_j) - \pi_j(b_0^+, s_j)$ is strictly SC in s_j, so the claim implies that also β_j is monotone at each $b \in (b_0, b_0 + \varepsilon'')$, for some $\varepsilon'' > 0$. Take $\varepsilon = \min\{\varepsilon', \varepsilon''\}$ and the statement follows. □

A variation of the preceding argument allows us to restrict equilibrium behavior at the infimum of the support of winning bids.

Lemma 10. *Both β_1 and β_2 are weakly quasi-monotone at $\underline{b}$. At least one of them is actually monotone at $\underline{b}$.*

Combining Lemma 9 and Lemma 10, we reach the conclusion that the equilibrium strategies are monotone increasing over the whole interior of the support of winning bids. Our main result follows.

Proposition 1. *Let β denote an equilibrium and also denote $S_i^0 = \{s_i \in S_i : s_i > \sup \beta_i^{-1}(\underline{b})\}$. Then, for $i = 1, 2$, we have that $\beta_i(s_i) \leq \underline{b}$ with probability 1 when $s_i \notin S_i^0$, and also that the restriction of β_i to S_i^0 is strictly increasing. In other words, at every equilibrium, the strategy employed by each bidder is monotone at each winning bid, and atomless at each $b > \underline{b}$.*

Proof: Denote $m = \inf\{b > \underline{b}$: *either β_1 or β_2 is not monotone at b*$\}$. First, I rule out the possibility that $m = \underline{b}$. In fact, Lemma 10 shows that both bidders' strategies are weakly quasi-monotone at $\underline{b}$ and that the strategy of at least one of them is monotone at $\underline{b}$. Then Lemma 9 implies that both bidders' strategies are monotone at each bid $b \in (\underline{b}, \underline{b} + \varepsilon)$, for some $\varepsilon > 0$. This implies that

$m > \underline{b}$. Now suppose that $\underline{b} < m \leq \bar{b}$. By definition of m, both β_1 and β_2 are monotone at each $b \in (\underline{b}, m)$, so they are quasi-monotone at m. Since $m > \underline{b}$, Lemma 4 implies that both β_1 and β_2 are monotone at m. Again, Lemma 9 implies that both β_1 and β_2 are monotone at each $b \in (\underline{b}, m + \varepsilon)$, for some $\varepsilon > 0$. Since this contradicts the definition of m, we must conclude that $m = \infty$, as desired. □

Remark 1: The Role of Assumption A_4. The need of restricting the analysis to piecewise monotone behavior strategies is connected to the crucial role played by Lemma 7 in the proof of Lemma 9. Lemma 7 is a local version of the single crossing property of incremental returns. Since such property concerns the effect of a change in a bidder's signal on his incremental returns, it is natural to focus on behavior strategies, which represent the bidders' behavior in terms of their respective signals. Moreover, since Lemma 7 only provides a *local* statement about incremental returns when the bid is increased from a *given* value b_0 to any larger value, say b_1, it only permits to rule out scenarios in which the graph of a selection $\tilde{\beta}_i$ from β_i has two cluster points (x_0, b_0) and (x_1, b_1) in $S_i \times \mathbb{R}_+$ such that $x_0 > x_1$ and $b_0 < b_1$, a parameter configuration that leads to contradiction when incremental returns are strictly single crossing locally. However, to rule out non-monotonic behavior right above b_0, we also need to be able to establish a necessary link between such behavior and the preceding parameter configuration. It turns out that when assumption A_4 is satisfied such link exists; more precisely, the proof of Lemma 9 shows that when a behavior strategy β_i is piecewise monotone, if this strategy is weakly quasi-monotone at b_0 but it is not monotone right above b_0, the desired parameter configuration must occur, and consequently, a contradiction is reached.

If we allow strategies that are not piecewise monotone, the preceding parameter configuration may not arise even when β_i is not monotone above b_0. An example is the case of a strategy $\hat{\beta}_i$ such that $\hat{\beta}_i(x) = \max\{x^2, x\sin(1/x)\}$ for $x > 0$ and $\hat{\beta}_i(0) = 0$, where $S_i = [0, 1]$. This strategy is monotone at $b_0 = 0$ but its value oscillates infinitely many times between x and x^2 in the interval $(0, \delta)$, for each $\delta > 0$, so it is not monotone above b_0.[5] Clearly, given any (x_1, b_1) such that $\hat{\beta}_i(x_1) = b_1 > 0$, the graph of $\hat{\beta}_i$ does not have any cluster point (x_0, b_0) with $x_0 > x_1$ and $b_0 = 0$, so the method of proof described above cannot be employed in this case.[6]

[5] This example has been suggested by a referee.

[6] Although assumption A_4 is not restrictive in the discrete case, it imposes substantial restrictions in the atomless case (see Footnote 2). One may wonder whether Proposition 1 can be generally extended to some *reasonable* class of mixtures of piecewice monotone strategies by using the present methods. However, since in the atomless case even mixtures of finitely many monotone pure strategies may give place to *realization equivalent* behavior strategies that are not piecewise monotone, we should not expect an extension on these lines. The following example illustrates this point. Consider a family of pure strategies $g_\alpha = (1-\alpha)g_0 + \alpha g_1$, where $g_0(x) = x$, $g_1(x) = x^2$ and $0 \leq \alpha \leq 1$. Let σ_i denote the mixed strategy that randomizes uniformly among the preceding strategies, and let β_i denote the realization equivalent behavior strategy that selects the value $g_\alpha(x)$, with α uniformly distributed on $[0, 1]$, when bidder i's signal is x. Although σ_i randomizes among strictly increasing strategies, the selections from β_i are not necessarily piecewise monotone; for instance, the pure strategy $\hat{\beta}_i$ mentioned in Remark 1 is one such selection. In fact, it is easy to see that even mixtures among the two monotone strategies g_0 and g_1 would run into a similar problem.

5. Uniqueness of the equilibrium

In this section I combine Proposition 1 with earlier results to investigate uniqueness of the equilibrium. Maskin and Riley (1986) establish existence and uniqueness within the class of non-decreasing strategies for the case in which the game is symmetric and the sets of bidders' signals are closed intervals. Lizzeri and Persico extend the analysis to asymmetric environments. They require the following additional assumptions.

(A_5) *f is continuously differentiable.*
(A_6) $u_i(b, s_i, s_j)$ *is strictly increasing in* s_j.
(A_7) $u_i(r, \underline{s}_i, \bar{s}_j) < 0$ *and* $u_i(b, \bar{s}_i, \bar{s}_j) < 0$ *for some b, where* $S_i = [\underline{s}_i, \bar{s}_i]$ *for* $i = 1, 2 \ldots$

where A_6 rules out the case of pure private values, and the first condition in A_7 implies that there is a positive probability that none of the buyers select a winning bid.

The method employed in the preceding papers relies on the fact that equilibria in non-decreasing strategies can be represented, within the support of winning bids, as solutions of a system of differential equations determined by the first order conditions of each buyer's optimization problem.

For an environment that satisfies assumptions A_1–A_7, Lizzeri and Persico show that there is a unique equilibrium in non-decreasing strategies for the case in which the set of signals is a closed interval. Their argument is based on the following "relative toughness logic". When signals are affiliated, a buyer finds it more profitable to win the higher the other buyer's signal is. Thus, with monotone strategies, if buyer j bids b when his signal is relatively high, buyer i will find bidding b convenient even when his signal is relatively low. Consequently, if there were two equilibria and the type of buyer i that bids b were relatively higher in the first equilibrium, the type of bidder j would be relatively lower (i.e.: for every winning bid b, $\tilde{\beta}_i^{-1}(b) > \hat{\beta}_i^{-1}(b)$ implies that $\tilde{\beta}_j^{-1}(b) < \hat{\beta}_j^{-1}(b)$). However, this would lead to a contradiction with the terminal equilibrium requirement that both buyers must bid the same amount when their signal is highest (i.e.: $\tilde{\beta}_i(\bar{s}_i) = \tilde{\beta}_j(\bar{s}_j)$ and $\hat{\beta}_i(\bar{s}_i) = \hat{\beta}_j(\bar{s}_j)$). Consequently, there cannot be two different equilibria. In fact, Lizzeri and Persico also establish existence by showing that there always exists a solution to the differential equations that satisfies the preceding terminal requirement as well as some relevant conditions at $\underline{b}$.

Combining these results with Proposition 1, one reaches the following important conclusion.

Proposition 2. *Suppose that assumptions A_1–A_7 are satisfied. Then an equilibrium exists and is effectively unique. Moreover, at the equilibrium both bidders employ pure strategies that are strictly increasing within the support of winning bids.*

Proof: Lizzeri and Persico (1997) establish existence of an equilibrium in monotone strategies and show that this equilibrium is effectively unique within the class of monotone non-decreasing strategies. Combining with Proposition 1, the desired conclusion follows. □

As an illustration, I also include an extension of the uniqueness argument in Maskin and Riley (1986) to the case of a symmetric auction with discrete-valued signals and no reserve price.[7] The symmetry requirement is summarized by the following assumption:

(A_8) (i) $S_1 = S_2 = S$
(ii) $u_i(b, s_i, s_j) = u_j(b, s_i, s_j) = u(b, s_i, s_j)$
(iii) *F is a symmetric distribution.*

The following lemma shows that symmetry implies additional restrictions at the infimun of the support of winning bids when there is no reserve price.

Lemma 11. *Suppose that $r = 0$ and that assumptions A_1–A_3 and A_8 are satisfied. Then, at every equilibrium, $\underline{b}_1 = \underline{b}_2 = \underline{b}$. Moreover, (i) when $S = [\underline{s}, \bar{s}]$, no bidder has a mass point at $\underline{b}$, and (ii) when $S = \{\underline{s}, \ldots, \bar{s}\}$, we have that $\beta_i^{-1}(\underline{b}) = \beta_j^{-1}(\underline{b}) = \{\underline{s}\}$ and that the type $\underline{s}$ of at least one player bids $\underline{b}$ with probability 1, where $u(\underline{b}, \underline{s}, \underline{s}) = 0$.*

Maskin and Riley's uniqueness result essentially is a particular version for symmetric environments of the preceding one.[8] It relies on the idea that if the type of buyer i that bids b is larger than the type of buyer j that bids b, then the type of buyer i must be larger for every bid higher than b. This contradicts the requirement that $\beta_i^{-1}(\bar{b}) = \beta_j^{-1}(\bar{b}) = \{\bar{s}\}$ and shows that the unique equilibrium in monotone strategies must be the symmetric one. Using this idea and Proposition 1, I obtain the following result.

Proposition 3. *Suppose that $r = 0$, that assumptions A_1–A_3 and A_8 are satisfied, and that $S = \{\underline{s}, \underline{s}+1, \ldots, \bar{s}\}$. Then the following (randomized) strategies, which are monotone increasing within the support of winning bids, constitute the unique equilibrium of this game.[9] For the lowest type of each bidder, we have that $Q_i(\underline{b}, \underline{s}) = 1/2$ and $Q_i(b, \underline{s}) = 0$ for $b > \underline{b}$. On the other hand, for $x > \underline{s}$, we have that:*

$$Q_i(b, x) = 0 \qquad b < b_{x-1}$$

$$Q_i(b, x) = \frac{\sum_{s=\underline{s}}^{x} u(b_x, x, s) f(s|x) - \sum_{s=\underline{s}}^{x-1} u(b, x, s) f(s|x)}{u(b, x, x) f(x|x)} \qquad b \in [b_{x-1}, b_x]$$

$$Q_i(b, x) = 1 \qquad b > b_x$$

[7] An equilibrium may fail to exist in asymmetric first price auctions with discrete-valued signals. See, for instance, Milgrom and Weber (1985).

[8] More precisely, Maskin and Riley (1986) establish that there is a unique equilibrium within the class of non-decreasing strategies in an environment that satisfies assumptions A_1–A_3 and A_8. It also requires that the support of F is a rectangle in $\mathbb{R}^2$, and that (i) $f(s_i|s_j)/F(s_i|s_j)$ is strictly decreasing in s_i, (ii) $\frac{\partial}{\partial b} u(b, s_i, s_j)$ is a non-increasing function, and (iii) $\frac{\partial^2}{\partial b \partial s_i} u(b, s_i, s_j) \leq \frac{\partial^2}{\partial b \partial s_j} u(b, s_i, s_j)$. Under these assumptions, we can also combine their result with Proposition 1 to establish uniqueness in piecewise monotone strategies for this case as well.

[9] Note that for every $b > \underline{b}$, $Q_i(b, x) = Prob.(\beta_j(x) < b)$ defines a randomized strategy for bidder j.

where $\underline{b} = b_{\underline{s}}$, *and* $b_{\underline{s}}, b_{\underline{s}+1}, \ldots, b_{\bar{s}}$ *is the unique solution to the system:*

$$u(b_{\underline{s}}, \underline{s}, \underline{s}) = 0$$

$$\sum_{s=\underline{s}}^{x} u(b_x, x, s) f(s|x) - \sum_{s=\underline{s}}^{x-1} u(b_{x-1}, x, s) f(s|x) = 0 \quad \textit{for } x = \underline{s}+1, \ldots, \bar{s}$$

Proof: Appendix.

Remark 2: All the results in this paper can also be stated in the context of a formulation with unconstrained strategy sets. The corresponding version of the results for this case is as follows: (i) every equilibrium in which the bidders use piecewise monotone strategies must be strictly increasing within the support of winning bids (Proposition 1), and (ii) there is an effectively unique equilibrium in which the bidders use piecewise monotone strategies (Proposition 2).[10] To be precise, let E denote the equilibrium set of the constrained game considered in this paper, and let E^* denote the set of equilibria in piecewise monotone strategies of the corresponding unconstrained game. Clearly, $E^* \subset E$. Lizzeri and Persico (1997) and Athey (1997) show that E^* is non-empty, and Proposition 2 shows that E is a singleton. We conclude that $E = E^*$.

6. Final remarks

The monotonicity result derived in this paper provides the missing link for the analysis of uniqueness in two-person first price auctions. At least in the two-person case the assumption of affiliation provides sufficient structure to ensure a very general monotonicity result, which does not depend on any symmetry considerations, on the absence of income effects or on the bidders' attitudes toward risk. Combining this result with existing work, it then follows a rather general uniqueness result which virtually settles the matter for the two-bidder case.

The two main limitations of the results reported here are related to our focus on *piecewise monotone* strategies and on *two-bidder* auctions. The restriction to piecewise monotone strategies is of a technical nature but may not be such a demanding requirement from the economic point of view. Moreover, it is not restrictive at all when the signals are discrete valued. The second qualification is of a more fundamental nature and, in fact, our results can be considered only a preliminary step in the analysis of the *n-person* case. However, the extension of the methods used here to the general case does not seem to be straightforward. In particular, Lemma 6 does not generalize to the *n*-dimensional case. The following counterexample establishes this point. Consider a probability measure p defined on the set Θ of the *3-tuples* (i, j, k) such that $i, j, k \in \{0, 1\}$. Let p_{ijk} denote the probability of the element (i, j, k) and

[10] The statement of Proposition 3 remains unchanged since assumption A_4 is not restrictive when signals are discrete valued.

let $p_{ij}(k)$ denote the conditional probability that the first two elements take values i and j given the fact that the third one takes the value k. Suppose that $p_{00}(0) = p_{01}(0) = 1/3$, $p_{10}(0) = p_{11}(0) = 1/6$, $p_{00}(1) = 4/15$, $p_{01}(1) = 4/10$, $p_{10}(1) = 2/15$ and $p_{11}(1) = 2/10$. Define a function $\varphi(i,j) = \varphi_{ij}$ such that $\varphi_{00} = 0$, $\varphi_{01} = -0.9$ and $\varphi_{10} = \varphi_{11} = 0.9$. It can be easily verified that p is TP_2 and that φ is a weakly QSC mapping. Finally, notice that $E(\varphi|k=0) = 0$ and that $E(\varphi|k=1) = -0.06$. Thus, $E(\varphi|k)$ is not QSC.

Appendix

Proof of Lemma 2: Using the continuity of u_i, we can write

$$\pi_i(\hat{b}, x) - \pi_i(\hat{b}^-, x) = E(u_i(\hat{b}, \mathbf{s}_i, \mathbf{s}_j)[Q_i(\hat{b}, \mathbf{s}_j) - Q_i(\hat{b}^-, \mathbf{s}_j)] \,|\, \mathbf{s}_i = x)$$

$$\pi_i(\hat{b}^+, x) - \pi_i(\hat{b}, x) = E(u_i(\hat{b}, \mathbf{s}_i, \mathbf{s}_j)[Q_i(\hat{b}^+, \mathbf{s}_j) - Q_i(\hat{b}, \mathbf{s}_j)] \,|\, \mathbf{s}_i = x)$$

Note that $Q_i(\hat{b}, s_j) - Q_i(\hat{b}^-, s_j) = Q_i(\hat{b}^+, s_j) - Q_i(\hat{b}, s_j)$ because the tie-breaker is symmetric. Thus, $\pi_i(\hat{b}, x) - \pi_i(\hat{b}^-, x) = \pi_i(\hat{b}^+, x) - \pi_i(\hat{b}, x)$. On the other hand, $\pi_i(\hat{b}, x) - \pi_i(\hat{b}^-, x) \geq 0$ and $\pi_i(\hat{b}^+, x) - \pi_i(\hat{b}, x) \leq 0$ because of the optimality of $\hat{b}$ for the bidder i with signal x. We conclude that $\pi_i(\hat{b}, x) = \pi_i(\hat{b}^-, x) = \pi_i(\hat{b}^+, x)$, as desired. □

Proof of Lemma 3: I argue by contradiction. Suppose that the bid $\hat{b} \in (\underline{b}, \overline{b}]$ is optimal for bidder i with signal x and that bidder j bids with probability zero in some interval $[\hat{b} - \varepsilon, \hat{b})$. Since $\pi_i(b, x)$ is continuous in $\hat{b}$ by Lemma 2, an arbitrarily small reduction in bidder i's bid has an arbitrarily small effect on his expected payoff. However, an additional reduction in his bid to $\hat{b} - \varepsilon$ means a definite reduction in what he expects to pay if he wins and does not affect his chance of winning. Thus, $\pi_i(\hat{b}, x) < \pi_i(\hat{b} - \varepsilon, x)$, contradicting the optimality of $\hat{b}$. □

Proof of Lemma 4: Suppose that β_j has a mass point and is quasi-monotone at b. Consider a sequence $(b^n, x^n)_{n\to\infty}$ such that $b^n \uparrow b$ and $x^n \in \beta_i^{-1}(b^n)$ (existence of such sequence follows from Lemma 3). Denote $x = \lim_{n\to\infty} x^n$ and $\tilde{s}_j = \sup\{s_j \in S_j : Q_i(b^-, s_j) > 0\}$. The fact that β_j is quasi-monotone at b implies that $Q_i(b^-, s_j) = 0$ for $s_j > \tilde{s}_j$. Thus, $\lim_{n\to\infty} \pi_j(b^n, x^n) = \pi(b^-, x) = E(u_i(b, \mathbf{s}_i, \mathbf{s}_j) Q_i(b^-, \mathbf{s}_j) \,|\, \mathbf{s}_i = x, \mathbf{s}_j \leq \tilde{s}_j) P(\mathbf{s}_j \leq \tilde{s}_j \,|\, \mathbf{s}_i = x)$. Note that $\pi(b^-, x) \geq 0$ as a consequence of the fact that $\pi_j(b^n, x^n) \geq 0$ for every n. Also note that $P(\mathbf{s}_j \leq \tilde{s}_j \,|\, \mathbf{s}_i = x) > 0$, due to the fact that $b > \underline{b}$. Hence, we conclude that $u_i(b, x, \tilde{s}_j) \geq 0$.

If $u_i(b, x, \tilde{s}_j) = 0$, then $\pi_i(b^-, x) = 0$ and $\pi_i(b - \varepsilon, x) > 0$ for some small $\varepsilon > 0$. Thus, for n sufficiently large, $\pi_i(b - \varepsilon, x^n) > \pi_i(b^n, x^n) \cong 0$, which implies that $x^n \notin \beta_i^{-1}(b^n)$, a contradiction.

If $u_i(b, x, \tilde{s}_j) > 0$, then $u_i(b, x, s_j) > 0$ for every $s_j > \tilde{s}_j$. Thus,

$$E(u_i(b, \mathbf{s}_i, \mathbf{s}_j) Q_i(b, \mathbf{s}_j) \,|\, \mathbf{s}_i = x, \mathbf{s}_j > \tilde{s}_j) > 0 \tag{2}$$

Note that $Q_i(b^-, s_j) = 0$ for $s_j > \tilde{s}_j$ and $Q_i(b, s_j) = Q_i(b^-, s_j)$ for $s_j < \tilde{s}_j$ as a consequence of the fact that β_j is quasi-monotone at b. Using also the continuity of u_i in b we can write

$$\pi_i(b,x) - \pi_i(b^-,x) = E(u_i(b,\mathbf{s}_i,\mathbf{s}_j)[Q_i(b,\mathbf{s}_j) - Q_i(b^-,\mathbf{s}_j)] \mid \mathbf{s}_i = x)$$

$$= E(u_i(b,\mathbf{s}_i,\mathbf{s}_j)[Q_i(b,\mathbf{s}_j) - Q_i(b^-,\mathbf{s}_j)] \mid \mathbf{s}_i = x, \mathbf{s}_j > \tilde{s}_j)P(\mathbf{s}_j > \tilde{s}_j \mid \mathbf{s}_i = x)$$

$$= E(u_i(b,\mathbf{s}_i,\mathbf{s}_j)Q_i(b,\mathbf{s}_j) \mid \mathbf{s}_i = x, \mathbf{s}_j > \tilde{s}_j)P(\mathbf{s}_j > \tilde{s}_j \mid \mathbf{s}_i = x) > 0$$

where the last inequality follows from (2) and the fact that $P(\mathbf{s}_j > \tilde{s}_j \mid \mathbf{s}_i = x) > 0$ because of the mass point at b. Since $\pi_i(b,x^n) - \pi_i(b^n,x^n) \to \pi_i(b,x) - \pi_i(b^-,x) > 0$ as $n \to \infty$, we conclude that $x^n \notin \beta_i^{-1}(b^n)$ also in this case. This contradiction establishes the lemma. □

Proof of Lemma 5:
(i) Follows directly from equation (1).
(ii) First, I claim that $\delta_i(s) > 0$ implies that (a) $Q_i(b_0^+, s_j) < 1$ and (b) $u_i(b_1,s) > 0$. To see this, note that $Q_i(b_0^+,s_j) = 1$ implies that $Q_i(b_1,s_j) = 1$ and therefore that $\delta_i(s) = u_i(b_1,s) - u_i(b_0,s) < 0$, so (a) follows. Now suppose that $u_i(b,s) \le 0$. Since $|u_i(b_1,s)| > |u_i(b_0,s)|$ when $u_i(b_0,s) \le 0$, we have that $\delta_i(s) = -|u_i(b_1,s)|Q_i(b_1,s_j) - u_i(b_0,s)Q_i(b_0^+,s_j) \le 0$. This establishes the claim.

Now suppose that $\delta_i(s) > 0$ and consider a signal $s_j' > s_j$. Since β_j is monotone at b_0, Lemma 1 and the preceding claim imply that $Q_i(b_0^+,s_j') = 0$. Thus, we have that $\delta_i(s_i,s_j') = Q_i(b_1,s_j')u_i(b_1,s_i,s_j') \ge 0$, where the inequality follows from the preceding claim and the fact that $u_i(b_1,s_i,s_j') \ge u_i(b_1,s_i,s_j)$ by assumption A_2. We conclude that $\delta_i(s_i,s_j) > 0$ implies that $\delta_i(s_i,s_j') \ge 0$, as desired. □

Proof of Lemma 6: Since φ is weakly QSC, there must be a parameter value $s_j^0 \in S_j$ such that $\varphi(s_j)(s_j - s_j^0) \ge 0$ for every $s_j \in S_j$. Given $x, x' \in S_i$ such that $x < x'$, denote $\Delta(s) = f_j(s_j|x')/f_j(s_j^0|x') - f_j(s_j|x)/f_j(s_j^0|x)$, where f_j denotes the density of the conditional distribution of $\mathbf{s}_j$ with respect to $\mathbf{s}_i$. I claim that $\varphi(s_j)\Delta(s_j) \ge 0$ for every $s_j \in S_j$. Note that when $s_j < s_j^0$ we have that $\Delta(s_j) \le 0$ because f is TP_2 and that $\varphi(s_j) \le 0$ because φ is weakly QSC. Similarly, $\Delta(s_j) \ge 0$ and $\varphi(s_j) \ge 0$ when $s_j > s_j^0$. The claim follows. Finally, note that $\psi(x')/f_j(s_j^0|x') - \psi(x)/f_j(s_j^0|x) = \int_{S_j} \varphi(\eta)\Delta(\eta)\, d\eta \ge 0$. Thus, ψ is QSC. □

Proof of Lemma 10: First I claim that both strategies are weakly quasi-monotone at $\underline{b}$. Consider, for instance, β_j. If $\underline{b}_j = \underline{b}$, then β_j is clearly quasi-monotone at $\underline{b}$. Suppose that $\underline{b}_j < \underline{b}$. If a winning bid $b > \underline{b}$ is optimal for j when his signal is s_j, we must have that $\pi_j(b,s_j) \ge 0$. Thus, Lemma 8 implies that $\pi_j(b,s_j') > 0$ for $s_j' > s_j$. Since bids lower than $\underline{b}$ yield zero expected profits to bidder j (either because $\underline{b} = r$ or because $\underline{b} = \underline{b}_i$), we conclude that when bidder j's signal is larger than s_j, he bids at least $\underline{b}$. The claim follows.

For concreteness, suppose now that $\underline{b}_j \ge \underline{b}_i$. Since $\underline{b}$ is a winning bid for bidder j when $\underline{b}_j < \underline{b}$, we can also apply the preceding argument to the case of a bid $b = \underline{b}$ to show that β_j actually is quasi-monotone at $\underline{b}$. Thus, the lemma follows trivially if β_j is atomless at $\underline{b}$. Consider the case in which β_j has a mass point at $\underline{b}$. Since in this case $\underline{b}$ is a winning bid for i and β_j is quasi-monotone at $\underline{b}$, Lemma 8 and the argument employed in the preceding paragraph imply that β_i also is quasi-monotone at $\underline{b}$. To show that β_i is actually monotone at $\underline{b}$, we only need to show that $\beta_i^{-1}(\underline{b})$ is a singleton. We argue by contradiction. Suppose that $\beta_i^{-1}(\underline{b})$, contains two signals s_i and s_i' such that $s_i' > s_i$. Since $\pi_i(\underline{b}, s_i) \geq 0$, Lemma 8 implies that $\pi_i(\underline{b}, s_i') > 0$. Moreover, since β_j is quasi-monotone at $\underline{b}$, the types of bidder j that bid $\underline{b}$ are larger than the ones that bid less than $\underline{b}$. Using these observations and the fact that u_i is non-decreasing in s_j, an argument similar to the one used in the proof of Lemma 4 implies that $E(u_i(b, \mathbf{s}_i, \mathbf{s}_j)Q_i(b, \mathbf{s}_j)|\mathbf{s}_i = s_j', \mathbf{s}_j > \tilde{s}_j) > 0$, where $\tilde{s}_j = \sup\{s_j \in S_j : Q_i(b^-, s_j) > 0\}$. Thus,

$$\begin{aligned}\pi_i(b^+, s_j') - \pi_i(b, s_j') &= E(u_i(b, \mathbf{s}_i, \mathbf{s}_j)[Q_i(b^+, \mathbf{s}_j) - Q_i(b, \mathbf{s}_j)] \mid \mathbf{s}_i = s_j') \\ &= E(u_i(b, \mathbf{s}_i, \mathbf{s}_j)[Q_i(b^+, \mathbf{s}_j) - Q_i(b, \mathbf{s}_j)] \mid \mathbf{s}_i = s_j', \mathbf{s}_j > \tilde{s}_j)P(\mathbf{s}_j > \tilde{s}_j \mid \mathbf{s}_i = s_j') \\ &= E(u_i(b, \mathbf{s}_i, \mathbf{s}_j)[Q_i(b, \mathbf{s}_j) - Q_i(b^-, \mathbf{s}_j)] \mid \mathbf{s}_i = s_j', \mathbf{s}_j > \tilde{s}_j)P(\mathbf{s}_j > \tilde{s}_j \mid \mathbf{s}_i = s_j') \\ &= E(u_i(b, \mathbf{s}_i, \mathbf{s}_j)Q_i(b, \mathbf{s}_j) \mid \mathbf{s}_i = s_j', \mathbf{s}_j > \tilde{s}_j)P(\mathbf{s}_j > \tilde{s}_j \mid \mathbf{s}_i = s_j') > 0 \qquad (3)\end{aligned}$$

where we used the facts that $Q_i(\hat{b}, s_j) - Q_i(\hat{b}^-, s_j) = Q_i(\hat{b}^+, s_j) - Q_i(\hat{b}, s_j)$ because the tie-breaker is symmetric, and that $Q_i(b^-, s_j) = 0$ for $s_j > \tilde{s}_j$ and $Q_i(b, s_j) = Q_i(b^+, s_j)$ for $s_j < \tilde{s}_j$ as a consequence of the fact that β_j is quasi-monotone at b. But (3) implies that $s_j' \notin \beta_i^{-1}(\underline{b})$. This contradiction completes the proof. $\square$

Proof of Lemma 11: I argue by contradiction. Let $\underline{b}_i < \underline{b}_j = \underline{b}$ and denote $s_i^0 = \sup \beta_i^{-1}(\underline{b})$. First I claim that $u(\underline{b}, s_i^0, \underline{s}) \leq 0$. To establish the claim suppose that $u(\underline{b}, s_i^0, \underline{s}) > 0$. Then $\pi_i(\underline{b} + \varepsilon, s_i^0) > \pi_i(\underline{b}, s_i^0)$ for a sufficiently small $\varepsilon > 0$, which contradicts the optimality of $\underline{b}$ for i when his signal is s_i^0. The claim follows.

Take $\underline{b}_i < b < \underline{b}$. Then

$$\begin{aligned}&\pi_j(\underline{b}, \underline{s}) - \pi_j(b, \underline{s}) \\ &\quad = E(u(\underline{b}, \underline{s}, \mathbf{s}_i)[Q_j(\underline{b}, \mathbf{s}_i) - Q_j(b, \mathbf{s}_i)] + Q_j(b, \mathbf{s}_i)[u(\underline{b}, \underline{s}, \mathbf{s}_i) - u(b, \underline{s}, \mathbf{s}_i)] \mid \mathbf{s}_i \leq s_i^0) \qquad (4)\end{aligned}$$

The preceding claim and assumption A_8 imply that $u(\underline{b}, \underline{s}, s_i^0) \leq 0$. By assumption A_2, we have that $u(\underline{b}, \underline{s}, \mathbf{s}_i) - u(b, \underline{s}, \mathbf{s}_i) < 0$. Moreover, $Q_j(b, \mathbf{s}_i) > 0$ with positive probability since $\underline{b}_i < b$. Thus, equation 4 implies that $\pi_j(\underline{b}, \underline{s}) - \pi_j(b, \underline{s}) < 0$, so $\underline{s} \notin \beta_j^{-1}(\underline{b})$. This contradicts Proposition 1 and establishes the first assertion.

To establish the part (i) of the last assertion, suppose that $S = [\underline{s}, \bar{s}]$. Also suppose that β_j has a mass point at $\underline{b}$. Since $\underline{b}$ is a winning bid for bidder i, we have that $\pi_i(\underline{b}^+, \underline{s}) \geq 0$. This implies that $u(\underline{b}, \underline{s}, \hat{s}_j) > 0$ for some $\hat{s}_j \in \beta_j^{-1}(\underline{b})$. Thus, assumption A_8 implies that $u(\underline{b}, \hat{s}_j, \underline{s}) > 0$. But then $\pi_j(\underline{b} + \varepsilon, \hat{s}_j) > \pi_j(\underline{b}, \hat{s}_j)$ for a sufficiently small $\varepsilon > 0$, a contradiction that establishes (i).

Finally, consider part (ii). Suppose that $S = \{\underline{s}, \ldots, \bar{s}\}$. First I claim that both bidders' strategies have a mass point at $\underline{b}$. To show this, I argue by contradiction. Suppose β_j is atomless at $\underline{b}$. Since the strategies are monotone, there must exist a type s and some $\varepsilon > 0$ such that $\beta_i^{-1}(b) = \{s\}$ and $\beta_j^{-1}(b) = \{\underline{s}\}$ for every $b \in (\underline{b}, \underline{b} + \varepsilon)$. Since $\pi_i(\underline{b}^+, s) = 0$, we have that $\pi_i(b, s) =$

$u(b,s,\underline{s})Q_i(b,\underline{s})f(\underline{s}|s) = 0$ for every $b \in (\underline{b}, \underline{b}+\varepsilon)$., Thus, $u(b,s,\underline{s}) = 0$ for every $b \in (\underline{b}, \underline{b}+\varepsilon)$, a contradiction that establishes the claim.

Since both bidders' strategies are monotone at $\underline{b}$, the preceding claim and the proof of Lemma 10 imply that $\beta_i^{-1}(\underline{b}) = \beta_j^{-1}(\underline{b}) = \{\underline{s}\}$ and that $u(\underline{b},\underline{s},\underline{s}) = 0$. Thus, if the type $\underline{s}$ of bidder j bids with positive probability in some interval $(\underline{b}, \underline{b}+\varepsilon)$, we have that $\pi_i(b,\underline{s}) = u(b,\underline{s},\underline{s})Q_i(b,\underline{s})f(\underline{s}|\underline{s}) < 0$ for $b \in (\underline{b}, \underline{b}+\varepsilon)$. We conclude that the type $\underline{s}$ of bidder i bids $\underline{b}$ with probability 1. Part (ii) follows. □

Proof of Proposition 3:

(i) *Existence.* Note that a solution to the system in the statement exists and is unique since, given $b_{\underline{s}}, \dots, b_{x-1}$, the value of b_x is uniquely determined because u is strictly decreasing in b. Thus, the symmetric, monotone strategy profile described in the statement is well defined. Note that if bidder j employs this strategy, bidder i's expected payoff conditional on his signal is $\pi_i(b,x) = \sum_{s=\underline{s}}^{x-1} u(b,x,s)f(s|x) + u(b,x,x)f(x|x)Q_i(b,x)$. It is easy to verify that $\pi_i(b,x) = \pi_i(b_x,x)$ for every $b \in [b_{x-1}, b_x]$. Moreover, using Lemma 7, for every $b \in [b_{y-1}, b_y]$ and $y < x$, we have that $\pi_i(b_y,y) = \pi_i(b,y)$ implies that $\pi_i(b_y,x) > \pi_i(b,x)$. Similarly, for every $b \in [b_{z-1}, b_z]$ and $z > x$, we have that $\pi_i(b,x) \geq \pi_i(b_{z-1},x)$ implies that $\pi_i(b,z) > \pi_i(b_{z-1},z)$. This contradiction shows that actually $\pi_i(b,x) < \pi_i(b_{z-1},x)$ in this case. We conclude from the preceding inequalities that the strategy profile in consideration is an equilibrium. Since this profile is symmetric, it is consistent with Lemma 11 if and only if each buyer bids $\underline{b}$ with probability 1 when his signal is $\underline{s}$. Thus, $Q_i(\underline{b},\underline{s}) = 1/2$ for $i = 1,2$, as a consequence of the tie-breaker.

(ii) *Uniqueness.* Proposition 1 and Lemma 11 imply that the equilibrium strategies are monotone over the set S. Now, I claim that the equilibrium is symmetric. I argue by contradiction. Suppose that there is a bid b_* and a type s_* such that $Q_j(b_*,s_*) > Q_i(b_*,s_*)$. Since in equilibrium $Q_j(\bar{b},\bar{s}) = Q_i(\bar{b},\bar{s}) = 1$, by Lemma 3, and the strategies are monotone, there must exist some bids b_0 and b_1, with $b_0 < b_1$, and some type $x \in S$ such that $\beta_i^{-1}(b) = \beta_j^{-1}(b) = \{x\}$ for every $b \in [b_0, b_1]$ and

$$Q_j(b_0,x) > Q_i(b_0,x)$$

$$Q_j(b_1,x) \leq Q_i(b_1,x) \tag{5}$$

Note that $\pi_i(b,x) = u(b,x,x)Q_i(b,x)f(x|x) + \sum_{s=\underline{s}}^{x-1} u(b,x,s)f(s|x)$, where f is a probability mass function. Thus,

$$[\pi_i(b_1,x) - \pi_i(b_0,x)] - [\pi_j(b_1,x) - \pi_j(b_0,x)]$$

$$= \{u(b_1,x,x)[Q_i(b_1,x) - Q_j(b_1,x)] - u(b_0,x,x)[Q_i(b_0,x) - Q_j(b_0,x)\}f(x|x) \tag{6}$$

Note that $\pi_i(b_1,x) \geq 0$ implies that $u(b_1,x,x) \geq 0$ and that $u(b_0,x,x) > 0$. Thus, (5) implies that the expression (6) is strictly positive. Then either $\pi_i(b_1,x) - \pi_i(b_0,x) > 0$ or $\pi_j(b_1,x) - \pi_j(b_0,x) > 0$, contradicting the definition of x. This establishes the claim.

Finally, Lemma 11(ii) shows that at a symmetric equilibrium both players bid $\underline{b}$ with probability 1 when their type is $\underline{s}$, where $\underline{b}$ is determined by the condition $u(\underline{b}, \underline{s}, \underline{s}) = 0$. Thus, monotonicity and the requirement that $Q_j(\bar{b}, \bar{s}) = Q_i(\bar{b}, \bar{s})$ uniquely determine the equilibrium strategies. □

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