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Strategy-proof and efficient allocation of an indivisible good on finitely restricted preference domains*

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Abstract. We consider allocation mechanisms in economies with a single indivisible good and money. First, we show that there is no strategy-proof and Pareto efficient mechanism on some preference domains which consist of a sufficiently large but finite number of quasi-linear preferences. Second, we show that there is no strategy-proof, Pareto efficient, and equally compensatory mechanism on arbitrary preference domains which consist of more than three quasi-linear preferences.

Key words: Strategy-proofness, efficiency, domain restriction

1. Introduction

We consider economies with a single indivisible good and a transferable good. The indivisible good can be consumed by only one agent. The transferable good, regarded as money, is used for compensation. We consider allocation mechanisms which determine who consumes the indivisible good and how much compensation the other agents receive on the basis of preferences of agents. We regard the following axioms as desiderata for mechanisms. The first axiom is strategy-proofness. A mechanism is strategy-proof if truthful revelation of preferences is a dominant strategy. The second one is Pareto

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efficiency. A mechanism is Pareto efficient if it always chooses a Pareto efficient allocation. We study the possibility of designing strategy-proof and Pareto efficient mechanisms.

The possibility of designing strategy-proof mechanisms depends on the size of the preference domain of the mechanisms. In a social choice framework, Gibbard (1973) and Satterthwaite (1975) establish the impossibility of strategy-proof mechanisms when the preference domain is "unrestricted", whereas Moulin (1980) and Barberà and Jackson (1994) characterize a rich class of strategy-proof, Pareto efficient, and anonymous mechanisms when the preference domain is restricted to "single peaked" preferences.

In two-agent pure exchange economies, Zhou (1991) shows that there is no strategy-proof, Pareto efficient, and non-dictatorial mechanism on the usual economic preference domain, and Schummer (1997) proves the same impossibility result even when the preference domain is restricted to (i) "homothetic" preferences, or (ii) more than three "linear" preferences. Therefore, the impossibility of strategy-proof and Pareto efficient mechanisms is well established in two-agent case.

However, when we consider economies with private goods, there is a crucial difference between the two-agent case and the case of more than two agents. Satterthwaite and Sonnenschein (1981) point out that there exist strategy-proof, Pareto efficient, and non-dictatorial mechanisms in the case of more than two agents. However, it is very difficult to characterize such strategy-proof mechanisms because of the concept of strategy-proofness and the presence of private goods. When some agent (e.g. agent 1) changes his preference and others remain unchanged, strategy-proofness puts constraint on agent 1's consumption bundle directly, but on other agents' consumption bundles indirectly (e.g. through budget balance). Satterthwaite and Sonnenschein (1981) introduce the non-bossiness condition to overcome this difficulty. Barberà and Jackson (1995) also use non-bossiness in order to characterize the set of strategy-proof and anonymous mechanisms in the case of more than two agents. However, we do not invoke non-bossiness in this paper since the economic interpretation of non-bossiness is not so clear.

We consider the possibility of strategy-proof and Pareto efficient mechanisms in economies with an indivisible good and money. A general result of Holmström (1979) implies that there is no strategy-proof and Pareto efficient mechanism on the set of all quasi-linear preferences.[1] First, we consider some finite restrictions of the preference domain in order to understand how strong the impossibility result is. In Theorem 1, we show that *there is no strategy-proof and Pareto efficient mechanism on a sufficiently large but finite number of quasi-linear preferences*. The impossibility result holds true even on finitely restricted preference domains. A possible drawback of the above theorem is that the preference domains contain a large number of preferences when the number of agents is large. Next, we impose an additional axiom "equal compensation" and consider the possibility of such mechanisms on small preference domains. In Theorem 2, we show that *there is no strategy-proof, Pareto efficient, and equally compensatory mechanism on arbitrary preference domains*

[1] To escape the impossibility result, one may weaken the incentive criterion from strategy-proofness to Bayesian incentive compatibility (d'Aspremont and Gérard-Varet, 1979; Myerson and Satterthwaite, 1983).

consisting of more than three quasi-linear preferences. Finally, we describe the structure of strategy-proof and Pareto efficient mechanisms on very small preference domains (consisting of two or three quasi-linear preferences). We conclude that the impossibility of strategy-proof and Pareto efficient mechanisms is inevitable since such small preference domains are very unrealistic.

2. Notation and definitions

We consider economies with a single indivisible good and a transferable good. The indivisible good can be consumed by only one agent. The transferable good, regarded as money, is used for compensation. Let $N = \{1, \ldots, n\}$ $(n \geq 2)$ be the set of agents. For each $i \in N$, the consumption space of agent i is the set of pairs $(t_i, x_i) \in R \times \{0, 1\}$, where t_i denotes money he receives and x_i denotes his consumption of the indivisible good. The amount of money each agent receives may be negative.

Each agent has a quasi-linear preference on his consumption space. Let U_A be the set of all quasi-linear preferences on $R \times \{0, 1\}$ which can be represented by a quasi-linear utility function $u_i(t_i, x_i) = t_i + v_i(x_i)$. For each $u_i \in U_A$, let $\lambda(u_i)$ denote agent i's valuation of the indivisible good, that is, $u_i(t_i + \lambda(u_i), 0) = u_i(t_i, 1)$ for all $t_i \in R$. We will consider an arbitrary preference domain U which is a finite subset of U_A. Let $\# U$ denote the number of preferences in U. A preference profile is a list $u = (u_1, \ldots, u_n) \in U^n$. Let M be the amount of money which is allocated to agents. We assume that M is known and fixed. The set of feasible allocations is $Z = \{z = (t_1, \ldots, t_n; x_1, \ldots, x_n) \in R^n \times \{0, 1\}^n \mid \sum_{i \in N} t_i = M \text{ and } \sum_{i \in N} x_i = 1\}$. The set of feasible transfer allocations is $Z_T = \{t = (t_1, \ldots, t_n) \in R^n \mid \sum_{i \in N} t_i = M\}$.

A mechanism (defined on U^n) is a function $f : U^n \to Z$, which associates a feasible allocation with each preference profile. Let $F(U^n)$ be the set of mechanisms (defined on U^n). Given $f \in F(U^n)$ and $u \in U^n$, we write as $f(u) = (t_1(u), \ldots, t_n(u); x_1(u), \ldots, x_n(u))$, $f_i(u) = (t_i(u), x_i(u))$, and $f_t(u) = (t_1(u), \ldots, t_n(u))$. Given $f \in F(U^n)$, let $C_f(u) = \{i \in N \mid x_i(u) = 1\}$ denote the consumer of the indivisible good at $u \in U^n$. Given $u \in U^n$, $i \in N$, and $\bar{u}_i \in U$, the notation $(\bar{u}_i, u_{-i})$ represents the preference profile obtained from u after the replacement of u_i by $\bar{u}_i$.

We introduce the main axioms.

Definition 1. A mechanism $f \in F(U^n)$ satisfies *strategy-proofness* iff for all $u \in U^n$, $i \in N$, and $\bar{u}_i \in U$, $u_i(f_i(u)) \geq u_i(f_i(\bar{u}_i, u_{-i}))$.

Strategy-proofness states that truthful revelation of preferences is a dominant strategy for each agent. If a mechanism $f \in F(U^n)$ does not satisfy strategy-proofness, then there exist $u \in U^n$, $i \in N$, and $\bar{u}_i \in U$ such that $u_i(f_i(\bar{u}_i, u_{-i})) > u_i(f_i(u))$. We then say that agent i can manipulate f at u via $\bar{u}_i$.

Definition 2. A mechanism $f \in F(U^n)$ satisfies *Pareto efficiency* iff for all $u \in U^n$, there is no $z \in Z$ such that [for all $i \in N$, $u_i(t_i, x_i) \geq u_i(f_i(u))$] and [for some $i \in N$, $u_i(t_i, x_i) > u_i(f_i(u))$].

Definition 3. A mechanism $f \in F(U^n)$ satisfies *equal compensation* iff for all $u \in U^n$ and $i, j \notin C_f(u)$, $t_i(u) = t_j(u)$.

Equal compensation requires that the non-consumers of the indivisible good should receive the same amount of money.

The following lemma is a well known result and the proof will be omitted.[2]

Lemma 1. *A mechanism $f \in F(U^n)$ satisfies Pareto efficiency if and only if for all $u \in U^n$, $C_f(u) \subset \text{Argmax}_{i \in N}\{\lambda(u_i)\}$.*

3. Sufficiently large but finite preference domains

A general result of Holmström (1979) implies that there is no strategy-proof and Pareto efficient mechanism on the set of all quasi-linear preferences.[3]

Theorem. (Holmström, 1979). *There is no strategy-proof and Pareto efficient mechanism $f \in F(U_A^n)$.*

Notice that the preference domain considered in Holmström (1979) contains an infinite number of preferences. We consider the possibility of strategy-proof and Pareto efficient mechanisms on some finite subsets of quasi-linear preferences. Given two integers a, b arbitrarily, let $[a, \dots, b]$ denote the set of integers between a and b inclusive. Let $U_{[a,b]} = \{u_i \in U_A \mid \lambda(u_i) \in [a, \dots, b]\}$.

The following lemma presents a necessary condition of strategy-proof and Pareto efficient mechanisms when the preference domain is restricted to $U_{[a,b]}$. It states that if agent j consumes the indivisible good at preference profile u, and if the other agent i can consume it by changing his preference, then the amount of money agent i receives decreases by $\lambda(u_j) - 1$ at least and $\lambda(u_j) + 1$ at most. In other words, agent i must pay $\lambda(u_j) - 1$ at least and $\lambda(u_j) + 1$ at most in order to consume the indivisible good.

Lemma 2. *Assume that a mechanism $f \in F(U^n_{[a,b]})$ satisfies strategy-proofness and Pareto efficiency. For all $u \in U^n_{[a+1,b-1]}$, $i \in N$, and $\bar{u}_i \in U_{[a,b]}$, if $C_f(u) = \{j\} \neq \{i\} = C_f(\bar{u}_i, u_{-i})$, then $\lambda(u_j) - 1 \leq t_i(u) - t_i(\bar{u}_i, u_{-i}) \leq \lambda(u_j) + 1$.*

Proof: Suppose first that for some $u \in U^n_{[a+1,b-1]}$, $i \in N$, and $\bar{u}_i \in U_{[a,b]}$, $C_f(u) = \{j\} \neq \{i\} = C_f(\bar{u}_i, u_{-i})$ and $t_i(u) - t_i(\bar{u}_i, u_{-i}) < \lambda(u_j) - 1$. Let $\hat{u}_i \in U_{[a,b]}$ be such that $\lambda(\hat{u}_i) = \lambda(u_j) - 1$. It follows from Lemma 1 that $C_f(\hat{u}_i, u_{-i}) \neq \{i\}$. By strategy-proofness, $f_i(\hat{u}_i, u_{-i}) = (t_i(u), 0)$. Since $\hat{u}_i(t_i(\bar{u}_i, u_{-i}), 1) = \hat{u}_i(t_i(\bar{u}_i, u_{-i}) + \lambda(\hat{u}_i), 0) > \hat{u}_i(t_i(u) - \lambda(u_j) + 1 + \lambda(\hat{u}_i), 0) = \hat{u}_i(t_i(u), 0)$, agent i can manipulate f at $(\hat{u}_i, u_{-i})$ via $\bar{u}_i$.

Suppose next that for some $u \in U^n_{[a+1,b-1]}$, $i \in N$, and $\bar{u}_i \in U_{[a,b]}$, $C_f(u) = \{j\} \neq \{i\} = C_f(\bar{u}_i, u_{-i})$ and $t_i(u) - t_i(\bar{u}_i, u_{-i}) > \lambda(u_j) + 1$. Let $\hat{u}_i \in U_{[a,b]}$ be such that $\lambda(\hat{u}_i) = \lambda(u_j) + 1$. It follows from Lemma 1 that $C_f(\hat{u}_i, u_{-i}) = \{i\}$.

[2] See e.g. Mas-Colell, Whinston, and Green (1995), Example 23.B.4, p. 862.

[3] By using a general result of Holmström (1979), Schummer (2000) proves that there is no strategy-proof and Pareto efficient mechanism on the set of all quasi-linear preferences in economies with multiple indivisible goods and money.

By strategy-proofness, $f_i(\widehat{u}_i, u_{-i}) = (t_i(\overline{u}_i, u_{-i}), 1)$. Since $\widehat{u}_i(t_i(u), 0) = \widehat{u}_i(t_i(u) - \lambda(\widehat{u}_i), 1) > \widehat{u}_i(t_i(\overline{u}_i, u_{-i}) + \lambda(u_j) + 1 - \lambda(\widehat{u}_i), 1) = \widehat{u}_i(t_i(\overline{u}_i, u_{-i}), 1)$, agent i can manipulate f at $(\widehat{u}_i, u_{-i})$ via u_i. **Q.E.D.**

We show the non-existence of strategy-proof and Pareto efficient mechanisms on a sufficiently large but finite number of quasi-linear preferences.

Theorem 1. *Let* $U_{[a,b]}$ *be such that* $b - a > \dfrac{2^n + 2n^2 - 2n - 2}{n-1}$. *There is no strategy-proof and Pareto efficient mechanism* $f \in F(U^n_{[a,b]})$.

Proof: Suppose that there is a strategy-proof and Pareto efficient mechanism $f \in F(U^n_{[a,b]})$. For each $i \in N$, let $u_i^-, u_i^+ \in U_{[a,b]}$ be such that $\lambda(u_i^-) = a + i$ and $\lambda(u_i^+) = b - i$. Then, $\lambda(u_1^-) < \cdots < \lambda(u_n^-) < \lambda(u_n^+) < \cdots < \lambda(u_1^+)$ and $\lambda(u_n^+) - \lambda(u_n^-) > \dfrac{2^n - 2}{n-1}$. Let $U^* = \times_{i \in N}\{u_i^-, u_i^+\}$. Let $\overline{U}^* = \{u \in U^* \mid$ there are an even number of agents who reveal u_i^+ at $u\}$ and $\hat{U}^* = \{u \in U^* \mid$ there are an odd number of agents who reveal u_i^+ at $u\}$. By budget balance, we have that

$$\sum_{i \in N} t_i(u) = M \quad \text{for all } u \in \overline{U}^*, \text{ and} \tag{1}$$

$$\sum_{i \in N} t_i(u) = M \quad \text{for all } u \in \hat{U}^*. \tag{2}$$

We will provide necessary conditions on $t_i(\cdot)$ for any pair of preference profiles where only agent i reveals different preferences, that is, (u_i^-, u_{-i}), $(u_i^+, u_{-i}) \in U^*$. We consider the following six cases.

Case 1: Let $(u_i^-, u_{-i}) \in \hat{U}^*$ and $(u_i^+, u_{-i}) \in \overline{U}^*$ be such that $C_f(u_i^-, u_{-i}) = \{j\}$ with $j \leq i$. It follows from Lemma 1 that $C_f(u_i^+, u_{-i}) = \{j\}$. By strategy-proofness,

$$t_i(u_i^-, u_{-i}) - t_i(u_i^+, u_{-i}) = 0. \tag{3}$$

Case 2: Let $(u_i^-, u_{-i}) \in \overline{U}^*$ and $(u_i^+, u_{-i}) \in \hat{U}^*$ be such that $C_f(u_i^-, u_{-i}) = \{j\}$ with $j \leq i$. It follows from Lemma 1 that $C_f(u_i^+, u_{-i}) = \{j\}$. By strategy-proofness,

$$t_i(u_i^-, u_{-i}) - t_i(u_i^+, u_{-i}) = 0. \tag{4}$$

Case 3: Let $(u_i^-, u_{-i}) \in \hat{U}^*$ and $(u_i^+, u_{-i}) \in \overline{U}^*$ be such that $C_f(u_i^-, u_{-i}) = \{j\}$ with $i < j < n$. Suppose that agent j reveals u_j^- at (u_i^-, u_{-i}). Since $\lambda(u_j^-) < \lambda(u_n^-) < \lambda(u_n^+)$, it follows from Lemma 1 that $C_f(u_i^-, u_{-i}) \neq \{j\}$. This is a contradiction. Hence, agent j reveals u_j^+ at (u_i^-, u_{-i}). It follows from Lemma 1 that $C_f(u_i^+, u_{-i}) = \{i\}$. It follows from Lemma 2 that

$$\lambda(u_j^+) - 1 \leq t_i(u_i^-, u_{-i}) - t_i(u_i^+, u_{-i}) \leq \lambda(u_j^+) + 1. \tag{5}$$

Case 4: Let $(u_i^-, u_{-i}) \in \overline{U}^*$ and $(u_i^+, u_{-i}) \in \hat{U}^*$ be such that $C_f(u_i^-, u_{-i}) = \{j\}$ with $i < j < n$. Suppose that agent j reveals u_j^- at (u_i^-, u_{-i}). Since $\lambda(u_j^-) < \lambda(u_n^-) < \lambda(u_n^+)$, it follows from Lemma 1 that $C_f(u_i^-, u_{-i}) \neq \{j\}$. This is a contradiction. Hence, agent j reveals u_j^+ at (u_i^-, u_{-i}). It follows from Lemma 1 that $C_f(u_i^+, u_{-i}) = \{i\}$. It follows from Lemma 2 that

$$\lambda(u_j^+) - 1 \leq t_i(u_i^-, u_{-i}) - t_i(u_i^+, u_{-i}) \leq \lambda(u_j^+) + 1. \tag{6}$$

Case 5: Let $(u_i^-, u_{-i}) \in \hat{U}^*$ and $(u_i^+, u_{-i}) \in \overline{U}^*$ be such that $C_f(u_i^-, u_{-i}) = \{j\}$ with $i < j = n$. That is, $(u_i^-, u_{-i}) = (u_1^-, \dots, u_{n-1}^-, u_n^+)$. It follows from Lemma 1 that $C_f(u_i^+, u_{-i}) = \{i\}$. It follows from Lemma 2 that

$$\lambda(u_n^+) - 1 \leq t_i(u_i^-, u_{-i}) - t_i(u_i^+, u_{-i}) \leq \lambda(u_n^+) + 1. \tag{7}$$

Case 6: Let $(u_i^-, u_{-i}) \in \overline{U}^*$ and $(u_i^+, u_{-i}) \in \hat{U}^*$ be such that $C_f(u_i^-, u_{-i}) = \{j\}$ with $i < j = n$. That is, $(u_i^-, u_{-i}) = (u_1^-, \dots, u_n^-)$. It follows from Lemma 1 that $C_f(u_i^+, u_{-i}) = \{i\}$. It follows from Lemma 2 that

$$\lambda(u_n^-) - 1 \leq t_i(u_i^-, u_{-i}) - t_i(u_i^+, u_{-i}) \leq \lambda(u_n^-) + 1. \tag{8}$$

We count the number of inequalities derived in Cases 3 and 4. Fix any $j(\neq 1, n)$. The condition $C_f(u_i^-, u_{-i}) = \{j\}$ requires that $u_1 = u_1^-, \dots, u_{j-1} = u_{j-1}^-, u_j = u_j^+$. Each agent $k = 1, \dots, j-1$ is a possible candidate for agent i. Each agent $l = j+1, \dots, n$ reveals either u_l^- or u_l^+. Thus, for any $j(\neq 1, n)$, we derive $(j-1) \cdot 2^{n-j}$ inequalities. By the summation through j, we have that

$$\begin{aligned}
\sum_{j=2}^{n-1}(j-1) \cdot 2^{n-j} &= 2\left\{\sum_{j=2}^{n-1}(j-1) \cdot 2^{n-j}\right\} - \sum_{j=2}^{n-1}(j-1) \cdot 2^{n-j} \\
&= 2^{n-1} + 2^{n-2} + \cdots + 2^2 - 2(n-2) \\
&= \sum_{j=2}^{n-1} 2^j - 2(n-2) = 2\left\{\sum_{j=2}^{n-1} 2^j\right\} - \sum_{j=2}^{n-1} 2^j - 2(n-2) \\
&= 2^n - 2^2 - 2(n-2) = 2^n - 2n.
\end{aligned}$$

Notice that each case provides the same number of inequalities for any $j(\neq 1, n)$. Therefore, each case provides $2^{n-1} - n$ inequalities.

We count the number of inequalities derived in Cases 5 and 6. The condition $C_f(u_i^-, u_{-i}) = \{n\}$ requires that $u_1 = u_1^-, \dots, u_{n-1} = u_{n-1}^-$. Each agent $k = 1, \dots, n-1$ is a possible candidate for agent i. Agent n reveals either u_n^- or u_n^+. Thus, we derive $2(n-1)$ inequalities. Notice that each case provides the same number of inequalities. Therefore, each case provides $n-1$ inequalities.

We consider the summation of all the equations (or inequalities) in (1), (3), (5), (7) and all the equations (or inequalities) multiplied -1 in (2), (4), (6), (8). For each $u \in \overline{U}^*$ and $i \in N$, the term $t_i(u)$ appears once in (1) and once in one of (3)–(8). For each $u \in \hat{U}^*$ and $i \in N$, the term $t_i(u)$ appears once in (2) and once in one of (3)–(8). Notice that the terms $t_i(u)$ cancel out each other in the summation process. Since (1) and (2) provide the same number of equations, the terms M cancel out each other in the summation process. Since (5) and (6) provide the same number of inequalities for any $j(\neq 1, n)$, the terms $\lambda(u_j^+)$ cancel out each other in the summation process. Therefore, the summation provides the inequality $-2^n + 2 + (n-1)\{\lambda(u_n^+) - \lambda(u_n^-)\} \leq 0 \leq 2^n - 2 + (n-1)\{\lambda(u_n^+) - \lambda(u_n^-)\}$. Since $\lambda(u_n^+) - \lambda(u_n^-) > \dfrac{2^n - 2}{n-1}$, the left-hand inequality is a contradiction. **Q.E.D.**

4. Small preference domains

In this section we consider the possibility of strategy-proof, Pareto efficient, and equally compensatory mechanisms on small preference domains. That is, we tackle the question whether or not, given any restriction of the preference domain, such mechanisms exist.

We describe a fundamental structure of strategy-proof, Pareto efficient, and equally compensatory mechanisms. We show that those mechanisms *almost* satisfy the constant transfer property: transfer allocations depend only on who consumes the indivisible good. We introduce some formal notation and definitions. A *transfer allocation function* is a function $\pi : N \to Z_T$, which associates a feasible transfer allocation with each consumer of the indivisible good. For each $i \in N$, we let $\pi(i) = (\pi_1(i), \ldots, \pi_j(i), \ldots, \pi_n(i))$, where $\pi_j(i)$ represents the amount of money agent j receives when agent i consumes the indivisible good. Let Π denote *the set of transfer allocation functions*. A mechanism $f \in F(U^n)$ satisfies *the constant transfer property on* $\mathscr{U}$ ($\mathscr{U} \subset U^n$) *relative to* $\pi \in \Pi$ iff for each $u \in \mathscr{U}$, $[C_f(u) = \{i\} \Rightarrow f_t(u) = \pi(i)]$.

Given an arbitrary preference domain U, we let u_i^h, $u_i^l \in U$ be such that $\lambda(u_i^h) \geq \lambda(u_i) \geq \lambda(u_i^l)$ for all $u_i \in U$. Such u_i^h and u_i^l exist uniquely since U is a finite subset of quasi-linear preferences. Given the Cartesian product of the preference domain U^n, we let $\Gamma(U^n) = \{u \in U^n \mid \text{there is at most one agent who reveals } u_i^h \text{ at } u\}$.

Ohseto (1999) shows that any strategy-proof, Pareto efficient, and equally compensatory mechanism $f \in F(U_A^n)$ satisfies the constant transfer property on U_A^n relative to some $\pi \in \Pi$. When the preference domain is finitely restricted, we can show the following limited version of that result.[4]

Lemma 3. *If a mechanism $f \in F(U^n)$ satisfies strategy-proofness, Pareto efficiency, and equal compensation, then f satisfies the constant transfer property on $\Gamma(U^n)$ relative to some $\pi \in \Pi$.*

Proof: It is sufficient to show that $C_f(u) = C_f(\bar{u})$ implies $f(u) = f(\bar{u})$ for all u, $\bar{u} \in \Gamma(U^n)$. Without loss of generality, we assume that $C_f(u) = C_f(\bar{u}) = \{1\}$. It follows from Lemma 1 that only agent 1 may reveal u_1^h at u, $\bar{u}$. It follows from Lemma 1 that $C_f(u_1^h, u_{-1}) = C_f(u_1^h, \bar{u}_{-1}) = \{1\}$. By strategy-proofness,

[4] Let $U = \{u_i^a, u_i^b\}$, where $\lambda(u_i^a) = 1$ and $\lambda(u_i^b) = 2$. Let $n = 3$ and $f \in F(U^n)$ be the mechanism such that $f(u_1^a, u_2^a, u_3^a) = f(u_1^b, u_2^a, u_3^a) = (-1, 1/2, 1/2; 1, 0, 0)$, $f(u_1^a, u_2^b, u_3^a) = f(u_1^b, u_2^b, u_3^a) = (2/3, -4/3, 2/3; 0, 1, 0)$, $f(u_1^a, u_2^a, u_3^b) = f(u_1^b, u_2^a, u_3^b) = f(u_1^a, u_2^b, u_3^b) = (2/3, 2/3, -4/3; 0, 0, 1)$, and $f(u_1^b, u_2^b, u_3^b) = (-4/3, 2/3, 2/3; 1, 0, 0)$. Then, f satisfies strategy-proofness, Pareto efficiency, and equal compensation, but f does not satisfy the constant transfer property on U^n.

$f_1(u) = f_1(u_1^h, u_{-1})$ and $f_1(\bar{u}) = f_1(u_1^h, \bar{u}_{-1})$. By equal compensation, $f(u) = f(u_1^h, u_{-1})$ and $f(\bar{u}) = f(u_1^h, \bar{u}_{-1})$. It follows from Lemma 1 that $C_f(u_1^h, u_2^l, u_3, \ldots, u_n) = C_f(u_1^h, u_2^l, \bar{u}_3, \ldots, \bar{u}_n) = \{1\}$. By strategy-proofness, $f_2(u_1^h, u_{-1}) = f_2(u_1^h, u_2^l, u_3, \ldots, u_n)$ and $f_2(u_1^h, \bar{u}_{-1}) = f_2(u_1^h, u_2^l, \bar{u}_3, \ldots, \bar{u}_n)$. By equal compensation, $f(u_1^h, u_{-1}) = f(u_1^h, u_2^l, u_3, \ldots, u_n)$ and $f(u_1^h, \bar{u}_{-1}) = f(u_1^h, u_2^l, \bar{u}_3, \ldots, \bar{u}_n)$. Repeatedly applying the same argument to the remaining agents, we have that $f(u) = f(u_1^h, u_{-1}^l)$ and $f(\bar{u}) = f(u_1^h, u_{-1}^l)$. Therefore, $f(u) = f(\bar{u})$. **Q.E.D.**

We show the non-existence of strategy-proof, Pareto efficient, and equally compensatory mechanisms on arbitrary preference domains which consist of more than three quasi-linear preferences.

Theorem 2. *Let* $\# U \geq 4$. *There is no strategy-proof, Pareto efficient, and equally compensatory mechanism* $f \in F(U^n)$.

Proof: Let u_i^a, u_i^b, u_i^c, u_i^d be preferences in U such that $\lambda(u_i^a) < \lambda(u_i^b) < \lambda(u_i^c) < \lambda(u_i^d)$. Suppose that there is a strategy-proof, Pareto efficient, and equally compensatory mechanism $f \in F(U^n)$. It follows from Lemma 3 that f satisfies the constant transfer property on $\Gamma(U^n)$ relative to some $\pi \in \Pi$. Notice that $(u_1^a, \ldots, u_n^a)$, $(u_1^c, \ldots, u_n^c) \in \Gamma(U^n)$, and for all $i \in N$, (u_i^b, u_{-i}^a), $(u_i^d, u_{-i}^c) \in \Gamma(U^n)$. First, we assume that $C_f(u_1^a, \ldots, u_n^a) = \{j\}$. It follows from Lemma 1 that for all $i \neq j$, $C_f(u_i^b, u_{-i}^a) = \{i\}$. By strategy-proofness, $u_i^a(\pi_i(j), 0) \geq u_i^a(\pi_i(i), 1)$ and $u_i^b(\pi_i(i), 1) \geq u_i^b(\pi_i(j), 0)$. Hence, $\pi_i(i) + \lambda(u_i^a) \leq \pi_i(j) \leq \pi_i(i) + \lambda(u_i^b)$ for all $i \neq j$. Adding up these inequalities for all $i \neq j$, we have that $\sum_{i \neq j} \pi_i(i) + (n-1)\lambda(u_i^a) \leq \sum_{i \neq j} \pi_i(j) \leq \sum_{i \neq j} \pi_i(i) + (n-1)\lambda(u_i^b)$. By budget balance $\sum_{i \in N} \pi_i(j) = M$, we have that $(n-1)\lambda(u_i^a) \leq M - \sum_{i \in N} \pi_i(i) \leq (n-1)\lambda(u_i^b)$. Next, we assume that $C_f(u_1^c, \ldots, u_n^c) = \{k\}$. It follows from Lemma 1 that for all $i \neq k$, $C_f(u_i^d, u_{-i}^c) = \{i\}$. By strategy-proofness, $u_i^c(\pi_i(k), 0) \geq u_i^c(\pi_i(i), 1)$ and $u_i^d(\pi_i(i), 1) \geq u_i^d(\pi_i(k), 0)$. Hence, $\pi_i(i) + \lambda(u_i^c) \leq \pi_i(k) \leq \pi_i(i) + \lambda(u_i^d)$ for all $i \neq k$. Adding up these inequalities for all $i \neq k$, we have that $\sum_{i \neq k} \pi_i(i) + (n-1)\lambda(u_i^c) \leq \sum_{i \neq k} \pi_i(k) \leq \sum_{i \neq k} \pi_i(i) + (n-1)\lambda(u_i^d)$. By budget balance $\sum_{i \in N} \pi_i(k) = M$, we have that $(n-1)\lambda(u_i^c) \leq M - \sum_{i \in N} \pi_i(i) \leq (n-1)\lambda(u_i^d)$. Since $\lambda(u_i^b) < \lambda(u_i^c)$, this is a contradiction.

Q.E.D.

We have two corollaries to Theorem 2. The first one is equivalent to Theorem 5 in Schummer (2000). It follows from the fact that equal compensation is vacuously true when $n = 2$. The second one is an extension of a corollary to Theorem 1 in Tadenuma and Thomson (1995).[5] It follows from the fact that "envy-freeness" (Foley, 1967) implies Pareto efficiency (Svensson, 1983) and equal compensation in our model.

Corollary 1. (Schummer, 2000). *Let* $n = 2$ *and* $\# U \geq 4$. *There is no strategy-proof and Pareto efficient mechanism* $f \in F(U^n)$.

[5] It follows from Theorem 1 in Tadenuma and Thomson (1995) that there is no strategy-proof and envy-free mechanism $f \in F(U_A^n)$.

Corollary 2. *Let $\# U \geq 4$. There is no strategy-proof and envy-free mechanism $f \in F(U^n)$.*

Next, we characterize the set of strategy-proof and Pareto efficient mechanisms on very small preference domains in the two-agent case. The following two theorems show that strategy-proofness puts some constraint on transfer allocation functions. It turns out that there is a trade-off between the restriction of the preference domain and the constraint on transfer allocation functions. The arguments are much the same as Lemma 3 and Theorem 2, and the proofs will be omitted.

Lemma 4. *Let $n = 2$. If a mechanism $f \in F(U^n)$ satisfies strategy-proofness and Pareto efficiency, then f satisfies the constant transfer property on U^n relative to some $\pi \in \Pi$.*

Theorem 3. *Let $n = 2$ and $\# U = 3$. Assume that $U = \{u_i^a, u_i^b, u_i^c\}$, where $\lambda(u_i^a) < \lambda(u_i^b) < \lambda(u_i^c)$. A mechanism $f \in F(U^n)$ satisfies strategy-proofness and Pareto efficiency if and only if (i) for all $u \in U^n$, $C_f(u) \subset \mathrm{Argmax}_{i \in N}\{\lambda(u_i)\}$, and (ii) f satisfies the constant transfer property on U^n relative to some $\pi \in \Pi$, where $\sum_{i \in N} \pi_i(i) = M - \lambda(u_i^b)$.*

Theorem 4. *Let $n = 2$ and $\# U = 2$. Assume that $U = \{u_i^a, u_i^b\}$, where $\lambda(u_i^a) < \lambda(u_i^b)$. A mechanism $f \in F(U^n)$ satisfies strategy-proofness and Pareto efficiency if and only if (i) for all $u \in U^n$, $C_f(u) \subset \mathrm{Argmax}_{i \in N}\{\lambda(u_i)\}$, and (ii) f satisfies the constant transfer property on U^n relative to some $\pi \in \Pi$, where $M - \lambda(u_i^b) \leq \sum_{i \in N} \pi_i(i) \leq M - \lambda(u_i^a)$.*

As in the two-agent case, we can construct strategy-proof, Pareto efficient, equally compensatory mechanisms on very small preference domains in the n-agent case.

Example 1. Let $\# U = 3$. Assume that $U = \{u_i^a, u_i^b, u_i^c\}$, where $\lambda(u_i^a) < \lambda(u_i^b) < \lambda(u_i^c)$. Consider mechanisms $f \in F(U^n)$ such that for all $u \in U^n$, (i) $C_f(u) \subset \mathrm{Argmax}_{i \in N}\{\lambda(u_i)\}$, (ii) $t_i(u) = \dfrac{M - (n-1)\lambda(u_i^b)}{n}$ for $i \in C_f(u)$, and (iii) $t_j(u) = \dfrac{M + \lambda(u_i^b)}{n}$ for all $j \notin C_f(u)$. These mechanisms satisfy strategy-proofness, Pareto efficiency, and equal compensation.

Example 2. Let $\# U = 2$. Assume that $U = \{u_i^a, u_i^b\}$, where $\lambda(u_i^a) < \lambda(u_i^b)$. Consider mechanisms $f \in F(U^n)$ such that for all $u \in U^n$, (i) $C_f(u) \subset \mathrm{Argmax}_{i \in N}\{\lambda(u_i)\}$, (ii) $t_i(u) = \dfrac{M - (n-1)\lambda'}{n}$ for $i \in C_f(u)$, and (iii) $t_j(u) = \dfrac{M + \lambda'}{n}$ for all $j \notin C_f(u)$, where $\lambda(u_i^a) \leq \lambda' \leq \lambda(u_i^b)$. These mechanisms satisfy strategy-proofness, Pareto efficiency, and equal compensation.

5. Concluding remarks

We studied the problem of allocating a single indivisible good when monetary compensation is possible. A general result of Holmström (1979) implies that there is no strategy-proof and Pareto efficient mechanism on the set of all quasi-linear preferences. We considered some finite restrictions of the preference domain in order to understand how strong the impossibility result is. We proved that there is no strategy-proof and Pareto efficient mechanism when (i) the preference domain consists of a sufficiently large but finite number of quasi-linear preferences (Theorem 1), or (ii) the preference domain consists of more than three quasi-linear preferences and equal compensation is imposed on mechanisms (Theorem 2). We conclude that the impossibility result is very strong since such drastic restrictions of the preference domain are very unrealistic.

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