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ORIGINAL PAPER

Winning strategies of (*i*, *j*) multimove games on Thai, Japanese, and Chinese chess games

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Abstract
Let i and j be positive integers. An (i, j) multimove game is a two-player game in which the first player (White/female) has i moves per turn, and the second player (Black/male) has j moves per turn. In this paper, we present winning strategies for the (i, j) multimove games in Thai, Japanese, and Chinese chess games for all (i, j) not equal to (1,1) or (2,2). In addition, we demonstrate that Black does not have a winning strategy in a (2,2) multimove game in Thai, Japanese, or Chinese chess.

Keywords Multimove game · Makruk · Thai chess · Shogi · Japanese chess · Xiang Qi · Chinese chess

Mathematics Subject Classification 91A68 · 91A05 · 91A22

1 Introduction

The western international chess (Chess) is a two-person game played on a chessboard. Each player has various types of pieces, which are set up on the chessboard. In this game, each player moves his/her own movable piece to an empty square or to replace an opponent's piece (capture). If a player can move some pieces to capture the king in his/her next turn, then we say that the opponent is in check. Moreover,

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if the opponent cannot block, capture, or move his/her king to get away from the pieces, we say that he/she is in checkmate. The players alternately play the game until one can capture the opponent's king, and that player is the winner.

There are many types of chess games for different cultures in addition to Chess such as Makruk (Thai chess), Xiangqi (Chinese chess), and Shogi (Japanese chess). Ma (2020) compared Chess and Chinese chess by considering the differences in social structures, thinking patterns, and geographic. Yen et al. (2004) compared Chess and Chinese chess to develop Chinese chess game programming. Matsubara et al. (1996) compared Chess and Japanese chess to develop Japanese chess game programming especially on game complexity and decision complexity.

Chess is one of the most well-known and popular games, and it has been studied widely. In graph theory, for example, Laskar and Wallis (1999) determined the domination parameters of a rook chessboard graph and its line graph. Chia and Ong (2005) considered an $m \times n$ chessboard which has no closed knight's tours, and they gave open knight's tours for some $m \times n$ chessboards where m and n are positive integers. Ionascu et al. (2008) studied about the maximum number of kings that can be placed on an $m \times n$ chessboard such that any two kings cannot capture each other, where $1 \leq m, n \leq 8$. In computer science, Spoerer et al. (2013) studied on groups of Chess programs and discovered that the winning rate rises with the size of the group. Wu and Beal (2001) used the endgames database of Chinese chess to improve their algorithms. In game theory, Boros et al. (2012) considered Chess in term of an n-person positional game and determined existence of Nash equilibria. Wagon (2014) considered Chess played on the triangular hexagon board. He studied how to place a few pieces (not necessary identical) that can attack all hexagonal cells. Laisin et al. (2020) considered Chess played on a three-dimensional board and showed that there are non-attacking bishops in the board.

Nowadays, no one can find a winning strategy for Chess; however, there are some optimal strategies for playing Chess in some situations. Van Reek et al. (1998) gave plannings in short, medium, and long terms to develop Chess game programming. Donovan (2015) presented openings, endgames, quick checkmate strategies, etc. Berger and Dubbs (2015) determined (i, j) multimove games on Chess and gave winning stategies for all $(i,j) \notin \{(1,1),(2,2)\}$. The paper of Berger and Dubbs motivates us to study (i, j) multimove games on Thai, Japanese, and Chinese chesses. Here we present winning strategies of these (i, j) multimove games for all $(i,j) \notin \{(1,1),(2,2)\}$.

2 Thai chess (Makruk)

Thai chess is similar to Chess in many ways: the number of pieces, the role of pieces, etc. However, there are some differences between Thai chess and Chess as follows.

1. In Chess, the positions of White's king and Black's king are in the same column while the kings in Thai chess are in different columns.

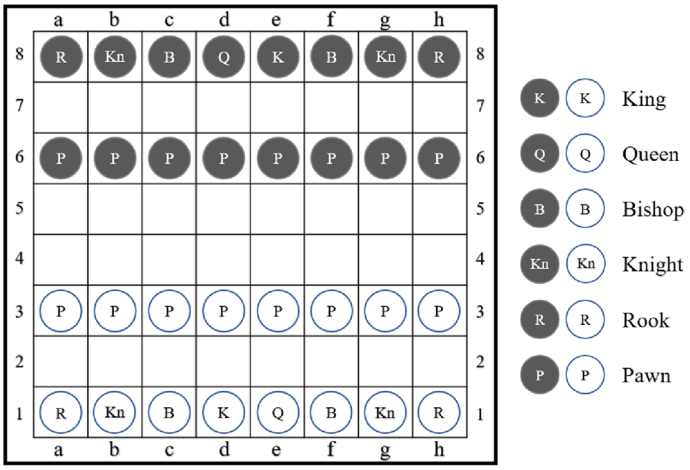

Fig. 1 The positions of pieces in Thai chess

Table 1 Movements of pieces in Thai chess

Pieces	Movements of pieces
King (Khun)	King moves one square in any direction
Queen (Med)	Queen moves one square diagonally
Bishop (Khon)	Bishop moves one square diagonally or forward
Rook (Rua)	Rook moves any number of squares orthogonally (forward, backward, left or right) without jumping over any piece in its way
Knight (Ma)	Knight moves exactly two squares orthogonally and then moves one square at a right angle. Moreover, the knight leaps freely over any piece in its path
Pawn (Bia)	Pawn moves one square forward, but to capture a piece, it moves one square forward diagonally. When a pawn reaches the opponent player's third row (the sixth row for White, or the third row for Black), it has been promoted to be a queen

2. Pawns in Chess are in the second row on each side, but in Thai chess, pawns are in the third row on each side. Furthermore, unlike in Chess, a pawn in Thai chess cannot move two squares in its first move.
3. The Queen is the most powerful piece in Chess. She can move many squares at one time; however, the Queen in Thai chess can only move one square.
4. The bishop in Chess can move any number of squares diagonally, but the bishop in Thai chess can move only one square diagonally or one square forward.

Thai chess (Makruk) is played on the 8×8 chessboard. Each player has 16 pieces, which consist of 1 king, 1 queen, 2 bishops, 2 knights, 2 rooks, and 8 pawns. The positions of pieces at the beginning of the game are shown in Fig. 1, and their movements are shown in Table 1. White moves first, and then the players alternate move. To move a piece, a player moves a movable own piece to an empty

Table 2 The player who has a winning strategy for (i, j) multimove Thai chess

White/Black	$j = 1$	$j = 2$	$j = 3$	$j \geq 4$
$i = 1$	?	Black	Black	Black
$i = 2$	White	?	Black	Black
$i = 3$	White	White	White	Black
$i \geq 4$	White	White	White	White

Table 3 Moving of the b8 or g8 black knight to capture the white king

Position of the king	Moving of the knight	Position of the king	Moving of the knight
a2	g8-e7-d5-c3-a2	d3	b8-d7-c5-d3
b1	g8-e7-d5-c3-b1	e1	b8-d7-e5-d3-e1
b2	b8-d7-e5-d3-b2	e2	g8-e7-d5-c3-e2
b3	b8-d7-c5-b3	e3	g8-e7-f5-e3
c1	b8-d7-e5-d3-c1	f1	g8-e7-f5-e3-f1
c2	g8-e7-f5-e3-c2	f2	b8-d7-e5-d3-f2
c3	g8-e7-d5-c3	f3	b8-d7-e5-f3
d2	b8-d7-e5-c4-d2	g2	g8-e7-f5-e3-g2

square, or replaces an opponent piece (capture). A player who captures the opponent king is the winner.

Theorem 1 *For* (i, j) *multimove Thai chess with* $(i, j) \notin \{(1, 1), (2, 2)\}$, *Black has a winning strategy if* $i < 4$ *and* $i < j$; *otherwise*, *White has a winning strategy*.

Table 2 shows the results of Theorem 1, and it can be proved by the following lemmas.

Lemma 2 *For* (i, j) *multimove Thai chess with* $i \geq 4$, *White has a winning strategy*.

Proof To capture the Black's king, White moves the knight b1-d2-c4-d6-e8. □

Lemma 3 *For* (i, j) *multimove Thai chess with* $i < 4 \leq j$, *Black has a winning strategy*.

Proof Note that White cannot capture the Black's king (e8) or the g8 knight in three moves. If White does not move the white king in the first turn, then Black moves the knight g8-e7-d5-e3-d1 to capture the White's king. Next, we consider the case that White moves the White's king in the first turn. In at most three moves, there are sixteen positions where the White's king can go to, which are a2, b1, b2, b3, c1, c2, c3, d2, d3, e1, e2, e3, f1, f2, f3 and g2. We show that Black can capture the White's king within four moves as shown in Table 3. □

Lemma 4 *For* (3, *j*) *multimove Thai chess with* $j \leq 3$, *White has a winning strategy*.

Proof White first moves the g1 knight to e2, and the a1 rook to a2. Also, White moves the b1 knight to d2 for check. Note that Black cannot capture the d1 king or the two knights (d2 and e2) within three moves. If Black does not move the king, White moves the knight d2-c4-d6-e8 to capture the Black's king. Otherwise, there are sixteen positions where the Black's king can go to within *j* moves, which are b7, c6, c7, c8, d6, d7, d8, e6, e7, f6, f7, f8, g6, g7, g8, and h7. By symmetry of Thai Chess's board, White can easily apply the algorithm shown in Table 3 to capture the Black's king. □

Lemma 5 *For* (2, 1) *multimove Thai chess*, *White has a winning strategy*.

Proof White moves the knight b1-d2-e4 for check. Note that Black cannot capture the e4 knight or the d1 king in one move. Thus, Black needs to move the e8 king to d7, e7 or f7. If Black moves the e8 king to d7(f7), then White moves the knight e4-c5-d7 (e4-g5-f7) to capture the king. Thus, Black should move the king to e7. Then White moves the knight g1-e2-d4 for check. Note that Black still cannot capture any knight or the king in one move. No matter how Black moves, White can easily apply Table 3 to capture the king. □

Lemma 6 *For* (2, 3) *multimove Thai chess*, *Black has a winning strategy*.

Proof Note that Black can move the rook a8-a7, the knight b8-d7, and the knight g8-e7 in his first turn to make White be in check in her second turn. Furthermore, there are twenty-nine positions to which White's king can go in two turns (see Fig. 2). If White does not capture one of the Black's knights or the Black's king in the first two turn, whenever White move the king, Black can capture it by using the moving shown in Tables 3 and 4. Then White has three options in her first turn to counter the Black's strategy.

Case 1: White moves the knight b1-d2-c4 or b1-d2-e4 for check or prepares to capture the e7 knight by moving the knight g1-e2-d4 or g1-e2-f4.

Black captures the white knight by a pawn in two moves, and then moves the g8 knight to e7 for check. Note that White cannot capture or block the e7 knight. Hence, White must move the d1 king to another position. If White moves the king to b1, c2, or e2, then Black captures the king by the e7 knight; otherwise (the White's king is in b2, c1, d2, e1, or f2), Black can move the b8 knight to d7 for checkmate (see Tables 3 and 4). Even though Black has two more moves, he may move some other pieces that do not affect the game such as moving the rook a8-a7-a8.

Case 2: White prepares to capture the e7 knight by moving a pawn c3-c4-c5, e3-e4-e5, or g3-g4-g5.

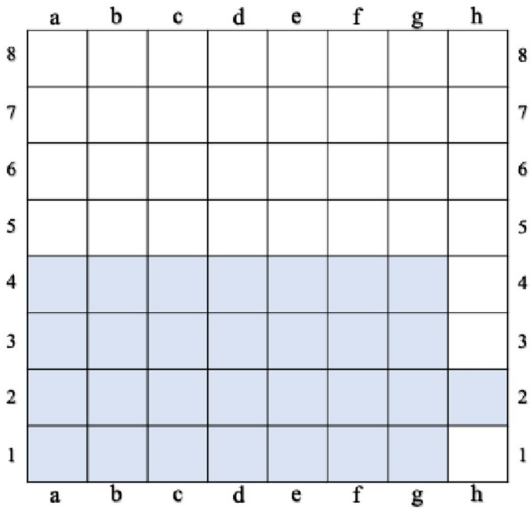

Fig. 2 The positions where the white king can go to within four moves

Table 4 Moving of the d7 or e7 black knight to capture the white king

Position of the king	Moving of the knight	Position of the king	Moving of the knight
a1	d7-c5-b3-a1	e4	d7-c5-e4
a3	d7-e5-c4-a3	f4	e7-d5-f4
a4	d7-c5-a4	g1	d7-e5-f3-g1
b4	e7-d5-b4	g3	e7-f5-g3
c4	d7-e5-c4	g4	d7-e5-g4
d1	e7-d5-c3-d1	h2	d7-e5-f3-h2
d4	e7-f5-d4		

Black moves a pawn to capture the white pawn, and then moves the two knights to d7 and e7 for checkmate (see Tables 3 and 4).

Case 3: White prepares to capture the e7 knight by moving the pawn c3-c4 and the knight b1-c3.

Black captures the c3 knight by moving the d6 pawn in three moves. We consider the following situations. If White moves the knight g1-e2-d4, then Black captures it by the promoted c3 pawn. Also, Black moves two knights to d7 and e7 for checkmate. If White moves the knight g1-e2-f4, then Black captures it by the e6 pawn, and Black moves the knight g8-e7 for check. Thus, White must move the king to b1. Black moves the promoted pawn c3-d4 and the knight b8-d7 for checkmate. If White moves a pawn e3-e4-e5, or g3-g4-g5, then Black captures it by a pawn, and moves the two knights to d7 and e7 for check. Thus, White must move the king to b1. Black moves the promoted pawn c3-d4 for checkmate. Otherwise, Black moves the b8 knight to d7, and moves the g8 knight to e7 for checkmate. □

For (1, 3) multimove Thai chess, Black can easily apply the strategy in Lemma 6 since White is less able to block or attack. Hence, we get the following lemma.

Lemma 7 *For* (1, 3) *multimove Thai chess*, *Black has a winning strategy*.

Lemma 8 *For* (1, 2) *multimove Thai chess*, *Black has a winning strategy*.

Proof Black uses the following algorithm with the condition that whenever White moves a pawn d3-d4 or e3-e4, Black captures the pawn immediately with his pawn.

If White moves the d1 king to d2, then Black moves the knight b8-d7-e5 for check. Thus, White must move the king to c2, d1, or e2. Then Black moves the knight g8-e7-d5 for checkmate.

If White moves the c3 (g3) pawn, Black moves the knight g8-e7-f5 (d5) for check. Then White must move the king to d2. After that, Black moves the knight b8-d7-c5 for checkmate.

If White moves the b3 (f3) pawn, Black move the knight g8-e7-d5 for check. Thus, White must move the king to d2. Then Black moves the knight b8-d7-e5 (c5) for checkmate.

Otherwise, Black moves the knight g8-e7-d5 for check. If White moves the king to c2 or e2, then Black can capture the king by d5 knight in the coming turn; otherwise (White moves the king to b2, c1, d2, e1, or f2), Black moves the knight b8-d7-e5 for checkmate. □

3 Japanese chess (Shogi)

Japanese chess (Shogi) is played on the 9×9 chessboard. Each player has 20 pieces, which consist of 1 king, 2 golds, 2 silvers, 2 knights, 2 lances, 1 rook, 1 bishop, and 9 pawns. The positions of pieces at the beginning of the game are shown in Fig. 3, and their movements are shown in Table 5. The goal of Japanese chess is capturing the opponent's king, which is the same as other chesses. In this chess, White moves first, and then the players alternate move. To move a piece, a player moves a movable own piece to an empty square or captures an opponent piece. If a piece (except a king or a gold) lives in the promotion zone (the seventh, eighth, and ninth rows for White and the first, second, and third rows for Black), the player can promote that piece after its move (the player does not need to promote the piece). In the (*i*, *j*) multimove Japanese chess, a promotion is not considered a move. Moreover, a player holding one or more captured pieces can return a captured piece onto the board instead of moving (drop). The dropped piece can be returned to an empty square under the following restrictions:

1. A dropped piece in the promotion zone must be an unpromoted piece; however, it can be promoted after it moves to or from the promotion zone.

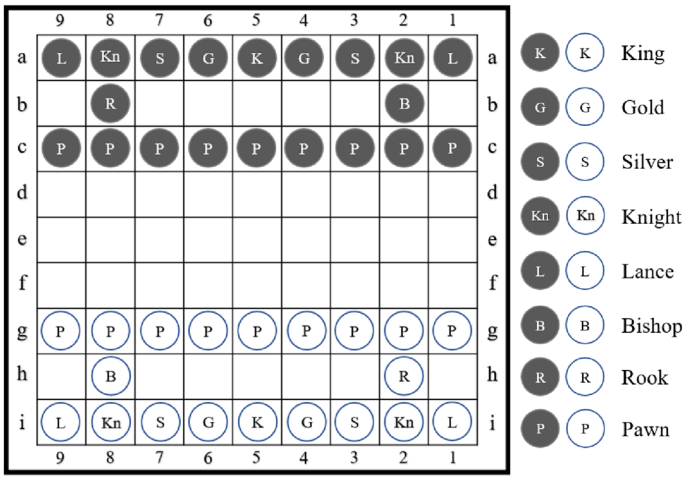

Fig. 3 The position of each piece in Japanese chess

2. A player cannot drop a pawn into a column that has another his/her unpromoted pawn.
3. Dropping a pawn cannot make the opponent player in checkmate.
4. A piece cannot be dropped at a square from which it has no possible future move.

Theorem 9 *For* (i, j) *multimove Japanese chess with* $(i,j) \notin \{(1,1),(2,2)\}$, *Black has a winning strategy if* $i < 3$ *and* $i < j$; *otherwise*, *White has a winning strategy*.

The results of Theorem 9 are shown in Table 6, and the theorem can be proved by the following lemmas.

Lemma 10 *For* (i, j) *multimove Japanese chess with* $i \geq 3$, *White has a winning strategy*.

Proof White moves the pawn g7-f7 and moves the bishop h8-c3-a5. □

Lemma 11 *For* (1, 3) *multimove Japanese chess*, *Black has a winning strategy*.

Proof Note that White is in check at the beginning of the game since Black can move the pawn c3-c4 and the bishop b2-g7-i5 to capture the i5 king in his first turn. Since White cannot capture the bishop, White must do one of the following cases.

Case 1 White moves the pawn g6-f6.
Black moves the pawn c1-d1 and the bishop b2-c1-g5 for check. Since White cannot capture the g5 bishop in one move, White is in checkmate.

Case 2 White moves either the silver i7-h6, the gold i6-h6, or the rook h2-h6.

Table 5 Movements of pieces in Japanese chess

Pieces	Movements of pieces	Movements of promoted pieces
King (Gyokusho)	King moves one square in any directions	They cannot be promoted
Gold (Kinsho)	Gold moves one square in one of six directions, which are forwards, diagonally forwards, sideways, and backwards	
Rook (Hisha)	Rooks moves any number square orthogonally without jumping over another piece	Promoted Rook moves like a normal rook, but acquires the power to move exactly one square diagonally
Bishop (Kakugyo)	Bishop moves any number of square diagonally, not allowed to jump over any pieces	Promoted bishop moves like a normal bishop. Also, he can move one square vertically or horizontally
Silver (Ginsho)	Silver moves one square in one of five directions consisting of forwards, diagonally forwards, and diagonally backwards	They move exactly like a gold.
Knight (Keima)	Knight moves two square forward and then one square sideways. Note that knight is the only pieces that can jump over another pieces	
Lance (Kyosha)	Lance moves any number of squares forwards without jumping over another pieces	
Pawn (Fuhyo)	Pawn moves only one square forwards	

Table 6 The player who has a winning strategy for (i, j) multimove Japanese chess

White/Black	$j = 1$	$j = 2$	$j \geq 3$
$i = 1$	?	Black	Black
$i = 2$	White	?	Black
$i \geq 3$	White	White	White

Black moves the pawn c3-d3 and the bishop b2-g7-h8 to capture the White's bishop. Also, Black promotes the h8 bishop for check. If White moves either the silver i7-h8 or the rook h6-h8 to capture the bishop, then Black returns the bishop to h4 to captures the i5 king; otherwise, he captures the king by that bishop.

Case 3 White moves either the king i5-h4 or i5-h5.

Black moves the pawn c3-d3 and the bishop b2-g7-h8. No matter how White moves in the next turn, Black can return the bishop to capture the White's king.

□

Lemma 11 implies the following lemma.

Lemma 12 *For* (1, *j*) *multimove Japanese chess with* $j \geq 3$, *Black has a winning strategy*.

Lemma 13 *For* (2, 3) *multimove Japanese chess*, *Black has a winning strategy*.

Proof Note that White is in check at the beginning of the game since Black can move the pawn c3-d3 and the bishop b2-g7-i5 in the next turn. Thus, White must do one of the following.

Case 1 White moves the pawn g5-f5-e5.

Black moves the c1 pawn and moves the bishop b2-c1-i7 to capture the silver. White is in checkmate since Black can return the silver to capture the White's king in the next turn.

Case 2 White moves the pawn g6-f6.

If White also moves the pawn g7-f7, then Black moves the pawn c3-d3 and the bishop b2-f6-h8. White is now in checkmate since Black can return the bishop to capture the White's king in the next turn.

If White also moves the pawn g8-f8, then Black moves the pawn c3-d3 and the knight a2-c3-e4. Note that Black can move the knight e4-g5, and promotes it to capture the king by the remaining two moves. Thus, White is in check. Since White cannot block or move her king to get away from the knight, White should capture the e4 knight. If White does so, the White's king is still at i5. Then Black can capture the king by moving the bishop b2-f6-g7-i5. If White also moves the rook h2-h6, then Black moves the pawn c3-d3 and the knight a2-c3-e4. Now White is in check with the e4 knight. Note that White need use at least two moves

to capture the e4 knight. If White does so, Black can move the bishop b2-f6-g5-h6 and promote it to checkmate.

Otherwise, Black moves the pawn c3-d3 and the bishop b2-f6-d8. White is in checkmate since Black can capture the king by the d8 bishop in the next turn.

Case 3 White moves the pawn g7-f7 and the bishop from h8 to either d4, e5, or f6.

Black moves the pawn c3-d3, and the b2 bishop to captures the White's bishop and the i9 lance, so White is in checkmate.

Case 4 White moves either the i7 silver, i6 gold, or the h2 rook to h6 by using one or two moves.

If White moves the silver i7-h6 and the bishop h8-i7, Black moves the pawn c3-d3 and the bishop b2-g7-h6 for checkmate. Otherwise, Black moves the pawn c3-d3 and the bishop b2-g7-h8 for checkmate.

Case 5 White moves the king to h3.

Black moves the pawn c3-d3, and moves the bishop b2-e5-d6 for checkmate.

Case 6 White moves the king to either h4, h5, i4 or i6 by using one or two move. Black moves like Case 4 for checkmate.

□

We can easily apply Lemma 13 to get the following lemma.

Lemma 14 *For* (2, *j*) *multimove Japanese chess with* $j \geq 3$, *Black has a winning strategy*.

Lemma 15 *For* (2, 1) *multimove Japanese chess*, *White has a winning strategy*.

Proof White moves the pawn g7-f7, and moves the bishop h8-e5 for check. Since Black cannot capture the e5 bishop or i5 king, and he also cannot block the bishop, he must move the king a5-b5. Then White moves the knight i8-g7-e6 for checkmate. □

Lemma 16 *For* (1, 2) *multimove Japanese chess*, *Black has a winning strategy*.

Proof Since Black can move the pawn c3-d3 and the bishop b2-e5 in his first turn (to capture the White's king in his second turn), We consider the following White's options in the first turn to counter this Black's strategy.

Case 1 White prepares to capture the e5 bishop by moving the pawn g7-f7 (to make way for the h8 bishop).

Then Black moves the pawn c3-d3 and the bishop b2-h8 to capture the White's bishop. Also, he promotes his own bishop for check. If White moves the king i5-h5, then Black can move the the promoted bishop h8-i7-i6 to capture the silver and gold for checkmate; otherwise, Black can return the bishop to capture the White's king.

Case 2 White prepares to capture the e5 bishop in the first turn by moving the pawn g5-f5.

Black moves the pawn c1-d1 and the bishop b2-c1 for check. Since White cannot capture the c1 bishop or move the king to get away from the bishop in one move, White must move the pawn g4-f4 to block the c1 bishop. Then Black moves the bishop c1-f4 to capture the pawn. He also moves the bishop f4-g5 to promote for checkmate.

Case 3 White prepares to block the e5 bishop by moving either the pawn g6-f6, or the pawn g4-f4.

Case 3.1 White moves the pawn g6-f6.

Black moves the pawn c3-d3 and the bishop b2-f6. Then White is in check. Note that White cannot capture the f6 bishop or the king in her second turn. White should not move the king in her second turn since Black can still capture it by the f6 bishop. Hence White should move either the h2 rook, the i6 gold, or the i7 silver to h6 to block the f6 bishop. Then Black moves the bishop f6-g7-h6 to checkmate.

Case 3.2 White moves the pawn g4-f4.

Black moves the pawn c3-d3 and the bishop b2-e5 to check. Note that White cannot capture the e5 bishop or the king in her second turn.

If White moves the king to get away from the e5 bishop, Black moves the knight a2-c3 and the bishop e5-f4 to checkmate.

If White moves the pawn g6-f6 to block the e5 bishop, Black moves the bishop e5-f6-g5 and promote it to checkmate.

If White moves either the h2 rook, the i6 gold, or the i7 silver to block the e5 bishop in the h6 position, Black moves the bishop e5-g7-h6 to checkmate.

Case 4 White prepares to block the e5 bishop by moving either the rook h2-h6, the silver i7-h6, or the gold i6-h6.

Black moves the pawn c3-d3 and the bishop b2-e5 for check. Now, White can block the bishop by moving either the gold i4-h4, the silver i3-h4, the rook h2-h4, or the pawn g4-f4. Also, White can move the king i5-h5 or i5-i6 (if it is possible) to get away from the bishop. Anyway, Black moves the bishop e5-g7-h6 to capture the piece at h6 (silver, gold, or rook), and promotes it for check. If White does not capture the promoted bishop, Black can move it to capture the king. Assume this is not the case. If Black captured piece is gold or rook in his second turn, then he can return it to capture the king. We next consider the case that Black captured the silver in his second turn. If White captured the promoted bishop by her king, Black can return the silver to i5 to capture the king; otherwise, Black can return the silver to h5 or i6 to capture the king.

Case 5 White prepares to block the e5 bishop by moving either the rook h2-h4, the silver i3-h4, or the gold i4-h4.

We can apply the algorithm in Case 4.

Case 6 White moves the king.

Case 6.1 White moves the king i5-h4.

Black moves the pawn c3-d3 and the bishop b2-e5 for check. Thus, White must move either the king h4-h5 or the pawn g4-f4. If White moves the king h4-h5, then Black moves the bishop e5-g3-h2 to capture the rook. No matter how White moves, Black can return the rook to capture the White's king. If White moves the pawn, then Black moves the bishop e5-d6-f4 for checkmate.

Case 6.2 White moves the king i5-h5.

Black moves the pawn c1-d1 and the bishop b2-c1 for check. Thus, White must move either the king h5-i5 or the pawn g4-f4. If White moves the king h5-i5, then Black moves the bishop c1-f4-g5, and promotes it for checkmate. Otherwise, Black moves the bishop c1-e3-f4 for checkmate.

Case 6.3 White moves the king i5-h6.

Black moves the pawn c3-d3 and the bishop b2-e5 for check. Thus, White must move either the king h6-h5 or the pawn g6-f6. In either way, Black moves the bishop e5-g3-h2 for checkmate.

In other cases, no matter how White moves a piece in the first turn, Black moves the pawn c3-d3 and the bishop b2-e5 to check. If White moves the king in her second turn, Black moves the knight a2-c3-e4 to checkmate. Otherwise, Black can capture the king by the e5 bishop. □

4 Chinese chess (Xiangqi)

Chinese chess (Xiangqi) is played by placing a piece on an intersection of lines. The board of Chinese chess has ten lines across nine lines. The area between fifth horizontal line and sixth horizontal line is called the river. Each player has 16 pieces, which consist of 1 king, 2 guards, 2 elephants, 2 rooks, 2 knights, 2 cannons, and 5 pawns. The positions of pieces at the beginning of the game are shown in Fig. 4, and their movements are shown in Table 7. White moves first, and then the players alternate move. To move a piece, a player moves a movable own piece to an empty intersection or captures an opponent piece. A player cannot move a piece to make

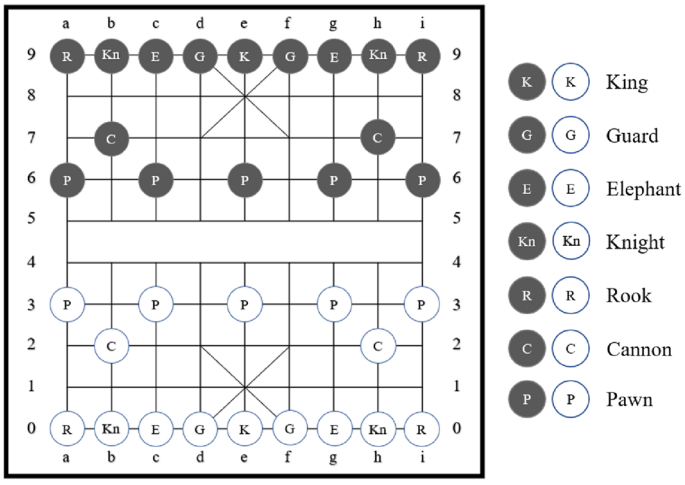

Fig. 4 The position of each piece in Chinese chess

Table 7 Movements of pieces in Chinese chess

Pieces	Movements of pieces
King (Shuai)	King moves one position orthogonally, and he must be in nine position of the palace (the positions on the square marked by X)
Guards (Shi)	Guard moves one position diagonally. Similar to the king, the guards must be in the palace
Elephants (Xiang)	Elephant moves exactly two positions diagonally without jumping over another piece, and it cannot move across the river (it must be on their half of the board)
Rooks (Ju)	Rook moves any number positions horizontally or vertically without a piece between the first and the last position
Knights (Ma)	Knight moves two positions orthogonally and followed by one position outward-diagonally
Cannons (Pao)	Cannon moves exactly like a rook, but it must jump over exactly one another piece (own or opponents) to capture a piece
Pawns (Bing)	Pawn moves only one position forward. After it has crossed the river, it may also move one step left or right. It always capture as it moves

the kings live in same vertical line without another piece between them. The player who captures the opponent king is the winner.

Theorem 17 *For* (i, j) *multimove Chinese chess with* $(i, j) \notin \{(1, 1), (2, 2)\}$, *Black has a winning strategy if* $i < 3$ *and* $i < j$; *otherwise*, *White has a winning strategy*.

Table 8 shows the results of Theorem 17, and it can be proved by the following lemmas.

Lemma 18 *For the* (i, j) *multimove Chinese chess with* $i \geq 3$, *White has a winning strategy*.

Proof White moves the cannon b2-b4-e4-e9 to capture the Black's King. □

Lemma 19 *For the* $(2, 3)$ *multimove Chinese chess*, *Black has a winning strategy*.

Proof White is in check at the beginning of the game since Black can move the cannon b7-b4-e4-e0, b7-b5-e5-e0, h7-h4-e4-e0, or h7-h5-e5-e0 to capture the White's king. Since White cannot capture the Black's king or any Black's cannon in two moves, White should move the king to get away from the Black's cannons or move

Table 8 The player who has a winning strategy for (i, j) multimove Chinese chess

White/Black	$j = 1$	$j = 2$	$j \geq 3$
$i = 1$	?	Black	Black
$i = 2$	White	?	Black
$i \geq 3$	White	White	White

some pieces to block them. If White moves the king to d0 or f0 in the first turn, then Black moves the cannon b7-b0-d0 or the cannon h7-h0-f0. If White moves the king to d1 or f1 in the first turn, then Black can capture the White's King by a rook in three moves. If White moves the king to e1 or e2 in the first turn, Black can move a cannon to capture the king within three moves. Hence, White should block the cannons as follows in the first turn.

Case 1 White moves the cannons to b5 and h5.

Then Black is in check. To block the White's cannons, Black moves the cannon b7-e7. Also, Black moves the rooks to a8 and i8 for check. Note that White has only one way to block the Black's rooks, that is, to move the cannons b5-d5 and h5-f5. If White does so, then Black moves the cannon h7-h4-e4-e0 to capture the king; otherwise, Black can capture the king by a rook.

Case 2 White moves either the cannon b2-b5 and the cannon h2-h6 or the cannon b2-b6 and the cannon h2-h5.

Without loss of generality, we may assume that White moves the cannon b2-b5 and the cannon h2-h6 for check. Then Black moves the cannon h7-d7-d5 to block the b5 cannon. Also, Black moves the rook a9-a8. Hence, White is in check (by the d5 cannon and the a8 rook). Note that White cannot capture both d5 cannon and a8 rook in two moves. Since the White's king cannot get away from the a8 rook, White should capture the a8 rook by the b5 cannon or block it by a rook, a guard, or a cannon. Then Black can capture the king by the d5 cannon or the b7 cannon.

Case 3 White moves the cannons to b6 and h6.

Black moves the rook a9-a8-d8 and the rook i9-i8 for checkmate.

Case 4 White moves a guard to e1, and moves an elephant or a cannon to e2.

Black moves the rooks a9-a8 and i9-i8, and moves the cannon b7-b5 for checkmate.

Case 5 White moves the pawn e3-e4-e5.

Black moves the pawn e6-e5 and the cannon b7-e7-e0.

□

For (i, j) multimove Chinese chess with $1 \leq i \leq 2$ and $j \geq 3$, Black can easily apply the above strategy. Hence, we get the following lemma.

Lemma 20 *For the* (i, j) *multimove Chinese chess with* $1 \leq i < 3 \leq j$, *Black has a winning strategy*.

Lemma 21 *For the* (2, 1) *multimove Chinese chess*, *White has a winning strategy*.

Proof White moves the cannon b2-b4 and the cannon h2-e2 for check. Since Black has only one move, his king must be in the column e and there are at most two other Black's pieces in this column. If there is one piece (two pieces), White can capture the king by the b4 (e2) cannon. □

Lemma 22 *For the* (1, 2) *multimove Chinese chess*, *Black has a winning strategy*.

Proof We first show that if White moves a guard or an elephant in the first turn, then Black can capture the White's king in his first turn. Without loss of generality, we may assume that White moves the d0 guard or the c0 elephant. Black then moves the cannon b7-b0-e0 to capture the White's king.

Suppose this is not the case. Note that White cannot capture the king or any cannon in the first turn. Also, White cannot move any piece for check. No matter what White does in the first turn, Black can move either the cannons b7-e7 and h7-h5, or the cannons h7-e7 and b7-b5 for check. If Black does so, then White cannot capture or get away from the cannons in one move. Clearly, White cannot capture the king or a cannon in the first turn. Also, White cannot move a piece for check, so we consider the following cases.

Case 1 White moves the king e0-e1.
Black moves the cannons b7-e7 and h7-h5 for check. Since White cannot block the cannons in one moves, White must move the king to d1 or f1. If White moves the king to d1, then Black moves the rook a9-a8-d8 for checkmate; otherwise, Black moves the rook a9-a8-f8 for checkmate.

Case 2 White moves the pawn e3-e4.
Black moves the cannons b7-e7 and h7-h5 for check. To block the h5 cannon, White may do one of the following, which are moving the e4 pawn or moving a piece (a cannon, an elephant, a guard) to the line e. If White moves the e4 pawn, Black moves the pawn e5 and the cannon e7-e0 to capture the White's King. If White moves some piece to the line e, then Black can capture the king by the e7 cannon.

Case 3 White moves the pawn c3-c4 or g3-g4.
Without loss of generality, we assume that White moves the c3 pawn. Then Black moves the cannons b7-e7 and h7-h5 for checkmate.

Case 4 White moves the cannon b2-e2 or h2-e2.
Without loss of generality, we assume that White moves the cannon b2-e2. Black moves the cannons b7-e7 and h7-h5 for check. White must move a guard to e1 to block the e7 cannon. Then Black moves a rook to d8 for checkmate.

Case 5 White moves a cannon b2-b5, b2-b6, h2-h5, or h2-h6.
Without loss of generality, we assume that White moves the cannon b2. Then Black moves the cannons b7-e7 and h7-h5 for checkmate. □

5 Discussion

The proof of Lemma 5 can be simplified since White can "pass" (for example, move the king from d1 to d2 and back to d1) in her first turn, and she can win by using Black's strategy as described in Lemma 8. Using similar reasoning, the proofs for Lemmas 15 and 21 can also be simplified. However, we have chosen to retain the proofs in those lemmas because we aim to provide readers with a short, constructive algorithm for each of them. Furthermore, if Black has a winning strategy in a (2,2) multimove

game in Thai, Japanese, or Chinese chess, the same idea implies that White would also have a winning strategy in the same game, leading to a contradiction and the subsequent result.

Corollary 23 *Black does not have a winning strategy in a* (2,2) *multimove game in Thai*, *Japanese*, *or Chinese chess*.

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