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Impartial games with decreasing Sprague–Grundy function and their hypergraph compound

Endre Boros[1] · Vladimir Gurvich[2] · Nhan Bao Ho[3] · Kazuhisa Makino[4] · Peter Mursic[1]

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Abstract
The Sprague–Grundy (SG) theory reduces the disjunctive compound of impartial games to the classical game of *NIM*. We generalize this concept by introducing hypergraph compounds of impartial games. An impartial game is called SG-decreasing if its SG value is decreased by every move. Extending the SG theory, we reduce hypergraph compounds of SG-decreasing games to hypergraph compounds of single-pile *NIM* games. We show that this reduction works only if all games involved in the compound are SG-decreasing. A hypergraph is called SG-decreasing if the corresponding hypergraph compound of single-pile *NIM* games is an SG-decreasing game. We provide some necessary and some sufficient conditions for a hypergraph to be SG-decreasing. In particular, for hypergraphs with hyperedges of size at most 3 we obtain a necessary and sufficient condition verifiable in polynomial time.

Keywords Impartial game · Sprague-Grundy function · *NIM* · Hypergraph *NIM* · SG-decreasing hypergraph

1 Introduction

In this paper we consider two-person impartial games. In such a game positions are not repeated, the same moves are available for both players, they take turns alternately, and one who makes the last move is the winner. It is also assumed that there are only finitely many moves from each position, and every play terminates in a finite number of moves. We shall refer to these in the sequel simply as games. Let us add that such a game is in fact a family of games and it becomes a game in the every

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✉ Nhan Bao Ho
nhan.ho@latrobe.edu.au; nhanbaoho@gmail.com

Extended author information available on the last page of the article

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day sense only after the players agree on an initial position and the order they take turns. For an overview of such games see, e.g., Albert et al. (2007), Berlekamp et al. (2004), Conway (1976), Siegel (2013).

Sprague (1935), Sprague (1937) and Grundy (1939) considered the disjunctive compound (named also as disjunctive sum) of games, and developed an effective theory to handle such composite games. Smith (1966) considered additionally operations, called conjunctive and selective compounds. Some other game compositions were also considered in the literature, see e.g., Berge (1953), Milnor (1953).

In this paper we introduce *hypergraph compounds* generalizing disjunctive, conjunctive and selective compounds. Given n games Γ_1, ..., Γ_n and a hypergraph $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$, where $[n] = \{1, 2, \dots, n\}$, the $\mathcal{H}$-compound $\Gamma_{\mathcal{H}}$ of these games is played as follows: a player on its turn chooses a hyperedge $H \in \mathcal{H}$ and makes a move in every game Γ_i, $i \in H$. Note that $\emptyset \notin \mathcal{H}$ means that players are not allowed to pass. For instance, the hypergraph compounds with $\mathcal{H} = \{\{1\}, \{2\}, \dots, \{n\}\}$, $\mathcal{H} = \{\{1, 2, \dots, n\}\}$, and $\mathcal{H} = 2^{[n]} \setminus \{\emptyset\}$ can be seen to correspond to disjunctive, conjunctive and selective compounds, respectively. In the special case when all games Γ_i, $i \in [n]$, are single-pile NIM games, their hypergraph compound, denoted by $NIM_{\mathcal{H}}$, was already considered in Boros et al. (2019a), Boros et al. (2019b) and called hypergraph-NIM. This family contains classical NIM (11) and several variants considered in Moore (1910), Jenkyns and Mayberry (1980), Boros et al. (2021), Boros et al. (2018).

Given a game $\Gamma = (X, E)$, where X and E denote its sets of positions and moves, respectively, Sprague and Grundy introduced a nonnegative integer valued mapping $\mathcal{G}_\Gamma : X \to \mathbb{Z}_+$, which was later called the SG-function of Γ. They showed that $\mathcal{G}_\Gamma$ not only helps to play Γ, but also reduces disjunctive compound of games to NIM. For a position $x \in X$ the value $\mathcal{G}_\Gamma(x)$ is called the SG-value of x.

Interestingly, such a functional relation may not exist for other compounds, see e.g. Beideman et al. (2020). However, for several cases it does exist. Assume that $\Gamma_i = (X_i, E_i)$ are games and $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$ is a given hypergraph. Then, equations

$$\mathcal{G}_{\Gamma_{\mathcal{H}}}(x_1, \dots, x_n) \; = \; \mathcal{G}_{NIM_{\mathcal{H}}}\big(\mathcal{G}_{\Gamma_1}(x_1), \dots, \mathcal{G}_{\Gamma_n}(x_n)\big) \quad \forall\, x_i \in X_i,\; i \in [n] \tag{1}$$

hold for all hypergraphs $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$ if Γ_i, $i \in [n]$ are all single-pile[1] NIM games, by the definition of $NIM_{\mathcal{H}}$. By the Sprague-Grundy theory it also holds for all games Γ_i, $i \in [n]$ if $\mathcal{H} = \{\{1\}, \dots, \{n\}\}$, that is for disjunctive compounds. Obviously, the above equality always holds if $n = 1$, and hence in the sequel, we assume that $n \geq 2$. In our paper we provide a more general characterization for the above equality to hold and study the related hypergraph structures.

Note that $\mathcal{G}_\Gamma(y) > \mathcal{G}_\Gamma(x)$ may hold for a move $x \to y$, in general. A game Γ is called *SG-decreasing* if the SG value is strictly decreased by every move. A single-pile NIM is the simplest example of an SG-decreasing game.

[1] Some authors prefer to use heaps instead of piles. Since heaps have a more specific meaning in computer science, we prefer to use piles.

Theorem 1 *Equation* (1) *holds for an arbitrary hypergraph* $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$ *and SG-decreasing games* Γ_i, $i \in [n]$.

Given a game Γ and a position x of it, we denote by $h_\Gamma(x)$ the length of the longest play starting at x and call it the *height* of x in the game. A game Γ is called *h-unbounded* if for any nonnegative integer $k \in \mathbb{Z}_+$ there exists a position x in Γ such that $h_\Gamma(x) = k$. Note that all hypergraph NIM games including classical NIM are h-unbounded.

Our next statement shows that Eq. (1) fails unless all games involved in the compound are SG-decreasing. Here a hypergraph $\mathcal{H} \subseteq 2^{[n]}$ is called *spanning* if $\cup_{H \in \mathcal{H}} H = [n]$.

Theorem 2 *Let* Γ_i, $i \in [n]$, *be h-unbounded games. Then Eq.* (1) *holds for an arbitrary hypergraph* $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$ *if and only if all games* Γ_i, $i \in [n]$, *are SG-decreasing. Otherwise, Eq.* (1) *fails for any hypergraph* $\mathcal{H}$ *whose* (*inclusion-wise*) *minimal hyperedges form a spanning hypergraph with no singleton.*

We derive Theorem 2 from a characteristic property of conjunctive compounds shown in Sect. 3. The theorem shows that h-unbounded SG-decreasing games form the largest family of impartial games satisfying (1) for all compound operations. We note that $\mathcal{H} = \{[n]\}$ corresponding to the conjunctive compound satisfies the property of hypergraphs in the otherwise statement in Theorem 2.

Although playing an SG-decreasing game seems very simple, just like a single-pile NIM, nevertheless both recognizing SG-decreasing games and computing their SG-values are nontrivial computational problems, even in the subclass of hypergraph NIM games; see Sect. 5 for more details.

Interestingly, if we replace in Eq. (1) the SG function by the height function we always get equality.

Theorem 3 *Given games* $\Gamma_1 = (X_1, E_1), \ldots, \Gamma_n = (X_n, E_n)$ *and a hypergraph* $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$, *for all positions* $x_1 \in X_1, \ldots, x_n \in X_n$ *we have the identity*

$$h_{\Gamma_\mathcal{H}}(x_1, \ldots, x_n) \;=\; h_{NIM_\mathcal{H}}\big(h_{\Gamma_1}(x_1), \ldots, h_{\Gamma_n}(x_n)\big). \tag{2}$$

Note that a game Γ is SG-decreasing if and only if $\mathcal{G}_\Gamma = h_\Gamma$ (which will be shown in Lemma 4). Thus, the above two theorems immediately imply the following statement.

Corollary 1 *If* $NIM_\mathcal{H}$ *is SG-decreasing, then the* $\mathcal{H}$*-compound of SG-decreasing games is SG-decreasing.*

The above results motivate us to study SG-decreasing games and, in particular, SG-decreasing hypergraph NIM games. We call a hypergraph $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$ *SG-decreasing* if $NIM_\mathcal{H}$ is an SG-decreasing game.

Given a hypergraph $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$ and a subset $S \subseteq [n]$, denote by $\mathcal{H}_S$ the subhypergraph $\mathcal{H}_S \;=\; \{H \in \mathcal{H} \mid H \subseteq S\}$ of $\mathcal{H}$ induced by S.

We call a hypergraph $\mathcal{H}$ *totally transversal* if every induced subhypergraph is either empty or contains a hyperedge that intersects all others, i.e.

$$\forall S \subseteq [n] \text{ with } \mathcal{H}_S \neq \emptyset \;\; \exists H \in \mathcal{H}_S \text{ such that } \forall H' \in \mathcal{H}_S \; : \; H \cap H' \neq \emptyset. \tag{3}$$

We call $\mathcal{H}$ *intersecting* if any two hyperedges of it intersect i.e.,

$$\forall H, H' \in \mathcal{H} \; : \; H \cap H' \neq \emptyset. \tag{4}$$

Note that lines of a projective plane (or more generally, hyperplanes of projective geometries) are examples for intersecting hypergraphs.

Theorem 4 *Condition* (3) *is necessary, while* (4) *is sufficient for a hypergraph* $\mathcal{H} \subseteq 2^{[n]} \backslash \{\emptyset\}$ *to be SG-decreasing.*

As examples, let us consider hypergraph NIM games with $\mathcal{H} = \{H \subseteq [n] \mid |H| = k\}$, where $1 \leq k < n$, called *exact NIM* in Boros et al. (2018). These hypergraphs satisfy Condition (4) and thus are SG-decreasing, if $k > n/2$.

We can reformulate Theorem 4 as follows: intersecting hypergraphs are SG-decreasing and SG-decreasing ones are totally transversal. Note that both containments are strict.

The *dimension* of a hypergraph $\mathcal{H}$, denoted by $\dim(\mathcal{H})$, is defined as the size of the largest hyperedge in $\mathcal{H}$. In fact, totally transversal hypergraphs of dimension 4 may be not SG-decreasing; see an example in Sect. 4. Furthermore, there are many SG-decreasing hypergraphs that are not intersecting.

Let us call a position of a game *terminal* if there is no move from it. Total transversality can be characterized in the following way.

Theorem 5 *A hypergraph* $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$ *is totally transversal if and only if the SG-value in* $NIM_{\mathcal{H}}$ *is zero only in terminal positions.*

Our next (and perhaps most difficult to prove) result is that total transversality is sufficient for dimension at most 3.

Theorem 6 *A hypergraph* $\mathcal{H}$ *of dimension at most* 3 *is SG-decreasing if and only if it is totally transversal.*

Obviously, Condition (4) can be checked in polynomial time in the size of the hypergraph. We prove the same for Condition (3) for hypergraphs of bounded dimension. In addition we show that computing the height (as well as the SG) function values for $NIM_{\mathcal{H}}$ is an NP-complete problem, even for hypergraphs of dimension 3, while it is polynomial for hypergraphs of dimension 2.

The rest of the paper is organized as follows. In Sect. 2 we provide precise definitions and prove several basic properties of height and SG-functions. In Sect. 3 we prove Theorems 1, 2, and 3. In Sect. 4 we prove Theorems 4, 5, and provide a sketch of the proof of 6. In Sect. 5 we consider the computational complexity of recognizing total transversality and computing height and SG values. Finally, in the Appendix we include the full technical proof of Theorem 6.

2 Definitions, notation, and basic properties

It is well-known that in every game $\Gamma = (X, E)$ there exists a subset $\mathcal{P} \subseteq X$ of its positions satisfying (and uniquely defined by) two properties: (a) from every position $x \in X \backslash \mathcal{P}$ there is a move to the set $\mathcal{P}$; (b) every move from a position $x \in \mathcal{P}$ moves to $X \setminus \mathcal{P}$. Such positions are called $\mathcal{P}$-positions of Γ (Berlekamp et al. 2004; Siegel 2013). $\mathcal{P}$-positions help to play the game, since choosing a move to $\mathcal{P}$ from $X \setminus \mathcal{P}$ guarantees winning.

To play the disjunctive compound of two games, it is not sufficient to know $\mathcal{P}$-positions of the compounding games. Sprague-Grundy (Sprague 1935, 1937; Grundy 1939) introduced a mapping $\mathcal{G}_\Gamma : X \to \mathbb{Z}_+$ associated to game Γ, called the *SG-function* of Γ, that generalizes the concept of $\mathcal{P}$-positions:

$$\mathcal{G}_\Gamma(x) \;=\; \min\left\{\mathbb{Z}_+ \setminus \{\mathcal{G}_\Gamma(y) \mid (x \to y) \in E\}\right\}. \tag{5}$$

We call $\mathcal{G}_\Gamma(x)$ the *SG-value* of position $x \in X$. It is known that $x \in \mathcal{P}$ if and only if $\mathcal{G}_\Gamma(x) = 0$. Furthermore, the SG-function of a compound game is the NIM-sum (11) of the SG-functions of the compounding games, by the cited results of Sprague and Grundy.

Lemma 1 *Given a game $\Gamma = (X, E)$ and a mapping $f : X \to \mathbb{Z}_+$, we have $f = \mathcal{G}_\Gamma$ if and only if*

(i) *for any move $x \to y$ we have $f(x) \neq f(y)$, and*
(ii) *for any $x \in X$ and nonnegative integer $v < f(x)$ there exists a move $x \to y$ such that $f(y) = v$.*

Proof By its definition, the SG-function $\mathcal{G}_\Gamma$ satisfies (i) and (ii). The other direction can be shown by induction. For a terminal position $x \in X$ there are no moves from x and thus by (ii) we must have $f(x) = 0 = \mathcal{G}_\Gamma(x)$. Let us observe next that properties (i) and (ii) imply

$$f(x) \;=\; \min\left\{\mathbb{Z}_+ \setminus \{f(y) \mid (x \to y) \in E\}\right\}$$

for all $x \in X$. Thus, if for an $x \in X$ we assume that equality $f(y) = \mathcal{G}_\Gamma(y)$ holds for all moves $x \to y$, then $f(x) = \mathcal{G}_\Gamma(x)$ follows. □

Somewhat similarly the height functions can be uniquely characterized as follows.

Lemma 2 *Given a game $\Gamma = (X, E)$ and a mapping $f : X \to \mathbb{Z}_+$, we have $f = h_\Gamma$ if and only if the following three properties hold.*

(a) *Every move decreases the value of f.*
(b) *If f is positive in a position then there exists a move from this position that decreases f by exactly one.*
(c) *In each terminal position f takes value zero.*

Proof Note that games are regarded as acyclic directed graphs and, hence, the statement follows by basic graph-theoretic arguments. □

The next lemma compares both functions.

Lemma 3 *Given a game $\Gamma = (X, E)$ and a position $x \in X$*

(i) *for every move $x \to y$ we have $\mathcal{G}_\Gamma(x) \neq \mathcal{G}_\Gamma(y)$ and $h_\Gamma(x) > h_\Gamma(y)$;*
(ii) *for every position $x \in X$ and integer $0 \leq v < \mathcal{G}_\Gamma(x)$ there exists a move $x \to y$ such that $\mathcal{G}_\Gamma(y) = v$;*
(iii) *for every position $x \in X$ there exists a move $x \to y$ such that $h_\Gamma(y) = h_\Gamma(x) - 1$;*
(iv) *for every position $x \in X$ we have $\mathcal{G}_\Gamma(x) \leq h_\Gamma(x)$.*

Proof Properties (i) and (ii) follow directly by the definitions of the height and SG-functions. Property (iii) follows by the definition of a longest path. For (iv) note that for every position $x \in X$ there exists a move $x \to y$ by (ii) such that $\mathcal{G}_\Gamma(y) = \mathcal{G}_\Gamma(x) - 1$. By repeating such moves one can create a path of length $\mathcal{G}_\Gamma(x)$ starting from x, and it cannot be longer than the longest path from the same position. □

Note that $\mathcal{G}_\Gamma(x) < \mathcal{G}_\Gamma(y)$ is possible for a move $x \to y$. Recall that Γ is called SG-decreasing if $\mathcal{G}_\Gamma(x) > \mathcal{G}_\Gamma(y)$ for every move $x \to y$ in the game. The following statement provides characterizations of SG-decreasing games that will be instrumental in our proofs.

Lemma 4 *Given a game $\Gamma = (X, E)$ the following three statements are equivalent*:

(i) *Γ is SG-decreasing*;
(ii) $\mathcal{G}_\Gamma = h_\Gamma$;
(iii) *for every position $x \in X$ and integer $0 \leq v < h_\Gamma(x)$ there exists a move $x \to y$ in Γ such that $h_\Gamma(y) = v$.*

Proof To see (i)$\Longrightarrow$(ii) let us assume that $\mathcal{G}_\Gamma \neq h_\Gamma$ and consider a position $x \in X$ for which $\mathcal{G}_\Gamma(x) \neq h_\Gamma(x)$. By (iv) of Lemma 3 we get $\mathcal{G}_\Gamma(x) < h_\Gamma(x)$. Let $t = h_\Gamma(x)$ and consider the positions $x_0 = x$, x_1,..., x_t along a longest path starting from x, that is, $x_i \to x_{i+1}$ is a move in Γ and $h_\Gamma(x_{i+1}) = h_\Gamma(x_i) - 1$ for all $i = 0, \dots, t-1$. Since by (i) of Lemma 3 we have $\mathcal{G}_\Gamma(x_{i+1}) \neq \mathcal{G}_\Gamma(x_i)$ and since $t > \mathcal{G}_\Gamma(x_0)$ there must exists an index $0 \leq i < t$ such that $\mathcal{G}_\Gamma(x_{i+1}) > \mathcal{G}_\Gamma(x_i)$, completing the proof of our claim.

Assuming (ii) we get for every move $x \to y$ that $\mathcal{G}_\Gamma(x) = h_\Gamma(x) > h_\Gamma(y) = \mathcal{G}_\Gamma(y)$ by (i) of Lemma 3, implying (i).

(ii)$\Longrightarrow$(iii) follows by the definition of the SG-function.

Finally (iii)$\Longrightarrow$(ii) follows by Lemma 1. □

It is easily seen that both functions h and $\mathcal{G}$ can be computed in time linear in the size of Γ, whenever Γ is given explicitly, as an acyclic directed graph.

It is also not difficult to see that the height functions of hypergraph-NIM games satisfy monotonicity.

Lemma 5 *For a hypergraph $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$, and for all positions $x, y \in \mathbb{Z}_+^{[n]}$ such that $x \leq y$ we have $h_\mathcal{H}(x) \leq h_\mathcal{H}(y) \leq h_\mathcal{H}(x) + \sum_{i \in [n]}(y_i - x_i)$.* □

For a hyperedge $H \in \mathcal{H}$, and a position $x \in \mathbb{Z}_+^{[n]}$, we call a move $x \to x'$ in *NIM*$_\mathcal{H}$ an *H-move* if $\{i \in [n] \mid x'_i < x_i\} = H$. For a subset $H \subseteq [n]$ we denote by $\chi(H)$ its characteristic vector. For positions x with $x \geq \chi(H)$, we shall consider two special H-moves from x:

Slow H-move: $x \to x^{s(H)}$ defined by $x_i^{s(H)} = x_i - 1$ for $i \in H$, and $x_i^{s(H)} = x_i$ for $i \notin H$, that is by decreasing the value of every coordinate in H by exactly one.

Fast H-move: $x \to x^{f(H)}$ defined by $x_i^{f(H)} = 0$ for $i \in H$, and $x_i^{f(H)} = x_i$ for $i \notin H$, that is, by decreasing the value of every coordinate in H to zero.

For a position $x \in \mathbb{Z}_+^{[n]}$ we denote by $\mathrm{supp}\,(x) = \{i \mid x_i > 0\}$ the set of its support, and by $\mathcal{H}_{\mathrm{supp}\,(x)}$ the subhypergraph of $\mathcal{H}$ induced by $\mathrm{supp}\,(x)$.

Then the following property holds for the height function $h_\mathcal{H}$.

Lemma 6 *For a hypergraph $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$, let $x \in \mathbb{Z}_+^{[n]}$ be a position such that $h_\mathcal{H}(x) > 0$. Then for any hyperedge $H \in \mathcal{H}_{\mathrm{supp}\,(x)}$ we have*

$$h_\mathcal{H}(x) \; > \; h_\mathcal{H}(x^{s(H)}) \; \geq \; \max\{h_\mathcal{H}(x^{f(H)}),\; h_\mathcal{H}(x) - |H|\}.$$

Proof It follows from Lemmas 2 and 5. □

Lemma 7 *Let $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$ be a hypergraph. For a hyperedge $H \in \mathcal{H}$, let $x \in \mathbb{Z}_+^{[n]}$ be a position such that $x \geq \chi(H)$. Then for every integer v with $h_\mathcal{H}(x^{f(H)}) \leq v \leq h_\mathcal{H}(x^{s(H)})$ we have an H-move $x \to x'$ such that $h_\mathcal{H}(x') = v$.*

Proof We decrease the i-th component of $x^{s(H)}$, $i \in H$, in an arbitrary order, subtracting one in each step, until we get $x^{f(H)}$. Every time the height decreases by at most one by Lemma 5. Since $x_i^{s(H)} < x_i$ for all $i \in H$, all the positions encountered in the above process can be reached from x by a single H-move. □

The next claim follows directly from the definitions, and will be frequently used in our proofs.

Lemma 8 *Let $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$ be a hypergraph and $x \in \mathbb{Z}_+^{[n]}$ be a position such that $h_{\mathcal{H}}(x) > 0$. Then we have $h_{\mathcal{H}}(x^{f(H)}) = 0$ for a hyperedge $H \in \mathcal{H}$ if and only if H intersects all hyperedges of $\mathcal{H}_{\mathrm{supp}\,(x)}$.* ☐

3 Hypergraph compounds of games

In this section we formally define hypergraph-compound of games, and prove Theorems 1, 2, and 3.

Given games $\Gamma_i = (X_i, E_i)$, $i \in [n] = \{1, \dots, n\}$, and a hypergraph $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$, we define the $\mathcal{H}$-compound $\Gamma_{\mathcal{H}} = (X, E)$ of these games by setting

$$X = \prod_{i \in [n]} X_i \;\text{ and}$$

$$E = \left\{ (x, x') \in X \times X \;\middle|\; \exists\, H \in \mathcal{H} \text{ such that } \begin{array}{ll} (x_i, x'_i) \in E_i & \forall i \in H \\ x_i = x'_i & \forall i \notin H \end{array} \right\}.$$

Since the compound game $NIM_{\mathcal{H}}$ plays a special role in our statements, we introduce a simplified notation for the rest of the paper. Namely, we denote by $\mathcal{G}_{\mathcal{H}}$ the SG function, by $h_{\mathcal{H}}$ the height function, and by $\mathcal{P}_{\mathcal{H}}$ the set of P-positions of $NIM_{\mathcal{H}}$.

Proof of Theorem 1 Consider the $\mathcal{H}$-compound $\Gamma_{\mathcal{H}} = (X, E)$ as defined above. To prove the theorem, we show first the "if" part of the statement, that is that the function defined by the right hand side of Equation (1) satisfies (i) and (ii) in Lemma 1 if all games Γ_i, $i \in [n]$, are SG-decreasing.

For (i), consider a position $x = (x_1, \dots, x_n) \in X$ and denote by $g(x) = (\mathcal{G}_{\Gamma_i}(x_i) \mid i \in [n]) \in \mathbb{Z}_+^{[n]}$ the vector of SG values in n given games. Since $g(x) \in \mathbb{Z}_+^{[n]}$, it is a position of $NIM_{\mathcal{H}}$. Let us define a function $f : X \to \mathbb{Z}_+$ by

$$f(x) = \mathcal{G}_{\mathcal{H}}(g(x)) = \mathcal{G}_{NIM_{\mathcal{H}}}(g(x)).$$

Consider first a move $(x, x') \in E$, where $(x_i, x'_i) \in E_i$ for $i \in H$ for some hyperedge $H \in \mathcal{H}$. By the definition of E we must have $x'_i = x_i$ for all $i \notin H$. Denote by $g(x') \in \mathbb{Z}_+^{[n]}$ the corresponding vector of SG values. Note that $g(x')_i < g(x)_i$ for $i \in H$, since Γ_i is an SG-decreasing game for all $i \in [n]$, and $g(x')_i = g(x)_i$ for all $i \notin H$ since $x'_i = x_i$ for these indices. Consequently, $g(x) \to g(x')$ is a move in $NIM_{\mathcal{H}}$ and therefore $f(x) = \mathcal{G}_{\mathcal{H}}(g(x)) \neq \mathcal{G}_{\mathcal{H}}(g(x')) = f(x')$. This proves (i) in Lemma 1.

For (ii) in Lemma 1, let us next consider an integer $0 \le v < f(x)$. We are going to show that there exists a move $x \to x'$ in $\Gamma_{\mathcal{H}}$ such that $f(x') = v$. Let us consider again the corresponding integer vector $g = g(x) \in \mathbb{Z}_+^{[n]}$, for which we have $f(x) = \mathcal{G}_{\mathcal{H}}(g)$. By the definition of the SG function of $NIM_{\mathcal{H}}$, there exists a move $g \to g'$ such that $\mathcal{G}_{\mathcal{H}}(g') = v$. Let $H \in \mathcal{H}$ be the hyperedge used in this move, i.e., $g'_i < g_i$ for

$i \in H$ and $g'_i = g_i$ for $i \notin H$. Then for each $i \in H$, we must have moves $x_i \to x'_i$ in Γ_i such that $\mathcal{G}_{\Gamma_i}(x'_i) = g'_i$ for all $i \in H$. By setting $x'_i = x_i$ for $i \notin H$, we get a move $x \to x' = (x'_1, \dots, x'_n)$ in the $\mathcal{H}$-compound such that $f(x') = v$. This proves (ii) in Lemma 1. □

Our proof of Theorem 2 is based on a general criterion showing when Eq. (1) holds for conjunctive compounds (Theorem 7) which may be of independent interest.

We first note that the conjunctive compound of single-pile NIM games, $NIM^\wedge = NIM_{\mathcal{H}}$ with $\mathcal{H} = \{[n]\}$, satisfies the equality:

$$\mathcal{G}_{NIM^\wedge}(x) = \min_{i\in[n]} x_i \quad \forall\, x = (x_1, \dots, x_n) \in \mathbb{Z}_+^{[n]}, \tag{6}$$

which we will use in our proof.

For a game $\Gamma = (X, E)$ and a position $x \in X$, let $\Gamma[x] = (X[x], E[x])$ denote the subgame of Γ obtained by restricting the positions reachable from x by a sequence of moves. We call $\Gamma[x]$ a *truncated* game. Note that any game with a fixed initial position is a truncated game. Note also that both the SG-function and the height function values at a position x depend only the positions reachable from x, and thus we have for every $y \in X[x]$ the equality $\mathcal{G}_\Gamma(y) = \mathcal{G}_{\Gamma[x]}(y)$ and $h_\Gamma(y) = h_{\Gamma[x]}(y)$. Furthermore, we have $h_\Gamma(y) \le h_\Gamma(x)$ for all $y \in X[x]$ with strict inequality for $y \ne x$ by (i) of Lemma 3.

To a game $\Gamma = (X, E)$ let us associate the parameter

$$\kappa(\Gamma) = \min\{\mathcal{G}_\Gamma(x) \mid x \in X,\ \mathcal{G}_\Gamma(x) \ne h_\Gamma(x)\}$$

where by convention we have $\kappa(\Gamma) = +\infty$ if Γ is SG-decreasing. The following technical lemma will be instrumental to our proofs.

Lemma 9 *Any position x of $\Gamma = (X, E)$ with $h_\Gamma(x) \le \kappa(\Gamma) + 1$ satisfies $\mathcal{G}_\Gamma(x) = h_\Gamma(x)$.*

Proof Assume that this is not the case. Then we have a position $x \in X$ such that

$$\mathcal{G}_\Gamma(x) < h_\Gamma(x) \le \kappa(\Gamma) + 1.$$

Since we have $\kappa(\Gamma) \le \mathcal{G}_\Gamma(x)$ by the definition of $\kappa(\Gamma)$ we must have $\kappa(\Gamma) \le \mathcal{G}_\Gamma(x) < h_\Gamma(x) \le \kappa(\Gamma) + 1$, implying $\kappa(\Gamma) = \mathcal{G}_\Gamma(x) = h_\Gamma(x) - 1$. By (iii) of Lemma 3, there exists a move $x \to y$ such that $h_\Gamma(y) = h_\Gamma(x) - 1$. Then we have $\mathcal{G}_\Gamma(y) \le h_\Gamma(y) = h_\Gamma(x) - 1$. Furthermore, it follows from (i) of Lemma 3 that $\mathcal{G}_\Gamma(y) \ne \mathcal{G}_\Gamma(x) = h_\Gamma(x) - 1$. Consequently we have $\mathcal{G}_\Gamma(y) < h_\Gamma(y) = \kappa(\Gamma)$. This contradicts the definition of $\kappa(\Gamma)$, which completes the proof. □

The following statement can be viewed as a criterion showing when Equation (1) holds for the conjunctive compound. It will be used to prove Theorem 2.

Theorem 7 *Let n be a positive integer with $n \geq 2$. For $i \in [n]$, let $\Gamma_i = (X_i, E_i)$ be games. Then Equation (1) holds for their conjunctive compound if and only if*

$$\min_{i\in[n]} \sup_{x_i\in X_i} h_{\Gamma_i}(x_i) \;\leq\; \min_{i\in[n]} \kappa(\Gamma_i). \tag{7}$$

Proof For the if direction, let us assume that (7) holds. For a position $x = (x_1, \dots, x_n)$ with $x_i \in X_i$, $i \in [n]$, let us denote the right hand side of (1) by

$$f(x) \;=\; \min_{i\in[n]} \mathcal{G}_{\Gamma_i}(x_i). \tag{8}$$

We are going to prove using Lemma 1 that $f(x)$ is the SG-function of the conjunctive compound of the games Γ_i, proving the equality in (1) under the condition (7).

To this end, we first claim that for any position $x = (x_1, \dots x_n)$, there exists a component $r \in [n]$ such that $f(x) = \mathcal{G}_{\Gamma_r}(x_r) = h_{\Gamma_r}(x_r)$. Assume that any component r with $f(x) = \mathcal{G}_{\Gamma_r}(x_r)$ satisfies $\mathcal{G}_{\Gamma_r}(x_r) < h_{\Gamma_r}(x_r)$ by Lemma 3 (iv). Then by (7), definition of κ, and Lemma 3 (iv), there exists a component $i \in [n]$ such that

$$\mathcal{G}_{\Gamma_i}(x_i) \leq h_{\Gamma_i}(x_i) \leq \min_{i\in[n]} \kappa(\Gamma_i) \leq \mathcal{G}_{\Gamma_r}(x_r)\,(= f(x)).$$

This together with (8) implies that $\mathcal{G}_{\Gamma_i}(x_i) = h_{\Gamma_i}(x_i) = f(x)$, which contradicts the assumption.

By the claim, let r be a component with $f(x) = \mathcal{G}_{\Gamma_r}(x_r) = h_{\Gamma_r}(x_r)$. Then any move $x_r \to y_r$ in X_r satisfies $\mathcal{G}_{\Gamma_r}(y_r) \leq h_{\Gamma_r}(y_r) < f(x)$ by Lemma 3 (i) and (iv). This together with (8) implies that any move $x \to y$ in the compound game satisfies $f(y) < f(x)$, which proves the property (i) in Lemma 1. To prove the property (ii) in Lemma 1, let us consider an arbitrary nonnegative integer $v < f(x)$. By (ii) of Lemma 3, any component $i \in [n]$ has a move $x_i \to y_i$ such that $\mathcal{G}_{\Gamma_i}(y_i) = v$, since $\mathcal{G}_{\Gamma_i}(x_i) \geq f(x)$ by (8). Therefore, $y = (y_1, \dots, y_n)$ satisfies $f(y) = \min_{i\in[n]} \mathcal{G}_{\Gamma_i}(y_i) \;=\; v$, which proves (ii) in Lemma 1.

For the only-if direction, let us denote by h^* and κ^* the minimum values on the left and right hand sides of inequality (7), and denote by $q \in [n]$ a component such that $\kappa^* = \kappa(\Gamma_q)$. Let us assume that (7) does not hold, that is,

$$\sup_{x_i\in X_i} h_{\Gamma_i}(x_i) \;\geq\; h^* \;>\; \kappa^* = \kappa(\Gamma_q) = \min_{\ell\in[n]} \kappa(\Gamma_\ell) \tag{9}$$

holds for all $i \in [n]$. Let us choose a position $x_q \in X_q$ such that $\kappa^* = \mathcal{G}_{\Gamma_q}(x_q) < h_{\Gamma_q}(x_q)$. By the definition of κ such a position exists in game Γ_q. For each other game Γ_i, $i \neq q$, we choose a position $x_i \in X_i$ such that $h_{\Gamma_i}(x_i) = \kappa^* + 1$. Since $\kappa^* + 1 \leq h^* \leq \sup_{z_i\in X_i} h_{\Gamma_i}(z_i)$, such a position exists by (iii) of Lemma 3. By Lemma 9, the truncated games $\Gamma_i[x_i]$, $i \neq q$ are all SG-decreasing, implying $\mathcal{G}_{\Gamma_i}(x_i) = \kappa^* + 1$ for all $i \neq q$. Note that on the right hand side of (1) we have $f(x) = \min_{\ell\in[n]} \mathcal{G}_{\Gamma_\ell}(x_\ell) \;=\; \kappa^*$ by (6). We prove a violation of the equality in (1) by exhibiting a move $x \to y$ in the conjunctive compound for which we have $f(y) = \kappa^*\,(= f(x))$. Since by (i) of Lemma 3 the SG-value must change in a move, equality in (1) does not hold for at least one of x and y.

In the game Γ_q, we choose a move $x_q \to y_q$ such that $h_{\Gamma_q}(y_q) = h_{\Gamma_q}(x_q) - 1 \geq \kappa^*$ according to (iii) of Lemma 3. Note that $\mathcal{G}_{\Gamma_q}(y_q) < \kappa^*$ would then contradict the definition of $\kappa(\Gamma_q)$, and therefore we also have $\mathcal{G}_{\Gamma_q}(y_q) \geq \kappa^*$. In each other game Γ_i, $i \neq q$, we choose a move $x_i \to y_i$ such that $h_{\Gamma_i}(y_i) = h_{\Gamma_i}(x_i) - 1 = \kappa^*$, in accordance with (iii) of Lemma 3. Note that such an i exists by $n \geq 2$. Since the truncated games $\Gamma_i[x_i]$, $i \neq q$ are all SG-decreasing, the equations $f(y) = \kappa^* = f(x)$ follow as claimed, completing the proof of the theorem. □

Proof of Theorem 2 Since the if part is proven in Theorem 1, we here show the only-if part. Assume that game Γ_i is not SG-decreasing for at least one $i \in [n]$. Define $\kappa^* = \min_{\ell \in [n]} \kappa(\Gamma_\ell)$, and let z_i be a position in X_i such that $\mathcal{G}_{\Gamma_i}(z_i) = \kappa^* < h_{\Gamma_i}(z_i)$, Take a minimal hyperedge $H \in \mathcal{H}$ be that contains i. For $j \in H \setminus \{i\}$, we arbitrarily choose a position z_j that satisfies $h_{\Gamma_j}(z_j) = \kappa^* + 1$. Note that such a position exists, since Γ_j is assumed to be h-unbounded. For $j \in [n] \setminus H$, let z_j a position with $h_{\Gamma_j}(z_j) = 0$.

Let us now consider $\mathcal{H}$-compound of the truncated games $\Gamma_j[z_j]$, $j \in [n]$. Since $h_{\Gamma_j[z_j]}(z_j) = 0$ for all $j \in [n] \setminus H$, it can be regarded as the conjunctive compound of $\Gamma_j[z_j]$, $j \in H$. Our statement then follows by Theorem 7, since condition (7) does not hold for the truncated games $\Gamma_j[z_j]$, $j \in H$. □

Finally, in this section we prove Theorem 3.

Proof of Theorem 3 Similarly to the proof of Theorem 1, we shall show that the function defined by the right hand side of Equation (2) satisfies the above properties (a), (b) and (c).

Consider a position $x = (x_1, \ldots, x_n) \in X$ and denote by $t(x) = (h_{\Gamma_i}(x_i) \mid i \in [n]) \in \mathbb{Z}_+^{[n]}$ the vector of height values in these n games. Notice that $t(x)$ is a position in the game $NIM_{\mathcal{H}}$. Let us denote by

$$f(x) = h_{\mathcal{H}}(t(x))$$

the function defined by the right hand side of Eq. (2).

For (a), consider a move $(x, x') \in E$, where $(x_i, x_i') \in E_i$ for $i \in H$ for some hyperedge $H \in \mathcal{H}$. By the definition of E we must have $x_i' = x_i$ for all $i \notin H$. Denote by $t' \in \mathbb{Z}_+^{[n]}$ the corresponding vector of height values, and note that $t_i' < t_i(x)$ for $i \in H$ since h_{Γ_i} satisfies property (a) for all $i \in [n]$, and $t_i' = t_i(x)$ for all $i \notin H$ since $x_i = x_i'$ for these indices. Consequently, $t \to t' = t(x')$ is a move in $NIM_{\mathcal{H}}$, and therefore $f(x) = h_{\mathcal{H}}(t(x)) > h_{\mathcal{H}}(t(x')) = f(x')$, since $h_{\mathcal{H}}$ satisfies property (a). This proves that f satisfies (a).

For (b), consider next an arbitrary position $x \in X$ such that $0 < f(x) = h_{\mathcal{H}}(t(x))$. Since $h_{\mathcal{H}}$ satisfies property (b), there exists a move $t(x) \to t'$ in $NIM_{\mathcal{H}}$ such that $h_{\mathcal{H}}(t') = h_{\mathcal{H}}(t(x)) - 1$. Then, by the definition of $NIM_{\mathcal{H}}$ we have $H = \{i \in [n] \mid t_i(x) > t_i'\} \in \mathcal{H}$. Since h_{Γ_i} satisfies property (b), there exists moves $x_i \to x_i'$ such that $h_{\Gamma_i}(x_i') = h_{\Gamma_i}(x_i) - 1 = t_i(x) - 1$ for $i \in H$. Define $x_i' = x_i$ for $i \notin H$.

Then we have $h_{\mathcal{H}}(t(x)) - 1 \geq h_{\mathcal{H}}(t(x')) \geq h_{\mathcal{H}}(t')$ by Lemmas 2 (b) and 5 for the height function $h_{\mathcal{H}}$. Consequently we have $f(x') = f(x) - 1$, which proves (b).

Finally, to see property (c), let us consider a terminal position $x \in X$ and its corresponding height vector $t(x)$. By the definition of $NIM_{\mathcal{H}}$ this is a terminal position if and only if $\{i \in [n] \mid t_i(x) = 0\}$ intersects all hyperedges of $\mathcal{H}$, in which case we must have $f(x) = h_{\mathcal{H}}(t(x)) = 0$. □

4 SG-decreasing hypergraphs

In this section we prove Theorems 4, 5, and sketch the proof of Theorem 6.

Proof of Theorem 4 We show first that Condition (3) is necessary for a hypergraph to be SG-decreasing. By Lemma 4 for an SG-decreasing hypergraph we have $\mathcal{G}_{\mathcal{H}} = h_{\mathcal{H}}$, implying that for every position with a positive SG-value we have a move to a terminal position. Let us consider now an arbitrary subset $S \subseteq [n]$ such that $\mathcal{H}_S$ is non-empty, and consider an arbitrary position $x \in \mathbb{Z}_+^{[n]}$ for which $\{i \mid x_i > 0\} = S$. Since $\mathcal{H}_S$ is non-empty, we have $h_{\mathcal{H}}(x) = \mathcal{G}_{\mathcal{H}}(x) > 0$. Thus there exists an H-move $x \to y$ for some $H \in \mathcal{H}_S$ such that $\mathcal{G}_{\mathcal{H}}(y) = h_{\mathcal{H}}(y) = 0$. This implies that H intersects all other hyperedges of $\mathcal{H}_S$. Since this is true for all non-empty induced subhypergraphs, $\mathcal{H}$ must be totally transversal.

To see that Condition (4) is sufficient let us consider an arbitrary position $x \in \mathbb{Z}_+^{[n]}$. If $h_{\mathcal{H}}(x) = 0$ then the claim holds by definition. Assume that $h_{\mathcal{H}}(x) > 0$ and consider a hyperedge $H \in \mathcal{H}$ such that $h_{\mathcal{H}}(x - \chi_H) = h_{\mathcal{H}}(x) - 1$. Such a hyperedge exists, since $h_{\mathcal{H}}(x) > 0$. Let us consider positions $x^{s(H)}$ and $x^{f(H)}$. By our choice of H, we have $h_{\mathcal{H}}(x^{s(H)}) = h_{\mathcal{H}}(x) - 1$. Since the hypergraph is intersecting, we also have $h_{\mathcal{H}}(x^{f(H)}) = 0$ by Lemma 8. Thus, by Lemma 7, for all values $0 \leq v \leq h_{\mathcal{H}}(x) - 1$ there exists an H-move $x \to x'$ such that $h_{\mathcal{H}}(x') = v$. Since this holds for all positions, we get $\mathcal{G}_{\mathcal{H}}(x) = h_{\mathcal{H}}(x)$, by Lemma 4. This completes the proof of the theorem. □

Let us associate to a hypergraph $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$ the set of positions $\mathcal{Z}_{\mathcal{H}} \subseteq \mathbb{Z}_+^{[n]}$ which have zero height:

$$\mathcal{Z}_{\mathcal{H}} = \{x \in \mathbb{Z}_+^{[n]} \mid h_{\mathcal{H}}(x) = 0\}.$$

Obviously, we have

$$\mathcal{Z}_{\mathcal{H}} \subseteq \mathcal{P}_{\mathcal{H}}, \tag{10}$$

since there is no move from x by the definition of the height function. We shall show next that in fact all P-positions of $NIM_{\mathcal{H}}$ are in $\mathcal{Z}_{\mathcal{H}}$ if and only if $\mathcal{H}$ is totally transversal.

Lemma 10 *For a hypergraph $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$, we have $\mathcal{P}_{\mathcal{H}} = \mathcal{Z}_{\mathcal{H}}$ if and only if $\mathcal{H}$ is totally transversal.*

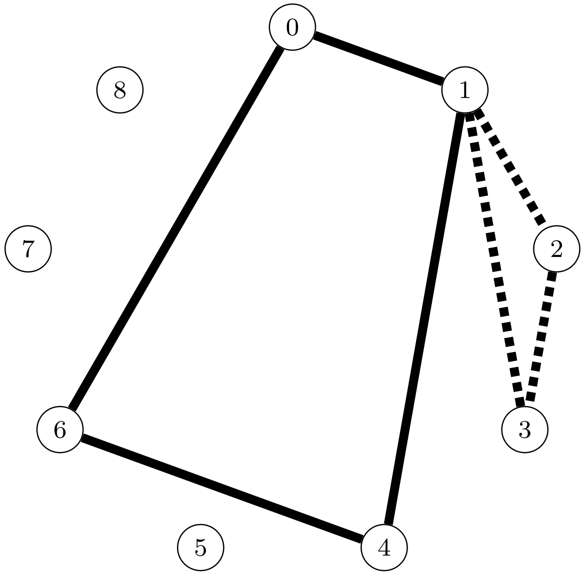

Fig. 1 A hypergraph $\mathcal{H}$ on the ground set $\mathbb{Z}_9 = \{0, \ldots, 8\}$, with hyperedges $T_i = \{i, i+1, i+2\}$ and $F_i = \{i, i+1, i+4, i+6\}$ for $i \in \mathbb{Z}_9$, where additions are modulo 9, that is, $\mathcal{H} = \{T_i,\ F_i \mid i \in \mathbb{Z}_9\}$. The figure shows T_1 and F_0 in dotted and solid lines, respectively

Proof Since $\mathcal{Z}_{\mathcal{H}} \subseteq \mathcal{P}_{\mathcal{H}}$ holds for any hypergraph $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$, we first assume the converse inclusion $\mathcal{P}_{\mathcal{H}} \subseteq \mathcal{Z}_{\mathcal{H}}$, and consider a subset $S \subseteq [n]$ such that $\mathcal{H}_S \neq \emptyset$. Let us then choose a position $x \in \mathbb{Z}_+^{[n]}$ such that $supp(x) = S$. Since $\mathcal{H}_S \neq \emptyset$, we have $h_{\mathcal{H}}(x) > 0$ implying $\mathcal{G}_{\mathcal{H}}(x) > 0$ by our assumption. Furthermore, by the definition of the SG function, we must have a hyperedge $H' \in \mathcal{H}$ and an H'-move $x \to x'$ such that $\mathcal{G}_{\mathcal{H}}(x') = 0$. Again by our assumption, we have $h_{\mathcal{H}}(x') = 0$, which implies $h_{\mathcal{H}}(x^{f(H')}) = 0$ by Lemma 5. Hence this $H' \subseteq \text{supp}\,(x) = S$ intersects all hyperedges of $\mathcal{H}_S$, which prove the total transversality.

For the other direction, assume that $\mathcal{H}$ is totally transversal, and consider a position $x \in \mathbb{Z}_+^{[n]}$ with $h_{\mathcal{H}}(x) > 0$. Then we have $\mathcal{H}_{\text{supp}\,(x)} \neq \emptyset$, By totally transversality of $\mathcal{H}$, we have a hyperedge $H \in \mathcal{H}_{\text{supp}\,(x)}$ that intersects all other hyperedges of this induced subhypergraph, i.e., $h_{\mathcal{H}}(x^{f(H)}) = 0$ by Lemma 8. This implies $\mathcal{G}_{\mathcal{H}}(x^{f(H)}) = 0$ by (iv) of Lemma 3. Since $x \to x^{f(H)}$ is an H-move, $\mathcal{G}_{\mathcal{H}}(x) \neq 0$ is implied by the definition of the SG function, which concludes that $\mathcal{P}_{\mathcal{H}} \subseteq \mathcal{Z}_{\mathcal{H}}$. This together with (10) proves the if-direction of the theorem. □

Proof of Theorem 5 Follows by Lemma 10. □

The following example demonstrates that total transversality alone is not enough, generally, to guarantee that a hypergraph is SG-decreasing.

Lemma 11 *The hypergraph $\mathcal{H}$ in Fig. 1 is totally transversal, but not SG-decreasing.*

Proof To see this, let us set T_j and F_j for $j \in \mathbb{Z}_9$ as in the caption of Fig. 1, where additions are modulo 9. Let us observe first that

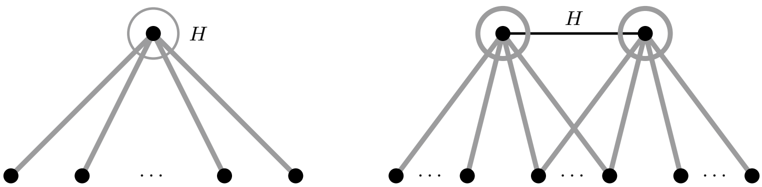

Fig. 2 Possible structures of totally transversal hypergraphs of dimension 2

$$T_j \cap F_i \neq \emptyset \text{ and } F_j \cap F_i \neq \emptyset \text{ for all } i, j \in \mathbb{Z}_9.$$

An easy case-analysis shows that $\mathcal{H}$ is totally transversal.

On the other hand, for the position $x = (1, \dots, 1) \in \mathbb{Z}_+^{[n]}$ we have $h_{\mathcal{H}}(x) = 3$, since T_0, T_3, and T_6 provides consecutive moves from x to $(0, \dots, 0)$. Furthermore, $h_{\mathcal{H}}(x - \chi(T_j)) = 2$ and $h_{\mathcal{H}}(x - \chi(F_j)) = 0$ for all $j \in \mathbb{Z}_9$. Thus, there exists no move $x \to x'$ with $h_{\mathcal{H}}(x') = 1$, which by (iii) of Lemma 4 implies that $\mathcal{H}$ is not SG-decreasing. □

Note that the above hypergraph is of dimension 4. In Apendix we show that there is no such example among the hypergraphs of dimension at most 3, as claimed in Theorem 6. In fact, for hypergraphs of dimension at most 2, the equivalence can be shown easily, since totally transversality (i.e., Condition (3)) can be substantially simplified.

Lemma 12

(i) *A hypergraph* $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$ *of dimension* 1 *is totally transversal if and only if it consists of a single hyperedge, i.e.,* $\mathcal{H} = \{\{i\}\}$ *for some* $i \in [n]$.

(ii) *A hypergraph* $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$ *of dimension* 2 *is totally transversal if and only if there is a hyperedge* $H \in \mathcal{H}$ *such that* $H \cap H' \neq \emptyset$ *for all* $H' \in \mathcal{H}$.

Proof It follows from Condition (3). □

Figure 2 shows the possible structures of such hypergraphs of dimension 2. On the left any hyperedge satisfies the condition in Lemma 12 (ii), while on the right it is a 2-element set described in black thin edge. Circles in both pictures indicate possible singletons (i.e., 1-element hyperedges).

Lemma 12 together with Fig. 2 implies the following lemma.

Lemma 13 *A hypergraph* $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$ *of dimension at most* 2 *is SG-decreasing if and only if it is totally transversal. Furthermore, it can be checked in linear time in the size of* $\mathcal{H}$ (*i.e.,* $\sum_{H \in \mathcal{H}} |H|$). □

By Theorem 4, total transversality is necessary for a hypergraph of any dimension to be SG-decreasing. Next, we prove that for hypergraphs of dimension 3, it is also sufficient.

Given a hypergraph $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$ and a position $x \in \mathbb{Z}_+^{[n]}$, let us call an integer vector $m \in \mathbb{Z}_+^{\mathcal{H}}$ an x-packing if

$$\sum_{H \in \mathcal{H}} m_H \chi(H) \;\leq\; x \quad \text{and} \quad \sum_{H \in \mathcal{H}} m_H \;=\; h_{\mathcal{H}}(x). \tag{11}$$

Let us denote by $M(x) \subseteq \mathbb{Z}_+^{\mathcal{H}}$ the family of x-packings. Let us further define

$$\mathcal{H}^{x\text{- }pack} \;=\; \{H \in \mathcal{H} \mid \exists m \in M(x) \text{ s.t. } m_H > 0\}, \tag{12}$$

that is, $\mathcal{H}^{x\text{- }pack}$ is the subfamily of $\mathcal{H}$ of those hyperedges that participate with a positive multiplicity in some x-packing of $\mathcal{H}$. Let us recall that $\mathcal{H}_{\operatorname{supp}(x)}$ is the subhypergraph induced by the support of x, and define a subhypergraph of $\mathcal{H}_{\operatorname{supp}(x)}$ by

$$\mathcal{H}^{x\text{- }all} \;=\; \{H \in \mathcal{H}_{\operatorname{supp}(x)} \mid \forall H' \in \mathcal{H}_{\operatorname{supp}(x)} \;:\; H \cap H' \neq \emptyset\}, \tag{13}$$

which consists of those hyperedges that intersect all others in $\mathcal{H}_{\operatorname{supp}(x)}$.

Lemma 14 *Let $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$ be a totally transversal hypergraph. For a position $x \in \mathbb{Z}_+^{[n]}$ such that $h_{\mathcal{H}}(x) > 0$, both $\mathcal{H}^{x\text{- }all}$ and $\mathcal{H}^{x\text{- }pack}$ are nonempty. Furthermore, we have*

(i) $h_{\mathcal{H}}(x^{s(H)}) \;=\; h_{\mathcal{H}}(x) - 1$ *for all* $H \in \mathcal{H}^{x\text{- }pack}$;
(ii) $h_{\mathcal{H}}(x^{f(H)}) \;=\; 0$ *if and only if* $H \in \mathcal{H}^{x\text{- }all}$.

Proof Let us first recall that $h_{\mathcal{H}}(x) = k > 0$ means that we can make k consecutive moves from x before arriving to a terminal position. All these moves can be assumed to be slow moves, furthermore they can be executed in any order.

Since $H \in \mathcal{H}^{x\text{- }pack}$ implies by definition that there exists a sequence of k consecutive moves starting from x such that one of these moves is an H-move, by the above remark we can assume that this slow H-move is the first one. Thus we can still make $k-1$ moves from $x^{s(H)}$, proving that $h_{\mathcal{H}}(x^{s(H)}) \;\geq\; h_{\mathcal{H}}(x) - 1$, from which (i) follows by Lemma 6.

To see (ii) observe that $h_{\mathcal{H}}(x^{f(H)}) \;=\; 0$ implies $\mathcal{H}_{\operatorname{supp}(x^{f(H)})} = \emptyset$, which can happen if and only if $H \in \mathcal{H}^{x\text{- }all}$. ☐

4.1 Plan of the proof of theorem 6

We assume that $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$ is a totally transversal hypergraph of dimension 3 that is not SG-decreasing and derive a contradiction from these assumptions.

First we observe by Lemma 4 the existence of a position $x \in \mathbb{Z}_+^{[n]}$ and an integer value $v \in \mathbb{Z}_+$ such that $h_{\mathcal{H}}(x) > v > 0$ and there is no move $x \to y$ with $h_{\mathcal{H}}(y) = v$. This allows us to show that $h_{\mathcal{H}}(x) = 3$, $v = 1$ and obtain a partition $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$

such that $h_{\mathcal{H}}(x^{s(H)}) = 0$ for all $H \in \mathcal{H}_1$ and $h_{\mathcal{H}}(x^{f(H)}) = 2$ for all $H \in \mathcal{H}_2$. We also prove that the subhypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ have a special structure, as summarized in Lemma 24.

Next we prove that $\mathcal{H}_1$ contains three hyperedges such any two intersects in exactly one point, and derive from this a partition of $\mathcal{H}_2$ into three non-empty subhypergraphs.

Finally we show that $\mathcal{H}$ must contain a special substructure that in fact is impossible to realize.

The full proof is quite long and technical, and therefore we included it in the Appendix of this paper.

5 Computational issues

In this section we consider the computational complexity of recognizing total transversality and computing height ad SG-values.

Let us call a hyperedge of a hypergraph $\mathcal{H}$ *intersecting* if it intersects all other hyperedges of $\mathcal{H}$. Recall that $\mathcal{H}$ is called totally transversal if every non-empty induced subhypergraphs of it contains an intersecting hyperedge. Let us call $\mathcal{H}$ *minimal transversal-free* if it does not contain an intersecting hyperedge, but every non-empty proper induced subhypergraph of it does.

Lemma 15 (Bogdanov 2017) *If $\mathcal{F} \subseteq 2^U$ is a minimal transversal-free hypergraph of dimension at most k, then*

$$|U| \leq k\binom{2k}{k}.$$

Proof Our assumptions imply that for every $i \in U$ we have a hyperedge $F_i \in \mathcal{F}_{U\setminus\{i\}}$ such that $F_i \cap F' \neq \emptyset$ for all $F' \in \mathcal{F}_{U\setminus\{i\}}$. Let us denote by $\mathcal{F}' = \{F_i \mid i \in U\}$ the family of these hyperedges. Since $\mathcal{F}$ does not have an intersecting hyperedge, for every $F \in \mathcal{F}'$ there exists a hyperedge $B(F) \in \mathcal{F}$ disjoint from F. Let us choose a minimal subhypergraph $\mathcal{B} \subseteq \mathcal{F}$ such that

$$\forall F \in \mathcal{F}' \;\; \exists B \in \mathcal{B}: \quad F \cap B = \emptyset. \tag{14}$$

Let us note first that such a $\mathcal{B}$ must form a cover of U, that is, $U = \bigcup_{B\in\mathcal{B}} B$. This is because for all $F_i \in \mathcal{F}'$ there exists a $B \in \mathcal{B}$ such that $F_i \cap B = \emptyset$ and, consequently, $i \in B$. Let us observe next that for all $B \in \mathcal{B}$ there exists at least one $A(B) \in \mathcal{F}'$ such that $A(B) \cap B = \emptyset$ and $A(B) \cap B' \neq \emptyset$ for all $B' \in \mathcal{B}\setminus\{B\}$. This is because we choose $\mathcal{B}$ to be a minimal family with respect to (14). Let us now define $\mathcal{A} = \{A(B) \in \mathcal{F}' \mid B \in \mathcal{B}\}$. The pair $\mathcal{A}$, $\mathcal{B}$ of hypergraphs now satisfies the conditions of the classical Bollobás' Lemma (Bollobás 1965), page 448, implying that

$$|\mathcal{A}| = |\mathcal{B}| \leq \binom{2k}{k}.$$

Since $\dim(\mathcal{B}) \leq k$ and it covers U, our claim follows. ◻

Theorem 8 *Given a hypergraph* $\mathcal{H} \subseteq 2^{[n]}$ *and a constant* k, *it can be tested in* $poly(n, |\mathcal{H}|)$ *time if* $\mathcal{H}$ *is totally transversal or not*, *if* $dim(\mathcal{H}) \leq k$.

Proof If Condition (3) does not hold for $\mathcal{H}$ then there is a minimal subset $U \subseteq [n]$ such that $\mathcal{H}_U \neq \emptyset$ and $\mathcal{H}_U$ is minimal transversal-free. Then, by Lemma 15, we have $|U| \leq k\binom{2k}{k}$. Since k is a fixed constant, we will need to check only polynomially many induced subhypergraphs and that can be accomplished in polynomial time. □

We consider next the complexity of computing the height function of a hypergraph NIM game.

Theorem 9 *Given a hypergraph* $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$ *and a position* $x \in \mathbb{Z}_+^{[n]}$, *computing* $h_{\mathcal{H}}(x)$ *is*

(i) *NP-hard for intersecting hypergraphs*, *already for dimension* 4;
(ii) *NP-hard for hypergraphs of dimension at most* 3;
(iii) *polynomial for hypergraphs of dimension at most* 2 (*that is*, *for graphs*).
(iv) *polynomial for fixed* n.

Proof Let us consider an arbitrary hypergraph $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$. Its matching number $\mu(\mathcal{H})$ is the maximum number of pairwise disjoint hyperedges of $\mathcal{H}$ and is known to be NP-hard to compute already for the hypergraphs of dimension 3 (Karp 1972).

Let us consider $w \notin [n]$ and define $\mathcal{H}^* = \{H \cup \{w\} \mid H \in \mathcal{H}\}$. Also consider position $x \in \mathbb{Z}_+^{[n]\cup\{w\}}$ defined by $x_i = 1$ for $i \in [n]$ and $x_w = |\mathcal{H}|$. Then $\mathcal{H}^*$ is an intersecting hypergraph and we have $h_{\mathcal{H}^*}(x) = \mu(\mathcal{H})$. This equality still holds when $\mathcal{H}$ is of dimension 3 and $x_i = 1$ for all $i \in [n]$.

Yet, if $\mathcal{H}$ is of dimension at most 2, then $h_{\mathcal{H}}(b)$ for a position $b \in \mathbb{Z}_+^{[n]}$ is the so called b-matching number of the underlying graph and is known to be computable in polynomial time (Edmonds 1965; Tutte 1954).

For a position x, computing $h_{\mathcal{H}}(x)$ is an integer programming problem, which is polynomial when the number of variables is a fixed constant (Lenstra 1983). □

Corollary 2 *Given a hypergraph* $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$ *and a position* $x \in \mathbb{Z}_+^{[n]}$, *computing* $\mathcal{G}_{\mathcal{H}}(x)$ *is NP-hard*, *even for intersecting hypergraphs*.

Proof Since intersecting hypergraphs satisfy Condition (3), Theorem 6 implies $h_{\mathcal{H}} = \mathcal{G}_{\mathcal{H}}$. Thus, the claim follows, by Theorem 9 (i). □

Let us finally remark that the complexity of computing the height of a position for hypergraphs of dimension 3 is open under Condition (3).

Appendix: Proof of Theorem 6

In order to prove Theorem 6, we assume for a contradiction that there exists a hypergraph $\mathcal{H} \subseteq 2^{[n]} \setminus \{\emptyset\}$ of $\dim(\mathcal{H}) = 3$ which is totally transversal, but not SG-decreasing. We first provide the properties of such hypergraphs, and derive a contradiction at the end.

By Lemma 4 our indirect assumption implies the existence of $x \in \mathbb{Z}_+^{[n]}$ and $v \in \mathbb{Z}_+$ such that $h_{\mathcal{H}}(x) > v \geq 0$ and there exists no move $x \to x'$ with $h_{\mathcal{H}}(x') = v$. Since Condition (3) applies to all induced subhypergraphs, we can assume without any loss of generality that

$$[n] = \operatorname{supp}(x).$$

Then it follows from Lemma 7 that any $H \in \mathcal{H}$ satisfies

$$\text{either} \quad h_{\mathcal{H}}(x^{s(H)}) \leq v - 1, \tag{15a}$$

$$\text{or} \quad h_{\mathcal{H}}(x^{f(H)}) \geq v + 1. \tag{15b}$$

By Lemma 6, no hyperedge $H \in \mathcal{H}$ satisfies both (15a) and (15b). Thus, the above defines a unique partition of $\mathcal{H}$:

$$\begin{aligned} \mathcal{H}_1 &= \{H \in \mathcal{H} \mid h_{\mathcal{H}}(x^{s(H)}) \leq v - 1\} \quad \text{and} \\ \mathcal{H}_2 &= \{H \in \mathcal{H} \mid h_{\mathcal{H}}(x^{f(H)}) \geq v + 1\}. \end{aligned} \tag{16}$$

By Lemma 14, both $\mathcal{H}_1$ and $\mathcal{H}_2$ are nonempty. For $H \in \mathcal{H}_1$ we get by Lemma 6 that

$$h_{\mathcal{H}}(x) - 3 \;\leq\; h_{\mathcal{H}}(x^{s(H)}) \;\leq\; v - 1,$$

while for $H \in \mathcal{H}_2$ we get by Lemmas 6 and 14 that

$$h_{\mathcal{H}}(x) - 1 \;\geq\; h_{\mathcal{H}}(x^{s(H)}) \;\geq\; h_{\mathcal{H}}(x^{f(H)}) \;\geq\; v + 1.$$

These inequalities together imply that $v = h_{\mathcal{H}}(x) - 2 > 0$,

$$h_{\mathcal{H}}(x^{s(H)}) = h_{\mathcal{H}}(x) - 3 \;\text{ for } H \in \mathcal{H}_1 \;\text{ and} \tag{17a}$$

$$h_{\mathcal{H}}(x^{f(H)}) = h_{\mathcal{H}}(x) - 1 \;\text{ for } H \in \mathcal{H}_2. \tag{17b}$$

The next series of claims help us to prove Lemma 24 which summarizes structural properties of non-SG-decreaing, totally transversal hypergraphs of dimension 3.

Lemma 16 *We have* $\mathcal{H}^{x\text{-}all} \subseteq \mathcal{H}_1$.

Proof This follows from Lemma 14. □

Lemma 17 *For all* $H \in \mathcal{H}_1$ *we have* $|H| = 3$.

Proof The claim follows by the definition of $\mathcal{H}_1$, Lemma 6, and the assumption that $\dim(\mathcal{H}) = 3$. □

Lemma 18 *We have* $\mathcal{H}_2 = \mathcal{H}^{x\text{- } pack}$.

Proof By Lemma 14, for all $H \in \mathcal{H}^{x\text{- } pack}$ we have

$$h_{\mathcal{H}}(x^{s(H)}) = h_{\mathcal{H}}(x) - 1 > h_{\mathcal{H}}(x) - 3,$$

implying $H \in \mathcal{H}_2$. For $H \in \mathcal{H}_2$, it follows from Lemma 6 that

$$h_{\mathcal{H}}(x) > h_{\mathcal{H}}(x^{s(H)}) \geq h_{\mathcal{H}}(x^{f(H)}) = h_{\mathcal{H}}(x) - 1,$$

implying $h_{\mathcal{H}}(x^{s(H)}) = h_{\mathcal{H}}(x) - 1$. Let us choose an arbitrary $m \in M(x^{s(H)})$ and define $m'_H = m_H + 1$ and $m'_{H'} = m_{H'}$ for all $H' \in \mathcal{H}$ with $H' \neq H$. Then we have $m' \in M(x)$ and $m'_H > 0$ implying $H \in \mathcal{H}^{x\text{- } pack}$ by (12). □

Lemma 19 *For all* $m \in M(x)$ *and* $H \in \mathcal{H}$ *we have* $m_H \leq 1$.

Proof If $m_H \geq 2$ for some $H \in \mathcal{H}$ then for position $x' = x - 2\chi(H)$ we have that $h_{\mathcal{H}}(x') = h_{\mathcal{H}}(x) - 2$ and $x \to x'$ is a move, contradicting our assumption that there exists no such move. □

Lemma 20 *For all* $H_1 \in \mathcal{H}^{x\text{- } all}$ *and* $H_2 \in \mathcal{H}^{x\text{- } pack}(= \mathcal{H}_2)$, *we have* $|H_1 \cap H_2| = 1$.

Proof Since $H_1 \in \mathcal{H}^{x\text{- } all}$, we have $H_1 \cap H_2 \neq \emptyset$. We thus assume to the contrary that $|H_1 \cap H_2| \geq 2$. By Lemmas 16 and 17 we have $|H_1| = 3$.

Assume without loss of generality that $H_1 = \{i, j, k\}$ and $\{i, j\} \subseteq H_2$. Let us define the position x' by $x'_\ell = x_\ell - 1$ for $\ell \in \{i, j\}$ and $x'_\ell = x_\ell$ for $\ell \notin \{i, j\}$. Then we have $x' \geq x^{s(H_2)}$, implying

$$h_{\mathcal{H}}(x') \geq h_{\mathcal{H}}(x^{s(H_2)}) = h_{\mathcal{H}}(x) - 1$$

by Lemmas 5, 14 (i), and 18. Furthermore, we have $x' - \chi(\{k\}) \leq x^{s(H_1)}$ implying by Lemma 5 that

$$h_{\mathcal{H}}(x') - 1 \;\leq\; h_{\mathcal{H}}(x' - \chi(\{k\})) \;\leq\; h_{\mathcal{H}}(x^{s(H_1)}).$$

From these inequalities, we obtain $h_{\mathcal{H}}(x^{s(H_1)}) \geq h_{\mathcal{H}}(x) - 2$. This contradicts (17a), which completes the proof of the lemma. □

For an x-packing $m \in M(x)$ let us associate the corresponding position $x(m)$ defined by

$$x(m) \;=\; \sum_{H\in\mathcal{H}} m(H)\chi(H). \tag{18}$$

Lemma 21 *For all $m \in M(x)$ and $i \in H^* \in \mathcal{H}^{x\text{-}all}$ we have $x(m)_i = x_i$.*

Proof Clearly, we must have $x(m) \le x$ for all $m \in M(x)$, by the definition of $M(x)$. Assume to the contrary that there exist an x-packing $m \in M(x)$ and a point $i \in H^* = \{i,j,k\}$ such that $x(m)_i < x_i$. Then we have $x(m) \le x - \chi(\{i\})$, implying by Lemma 5 that

$$h_{\mathcal{H}}(x) \;\ge\; h_{\mathcal{H}}(x-\chi(\{i\})) \;\ge\; \sum_{H\in\mathcal{H}} m(H) \;=\; h_{\mathcal{H}}(x),$$

from which $h_{\mathcal{H}}(x-\chi(\{i\})) = h_{\mathcal{H}}(x)$ follows. Thus, again by Lemma 5, we get

$$h_{\mathcal{H}}(x^{s(H^*)}) \;=\; h_{\mathcal{H}}((x-\chi(\{i\}))-\chi(\{j,k\})) \;\ge\; h_{\mathcal{H}}(x-\chi(\{i\}))-2 \;=\; h_{\mathcal{H}}(x)-2.$$

This contradicts (17a) and Lemma 16, which completes the proof. □

Lemma 22 *For all $H^* \in \mathcal{H}^{x\text{-}all}$ we have $h_{\mathcal{H}}(x) = \sum_{i\in H^*} x_i$.*

Proof By Lemmas 18, 20, and 21, $h_{\mathcal{H}}(x)$ can be restated as follow.

$$\begin{aligned} h_{\mathcal{H}}(x) &= \sum_{H\in\mathcal{H}} m(H) \\ &= \sum_{H\in\mathcal{H}_2} m(H) && (\text{ by Lemma 18}) \\ &= \sum_{i\in H^*} \sum_{\substack{H\in\mathcal{H}_2:\\ H\ni i}} m(H) && (\text{ by Lemma 20}) \\ &= \sum_{i\in H^*} x(m)_i \\ &= \sum_{i\in H^*} x_i && (\text{ by Lemma 21}), \end{aligned}$$

where proves the statement of the lemma. □

Lemma 23 *For all $H^* \in \mathcal{H}^{x\text{-}all}$ and all $i \in H^*$ we have $x_i = 1$.*

Proof By Lemmas 20 and 21, any hyperedge $H^* \in \mathcal{H}^{x\text{-}all}$ has a hyperedge $H_2 \in \mathcal{H}_2$ with $H_2 \cap H^* = \{i\}$ for all $i \in \mathcal{H}^*$. Let us fix an element i in H^* arbitrarily and let $H^* = \{i,j,k\}$ by Lemmas 16 and 17.

Let $m \in M(x^{f(H_2)})$ be an $x^{f(H_2)}$-packing. By (17b), it is not a zero vector, and for all $H \in \mathcal{H}$ with $m(H) > 0$, we have $H \subseteq \text{supp}\,(x^{f(H_2)}) \subseteq \text{supp}\,(x)$, and thus $H \cap (H^* \setminus H_2) \ne \emptyset$ by the definition of $\mathcal{H}^{x\text{-}all}$. Again by (17b), we can write

$$
\begin{aligned}
h_{\mathcal{H}}(x) - 1 \;=\; h_{\mathcal{H}}(x^{f(H_2)}) = & \sum_{H \in \mathcal{H}} m(H) \\
\le & \; x(m)_j + x(m)_k \quad (\text{ by } H^* \in \mathcal{H}^{x-all}) \\
\le & \; x_j^{f(H_2)} + x_k^{f(H_2)} \\
= & \; x_j + x_k \\
= & \; h_{\mathcal{H}}(x) - x_i \qquad (\text{ by Lemma } 22),
\end{aligned}
$$

which implies $x_i \le 1$. Since $H^* \subseteq \operatorname{supp}(x)$ implies $x_i \ge 1$, we have $x_i = 1$. □

Lemma 24 *A hypergraph $\mathcal{H}$ is partitioned into*

$$\mathcal{H}_1 = \mathcal{H}^{x-all} \text{ and } \mathcal{H}_2 = \mathcal{H}^{x-pack} \tag{19}$$

such that

(i)$|H| = 3$ *for every* $H \in \mathcal{H}_1$,
(ii)$|H_1 \cap H_2| = 1$ for every $H_1 \in \mathcal{H}_1$ and $H_2 \in \mathcal{H}_2$,
(iii)For every $H_1 \in \mathcal{H}_1$ and every $i \in H_1$, there exists $H_2 \in \mathcal{H}_2$ such that $H_1 \cap H_2 = \{i\}$,
(iv)$x_i = 1$ for every $i \in \bigcup_{H \in \mathcal{H}_1} H$,
(v)$h_{\mathcal{H}}$ *satisfies*

$$h_{\mathcal{H}}(x) = 3 \tag{20a}$$

$$h_{\mathcal{H}}(x^{s(H)}) = 0 \text{for all } H \in \mathcal{H}_1 \tag{20b}$$

$$h_{\mathcal{H}}(x^{f(H)}) = 2 \text{ for all } H \in \mathcal{H}_2. \tag{20c}$$

Proof By Lemmas 17, 18, 20, 23 we have $\mathcal{H}_2 = \mathcal{H}^{x-pack}$, (i), (ii), and (iv).

$\mathcal{H}_1 = \mathcal{H}^{x-all}$: We note that $h_{\mathcal{H}}(x^{s(H)}) = 0$ for every $H \in \mathcal{H}_1$ by (20a) and (17a), implying $\mathcal{H}_1 \subseteq \mathcal{H}^{x-all}$. This together with Lemma 16 proves the claim.

(iii): It follows from Lemmas 20 and 21.

(v): (20a) follows from Lemmas 22 and 23. (20b) and (20c) are obtained from (17a) and (17b). □

In the rest of the proof we show that $\mathcal{H}_1$ and $\mathcal{H}_2$ have some special structure, from which we can derive a contradiction at the end. To this end we show first that $\mathcal{H}_1$ includes three hyperedges such that any two of those intersect in exactly one point.

Lemma 25 *For all $H^* \in \mathcal{H}_1$ and $i \in H^*$, there exists a hyperedge $H^{**} \in \mathcal{H}_1$ such that $i \notin H^{**}$.*

Proof Since $|H^*| = 3$ by Lemma 24 (i), we have a point $j \in H^* \setminus \{i\}$. Lemma 24 (iii) implies the existence of a hyperdege $H \in \mathcal{H}_2$ such that $H \cap H^* = \{j\}$. This implies $\mathcal{H}_{[n]\setminus\{i\}} \neq \emptyset$, since this induced subhypergraph contains H. Therefore, by the total transversality of $\mathcal{H}$, there exists a hyperedge $H^{**} \in \mathcal{H}_{[n]\setminus\{i\}}$ that intersects all others in the induced subhypergraph. Consequently, all hyperedges $H' \in \mathcal{H}$ such that $H' \cap H^{**} = \emptyset$ contain the point i, which implies $h_{\mathcal{H}}(x^{f(H^{**})}) \leq x_i = 1$ by Lemma 24 (iv). By Lemma 24 (v), we get $H^{**} \in \mathcal{H}_1$, as claimed. □

Lemma 26 *Let H_1 and H_2 be hyperedges in $\mathcal{H}_1$ such that $i \in H_1 \cap H_2$. Then there exists no $H_3 \in \mathcal{H}_1$ such that $H_3 \subseteq (H_1 \cup H_2) \setminus \{i\}$.*

Proof By Lemma 24 (ii) and (iii), there exists a hyperedge $H' \in \mathcal{H}_2$ such that $H' \cap H_1 = H' \cap H_2 = \{i\}$. This implies that $H' \cap ((H_1 \cup H_2) \setminus \{i\}) = \emptyset$. Again by Lemma 24 (ii), we have $H' \cap H_3 \neq \emptyset$ for all $H_3 \in \mathcal{H}_1$, which completes the proof of the lemma. □

Lemma 27 *There exist three hyperedges $H_1, H_2, H_3 \in \mathcal{H}_1$ such that $H_1 \cap H_2 \cap H_3 = \emptyset$ and $|H_p \cap H_q| = 1$ for all $1 \leq p < q \leq 3$.*

Proof We first claim the existence of two hyperedges H_1 and H_2 in $\mathcal{H}_1$ with $|H_1 \cap H_2| = 1$. Note that $\mathcal{H}_1$ contains at least two hyperedges by Lemma 25, and any two distinct hyperedges of $\mathcal{H}_1$ intersect in one or two points by Lemma 24. Assume to the contrary that any two (distinct) hyperedges of $\mathcal{H}_1$ intersect in two points. Take arbitrarily two hyperedges in $\mathcal{H}_1$, say $H_1 = \{i,j,k\}$ and $H_2 = \{i,j,\ell\}$, where $|H_1| = |H_2| = 3$ is implied by Lemma 24 (i), and let $H_3 \in \mathcal{H}_1$ be a hypergraph with $i \notin H_3$. By Lemma 25, such an H_3 exists. Then $|H_1 \cap H_3| \geq 2$ and $|H_2 \cap H_3| \geq 2$ together with $i \notin H_3$ imply $H_3 = \{j,k,\ell\}$, that is, $H_3 \subseteq (H_1 \cup H_2)\setminus\{i\}$. This contradicts Lemma 26, which proves the claim.

Let H_1 and H_2 be such two hyperedges in $\mathcal{H}_1$, i.e., $H_1 \cap H_2 = \{i\}$ for some $i \in [n]$. Then by Lemma 25, there exists $H_3 \in \mathcal{H}_1$ such that $i \notin H_3$. Since $H_3 \nsubseteq (H_1 \cup H_2) \setminus \{i\}$ by Lemma 26, H_1, H_2, and H_3 satisfy the properties in the lemma. □

Corollary 3 *There exist six distinct points $U = \{a,b,c,d,e,f\} \subseteq [n]$ such that $H_1 = \{a,b,f\}$, $H_2 = \{b,c,d\}$ and $H_3 = \{c,a,e\}$ are all hyperedges in $\mathcal{H}_1$.* □

We show next that $\mathcal{H}_2$ has also a special form with respect to these six points.

Lemma 28 *Any hyperedge $H \in \mathcal{H}_2$ satisfies $H \cap U = \{a,d\}, \{b,e\}$, or $\{c,f\}$.*

Proof By Lemma 24 (ii), we have $|H \cap H_p| = 1$ for all $p = 1,2,3$. Thus either H satisfies the property in the lemma, or $H = \{d,e,f\}$. In the latter case, let us consider a hyperedge $H' \in \mathcal{H}_2$ such that $H' \cap H = \emptyset$. Such an H' must exist by (20c). This H' also intersects H_p, $p = 1,2,3$ in exactly one point. However, this is impossible without intersecting H, which completes the proof. □

Corollary 4 *Let* $\alpha = \{a, d\}$, $\beta = \{b, e\}$ *and* $\gamma = \{c, f\}$. *Then the subhypergraphs*

$$\mathcal{H}_{2,\alpha} = \{H \in \mathcal{H}_2 \mid \alpha = H \cap U\},$$
$$\mathcal{H}_{2,\beta} = \{H \in \mathcal{H}_2 \mid \beta = H \cap U\},$$
$$\mathcal{H}_{2,\gamma} = \{H \in \mathcal{H}_2 \mid \gamma = H \cap U\}$$

form a partition of $\mathcal{H}_2$. *In particular, none of these families is empty.*

Proof The first claim follows directly from Lemma 28. By Lemma 24 (iv), we have $x_a = x_b = x_c = x_d = x_e = x_f = 1$, and thus for any $m \in M(x)$ and $\mu \in \{\alpha, \beta, \gamma\}$ we have $\sum_{H \in \mathcal{H}_{2,\mu}} m(H) \leq 1$. On the other hand, since $h_{\mathcal{H}}(x) = 3$ by (20a), for any $m \in M(x)$ and for any $\mu \in \{\alpha, \beta, \gamma\}$ we have a hyperedge $H \in \mathcal{H}_{2,\mu}$ with $m(H) = 1$, completing the proof of the claim. □

In the rest of our proof, we show that $\mathcal{H}$ must contain a special small substructure that in fact cannot be realized.

Lemma 29 *No hyperedge* $H \in \mathcal{H}_1$ *contains* $\mu \in \{\alpha, \beta, \gamma\}$.

Proof By Corollary 4 there exists a hyperedge $H' \in \mathcal{H}_{2,\mu} \subseteq \mathcal{H}_2$, and thus $|H \cap H'| \geq |\mu| = 2$ holds if such a hypergraph H exists. This contradicts Lemma 24 (ii). □

Lemma 30 *For any two distinct* $\mu, \nu \in \{\alpha, \beta, \gamma\}$, *let* $H \in \mathcal{H}_{2,\mu}$ *and* $H' \in \mathcal{H}_{2,\nu}$ *be two sets such that* $H \cap H' = \emptyset$. *Then there exists a hyperedge* $H'' \in \mathcal{H}_{2,\mu} \cup \mathcal{H}_{2,\nu}$ *that intersects both* H *and* H'.

Proof By the total transversality, we have a set $H'' \in \mathcal{H}$ such that $H'' \subseteq H \cup H'$ and it intersects all hyperedges in the (nonempty) induced subhypergraph $\mathcal{H}_{H \cup H'}$. If $H'' \in \mathcal{H}_1$, then we have either $|H'' \cap H| \geq 2$ or $|H'' \cap H'| \geq 2$, since $|H''| = 3$ by Lemma 24 (i). This contradicts Lemma 24 (ii), which implies $H'' \in \mathcal{H}_2$, and therefore $H'' \in \mathcal{H}_{2,\mu} \cup \mathcal{H}_{2,\nu}$ by Corollary 4. □

Let us next introduce $N_\mu = \bigcup_{H \in \mathcal{H}_{2,\mu}} H$ for $\mu \in \{\alpha, \beta, \gamma\}$. Note that these sets are all disjoint from $X = H_1 \cup H_2 \cup H_3$, by definition of $\mathcal{H}_{2,\mu}$.

Corollary 5 *For any distinct* $\mu, \nu \in \{\alpha, \beta, \gamma\}$, *we have* $N_\mu \subseteq N_\nu$ *or* $N_\nu \subseteq N_\mu$.

Proof If there are points $u \in N_\mu \backslash N_\nu$ and $v \in N_\nu \backslash N_\mu$, then by Lemma 30 we have $\mu \cup \{v\} \in \mathcal{H}_{2,\mu}$ or $\nu \cup \{v\} \in \mathcal{H}_{2,\nu}$, which contradicts $u \notin N_\nu$ or $v \notin N_\mu$. □

Lemma 31 *For any distinct* $\mu, \nu \in \{\alpha, \beta, \gamma\}$, *there exist no two distinct points* $u, v \in [n] \setminus U$ *such that all four sets* $\mu \cup \{u\}$, $\mu \cup \{v\}$, $\nu \cup \{u\}$, *and* $\nu \cup \{v\}$ *are hyperedges in* $\mathcal{H}$.

Proof Assume to the contrary that such two points exist. Then by Lemma 29 these sets are all from $\mathcal{H}_2$. By the total transversality, $\mathcal{H}$ contains a hyperedge $H \subseteq \mu \cup \nu \cup \{u, v\}$ that intersects all these four hyperedges. Since H must intersect some of these four hyperedges in two points, $H \in \mathcal{H}_2$ holds by Lemma 24 (ii). Then by Corollary 4, we have $H \in \mathcal{H}_{2,\mu} \cup \mathcal{H}_{2,\nu}$. This is however impossible, since H is of size at most 3. □

Corollary 6 *For any distinct $\mu, \nu \in \{\alpha, \beta, \gamma\}$, we have $|N_\mu \cap N_\nu| \leq 1$.*

Proof Immediate from Lemma 31. □

Corollary 7 *By relabeling the points in $[n]$, we have $N_\alpha \subseteq N_\beta \subseteq N_\gamma$ with $|N_\alpha| \leq |N_\beta| \leq 1$.*

Proof Immediate from Corollaries 5 and 6. □

In the subsequent discussion, we assume that points in $[n]$ are relabeled as in Corollary 7.

Lemma 32 *At most one of α, β, and γ is a hyperedge in $\mathcal{H}$.*

Proof Suppose that two $\mu, \nu \in \{\alpha, \beta, \gamma\}$ belongs to $\mathcal{H}$. Then by total transversality, we have a hyperedge $H \in \mathcal{H}$ such that $H \subseteq \mu \cup \nu$ and it intersects both μ and ν. Since $\mu, \nu \in \mathcal{H}_2$, Lemma 24 (i) and (ii) implies that $H \in \mathcal{H}_2$. Then, by Corollary 4 we have $H \in \mathcal{H}_{2,\mu}$ or $H \in \mathcal{H}_{2,\nu}$. Since H intersects both μ and ν, we have $|H| = 3$, from which we derive a contradiction by Lemma 24 (ii) due to the structure of $\mathcal{H}_1$ within the set U. □

Lemma 33 $N_\alpha \neq \emptyset$.

Proof Assume to the contrary that $N_\alpha = \emptyset$. This implies that $\mathcal{H}_{2,\alpha} = \{\alpha\}$. Let us consider an arbitrary x-packing $m \in M(x)$. Since $h_{\mathcal{H}}(x) = 3$ by (20a), we have hyperedges $H_\mu \in \mathcal{H}_{2,\mu}$ for all $\mu \in \{\alpha, \beta, \gamma\}$ with $m(H_\mu) = 1$ by Lemma 24 (iv). In particular, we have $m(\alpha) = 1$ and $m(H) = 1$ for some $H \in \mathcal{H}_{2,\beta}$. Since $\alpha \cap H = \emptyset$, by the total transversality, $\mathcal{H}$ contains a hyperedge H' such that $H' \subseteq \alpha \cup H$ and it intersects both α and H. If $H' \in \mathcal{H}_1$ then we get a contradiction by Lemma 24 (ii). On the other hand if $H' \in \mathcal{H}_2$, then by Corollary 4, $\mathcal{H}_{2,\alpha} = \{\alpha\}$ implies that $H' \in \mathcal{H}_{2,\beta}$, which contradicts the fact that α is disjoint from all hyperedges in $\mathcal{H}_{2,\beta}$. □

Now we are ready to complete the proof of Theorem 6. By Corollary 7 and Lemma 33, we have $|N_\alpha| = |N_\beta| = 1$, that is, for some $u \in [n]$ we have $N_\alpha = N_\beta = \{u\} \subseteq N_\gamma$ and, therefore $H = \gamma \cup \{u\} \in \mathcal{H}_{2,\gamma}$. Let $x' = x^{f(H)}$, and consider an x'-packing $m \in M(x')$. By Lemma 24 (ii), we have $m(H^*) = 0$ for all $H^* \in \mathcal{H}_1$. Furthermore, any $H' \in \mathcal{H}_2$ with $u \in H'$ satisfies $m(H') = 0$. Consequently, only $H' \in \mathcal{H}_{2,\alpha} \cup \mathcal{H}_{2,\beta}$ with $u \notin H'$ can have $m(H') = 1$ by

Lemma 24 (iv). Since by Lemma 32 at most one of α and β can belong to $\mathcal{H}_2$, we have $h_{\mathcal{H}}(x') \leq 1$, which contradicts (20c). This completes the proof of the theorem. □

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Authors and Affiliations

Endre Boros[1] · Vladimir Gurvich[2] · Nhan Bao Ho[3] · Kazuhisa Makino[4] · Peter Mursic[1]

Endre Boros
endre.boros@rutgers.edu

Vladimir Gurvich
vgurvich@hse.ru

Kazuhisa Makino
makino@kurims.kyoto-u.ac.jp

Peter Mursic
peter.mursic@rutgers.edu

1 MSIS and RUTCOR, RBS, Rutgers University, 100 Rockafeller Road, Piscataway, NJ 08854, USA

2 National Research University Higher School of Economics, Moscow, Russia

3 Department of Mathematics and Statistics, La Trobe University, Melbourne 3086, Australia

4 Research Institute for Mathematical Sciences (RIMS), Kyoto University, Kyoto 606-8502, Japan