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ORIGINAL PAPER

The projective core of symmetric games with externalities

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Abstract
The purpose of this paper is to study which coalition structures have stable distributions. We employ the projective core as a stability concept. Although the projective core is often defined only for the grand coalition, we define it for every coalition structure. We apply the core notion to a variety of economic models including the public goods game, the Cournot and Bertrand competition, and the common pool resource game. We use a partition function to formulate these models. We argue that symmetry is a common property of these models in terms of a partition function. We offer some general results that hold for all symmetric partition function form games and discuss their implications in the economic models. We also provide necessary and sufficient conditions for the projective core of the models to be nonempty. In addition, we show that our results hold even in the presence of small perturbations of symmetry.

Keywords Core · Externalities · Oligopoly · Public goods

1 Introduction

Most economic/political situations include both competition and cooperation among players. Although competition and cooperation influence each other, they are often separately analyzed in two different models: noncooperative game theory and cooperative game theory. While dividing a situation into the two different models enables us to focus on their respective specialties, it eliminates rich interactions between cooperation and competition. Thrall (1961) and Thrall and Lucas (1963) are early attempts

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to address this problem. These authors introduced *partition function form games*, also known as games with externalities, to describe the competition among coalitions and cooperation within each coalition.

In partition function form games, the worth of a coalition depends on both the coalition and the coalition structure of the other coalitions. This complexity causes a problem when we generalize one of the solution concepts for cooperative games to games with externalities: the core. The core is a set of payoff allocations from which no coalition has an incentive to deviate, where a deviation means that a group of players forms their own coalition and splits off from an allocation to improve the payoff of the deviating coalition. In order to determine the worth of the deviating coalition, one must also know the coalition structure that the non-deviating players form after the deviation.

The literature handles this difficulty by various assumptions on the coalition structure. von Neumann and Morgenstern (1944) and Hart and Kurz (1983) consider that non-deviating players will be separate into one-person coalitions. Aumann (1967) and Hart and Kurz (1983) assume that the non-deviating players reorganize their coalitions to minimize the worth of the deviating coalition. Shenoy (1979) proposes maximizing the worth of the deviating coalition. Bloch and van den Nouweland (2014) offer a general function called an *expectation formation rule* that generalizes these assumptions and axiomatically characterize those expectation formation rules in a general framework. Moreover, Chwe (1994), Xue (1997), Ray and Vohra (1997) , and Diamantoudi and Xue (2003) developed the theory of farsightedness. They consider a sequence of deviations: a deviation causes another deviation, and the (second) deviation causes another deviation, and the sequence of deviations continues. Abe (2018) shows that the farsighted stable set and a certain type of myopic core coincide in some coalition structures in symmetric majority games. Kóczy (2007) proposes a recursive form of farsightedness and defines recursive optimism and pessimism. Kóczy (2018) also provides a comprehensive survey on these models. However, most authors say little about the projection core, which is studied in our paper.

In the projective core, a coalition S deviates from a coalition structure $\mathcal{P}$, projecting the original partition $\mathcal{P}$ onto $N\backslash S$. The projective core is special in the sense that it focuses on the coalition structure that forms immediately after the deviation, which represents the anticipations of a shortsighted player. One could argue that this is the only approach that deals with myopic players. Moreover, the projective core differs from other approaches in that the post-deviation coalition structure depends on the pre-deviation coalition structure. Because of this dependence, the partition function cannot be translated to the coalition function. This makes it difficult to offer general results. We address this problem by focusing on certain classes of symmetric games with important applications. We employ the symmetry notion proposed by de Clippel and Serrano (2008). The applications include the Cournot/Bertrand oligopoly, the public goods game (Ray and Vohra 1997; Yi 1997), and the common pool resource game (Funaki and Yamato 1999). Some general propositions that hold for every symmetric game are also offered.

The rest of the paper is organized as follows. In Sect. 2, we define games with externalities and the projective core. The class of symmetric games and its subclasses are also introduced. In Sect. 3, we offer some general results that hold for all symmetric games. The public goods game is discussed in this section. The class of the largest coalition games, which includes the Bertrand oligopoly, is analyzed in Sect. 4. In Sect. 5, we consider the class of games with partition cardinality properties; this class includes the Cournot oligopoly and the common pool resource game. We conclude this paper with some remarks in Sect. 6.

2 Preliminaries

Let $N = \{1, \ldots, n\}$ be the set of players. A *coalition* S is a subset of N. Let $|S|$ denote the number of players in S. We typically use $\mathcal{P}$ or $\mathcal{Q}$ to denote a partition of N. Let $|\mathcal{P}|$ denote the number of coalitions in $\mathcal{P}$. For any $i \in N$, $\mathcal{P}(i)$ denotes the coalition in $\mathcal{P}$ that contains player i. For any coalition $S \subseteq N$, let $\Pi(S)$ be the set of all partitions of S. For any $S \subseteq N$, let $\mathcal{P}_S$ be the *projection* of $\mathcal{P}$ on S, formally

$$\mathcal{P}_S = \{S \cap C | C \in \mathcal{P}, S \cap C \neq \emptyset\}.$$

Hence, $\mathcal{P}_S$ is a partition of S. For example, if $\mathcal{P} = \{\{1, 2\}, \{3, 4, 5\}\}$ and $S = \{2, 3, 4\}$, then $\mathcal{P}_S = \{\{2\}, \{3, 4\}\}$. For simplicity, we define

$$\mathcal{P}^S := \{S\} \cup (\mathcal{P}_{N \setminus S}).$$

For example, if $\mathcal{P} = \{\{1, 2\}, \{3, 4, 5\}\}$ and $S = \{2, 3, 4\}$, then we have $\mathcal{P}^S = \{\{2, 3, 4\}, \{1\}, \{5\}\}$. For any partition $\mathcal{P}$, $[\mathcal{P}]$ represents a multiset of cardinalities that admits multiple instances for each element. For example, if $\mathcal{P} = \{\{1, 2\}, \{3, 4\}, \{5\}\}$, then $[\mathcal{P}] = \{2, 2, 1\}$.

We define an *embedded coalition* of N by $(S, \mathcal{P})$ satisfying $\emptyset \neq S \subseteq N, \mathcal{P} \in \Pi(N)$, and $S \in \mathcal{P}$. The set of all embedded coalitions of N is given by

$$\mathcal{E}(N) = \{(S, \mathcal{P}) \mid \emptyset \neq S \subseteq N,\ \mathcal{P} \in \Pi(N),\ \text{and } S \in \mathcal{P}\}.$$

A *game with externalities* is a pair (N, v) in which a *partition function* v assigns a real number to each embedded coalition, namely, $v : \mathcal{E}(N) \to \mathbb{R}$. By convention, we define $v(\emptyset, \mathcal{P}) = 0$ for all $\mathcal{P} \in \Pi(N)$. We sometimes call a game with externalities simply a game. Let $\mathcal{G}_N$ be the set of all games with externalities whose player set is N.

We now introduce the notion of the projective core. In the literature on cooperative game theory, the formation of the grand coalition is often implicitly assumed, and the core is defined for the grand coalition. In this paper, we define the core for each partition. Greenberg (1994), Kóczy (2007, 2009) and Koczy and Lauwers (2004) consider a solution as a pair of payoff distributions and a partition. Our definition fits into this context. Let $v \in \mathcal{G}_N$. For each $\mathcal{P} \in \Pi(N)$, let $F(v, \mathcal{P})$ be the set of feasible allocations for partition $\mathcal{P}$, formally $F(v, \mathcal{P}) = \{x \in \mathbb{R}^N | \sum_{j \in S} x_j \leq v(S, \mathcal{P})$ for any $S \in \mathcal{P}\}$. We similarly define the set of efficient allocations for $\mathcal{P}$: $X(v, \mathcal{P}) = \{x \in \mathbb{R}^N | \sum_{j \in S} x_j = v(S, \mathcal{P})$ for any $S \in \mathcal{P}\}$. The *projective core* for $\mathcal{P}$

is defined as follows:

$$C^{\text{proj}}(v, \mathcal{P}) = \left\{ x \in X(v, \mathcal{P}) | \sum_{j \in S} x_j \geq v(S, \mathcal{P}^S) \text{ for any } S \subseteq N \right\}.$$

We say that a partition has a nonempty projective core if the projective core for $\mathcal{P}$ is nonempty.[1]

As described in Sect. 1, the projective core can be thought of as the most myopic core in the sense of reaction to a deviating coalition. The partition resulting from the deviation of coalition S is given as the combination of coalition S and the projection of $\mathcal{P}$ on the remaining players $N \backslash S$, namely $\mathcal{P}_{N \backslash S}$. Therefore, the resulting partition $\mathcal{P}^S (= \{S\} \cup \mathcal{P}_{N \backslash S})$ depends on S and the original partition $\mathcal{P}$.

Now, we introduce three classes of games.

- Let $\sigma : N \to N$ be a permutation. We define $\sigma(S) = \{\sigma(i) | i \in S\}$, and similarly, $\sigma(\mathcal{P}) = \{\sigma(S) | S \in \mathcal{P}\}$. A game v is **symmetric** if for any σ, $v(S, \mathcal{P}) = v(\sigma(S), \sigma(\mathcal{P}))$. A symmetric game can be also defined as follows: for any $(S, \mathcal{P})$ and $(T, \mathcal{Q})$ in $\mathcal{E}(N)$, if $|S| = |T|$ and $[\mathcal{P}] = [\mathcal{Q}]$, then $v(S, \mathcal{P}) = v(T, \mathcal{Q})$. Let $\mathcal{G}_N^S$ be the set of symmetric games.[2]
- A game v satisfies **partition cardinality property** (a PCP game) if for any $(S, \mathcal{P})$ and $(T, \mathcal{Q})$, if $|\mathcal{P}| = |\mathcal{Q}|$, then $v(S, \mathcal{P}) = v(T, \mathcal{Q})$. Let $\mathcal{G}_N^{PCP}$ be the set of PCP games. In a PCP game, v no longer depends on S. The number of coalitions in $\mathcal{P}$ determines the worth of the coalitions in $\mathcal{P}$ that is the same for all the coalitions in $\mathcal{P}$.
- A game is called a **largest coalition game** if there exists a function f such that for any $(S, \mathcal{P}) \in \mathcal{E}(N) \backslash (N, \{N\})$,

$$v(S, \mathcal{P}) = \begin{cases} f(|S|, [\mathcal{P}]) \geq 0 & \text{if } |S| \geq |S'| \text{ for every } S' \in \mathcal{P}, \\ 0 & \text{otherwise,} \end{cases}$$
$$v(N, \{N\}) > 0.$$

  In a largest coalition game, for each partition $\mathcal{P}$, the largest coalition in $\mathcal{P}$ obtains the worth $f(|S|, [\mathcal{P}])$. Each coalition that is not the largest in $\mathcal{P}$ obtains zero. If two or more coalitions have the largest cardinality, each of them obtains the same worth $f(|S|, [\mathcal{P}])$. Let $\mathcal{G}_N^{LC}$ be the set of largest coalition games.

[1] We define the core by inequalities, while one may use *domination*: for any partition $\mathcal{P} \in \Pi(N)$ and any allocation $x \in X(v, \mathcal{P})$, we say that $(y, \mathcal{Q})$ dominates x if there exists an $S \subseteq N$ such that (i) $y_j > x_j$ for any $j \in S$, (ii) $\mathcal{Q} = \{S\} \cup \mathcal{P}_{N \backslash S}$, and (iii) $y \in X(v, \mathcal{Q})$. The core for $\mathcal{P}$ is the set of allocations x in $X(v, \mathcal{P})$ that are not dominated by any such $(y, \mathcal{Q})$. Similar to the traditional core for a game without externalities, the inequality core becomes a subset of the dominance core.

[2] We say that a game satisfies *strong symmetry* if for any $(S, \mathcal{P})$ and $(T, \mathcal{Q})$,

$$|S| = |T| \text{ and } |\mathcal{P}| = |\mathcal{Q}| \Rightarrow v(S, \mathcal{P}) = v(T, \mathcal{Q}).$$

Note that the class of strong symmetry games $\mathcal{G}_N^{SS}$ is different from that of symmetry games: $\mathcal{G}_N^{SS} \subseteq \mathcal{G}_N^S$. For the relationship among some symmetry definitions, see Sect. 6.

We have $\mathcal{G}_N^{PCP} \subseteq \mathcal{G}_N^S$ and $\mathcal{G}_N^{LC} \subseteq \mathcal{G}_N^S$. Note that each game in the intersection $\mathcal{G}_N^{PCP} \cap \mathcal{G}_N^{LC}$ is described as follows: $v(N, \{N\}) > 0$ and $v(S, \mathcal{P}) = 0$ for any $(S, \mathcal{P}) \in \mathcal{E}(N) \backslash (N, \{N\})$.

In this paper, we consider public good games as applications of symmetric partition function form games. Largest coalition games are illustrated by Bertrand competition and a simple example of common goods competition. We also present two models in the class of PCP games: Cournot oligopoly and the common pool resource game of Funaki and Yamato (1999).

3 Symmetric games

3.1 General symmetric games

We begin with the basic property of a projection.

Lemma 3.1 *Let $\mathcal{P} \in \Pi(N)$ and $S^* \in \mathcal{P}$. Let permutation $\sigma^{S^*} : N \to N$ satisfy $\sigma(i) = i$ for any $i \in N \backslash S^*$. For any $S \subseteq N$,*

$$\mathcal{P}^{\sigma^{S^*}(S)} = \sigma^{S^*}(\mathcal{P}^S).$$

Lemma 3.1 simply shows that the projective partition resulting from the rearranged coalition coincides with the partition that is rearranged after the deviation. Now, we define the equal division of an arbitrary partition: for any game $v \in \mathcal{G}_N$, $e_i(v, \mathcal{P}) := \frac{v(\mathcal{P}(i), \mathcal{P})}{|\mathcal{P}(i)|}$ for every $i \in N$.

The following proposition is an extension of the necessary and sufficient condition for the core of symmetric games without externalities to be nonempty.

Proposition 3.2 *Let v be a symmetric game. Let $\mathcal{P} \in \Pi(N)$. Then,*

$$C^{proj}(v, \mathcal{P}) \neq \emptyset \iff e(v, \mathcal{P}) \in C^{proj}(v, \mathcal{P})$$

The necessary and sufficient condition above is equivalent to the following statement: for any $S \subseteq N$, $\sum_{j \in S} \frac{v(\mathcal{P}(j), \mathcal{P})}{|\mathcal{P}(j)|} \geq v(S, \mathcal{P}^S)$. Proposition 3.2 is a generalization in the following two senses. The first is regarding the scope of coalition structures. For a game without externalities, the core and its nonemptiness condition are provided for the grand coalition, whereas Proposition 3.2 is a condition for each partition. The second is regarding the multiplicity of cores. For games without externalities, the core is uniquely defined. However, in the presence of externalities, multiple reactions are studied because of the deviations, as introduced in Sect. 1. Multiplicity yields multiple definitions of the core. The pessimistic core, the optimistic core, the disintegrating core, and the merging core are well-known core concepts. Hafalir (2007) calls the disintegrating core the s-core (singleton-core) and the merging core the m-core. It is relatively straightforward to analyze these core concepts because they do not depend on the partition from which a coalition deviates. In contrast, the projective core inherits the feature of projective reaction and, unlike the cores above, depends on the partition for which the projective core is defined.

The following proposition is useful for finding a partition with an empty projective core and will play an important role in analyzing the economic applications.

Proposition 3.3 *Let v be a symmetric game. Let $\mathcal{P} \in \Pi(N)$. If there exist coalitions $S, S' \in \mathcal{P}$ such that $|S| > |S'|$ and $\frac{v(S,\mathcal{P})}{|S|} > \frac{v(S',\mathcal{P})}{|S'|}$, then $C^{proj}(v, \mathcal{P}) = \emptyset$.*

Proposition 3.3 shows that if a partition has two different coalitions in the senses of both size and average worth, then the partition has no core element. Therefore, this can be thought of as a necessary condition for a partition to have a nonempty core. Except for some "special" partitions such as the grand coalition and the partition that consists of n one-person coalitions, many partitions contain different sizes of coalitions. The proposition states that such normal partitions must obey an additional condition to have a nonempty projective core in terms of average worth. For example, let $\mathcal{P} = \{\{1\}, \{2, 3\}\}$ and $v(\{1\}, \mathcal{P}) = 1$. We must have $v(\{2, 3\}, \mathcal{P}) \leq 2$ for $\mathcal{P}$ to be able to have a nonempty projective core.

The intuition behind this result lies in the relationship between a deviation and symmetric partitions. For example, we consider a partition $\{\{1, 2, 3, 4\}, \{5, 6\}\}$. For this partition, there exists a four-person coalition that contains $\{5, 6\}$, *e.g.*, $\{3, 4, 5, 6\}$. After such a coalition deviates, its resulting partition, namely $\{\{3, 4, 5, 6\}, \{1, 2\}\}$, must be symmetric to the initial partition $\{\{1, 2, 3, 4\}, \{5, 6\}\}$. Similarly, some four-person coalitions exist and deviate from the resulting partition $\{\{3, 4, 5, 6\}, \{1, 2\}\}$, which leads to another symmetric partition that may be the first partition. In general, for each partition that contains different sizes of coalitions, there exists a coalition for which the deviation yields a partition that is symmetric to the original partition. Therefore, some cycles of deviations can be found among symmetric partitions. Such cycles make the projective cores empty for a group of symmetric partitions.

These general conditions become more informative in some specific subclasses. We first analyze the class of public goods games.

3.2 Public goods games

Some models of public goods games are provided by Ray and Vohra (1997) and Yi (1997) . In this paper, we introduce a variation of those models and analyze its projective core.

Every player $i \in N$ is endowed with one unit of private good, $e.g.$, time or labor. We focus on a coalition S in a partition $\mathcal{P}$. Every member $i \in S$ contributes $x_i \leq 1$ to produce public goods for S. Let y be the level of public goods and $\bar{y}$ be its maximal level. We use $c(y)$ to denote the cost of producing y public goods. Every member of S enjoys benefit $b(y)$ from consuming the public goods and equally shares the cost $c(y)$ within S. The members choose the optimal level of public goods and equally share the cost. Therefore, every member of S enjoys $I(|S|) := \max_{0 \leq y \leq \bar{y}} b(y) - \frac{c(y)}{|S|}$.

The benefit of the public goods produced by coalition S may spill over from S to other coalitions in $\mathcal{P}$. We use $E(|S|) \geq 0$ to denote the external benefit that each player in $N \backslash S$ derives from S. Therefore, for any $\mathcal{P} \in \Pi(N)$ and any $S \in \mathcal{P}$, player $i \in S$

enjoys $I(|S|) + \sum_{T \in \mathcal{P} \setminus \{S\}} E(|T|)$. A public goods game is defined as follows: for any $\mathcal{P} \in \Pi(N)$ and any $S \in \mathcal{P}$

$$v(S, \mathcal{P}) = |S| \cdot \left[I(|S|) + \sum_{T \in \mathcal{P} \setminus \{S\}} E(|T|) \right].$$

If one considers a model with purely local public goods (or a model without spillovers), then $E(k) = 0$ for any $k = 1, \dots, n$.

We can consider I to be a function given by $I : \{0, \dots, n\} \to \mathbb{R}_+$, and similarly E to be $E : \{0, \dots, n\} \to \mathbb{R}_+$. We assume that $I(0) = E(0) = 0$. We define $\Delta^I(k) = I(k) - I(k-1)$ and $\Delta^E(k) = E(k) - E(k-1)$ for any $k = 1, \dots, n$ and assume that a marginal internal effect is larger than a marginal external effect:

$$\Delta^I(k) > \Delta^E(k)$$

for any $k = 1, \dots, n$. This assumption indicates that a change in the size of a coalition affects the members more than the nonmembers.

Lemma 3.4 *Let v be a public goods game and let $\mathcal{P} \in \Pi(N)$. If there exist coalitions $S, S' \in \mathcal{P}$ such that $|S| > |S'|$, then $C^{proj}(v, \mathcal{P}) = \emptyset$.*

The proof is straightforward in view of Proposition 3.3. This lemma implies that for a partition to have a nonempty projective core, every coalition in the partition must have the same cardinality. Moreover, note that Lemma 3.4 does not depend on $I(\cdot)$ and $E(\cdot)$: the emptiness is valid for any form of internal/external effect functions. What condition guarantees the nonemptiness of the projective core? To determine this, we now focus on convex $I(\cdot)$. The following proposition shows that the convexity gives the grand coalition a nonempty projective core and gives the other partitions empty projective cores.

Proposition 3.5 *If $\Delta^I(k) \leq \Delta^I(k+1)$ for every $k = 1, \dots, n-1$, then $C^{proj}(v, \{N\}) \neq \emptyset$ and $C^{proj}(v, \mathcal{P}) = \emptyset$ for any $\mathcal{P} \in \Pi(N) \setminus \{N\}$.*

Note that spillover effect E is not conditioned in this proposition: E does not influence the nonemptiness of the core under the above condition of I. In the presence of a spillover, each member of every coalition benefits from their own public goods and the spillovers from other coalitions. Therefore, one might consider that a partition consisting of multiple coalitions can also be seen as a stable coalition structure. However, Proposition 3.5 shows that the grand coalition, in which there is no such spillover from another coalition, is the only stable coalition structure. This result occurs because of the convexity of I. In view of $\Delta^I(k) > \Delta^E(k)$, the internal effect surpasses the external effect, and the convexity of I benefits a larger coalition, which provides players with a larger incentive to cooperate and jointly produce public goods rather than enjoying the public goods as free riders.

If I is concave, namely, $\Delta^I(k) \geq \Delta^I(k+1)$ for every $k = 1, \dots, n-1$, and $\Delta^I(1) \leq \Delta^E(1)$, then the partition of the player set into singletons has a nonempty projective core in a way that is similar to Proposition 3.5. Given the results above, one may consider that if there is k^* such that $I(k)$ is convex for $1 \leq k \leq k^* - 1$ and is concave for $k^* \leq k \leq n - 1$, then a partition $\mathcal{P}$ satisfying $k^* = |S|$ for every $S \in \mathcal{P}$ has a nonempty projective core. However, this conjecture is not true. For example, let $n = 10$ and $\mathcal{P} = \{S_1, S_2\}$ with $S_1 = \{1, \ldots, 5\}$ and $S_2 = \{6, \ldots, 10\}$. Consider $T = \{2, \ldots, 8\}$. We suppose that $I(k) = k$ for $k = 1, \ldots, 5$, $I(k) = 5 + 3k/7$ for $k = 6, \ldots, 10$, and $E(k) = k/2$ for every $k = 1, \ldots, 10$. Then, $e_i(v, \mathcal{P}) = I(5) + E(5) = 7.5$, while $v(T, \{T\} \cup \mathcal{P}_{N\setminus T}) = I(7) + E(1) + E(2) = 9.5$. Hence, coalition T has an incentive to deviate from $\mathcal{P}$, and the projective core for the partition $\mathcal{P}$ is empty.

If I is convex, the public goods game should also satisfy a sort of convexity. Since the convexity of games with externalities is not unique, below we consider the convexity notion proposed by Hafalir (2007) . A game with externalities is convex if for any $S, T \subseteq N$ and any $\mathcal{P}' \in \Pi(N \setminus (S \cup T))$, we have $v(S \cup T, \{(S \cup T)\} \cup \mathcal{P}') + v(S \cap T, \{(S \setminus T), (S \cap T), (T \setminus S)\} \cup \mathcal{P}') \geq v(S, \{(S), (T \setminus S)\} \cup \mathcal{P}') + v(T, \{(T), (S \setminus T)\} \cup \mathcal{P}')$. As long as I is convex, the public goods game also satisfies this convexity definition. However, any game with this convexity notion does not necessarily satisfy the condition of Proposition 3.3 even if it is symmetric. In other words, even a symmetric convex game may have a partition $\mathcal{P}$ that contains a pair of coalitions $S, S' \in \mathcal{P}$ with $|S| > |S'|$ and $\frac{v(S,\mathcal{P})}{|S|} > \frac{v(S',\mathcal{P})}{|S'|}$. Therefore, a partition that is not the grand coalition may have a nonempty projective core. However, we can obtain the same result as Proposition 3.5 by assuming a strict efficiency requirement: for every $\mathcal{P} \in \Pi(N) \setminus \{\{N\}\}$, $v(N, \{N\}) > \sum_{S \in \mathcal{P}} v(S, \mathcal{P})$. If a symmetric game satisfies both the convexity notion above and this strict efficiency condition, then the equal division at the grand coalition lies in the projective core, and any other partition does not have a projective core allocation.

4 Largest coalition games

4.1 General largest coalition games

In this section, we focus on the class of largest coalition games defined in Sect. 2. Note that this is a subclass of the class of symmetric games. We briefly recall its intuition: for each partition $\mathcal{P}$, the largest coalition in $\mathcal{P}$ obtains the worth $f(|S|, [\mathcal{P}])$, while each coalition that is not the largest in $\mathcal{P}$ obtains zero. The importance of this subclass lies in the fact that some important economic scenes can be modeled as a largest coalition game. As preparation, we begin with the following general result that is similar to Lemma 3.4.

Corollary 4.1 *Let v be a largest coalition game and $\mathcal{P} \in \Pi(N)$. If there exist coalitions $S, S' \in \mathcal{P}$ such that $|S| > |S'|$, then $C^{proj}(v, \mathcal{P}) = \emptyset$.*

Although the resulting appearance is the same as Lemma 3.4, the reasoning behind Corollary 4.1 is different from that of Lemma 3.4. In a large coalition game, the second largest coalition in a partition obtains zero. Therefore, the second condition of Proposition 3.3, namely, $\frac{v(S,\mathcal{P})}{|S|} > \frac{v(S',\mathcal{P})}{|S'|}$, always holds for the largest coalition S and another coalition S' as long as $v(S, \mathcal{P})$ is positive. If partition $\mathcal{P}$ satisfies $v(S, \mathcal{P}) = 0$ for all $S \in \mathcal{P}$, then the partition (even if it contains some coalitions with different sizes) does not violate Proposition 3.3. However, all players have an incentive to deviate by forming N and obtain $v(N, \{N\}) > 0$. Hence, the projective core for partitions satisfying $v(S, \mathcal{P}) = 0$ for all $S \in \mathcal{P}$ is also empty. If we focus on the equal division on the grand coalition, coalition S has no incentive to deviate from the equal division of $v(N, \{N\})$ if and only if $\frac{1}{|N|} f(|N|, [n]) \geq \frac{1}{|S|} f(|S|, [|S|, |N \backslash S|])$ since a deviation from the grand coalition projects $\{N\}$ onto the set of the non-deviating players $N \backslash S$, which results in the partition $\{S, N \backslash S\}$.

The class of largest coalition games is suitable for describing competition among coalitions. To see this, we first apply Corollary 4.1 to the following simple example.

Example 1 (Common goods competition) Consider that some identical and divisible goods are to be distributed. Let $x > 0$ be the amount of the goods. For any $\mathcal{P} \in \Pi(N)$, the largest coalition in $\mathcal{P}$ wins all x goods. If two or more coalitions have the largest cardinality, they equally share x. Therefore, every common goods competition game is a largest coalition game.

In this game, the projective core is empty for all partitions including the grand coalition. To see this, given Corollary 4.1, we focus on partition $\mathcal{P}$, which consists of coalitions of the same size. Let $\mathcal{P} = \{S_1, \ldots, S_m\}$, with $|S_1| = \cdots = |S_m|$.

We first consider $\mathcal{P} \neq \{N\}$. Let $T \subseteq N$ be $|T| = |S_1| + 1$. Then, T is the largest coalition in $\mathcal{P}^T (= \{T\} \cup \mathcal{P}_{N \backslash T})$. Hence, $v(T, \mathcal{P}^T) = x$, and we have

$$\sum_{j \in T} e_j(v, \mathcal{P}) = |T| \frac{x}{n} < x = v(T, \mathcal{P}^T).$$

Thus, in view of Proposition 3.2, $C^{\text{proj}}(v, \mathcal{P}) = \emptyset$.

We now consider $\mathcal{P} = \{N\}$. For every $i \in N$, we have

$$\sum_{j \in N \backslash \{i\}} e_j(v, \{N\}) = (n-1) \frac{x}{n} < x = v(N \backslash \{i\}, \{\{i\}, N \backslash \{i\}\}).$$

Hence, similarly, $C^{\text{proj}}(v, \{N\}) = \emptyset$.

This example shows that the projective core can be empty even for such a simple setting. However, some economic games have a nonempty projective core. We now analyze Bertrand competition in the framework of largest coalition games. This must be the first formulation of Bertrand competition in partition function form.

4.2 Bertrand oligopoly with size-dependent cost functions

Let c be a cost function. Coalition S produces one unit of identical goods with cost $c(|S|)$: the cost function assigns a real number to each natural number $1, \ldots, n$. We assume that as a coalition becomes larger, the cost monotonically decreases: $c(k) > c(k+1)$ for any $k = 1, \ldots, n-1$. We assume $c(n) \geq 0$. The demand function is given by $q(p)$, where p is a price. We assume that $q'(p) \leq 0$ and $q(p) \geq 0$ for any $p \geq 0$.

Given partition $\mathcal{P}$, coalitions in $\mathcal{P}$ simultaneously determine the price of the good. The coalition that offers the lowest price, say p_*, obtains all the demand at that price, $q(p_*)$. Since the cost function depends on the size of the coalition, the largest unique coalition obtains all the demand $q(p_*)$. If some coalitions tie in the senses of size and cost, they receive zero profit, as indicated by the typical Bertrand oligopoly. Let S be the largest unique coalition in $\mathcal{P}$, and let S' be the second largest coalition in $\mathcal{P}$. As a result of the competition, coalition S chooses a price so as to maximize its profit:

$$\max_{p \leq c(|S'|)} q(p)(p - c(|S|)).$$

The largest coalition S must choose $p \leq c(|S'|)$ to win the price competition and $p > c(|S|)$ to obtain positive profit. We assume that a coalition withdraws from the competition if its profit is zero. We consider the profit above as the worth of coalition S in $\mathcal{P}$:

$$v(S, \mathcal{P}) = \begin{cases} \max\limits_{p \leq c(|S'|)} q(p)(p - c(|S|)) & \text{if } |S| > |T| \forall T \in \mathcal{P} \backslash \{S\}, \\ 0 & \text{otherwise.} \end{cases}$$

If $\mathcal{P} = \{N\}$, the grand coalition N obtains the monopoly profit, where the monopoly price p^{m} solves $q'(p^{\mathrm{m}}) \cdot (p^{\mathrm{m}} - c(n)) + q(p^{\mathrm{m}}) = 0$. Let π^{m} denote monopoly profit. We assume that $\pi^{\mathrm{m}} > 0$. Note that every Bertrand oligopoly with size-dependent cost functions is a large coalition game. We first obtain the following result.

Proposition 4.2 *For any* $\mathcal{P} \in \Pi(N) \backslash \{N\}$, $C^{proj}(v, \mathcal{P}) = \emptyset$.

The proof is straightforward in view of Corollary 4.1. In addition to partitions containing different sizes of coalitions, some partitions consisting of the same size coalitions also lack projective cores. Therefore, we restrict our attention to the grand coalition to find a nonempty projective core. We first define $h(n)$ as follows:

$$h(n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} + 1 & \text{if } n \text{ is even.} \end{cases}$$

For notational simplicity, we simply write h instead of $h(n)$. Now, for cost function c and demand function q, we define $d^{c,q}$ as follows: for any $s = 1, \ldots, n$,

$$d^{c,q}(s) = \max_{p \leq c(n-s)} q(p)(p - c(s)).$$

Proposition 4.3 *The projective core for the grand coalition* $C^{proj}(v, \{N\})$ *is nonempty if and only if for every* $s = h, \ldots, n-1$,

$$\frac{d^{c,q}(s)}{s} \leq \frac{\pi^m}{n}.$$

Proposition 4.3 is a necessary and sufficient condition for the projective core to be nonempty. Note that we do not have to check $d^{c,q}(s)$ for $s = 1, \ldots, h-1$. This result contributes to the literature on oligopolies in partition function form in three ways.

- One is the generality of the cost function and the demand function. Most preceding works employ a linear demand function and constant marginal costs, while the proposition holds for general demand and cost functions.
- Another novelty is that we formally show that the partitions other than the grand coalition are not stable in terms of the core. The formation of the grand coalition has been often assumed in this context. Proposition 4.2 offers the reasons for this implicit assumption.
- As elaborated in the next section, in a Cournot oligopoly, the projective core of the grand coalition is empty even under the linear setting, and it is also empty for the other partitions. Proposition 4.3 shows that the core can be nonempty for some demand functions and cost functions in a Bertrand oligopoly.

Below, as instances of the third issue discussed above, we offer two numerical examples. In Example 2, we show that the grand coalition can have a nonempty core with a simple (linear) demand function and a (linear) cost function. Example 3 describes an empty projective core.

Example 2 Let $N = \{1, 2, 3, 4, 5\}$. Consider $q(p) = \max\{12 - p, 0\}$ for $p \in \mathbb{R}^+$ and $c(s) = 6 - s$ for $s = 1, \ldots, 5$. Note that $h = 3$. The monopoly profit $v(N, \{N\}) = \pi^{\mathrm{m}}$ is given by

$$\pi^{\mathrm{m}} = \max_{p \geq 0} q(p)(p - c(5)) = \max_{p \geq 0}(12 - p)(p - 1) = 30.25.$$

As Proposition 4.3 shows, it suffices to check $d^{c,q}(3)$ and $d^{c,q}(4)$. We have

$$\begin{aligned} d^{c,q}(3) &= \max_{p \leq c(2)} q(p)(p - c(3)) = \max_{p \leq 4}(12 - p)(p - 3) = 8 \\ d^{c,q}(4) &= \max_{p \leq c(1)} q(p)(p - c(4)) = \max_{p \leq 5}(12 - p)(p - 2) = 21. \end{aligned}$$

Hence, we have $8/3 = 2.666\cdots < 6.05 = 30.25/5$ for $d^{c,q}(3)$ and $21/4 = 5.25 < 6.05 = 30.25/5$ for $d^{c,q}(4)$, and the projective core for the grand coalition is nonempty. The projective cores for the other partitions are empty, as Proposition 4.2 describes.

Example 3 We now consider another (decreasing) cost function with $c(5) = 1, c(3) = 2$, and $c(2) = 7$. We use the same demand function $q(p) = \max\{12 - p, 0\}$ for $p \in \mathbb{R}^+$. The monopoly profit is the same, $\pi^{\mathrm{m}} = 30.25$. For $d^{c,q}(3)$, we have $d^{c,q}(3) = \max_{p \leq c(2)} q(p)(p - c(3)) = \max_{p \leq 7}(12 - p)(p - 2) = 25$. Hence, we have $25/3 = 8.333\cdots > 6.05 = 30.25/5$ for $d^{c,q}(3)$. The necessary and sufficient condition is violated, and the projective core for the grand coalition is empty. Therefore, in this example, every partition has an empty projective core.

5 Games with partition cardinality property

We recall the definition of a PCP game and its intuition. We say that a game v is a PCP game if for any $(S, \mathcal{P})$ and $(T, \mathcal{Q})$, if $|\mathcal{P}| = |\mathcal{Q}|$, then $v(S, \mathcal{P}) = v(T, \mathcal{Q})$. Therefore, worth v depends on the number of coalitions in $\mathcal{P}$. The worth of the coalitions in $\mathcal{P}$ is the same for all the coalitions in $\mathcal{P}$.

Although a PCP game seems more restrictive than the other symmetric games, this class contains two important economic applications: the Cournot oligopoly proposed by Ray and Vohra (1997) and Yi (1997) and the common pool resource game studied by Funaki and Yamato (1999) and Abe and Funaki (2017). The rich results of these two games can be ascribed to the simplification of the partition function. A PCP game $v \in \mathcal{G}_N^{PCP}$ is represented by function f such that for any $(S, \mathcal{P}) \in \mathcal{E}(N)$,

$$f(|\mathcal{P}|) = v(S, \mathcal{P}).$$

In other words, the worth of an embedded coalition only depends on the cardinality of the partition. In this section, we use f instead of v to denote a PCP game.

As we have already seen in the previous sections, the projective core often becomes empty for partitions other than the grand coalition because of Proposition 3.3 and its corollaries. Moreover, in some specific instances, the projective core is empty even for the grand coalition. However, in the class of PCP games, every partition might have a nonempty projective core. Below, we offer a necessary and sufficient condition for a partition to have a nonempty projective core.

We define PCP game f and partition $\mathcal{P} \in \Pi(N)$. For every $k \geq 0$, we define

$$g^{f,\mathcal{P}}(k) = \begin{cases} (|\mathcal{P}| - k + 1) f(|\mathcal{P}|) & \text{if } k \leq |\mathcal{P}|, \\ \frac{1}{|S_{\max}(\mathcal{P})|} f(|\mathcal{P}|) & \text{if } k > |\mathcal{P}|, \end{cases}$$

where $S_{\max}(\mathcal{P})$ is one of the largest coalitions in $\mathcal{P}$: $|S_{\max}(\mathcal{P})| \geq |S'|$ for every $S' \in \mathcal{P}$.

Proposition 5.1 *Let f be a PCP game and set $\mathcal{P} \in \Pi(N)$. Assume $f(|\mathcal{P}|) \geq 0$. Then,*

$$C^{proj}(f, \mathcal{P}) \neq \emptyset \iff f(k) \leq g^{f,\mathcal{P}}(k) \text{ for every } k = 1, \ldots, |\mathcal{P}| + 1.$$

Game f is a function with an input of natural number and an output of real number, and so is $g^{f,\mathcal{P}}$. The proposition states that the projective core is nonempty if and only if $g^{f,\mathcal{P}}$ is located "above" f for each natural number k. Moreover, we can derive some implications from Proposition 5.1.

- If $f(k)$ is **nondecreasing** in k, partition $\{\{i\}|i \in N\}$ is the only partition that has a nonempty projective core. If $f(k)$ is **nonincreasing** in k, the nonemptiness of the core is more complicated. The following two games are the PCP games with nonicreasing f.
  - The common pool resource game has a nonincreasing f. Moreover, this game satisfies $k \cdot f(k) > k' \cdot f(k')$ if $k < k'$. In view of this inequality, Proposition 5.1 is not satisfied for each partition, which implies that every common pool resource game with $n \geq 3$ has an empty projective core for every partition.

  - The Cournot oligopoly is another example. This game is explicitly given as $f(k) = \frac{1}{(k+1)^2}$, which immediately violates the necessary and sufficient condition and causes there to be an empty projective core for every partition.
- The observations above show that if f is **constant**, the partition consisting of singletons $\{\{i\}|i \in N\}$ is the only partition that has a nonempty projective core. Moreover, if f is partially constant and increases at some k^*, a partition whose cardinality is k^* may have a nonempty projective core. For example, let $f(k) = c$ for $k \leq k^*$ and $f(k) = d$ for $k > k^*$ ($c \neq d$). If some partition $\mathcal{P}$ satisfies $|\mathcal{P}| = k^*$ and $c \cdot \frac{1}{|S_{\max}(\mathcal{P})|} \geq d$, then this partition $\mathcal{P}$ has a nonempty projective core.
- In regards to any **single-peaked** function, according to our proposition it readily follows that the left side of the peak (namely, the coarse partitions or the partitions for which worth is increasing) lacks a projective core. Similarly, for any **single-dipped** function, the right side of the dip (namely, the finer partitions) lacks a projective core, except for the partition $\{\{i\}|i \in N\}$.

Moreover, the following result follows for "adjacent" partitions.

Corollary 5.2 *For any pair of partitions $\mathcal{P}, \mathcal{Q} \in \Pi(N)$ with $|\mathcal{P}| + 1 = |\mathcal{Q}|$, if $C^{proj}(f, \mathcal{P}) \neq \emptyset$ and $C^{proj}(f, \mathcal{Q}) \neq \emptyset$, then $|S_{\max}(\mathcal{P})| = 2$ and $f(|\mathcal{P}|) = 2f(|\mathcal{Q}|)$.*

We call the partitions $\mathcal{P}, \mathcal{Q} \in \Pi(N)$, satisfying $|\mathcal{P}| + 1 = |\mathcal{Q}|$, adjacent partitions (in the sense of cardinality). This corollary indicates that the adjacent partitions seldom have nonempty projective cores simultaneously. For example, consider partition $\mathcal{Q}$. Assume that the core for $\mathcal{Q}$ is nonempty. Then, we have to find partition $\mathcal{P}$, which consists of only two-person coalitions and one-person coalitions and satisfies $|\mathcal{P}|+1 = |\mathcal{Q}|$. Furthermore, the worth of partition $\mathcal{P}$, $f(|\mathcal{P}|)$, must be exactly equal to $2f(|\mathcal{Q}|)$. In most games, such a partition $\mathcal{P}$ does not exist, which shows how difficult it can be for two adjacent partitions to simultaneously have nonempty projective cores.

6 Remarks

Remark 1 (Approximate symmetry) The results we discussed in the previous sections are *robust* to small perturbations: we allow $v(S, \mathcal{P})$ to vary within a certain value $\pm\epsilon$ for $\epsilon > 0$. We say that a game $v \in \mathcal{G}_N$ satisfies *ϵ-approximate symmetry* (an ϵ-AS game) if there is a symmetric game $\mu \in \mathcal{G}_N^S$ such that for every $(S, \mathcal{P}) \in \mathcal{E}(N)$, $\mu(S, \mathcal{P}) - \epsilon \leq v(S, \mathcal{P}) \leq \mu(S, \mathcal{P}) + \epsilon$. Considering Proposition 3.3, it should be clear that for every $\epsilon > 0$, every ϵ-AS game v and every $\mathcal{P} \in \Pi(N)$, if there are coalitions $S, S' \in \mathcal{P}$ such that $|S| > |S'|$ and $\frac{v(S,\mathcal{P})}{|S|} - \frac{v(S',\mathcal{P})}{|S'|} > \frac{2\epsilon}{|S'|}$, then $C^{\text{proj}}(v, \mathcal{P}) = \emptyset$. This observation shows that if ϵ is small enough, Proposition 3.3 holds even in the presence of perturbations, and we can straightforwardly apply the condition above to the two subclasses with perturbations. Therefore, this generalizes our result to a wider range of classes that admit asymmetry.

Remark 2 (Other symmetry definitions) In this paper, we define a symmetric game as a game v satisfying $v(S, \mathcal{P}) = v(\sigma(S), \sigma(\mathcal{P}))$ for any permutation σ. We now define *strong symmetry* (SS) as follows: for any $(S, \mathcal{P})$ and $(T, \mathcal{Q})$, $|S| = |T|$ and $|\mathcal{P}| =$

$|\mathcal{Q}| \Rightarrow v(S, \mathcal{P}) = v(T, \mathcal{Q})$. In view of the definition of strong symmetry, one may immediately define two weaker variations, coalition symmetry and partition symmetry, as follows. A game v satisfies *coalition symmetry* (CS) if for any $(S, \mathcal{P})$ and $(T, \mathcal{Q})$, $|S| = |T|$ and $\mathcal{P} = \mathcal{Q} \Rightarrow v(S, \mathcal{P}) = v(T, \mathcal{Q})$. A game v satisfies *partition symmetry* (PS) if for any $(S, \mathcal{P})$ and $(T, \mathcal{Q})$, $S = T$ and $|\mathcal{P}| = |\mathcal{Q}| \Rightarrow v(S, \mathcal{P}) = v(T, \mathcal{Q})$. From the definitions above, it readily follows that

$$SS \Rightarrow Symmetry \Rightarrow CS;\ \ SS \Rightarrow PS.$$

Note that symmetry does not imply PS as another equivalent definition of symmetry is given as $|S| = |T|$ and $[\mathcal{P}] = [\mathcal{Q}] \Rightarrow v(S, \mathcal{P}) = v(T, \mathcal{Q})$.

Appendix

Proof of Lemma 3.1 First, we have $\sigma^{S^*}(\mathcal{P}^S) = \sigma^{S^*}(\{S\} \cup \mathcal{P}_{N\setminus S}) = \{\sigma^{S^*}(S)\} \cup \sigma^{S^*}(\mathcal{P}_{N\setminus S})$. We now focus on $\sigma^{S^*}(\mathcal{P}_{N\setminus S})$. Then, we have

$$\begin{aligned} \sigma^{S^*}(\mathcal{P}_{N\setminus S}) &= \sigma^{S^*}(\{(N\setminus S) \cap C | C \in \mathcal{P}\}) \\ &= \{\sigma^{S^*}((N\setminus S) \cap C) | C \in \mathcal{P}\} \\ &= \{\sigma^{S^*}(N\setminus S) \cap \sigma^{S^*}(C) | C \in \mathcal{P}\} \\ &= \{(N\setminus \sigma^{S^*}(S)) \cap \sigma^{S^*}(C) | C \in \mathcal{P}\} \\ &\overset{S^*\in\mathcal{P}}{=} \{(N\setminus \sigma^{S^*}(S)) \cap C | C \in \mathcal{P}\} = \mathcal{P}_{N\setminus \sigma^{S^*}(S)}. \end{aligned}$$

Therefore, we obtain $\{\sigma^{S^*}(S)\} \cup \mathcal{P}_{N\setminus \sigma^{S^*}(S)} = \mathcal{P}^{\sigma^{S^*}(S)}$. □

Proof of Proposition 3.2 The proof of $\Leftarrow$ is clear. Below, we show $\Rightarrow$. Let $\mathcal{P} = \{S_1, \ldots, S_{|\mathcal{P}|}\}$. For every $S \in \mathcal{P}$, let σ^S satisfy $\sigma^S(i) = i$ for every $i \in N\setminus S$, *i.e.*, σ^S denotes a permutation only for the members in S. Let $x \in C^{\text{proj}}(v, \mathcal{P})$. From the definition of C^{proj}, it follows that $\sum_{j\in S} x_j \geq v(S, \mathcal{P}^S)$ for any $S \subseteq N$, and $\sum_{j\in S} x_j = v(S, \mathcal{P})$ for any $S \in \mathcal{P}$. For any σ, we define $\sigma(x)_i := x_{\sigma(i)}$ for any $i \in N$. Then, for any $S \subseteq N$, we have $\sum_{j\in S} \sigma^{S_1}(x)_j = \sum_{j\in S} x_{\sigma^{S_1}(j)} = \sum_{j\in\sigma^{S_1}(S)} x_j \geq v(\sigma^{S_1}(S), \mathcal{P}^{\sigma^{S_1}(S)})$, by Lemma 3.1, which equals $v(\sigma^{S_1}(S), \sigma^{S_1}(\mathcal{P}^S)) = v(S, \mathcal{P}^S)$. Similarly, it follows from Lemma 3.1, and $v \in \mathcal{G}_N^S$ that for every $S \in \mathcal{P}$, $\sum_{j\in S} \sigma^{S_1}(x)_j = v(S, \mathcal{P})$. Hence, $\sigma^{S_1}(x) \in C^{\text{proj}}(v, \mathcal{P})$. This holds for every permutation σ^{S_1}. Note that there are $|S_1|!$ permutations that arrange the members in S_1. We denote the set of the $|S_1|!$ permutations by A^{S_1}. We define $y := \frac{1}{|S_1|!}\sum_{\sigma^{S_1}\in A^{S_1}} \sigma^{S_1}(x)$. For any player $i \in S_1$, we have $y_i = \frac{1}{|S_1|!}\sum_{\sigma^{S_1}\in A^{S_1}} \sigma^{S_1}(x)_i = \frac{1}{|S_1|!}(|S_1| - 1)!\left(\sum_{j\in S_1} x_j\right)$. This is equivalent to $\frac{1}{|S_1|!}(|S_1| - 1)! v(S_1, \mathcal{P})$, and we obtain $\frac{v(S_1,\mathcal{P})}{|S_1|}$. Hence, allocation y is given as follows: $y_i := \frac{v(S_1,\mathcal{P})}{|S_1|}$ for every $i \in S_1$, $y_i := x_i$ for every $i \in N\setminus S_1$; and, in view of the convexity of the core, belongs to $C^{\text{proj}}(v, \mathcal{P})$, namely, with slightly abusing the notation, $y = (e^{S_1}, x^{S_2}, \ldots, x^{S_{|\mathcal{P}|}}) \in C^{\text{proj}}(v, \mathcal{P})$. We repeat this process for each $S_2, \ldots, S_{|\mathcal{P}|}$ and obtain $e(v, \mathcal{P}) \in C^{\text{proj}}(v, \mathcal{P})$. □

Proof of Proposition 3.3 We show that $e(v, \mathcal{P}) \notin C^{\text{proj}}(v, \mathcal{P})$. Let S and S' in $\mathcal{P}$ satisfy $|S| > |S'|$ and $\frac{v(S,\mathcal{P})}{|S|} > \frac{v(S',\mathcal{P})}{|S'|}$. There exists a coalition $T \subseteq N$ such that $|T| = |S|$ and $T = S' \cup R$ for some $\emptyset \neq R \subsetneq S$. We have $|S \backslash T| = |S'|$. Since $v \in \mathcal{G}_N^S$, we have $v(T, \{T\} \cup \mathcal{P}_{N \backslash T}) = v(T, \{T, S \backslash T\} \cup \mathcal{P}_{N \backslash (T \cup S)}\}) = v(S, \{S, S'\} \cup \mathcal{P}_{N \backslash (S \cup S')}) = v(S, \mathcal{P})$. Hence, we obtain

$$\begin{aligned}\sum_{j \in T} e_j(v, \mathcal{P}) &= |R| \frac{v(S, \mathcal{P})}{|S|} + |S'| \frac{v(S', \mathcal{P})}{|S'|} < |R| \frac{v(S, \mathcal{P})}{|S|} + |S'| \frac{v(S, \mathcal{P})}{|S|} \\ &= |T| \frac{v(S, \mathcal{P})}{|S|} \\ &= |T| \frac{v(T, \{T\} \cup \mathcal{P}_{N \backslash T})}{|T|} = v(T, \{T\} \cup \mathcal{P}_{N \backslash T}) = v(T, \mathcal{P}^T).\end{aligned}$$

Thus, $e(v, \mathcal{P}) \notin C^{\text{proj}}(v, \mathcal{P})$. □

Proof of Proposition 3.5 We first show that $C^{\text{proj}}(v, \mathcal{P}) = \emptyset$ for any $\mathcal{P} \in \Pi(N) \backslash \{N\}$. In view of Lemma 3.4, consider $\mathcal{P} \neq \{N\}$ satisfying $k^* := |S'|$ for every $S' \in \mathcal{P}$. We have $e_i(v, \mathcal{P}) = I(k^*) + (|\mathcal{P}| - 1)E(k^*)$ for every $i \in N$. Consider $S \subseteq N$ with $|S| = k^* + 1$. We have

$$\begin{aligned}&v(S, \mathcal{P}^S\}) - \sum_{j \in S} e_j(v, \mathcal{P}) \\ &= s[I(k^* + 1) + E(k^* - 1) + (|\mathcal{P}| - 2)E(k^*)] - s[I(k^*) + (|\mathcal{P}| - 1)E(k^*)] \\ &= s[I(k^* + 1) - I(k^*) - (E(k^*) - E(k^* - 1))] = s[\Delta^I(k^* + 1) - \Delta^E(k^*)] \\ &> s[\Delta^I(k^* + 1) - \Delta^I(k^*)] \geq 0.\end{aligned}$$

Hence, $e(v, \mathcal{P}) \notin C^{\text{proj}}(v, \mathcal{P})$. From Proposition 3.2, $C^{\text{proj}}(v, \mathcal{P}) = \emptyset$ follows.

Now, we show that $C^{\text{proj}}(v, \{N\}) \neq \emptyset$. For any $i \in N$, $e_i(v, \{N\}) = I(n)$. For any $S \subseteq N$, $v(S, \{S, N \backslash S\}) = s[I(s) + E(n - s)]$. Hence, for any $S \subseteq N$, we have

$$\begin{aligned}&\sum_{j \in S} e_j(v, \mathcal{P}) - v(S, \{S, N \backslash S\}) = s[I(n) - I(s) - E(n - s)] \\ &= s\left[\sum_{k=s+1}^{n} \Delta^I(k) - \sum_{k=1}^{n-s} \Delta^E(k)\right] \geq s\left[\sum_{k=1}^{n-s} \Delta^I(k) - \sum_{k=1}^{n-s} \Delta^E(k)\right] > 0,\end{aligned}$$

which implies $C^{\text{proj}}(v, \{N\}) \neq \emptyset$. □

Proof of Proposition 4.3 If-part: For any coalition S with $|S| =: s$, we have

$$v(S, \{S, N \backslash S\}) = \begin{cases} \max_{p \leq c(n-s)} q(p)(p - c(s)) & \text{for } h \leq s \leq n - 1, \\ 0 & \text{for } 1 \leq s \leq h - 1. \end{cases} \tag{1}$$

Hence, for any $S \subseteq N$ with $h \leq s \leq n-1$, we obtain $\sum_{j \in S} e_j(v, N) = s\frac{\pi^{\mathrm{m}}}{n} \geq d^{c,q}(s) \overset{(1)}{=} v(S, \{S, N \backslash S\})$. For any $S \subseteq N$ with $1 \leq s \leq h-1$, we have $\sum_{j \in S} e_j(v, N) = s\frac{\pi^{\mathrm{m}}}{n} \geq 0 \overset{(1)}{=} v(S, \{S, N \backslash S\})$. Thus, $e(v, N) \in C^{\mathrm{proj}}(v, \{N\})$.

Only-if-part: From Proposition 3.2, it follows that $e(v, N) \in C^{\mathrm{proj}}(v, \{N\})$. We have $v(S, \{S, N \backslash S\}) \leq |S|\frac{\pi^{\mathrm{m}}}{n}$ for every $S \subseteq N$. Hence, in view of (1), for every $s = h, \ldots, n-1$, $d^{c,q}(s) \leq s\frac{\pi^{\mathrm{m}}}{n}$. □

Proof of Proposition 5.1 We fix a PCP game f and a partition $\mathcal{P} \in \Pi(N)$. For convenience, we offer the definition of $g^{f,\mathcal{P}}$ again:

$$g^{f,\mathcal{P}}(k) = \begin{cases} (|\mathcal{P}| - k + 1) f(|\mathcal{P}|) & k \leq |\mathcal{P}|, \\ \frac{1}{|S_{\max}(\mathcal{P})|} f(|\mathcal{P}|) & k > |\mathcal{P}|. \end{cases} \tag{2}$$

Proof of $\Leftarrow$: Assume that $C^{\mathrm{proj}}(v, \mathcal{P}) = \emptyset$. Since $f(|\mathcal{P}|) \geq 0$, $X_+(v, \mathcal{P}) := \{x \in X(v, \mathcal{P}) | x_j \geq 0 \text{ for any } j \in N\}$ is not empty. Let $x \in X_+(v, \mathcal{P})$. As the projective core for $\mathcal{P}$ is empty, there exists a coalition $S \subseteq N$ such that

$$\sum_{j \in S} x_j < v(S, \mathcal{P}^S) = f(k), \tag{3}$$

where $k = |\mathcal{P}^S|$. For the coalition S, define $\mathcal{T}^S = \{T \in \mathcal{P} |\ T \in \mathcal{P}_S\}$. Note that $|\mathcal{T}^S| = |\mathcal{P}| - k + 1$. If $|\mathcal{T}^S| \geq 1$, then $k \leq |\mathcal{P}|$. We have $\sum_{j \in S} x_j = |\mathcal{T}^S| \cdot f(|\mathcal{P}|) + \sum_{j \in S \backslash (\cup_{T \in \mathcal{T}^S} T)} x_j \geq |\mathcal{T}^S| \cdot f(|\mathcal{P}|) = (|\mathcal{P}| - k + 1) f(|\mathcal{P}|)$. Hence, from (3), it follows that $f(k) > (|\mathcal{P}| - k + 1) f(|\mathcal{P}|)$. However, in view of (2), for any $k \leq |\mathcal{P}|$, $f(k) \leq g^{f,\mathcal{P}}(k)$ implies that $f(k) \leq (|\mathcal{P}| - k + 1) f(|\mathcal{P}|)$. This is a contradiction.

Next, if $|\mathcal{T}^S| = 0$, then $k = |\mathcal{P}| + 1$. Since for any $x \in X_+(v, \mathcal{P})$ there exists $S \subseteq N$ satisfying (3), such a coalition S also exists for the equal division $e(f, \mathcal{P}) \in X_+(v, \mathcal{P})$, *i.e.*, $e_j(f, \mathcal{P}) = \frac{f(|\mathcal{P}|)}{|\mathcal{P}(j)|}$ for every $j \in N$. Note that $e_j(f, \mathcal{P}) \geq 0$ for every $j \in N$ as $f(|\mathcal{P}|) \geq 0$. Hence, there exists a player $i \in S$ such that $e_i(f, \mathcal{P}) \leq \sum_{j \in S} e_j(f, \mathcal{P}) < f(k)$ by (3). Moreover, $e_i(f, \mathcal{P}) = \frac{f(|\mathcal{P}|)}{|\mathcal{P}(i)|} \geq \frac{f(|\mathcal{P}|)}{|S_{\max}(\mathcal{P})|}$. Hence, we have $\frac{f(|\mathcal{P}|)}{|S_{\max}(\mathcal{P})|} < f(k)$. However, in view of (2), for any $k > |\mathcal{P}|$, $f(k) \leq g^{f,\mathcal{P}}(k)$ implies that $f(k) \leq \frac{f(|\mathcal{P}|)}{|S_{\max}(\mathcal{P})|}$. This is a contradiction.

Proof of $\Rightarrow$: We show that if there exists $k \in \{1, \ldots, |\mathcal{P}| + 1\}$ such that $f(k) > g^{f,\mathcal{P}}(k)$, then $C^{\mathrm{proj}}(v, \mathcal{P}) = \emptyset$. Assume $x \in C^{\mathrm{proj}}(v, \mathcal{P})$. If $k \leq |\mathcal{P}|$, we have $f(k) > (|\mathcal{P}| - k + 1) f(|\mathcal{P}|) = \sum_{a=1}^{|\mathcal{P}|-k+1} \sum_{j \in S_a} x_j$, where $S_1, \ldots, S_{|\mathcal{P}|-k+1}$ are arbitrary $|\mathcal{P}| - k + 1$ coalitions in $\mathcal{P}$. Hence, $|\mathcal{P}| - k + 1$ coalitions in $\mathcal{P}$ have an incentive to jointly deviate by merging and obtain $f(k)$ in total after the deviation. If $k > |\mathcal{P}|$, then (2) implies $f(k) > \frac{1}{|S_{\max}(\mathcal{P})|} f(|\mathcal{P}|)$. Moreover, there exists $i \in S_{\max}(\mathcal{P})$ such that $x_i < f(k)$, because otherwise for any $j \in S_{\max}(\mathcal{P})$ we have $x_j \geq f(k)$, which implies $\sum_{j \in S_{\max}(\mathcal{P})} x_j \geq |S_{\max}(\mathcal{P})| f(k)$ and $\sum_{j \in S_{\max}(\mathcal{P})} x_j = f(|\mathcal{P}|)$: a contradiction. Hence, there exists $i \in S_{\max}(\mathcal{P})$ such that $x_i < f(k)$, and the player i has an incentive to deviate. □

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