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Generalized Coleman-Shapley indices and total-power monotonicity

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Abstract

We introduce a new axiom for power indices, which requires the total (additively aggregated) power of the voters to be nondecreasing in response to an expansion of the set of winning coalitions; the total power is thereby reflecting an increase in the collective power that such an expansion creates. It is shown that total-power monotonic indices that satisfy the standard semivalue axioms are probabilistic mixtures of generalized Coleman-Shapley indices, where the latter concept extends, and is inspired by, the notion introduced in Casajus and Huettner (Public choice, forthcoming, 2019). Generalized Coleman-Shapley indices are based on a version of the random-order pivotality that is behind the Shapley-Shubik index, combined with an assumption of random participation by players.

Keywords Simple games · Voting power · Shapley-Shubik index · Banzhaf index · Coleman-Shapley index · Semivalues · Power of collectivity to act · Total-power monotonicity axiom · Probabilistic mixtures

JEL classification numbers C71 · D72

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1 Introduction

The Shapley-Shubik power index [1] (henceforth, SSPI) and the Banzhaf power index [2] (henceforth, BPI) enjoy a near-universal recognition as valid measures of a priori voting power. The two indices quantify the power held by individual voters under a given decision rule by assigning each individual the *probability of being pivotal* in a certain mode of random voting. The SSPI views voters as "aligned in order of their enthusiasm for the proposal" over which the vote is held, with all orders being possible and equally likely a priori; an individual is pivotal in an order if "by joining his more enthusiastic colleagues, [he] brings [that] coalition up to winning strength."[3] In the BPI, the pivotal status of an individual is defined as his ability to affect the outcome of the vote in the random set of yes-voters, assuming that each individual votes "yes" with probability $\frac{1}{2}$, independently of anyone else. Thus, the assumption behind the SSPI is that all individuals ultimately vote "yes," in the order of their enthusiasm; [4] the pivotality of a voter then arms him with some bargaining advantage in demanding adjustments in the content of the proposal.[5] The BPI, on the other hand, views pivotality as being in a position to single-handedly push the proposal through.

Recently, Casajus and Huettner (2019) suggested a new power index, which they named the Coleman-Shapley index (henceforh, CSPI), with an underlying probability model that naturally combines the assumptions behind the SSPI and BPI. In this paper we will consider a family of indices that generalize the CSPI and are obtained by varying a single parameter in Casajus and Huettner's (2019) definition. For a given $q \in (0, 1]$, the *q -Coleman-Shapley power index* (or q-CSPI, for short) identifies the power of an individual voter with his probability of being pivotal in the following hybrid situation. Assume that each voter turns up for vote with probability q (in which case we shall call him q-active, or simply *active*), independently of others. In other words, we start from a Banzhaf-like scenario, except that an individual decision to be active need not necessarily amount to voting "yes," and the probability q that an individual is interested (or capable) to vote may be different from $\frac{1}{2}$. Next, as in the Shapley-Shubik scenario, think of the active voters as declaring their support for the proposal in the order of their enthusiasm. The pivotality of a voter is now defined with respect to a random (enthusiasm-based) order of all *active* voters; the q-CSPI of a voter is defined as the probability that he is pivotal, conditional on being active.[6]

1 Defined in Shapley and Shubik (1954).

2 As is often done in the literature, we use the term "Banzhaf power index" for brevity, although the origin of this power index lies in multiple works (Penrose (1946), Banzhaf (1965, 1966, 1968), Coleman (1971)). The specific variant of the BPI used in this work is referred to as the "Banzhaf measure" in Felsenthal and Machover (1998).

3 All quotations in this sentence are taken from Dubey and Shapley (1979, p. 103).

4 This may be weakened by noting that voting "yes" is not expected from players less enthusiastic than the pivot. Indeed, the proposal will be collectively approved (by virtue of the votes of the pivot and his predecessors in the order) *before* the less enthusiastic players will be asked to join in support.

5 This view of the SSPI may also be (more formally) supported by the fact that the underlying Shapley value arises as an equilibrium outcome of certain natural bargaining procedures (see, e.g., Hart and Mas-Colell (1996)).

6 The notion of pivotality here still implies the ability to affect the content of a proposal, but the passage of the proposal is now uncertain because the set of q-active voters may be losing.

The CSPI of Casajus and Huettner (2019) is a member of our family of generalized CSPIs, corresponding to $q = \frac{1}{2}$. This family also contains the SSPI, which corresponds to $q = 1$. The lower boundary of the parameter range, $q = 0$, will also be admitted; with an underlying assumption is that no one supports the proposal, the power of a voter in this case is equal to the winning status of his stand-alone coalition. As will be made clear in what follows, all q-CSPIs, for $q \in [0, 1]$, share a very intuitive and important property, and are, essentially, characterized by it.

All aforementioned power indices, in their narrow interpretation, measure the voting power of each *individual* voter. However, in practice and in theory, the individual power is often additively aggregated across individuals in order to compute the implied power of *sets* of voters. There is somewhat less clarity as to what an aggregation of power over a set represents, compared to the rather straightforward concept of individual power that is behind the SSPI and BPI, but such an aggregation is taken quite seriously. A need for comparison of power of different sets arises on various occasions [see, e.g., Brams (2013, Chapter 5)], and an axiomatic treatment of power indices often contains references to the *total*, or *combined*, power of voters. Indeed, following the discovery of the 2-efficiency of the BPI by Lehrer (1988), whereby the additively combined power of any two voters remains unchanged if the two voters "merge" and act as a single bloc, multiple axiomatizations of the BPI were offered based on relaxed versions of that property.[7] Axioms based on assumptions on the total, additively aggregated, power of the entire voter set (henceforth referred to simply as *total power*) are also common. In the earliest axiomatization of a power index, that of the SSPI by Dubey (1975), the efficiency axiom was imposed, whereby the total power is independent of the particular decision rule, and is equal to 1. Dubey and Shapley (1979), who axiomatized the BPI, assumed the total power to be equal to the expected number of swing voters, or "swingers," [8] in the voter set; this number is also known as the "sensitivity of the decision rule" (see Felsenthal and Machover (1998), Sect. 3.3). The CSPI of Casajus and Huettner (2019) is also axiomatized with a central property being the equality of the total power to (a multiple of) another well-recognized concept, the Coleman's (1971) "power of a collectivity to act," defined as the proportion of winning sets among all (sub)sets of voters.

The behavior of the total power as a function of the decision rule is quite distinct under BPI compared to the other indices that have been mentioned. The total power according to BPI, being the "sensitivity of a decision rule," quantifies "the ease with which [the decision rule] responds to voters' wishes."[9] It is therefore not surprising that the total BPI power favors rules where the outcome of the vote appears, a priori, to be very uncertain, as this is when individual voters have the best chance to be pivotal. Indeed, as shown in Dubey and Shapley (1979), the total BPI power is maximal for the simple majority rule, as that rule creates the greatest instability in the outcome of the vote under the assumption that votes are cast completely at random and independently across individuals.

7 See, e.g., Lehrer (1988), Nowak (1997), Casajus (2012), Haimanko (2018).

8 See Dubey et al. (1979, p. 103). A swinger is defined w.r.t. a random set of yes-voters (with the uniform distribution over all subsets of the voting body) by the requirement that the change in his vote affects the the voting outcome.

9 See Felsenthal and Machover (1998, p. 52).

The total power behaves in a notably different fashion under the other indices. The total SSPI power is fixed at 1, and hence the simple majority rule has the same standing as the rest. Under the CSPI, the total power (identifiable with the aforementioned Coleman power of collectivity to act) is at the intermediate level for the simple majority rule, *falling* with an increase in the majority quota. Indeed, Coleman's measure of the power of collectivity is concerned with the ease of a *collective* achievement, with higher quotas implying a lower number of winning sets, and, accordingly, lower collective power.

The above monotonicity feature of the total CSPI power obviously extends from the simple majority to general, not necessarily symmetric and quota-based, decision rules: the smaller is the set of winning sets (as in the particular case of a rising majority quota), the lower is the total power. This feature is, moreover, common to all generalized CSPIs. Indeed, it will be shown in this work that, for any $q \in (0, 1]$, the total q-CSPI power is a ($\frac{1}{q}$-scaled) modified version of the Coleman power of collectivity to act, defined as the probability that the set of all q -active voters is winning. Obviously, such a probability responds monotonically to an addition of winning sets.[10]

We will call the property whereby the total power is nondecreasing when winning sets are added *total-power monotonicity* of a power index, or TP-monotonicity. As has been claimed above, all generalized CSPIs are TP-monotonic. TP-monotonicity appears to be quite desirable if one wishes the total power to measure, or at least be highly correlated with, some form of *collective* power held by the voters. Indeed, as the set of winning coalitions expands, passing any proposal that is put to vote becomes easier, [11] indicating an increase in the collective power. [12] Thus, if the total power is to be regarded as a numerical proxy for the collective power of all voters, the total power should respond positively to an expansion of the set of winning coalitions. Accordingly, TP-monotonicity may be viewed as a necessary condition for a power index to satisfy if there is an intention to use additive aggregation of the individual power measured by that index in estimating the collective power of sets of voters.

This work will explore the implications of TP-monotonicity on the structure of power indices. Following the approach pioneered in Shapley and Shubik (1954) and adopted in much of the literature on power indices, we will model voting situations/decision rules as cooperative games known as *simple* (or *voting*) *games*, and view a power index as a map defined on the domain of simple games. The focus will be on power indices that are *semivalues*, a term that was borrowed by Einy (1987) from the realm of value maps considered by Dubey et al. (1981), and applied to power indices that satisfy four axioms that are quite standard and figure prominently in the

[10] Under the degenerate 0-CSPI, the total power aslo responds in a (weakly) monotonic fashion to an addition of winning sets.

[11] This is because no coalition of yes-voters can turn from winning to losing, and at least one such coalition turns from losing to winning, under such a change in the decision rule.

[12] It may be argued that if the winning coalitions become too numerous, then in some contexts (such as under symmetric majority rules with quotas below $\frac{1}{2}$) the collective power could suffer because any proposal that may be easy to pass with just a minority approval, can be subsequently overturned by a counter-proposal supported by an opposing minority. However, we take the view that the measurement of power concerns a *single decision*, namely, passage of a single (anticipated but a priori unknown) proposal. Under a scope restricted to the proposal at hand, it is natural to regard the power of collectivity as commensurate with the ease of passing the proposal.

literature on axiomatizations. These axioms are: *transfer* (or *valuation*), which has been a routine substitute for the additivity axiom for value maps in the context of simple games since its introduction in Dubey (1975); *non-negativity* of the power index; *anonymity*, which requires covariance under permutations of the player (voter) set; and *dummy*, whereby the power of a dummy player (which can only be a null player, or a dictator, in a simple game) equals to the payoff of his stand-alone coalition.

By standard arguments, all generalized CSPIs are semivalues. They are not the only TP-monotonic semivalues, but we will show, via a somewhat indirect approach, that they generate all such semivalues. Our main tool will be Einy's (1987) characterization of semivalues of simple games as probabilistic mixtures of x-values. For $x \in [0, 1]$, the x -value is a power index (in fact, a semivalue itself) that assigns each voter i in a simple game v the probability that he is pivotal[13] for a random coalition of other players, joined by each player with probability x independently of the rest. Einy's result states that any semivalue is obtained by integrating over x-values w.r.t. a uniquely determined probability distribution ξ. Our first result, Theorem 1, studies the effect of imposing the TP-monotonicity assumption on a semivalue in terms of the implied conditions on the representing distribution ξ. It turns out that a semivalue is TP-monotonic if and only if the c.d.f. of the distribution ξ is a concave function.

The structural implication of TP-monotonicity in Theorem 1 appears rather technical from first glance. However, it contains a much more explicit message, initially hidden from view. Our Theorem 2 uses the concavity of the c.d.f. of the representing distribution of a TP-monotonic semivalue to show that the latter is a probabilistic mixture of generalized CSPIs. This characterization of the TP-monotonic semivalues is a complete one: a semivalue is TP-monotonic if and only if it is obtained by integrating q -CSPIs over $q \in [0, 1]$ w.r.t. a uniquely defined probability measure on $[0, 1]$. In particular, any TP-monotonic semivalue that is not a convex combination of generalized CSPIs can be approximated by such combinations, since integrals in our characterization are approximable by weighted averages.

The paper is organized as follows. Section 2 recalls the basic definitions pertaining to games and power indices, lists the semivalue axioms, and calls attention to the known characterization of semivalues as mixtures of x -values. Generalized CSPIs are defined in Sect. 3, and are shown to be attainable by the random-arrival and random-order approaches. Section 3 also introduces the axiom of TP-monotonicity, and checks that it is satisfied by any q-CSPI. Section 4 contains our main results: Theorem 1, which characterizes TP-monotonic semivalues in terms of their underlying probability distribution, and Theorem 2, which represents TP-monotonic semivalues as mixtures of generalized CSPIs. Two extended remarks, on the extendibility of our results to value maps for general finite games, and the relation between general semivalues and generalized CSPIs via the notion of decomposition, appear at the end of Sect. 3. Some parts of our proofs that rely on later results appear in the Appendix.

[13] In the context of a simple game, a pivot for a coalition is a player whose presence switches that coalition from losing to winning.

2 Preliminaries

2.1 Finite games, Simple games, and Power indices

Let U be an infinite universe of *players* (or *voters*), and assume, w.l.o.g., that U includes the set $\mathbb{N}$ of positive integers. Denote the collection of all *coalitions* (subsets of U) by 2^U, and the empty coalition by $\emptyset$. A *game* on U is given by a map $v : 2^U \to \mathbb{R}$ with $v(\emptyset) = 0$. A coalition $N \subset U$ is called a *carrier* of v if $v(S) = v(S \cap N)$ for any $S \in 2^U$. We say that v is a *finite game* if it has a finite carrier; the minimal carrier of such v is, in effect, its true player set. The space of all finite games on U is denoted by $\mathcal{G}$. The domain $\mathcal{SG}$ of *simple* (or *voting*) *games* on U consists of all $v \in \mathcal{G}$ such that: (i) $v(S) \in \{0, 1\}$ for all $S \in 2^U$; (ii) $v(U) = 1$; and (iii) v is *monotonic*, i.e., if $S \subset T$ then $v(S) \leq v(T)$.[14] If $v \in \mathcal{SG}$, a coalition S is *winning* if $v(S) = 1$, and *losing* otherwise. Thus, as in Shapley and Shubik (1954), any $v \in \mathcal{SG}$ describes a voting system or a decision rule, with a full account of all possible coalitions of yes-voters that can win the vote.

The space $\mathcal{AG}$ of *additive games* consists of all $v \in \mathcal{G}$ satisfying $v(S \cup T) = v(S) + v(T)$ whenever $S \cap T = \emptyset$. Any $w \in \mathcal{AG}$ with a finite carrier N is identifiable with the vector[15] $\{w(i) \mid i \in N\}$, and thus may be thought of as a payoff vector to the players in N.

A *power index* φ is a map $\varphi : \mathcal{SG} \to \mathcal{AG}$, where $\varphi(v)(i)$ is interpreted as the voting power of player i in a simple game v. We will refer to $\varphi(v)(U)$ as the *total power* of players; since $\varphi(v) \in \mathcal{AG}$ is additive and has a finite carrier, the total power satisfies the equality $\varphi(v)(U) = \sum_{i \in N} \varphi(v)(i)$ for any finite $N \subset U$ that is a joint carrier of v and $\varphi(v)$.

2.2 Semivalues axioms

The following four axioms—plausible requirements that a general power index φ may be expected to obey—are quite routinely assumed in analyzing and designing power indices, either in their entirety or in part. As in Einy (1987), who was the first to look at the conjunction of these four axioms, we will use the term *semivalue*[16] in reference to any power index φ that satisfies all the axioms.[17]

Axiom I: Transfer.[18] For any $v, w \in \mathcal{SG}$, $\varphi(\max\{v, w\}) + \varphi(\min\{v, w\}) = \varphi(v) + \varphi(w)$.[19]

[14] This definition of simple games follows the convention set forth in Dubey and Shapley (1979), and used in much of the subsequent research.

[15] We shall henceforth omit braces when indicating one-player sets.

[16] The term "semivalue" was originally coined in Dubey et al. (1981) in the context of value maps on $\mathcal{G}$ (see Remark 1).

[17] Variants of semivalue axioms have been present in the original axiomatizations of the SSPI and BPI (see Dubey (1975) and Dubey and Shapley (1979)).

[18] The term **Transfer** is due to Weber (1988).

[19] max, min in the statement of **Transfer** refer to the maximum/minimum of functions on 2^U, and hence both $\max\{v, w\}$ and $\min\{v, w\}$ are well-defined games in $\mathcal{SG}$.

As was shown in Dubey et al. (2005, p. 24), **Transfer** can be restated in an equivalent but conceptually clearer form, amounting to a requirement that the change in power depends only on the change in the voting game.[20]

Axiom II: Anonymity. For any $v \in \mathcal{SG}$, $i \in U$, and a permutation π of U, $\varphi(\pi v)(i) = \varphi(v)(\pi(i))$, where $\pi v \in \mathcal{SG}$ is given by $(\pi v)(S) = v(\pi(S))$ for all $S \in 2^U$.

According to **Anonymity**, if players are relabeled in a game, their power indices will be relabeled accordingly. Thus, irrelevant characteristics of the players, outside of their role in the game v, have no influence on the power index.

Axiom III: Non-negativity. For any $v \in \mathcal{SG}$ and $i \in U$, $\varphi(v)(i) \geq 0$.

Non-negativity is natural because every $v \in \mathcal{SG}$ is monotonic by assumption, and hence no player that joins a coalition can affect its winning status negatively.

Axiom IV: Dummy. If $v \in \mathcal{SG}$ and i is a dummy player in v, i.e. $v(S \cup i) = v(S) + v(i)$ for every $S \subset U \setminus i$, then $\varphi(v)(i) = v(i)$.

A dummy player in a simple game can be either a dictator (if $v(i) = 1$), in which case $\{i\}$ is the minimal carrier of v, or a null player (if $v(i) = 0$), that does not belong to the minimal carrier of v. **Dummy** can be viewed as a normalization requirement, assigning power 1 to a dictator and power 0 to a null player.

2.3 Characterization of semivalues

Dubey et al. (1981) defined a family of semivalues[21] $(\phi_\xi)_\xi$, parameterized by $\xi \in M([0,1]) \equiv$the set of probability measures on [0, 1], as follows: given $\xi \in M([0,1])$, for every $v \in \mathcal{SG}$ with some finite carrier N,

$$\phi_\xi(v)(i) = \sum_{S \subset N \setminus i} p_{|S|}^{|N|}(\xi)\,[v(S \cup i) - v(S)] \tag{1}$$

if $i \in N$, where

$$p_s^n(\xi) = \int_0^1 x^s (1-x)^{n-s-1}\, d\xi(x); \tag{2}$$

and $\phi_\xi(v)(i) = 0$ if $i \in U \setminus N$. The definition is independent of the choice of a carrier N.

Einy (1987) showed that the set of semivalues on $\mathcal{SG}$ *coincides* with the family $(\phi_\xi)_{\xi \in M([0,1])}$:

Proposition 1 (Einy (1988)). *A power index φ is a semivalue if and only if $\varphi = \phi_\xi$ for some $\xi \in M([0,1])$, with ξ uniquely determined by φ.*

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[20] Specifically, if $v, w, v', w' \in \mathcal{SG}$ are such that $v \geq v'$, $w \geq w'$ and $v - v' = w - w'$, then $\varphi(v) - \varphi(v') = \varphi(w) - \varphi(w')$.

[21] Dubey et al. (1981) considered semivalues on $\mathcal{G}$ and not on $\mathcal{SG}$ (for further discussion, see Remark 1). The family of power indices with the forthcoming description is obtained by restricting those semivalues to games in $\mathcal{SG}$.

Relying on the equivalence in Proposition 1, the term *semivalue* will henceforth be used in reference to some member of the family $(\phi_\xi)_{\xi \in M([0,1])}$. Each semivalue ϕ_ξ has a simple probabilistic interpretation. Assume that player i believes that players other than himself have the same probability x of voting "yes" (thereby joining the coalition of yes-voters), and that they do so independently of each other; however, i may be uncertain about the parameter x, with his prior belief being the distribution ξ over x. Then $\phi_\xi(v)(i)$ represents i's a priori likelihood to switch a random coalition of yes-voters from losing to winning by joining it.

If the parameter x is known, one may refer to the corresponding semivalue, for which ξ is the Dirac measure concentrated on x, as x *-value,* which will be denoted ϕ_x for simplicity. A general ϕ_ξ is then a probabilistic mixture of x-values: the definition of ϕ_ξ implies that, for any $v \in \mathcal{SG}$ and $i \in U$,

$$\phi_\xi(v)(i) = \int_0^1 \phi_x(v)(i) d\xi(x). \tag{3}$$

The family $(\phi_\xi)_{\xi \in M([0,1])}$ includes the two best-known and widely used semivalues: the Banzhaf power index (BPI) $\phi_{\frac{1}{2}}$, corresponding to ξ that is the Dirac measure concentrated on $\frac{1}{2}$, and the Shapley-Shubik power index (SSPI), corresponding to the uniform distribution on $[0, 1]$. The Coleman-Shapley power index (CSPI), introduced in Casajus and Huettner (2019), is precisely ϕ_ξ for ξ that corresponds of the uniform distribution on $\left[0, \frac{1}{2}\right]$. Its probabilistic interpretation will be discussed in the next section, in a unifying set-up that will single out a subfamily of semivalues in $(\phi_\xi)_{\xi \in M([0,1])}$.

3 Generalized CSPIs and total-power monotonicity

The definition of the CSPI in Casajus and Huettner (2019) allows to conjure up a more general framework, in which the SSPI and CSPI are included as particular cases. We will define *generalized CSPIs* as a one-pramateric family of semivalues, and will then show how these indices arise in two related models of random voting.

3.1 Generalized CSPIs as semivalues

For any $0 \leq q \leq 1$, consider the probability measure $\xi_q \in M([0, 1])$ that is concentrated on the interval $[0, q]$ and, when $q > 0$, corresponds to the uniform distribution on $[0, q]$, i.e.,

$$d\xi_q(x) = \frac{1}{q} I_{x \leq q} dx, \tag{4}$$

where I_A denotes the indicator function of a set A. Denote $\varphi_q = \phi_{\xi_q}$, and call it q*-Coleman-Shapley power index*, or q*-CSPI* for short.

3.2 Random-arrival interpretation of q-CSPIs

When $q > 0$, the definition of the q-CSPI by means of (1), (2) and (4) lends itself to the following probabilistic interpretation, which is a version of the "random arrival times" view that has usually been reserved for the Shapley value and the weighted Shapley value (starting with Owen (1968)). Let $v \in \mathcal{SG}$ be a game with some finite carrier N, and let $\{X_i\}_{i \in N}$ be i.i.d. random variables with the uniform distribution on [0, 1]. Think of X_i as measuring the *dissatisfaction* of player i with a proposal that stands for vote; the given parameter q represents the cut-off value of dissatisfaction above which a player will never vote in favor of a proposal. Players whose dissatisfaction falls below or is equal to q will, on the other hand, ultimately vote "yes", but their turn to join the support of the proposal depends on their measure of dissatisfaction: the higher is X_i, the later will i join the other yes-voters. It stands to reason that, in such a scenario, the influence of player i over the vote should be quantified as the probability (conditional on i being a yes-voter, having $X_i \leq q$) that the coalition of the proposal supporters switches from losing to winning precisely when i's turn arrives and he declares his support for the proposal.

The measure of voting power given by $\varphi_q(v)(i) = \phi_{\xi_q}(v)(i)$ does exactly that. Formally, (1), (2) and (4) mean that

$$\varphi_q(v)(i) = \sum_{S \subset N \setminus i} \left(\int_0^q x^{|S|} (1-x)^{|N|-|S|-1} \frac{1}{q} dx \right) [v(S \cup i) - v(S)],$$

for every $i \in N$ (and $\varphi_q(v)(i) = 0$ for every $i \in U \setminus N$), which can be readily seen to be a restatement in terms of integrals of the equality

$$\varphi_q(v)(i) = E\left[v(\{j \in N \mid X_j \leq X_i\}) - v(\{j \in N \mid X_j < X_i\}) \mid X_i \leq q\right], \quad (5)$$

where E stands for the expectation operator. The last equality is itself equivalent to

$$\varphi_q(v)(i) = \Pr\left[v(\{j \in N \mid X_j < X_i\}) = 0 \text{ and } v(\{j \in N \mid X_j \leq X_i\}) = 1 \mid X_i \leq q\right]. \quad (6)$$

3.3 Random-order interpretation of q-CSPIs

The following alternative description of a q-CSPI can be derived from (5). Given $q \in [0, 1]$ and $v \in \mathcal{SG}$ with a finite carrier N, consider a random coalition $\overline{S}_N^q \subset N$ that satisfies

$$\Pr(\overline{S}_N^q = S) = q^{|S|}(1-q)^{|N \setminus S|} \quad (7)$$

for every $S \subset N$. Put differently, each $i \in N$ belongs to $\overline{S}_N^q$ with probability q, independently of the other players in N. We can think of $\overline{S}_N^q$ as the coalition of players who are interested in, or capable of, voting for a proposal. Call the players in $\overline{S}_N^q$ (q-)*active*. Additionally, let $\mathcal{R}_N$ be a random linear order of players in N, chosen w.r.t. to the uniform distribution over all such orders, and assume that the choice of order is made independently of the realization of $\overline{S}_N^q$. $\mathcal{R}_N$ can be thought of as the ranking of players w.r.t. their eagerness to vote in favor of a proposal; note that $\mathcal{R}_N$ ranks all players, including those who might not be active. For any such $\mathcal{R}_N$ and $i \in N$, denote by $S_i(\mathcal{R}_N)$ the (random) coalition of players in N that precede i in $\mathcal{R}_N$ (according to our interpretation, $S_i(\mathcal{R}_N)$ consists of players who like the proposal more than i). Then, for $q > 0$, (5) is equivalent to

$$\varphi_q(v)(i) = E\left[v((S_i(\mathcal{R}_N) \cup i) \cap \overline{S}_N^q) - v(S_i(\mathcal{R}_N) \cap \overline{S}_N^q) \mid i \in \overline{S}_N^q\right], \tag{8}$$

or

$$\varphi_q(v)(i) = \Pr\left[v(S_i(\mathcal{R}_N) \cap \overline{S}_N^q) = 0 \text{ and } v((S_i(\mathcal{R}_N) \cup i) \cap \overline{S}_N^q) = 1 \mid i \in \overline{S}_N^q\right], \tag{9}$$

for every $i \in N$.

Similarly to the random-arrival approach, here $\varphi_q(v)(i)$ is expressed as the probability that i switches from losing to winning the coalition of q-active voters who are ranked below i (i.e., are stronger than i) in their support, conditional on that i is himself active. In order to see how (8) is obtained from (5), take $\mathcal{R}_N$ be the order induced by the relative positions of the players in $\{X_i\}_{i \in N}$ (which were defined in Sect. 3.2), and let $\overline{S}_N^q = \{i \in N \mid X_i \leq q\}$ be the random coalition of players whose dissatisfaction does not exceed q. Notice that even though such $\mathcal{R}_N$ is *not* independent of $\overline{S}_N^q$, the random coalition $S_i(\mathcal{R}_N) \cap \overline{S}_N^q$ is distributed *as if* $\mathcal{R}_N$ is independent of $\overline{S}_N^q$ when there is a conditioning on $i \in \overline{S}_N^q$.[22]

Observe that the 1-CSPI (φ_1) is just the SSPI, as (6) or (9) boil down to its usual definition as the (unconditional) probability of being pivotal in a random order. Also, when $q = \frac{1}{2}$, (9) is, in effect, the definition of the CSPI in Casajus and Huettner (2019), and hence the $\frac{1}{2}$-CSPI ($\varphi_{\frac{1}{2}}$) is precisely that index.

3.4 The total power under *q*-CSPI

The total power of players under a given q-CSPI can be computed directly, but we will find it as an upshot of a more general exercise. It turns out, as has been already observed by Casajus and Huettner (2019) in the case of $\varphi_{\frac{1}{2}}$, that for *any* $q \in [0, 1]$ the q -CSPI of $v \in \mathcal{SG}$ can be expressed as the Shapley (1953) value of an appropriately modified game $v_q \in \mathcal{G}$. Indeed, fix a finite carrier N for v. For any $0 < q \leq 1$ and $S \in 2^U$, define

[22] In (8), $\mathcal{R}_N$ can be replaced by $\mathcal{R}_{\overline{S}_N^q}$ (a random, uniformly distributed order of players in $\overline{S}_N^q$), i.e., it suffices to rank only the active players. Such an equation would have been the reduced form of both (5) and (8), consistent with our description of the q-CSPI in the Introduction. The current (8) is preferable, however, as it is used in the proof of our upcoming Proposition 2.

$$v_q(S) := \frac{1}{q} E\left[v(S \cap \overline{S}_N^q)\right], \tag{10}$$

where (recall) $\overline{S}_N^q$ is the random coalition of q-active players that satisfies (7).[23] Thus, $v_q(S)$ is the $\frac{1}{q}$-scaled probability that the coalition of q-active players in S is winning in the given $v \in \mathcal{SG}$. For $q = 0$,

$$v_0(S) := \sum_{i \in S} v(i), \tag{11}$$

which is consistent with (10) when $q > 0$ tends to 0. Also recall that, for any game $w \in \mathcal{G}$ with a finite carrier N, its Shapley value $Sh(w) \in \mathcal{AG}$ is defined as $Sh\,(w)\,(i) = E\,[w(S_i(\mathcal{R}_N) \cup i) - w(S_i(\mathcal{R}_N))]$ for every $i \in N$ (and $Sh\,(w)\,(i) = 0$ for every $i \in U \setminus N$).

Proposition 2 *For any* $q \in [0, 1]$, $v \in \mathcal{SG}$ *and* $i \in U$, $\varphi_q\,(v)\,(i) = Sh(v_q)(i)$.

Proof Let N be a finite carrier of v, and take $i \in N$. When $q > 0$, by using the independence of $S_i(\mathcal{R}_N)$ and $\overline{S}_N^q$, (8) can be transformed into

$$\begin{aligned}
\varphi_q\,(v)\,(i) &= E_{\mathcal{R}_N}\left(E_{\overline{S}_N^q}\left[v((S_i(\mathcal{R}_N) \cup i) \cap \overline{S}_N^q) - v(S_i(\mathcal{R}_N) \cap \overline{S}_N^q) \mid i \in \overline{S}_N^q\right]\right)\\
&= E_{\mathcal{R}_N}\left(\frac{1}{q} E_{\overline{S}_N^q}\left[\left(v((S_i(\mathcal{R}_N) \cup i) \cap \overline{S}_N^q) - v(S_i(\mathcal{R}_N) \cap \overline{S}_N^q)\right) \cdot I_{i \in \overline{S}_N^q}\right]\right)\\
&= E_{\mathcal{R}_N}\left(\frac{1}{q} E_{\overline{S}_N^q}\left[v((S_i(\mathcal{R}_N) \cup i) \cap \overline{S}_N^q) - v(S_i(\mathcal{R}_N) \cap \overline{S}_N^q)\right]\right)\\
&= E_{\mathcal{R}_N}(v_q(S_i(\mathcal{R}_N) \cup i) - v_q(S_i(\mathcal{R}_N)) = Sh(v_q)(i).
\end{aligned}$$

Also note that, for $q = 0$, trivially $\varphi_0\,(v)\,(i) = v(i) = Sh(v_0)(i)$. Finally, when $i \in U \setminus N$, $\varphi_q\,(v)\,(i) = 0$ and $Sh(v_q)(i) = 0$ by definition, for any $q \in [0, 1]$. □

Proposition 2 and the efficiency of the Shapley value imply that, for $0 < q \leq 1$ and $v \in \mathcal{SG}$ with a finite carrier N,

$$\varphi_q\,(v)\,(U) = v_q(N) = \frac{1}{q} E\left[v(\overline{S}_N^q)\right], \tag{12}$$

which is equivalent to

$$\varphi_q\,(v)\,(U) = \frac{1}{q} \Pr\left[v(\overline{S}_N^q) = 1\right]. \tag{13}$$

That is, the total power in the game v, as measured by φ_q, is a constant multiple $(\frac{1}{q})$ of the probability that the coalition of all q-active players is winning. For $q = 0$, clearly

$$\varphi_0\,(v)\,(U) = v_0(N) = \sum_{i \in N} v(i). \tag{14}$$

[23] It is easy to see that the definition of v_q is independent of the choice of a carrier N.

When $q = 1$, (12) is the usual efficiency property of the SSPI. When $q = \frac{1}{2}$, (13) is precisely the 2CPCA-efficiency of Casajus and Huettner (2019), whereby the total power in v equals to twice the Coleman (1971) *power of a collectivity to act* ($\equiv$the proportion of the winning coalitions among all coalitions in a carrier N of v). Casajus and Huettner (2019) used the 2CPCA-efficiency to characterize the Coleman-Shapley index $\varphi_{\frac{1}{2}}$ using 2CPCA-efficiency as a replacement of efficiency in the set of axioms of Dubey (1975) (originally devised for the SSPI). Similar axiomatizations can be obtained for any φ_q, with (13) as a substitute for 2CPCA-efficiency, where the right-hand side is viewed as an alternative measure of the power of a collectivity to act.

3.5 The axiom of total-power monotonicity

When the set of winning coalitions in the game expands, there is no guarantee that the individual power of every (or even most) players will not be affected negatively. Indeed, for any given $\xi \in M([0, 1])$ that is not a Dirac measure concentrated on 0, think of the change in the semivalue $\varphi = \phi_\xi$ when a unanimity game $v = u_T$ (where $u_T(S) = 1$ if and only if $T \subset S$) sees its carrier T shrink to a strict subset, $T' \subsetneq T$, and the game becomes $u_{T'}$. The power of every $i \in T \backslash T'$ then falls from $\varphi(u_T)(i) = \int_0^1 x^{|T|-1} d\xi(x) > 0$ to $\varphi(u_{T'})(i) = 0$. That is to be expected because the players in $T \backslash T'$ become null in $u_{T'}$, despite there being *more* winning coalitions to which each $i \in T \backslash T'$ belongs in $u_{T'}$ compared to u_T. Only the members of the new minimal winning coalition, T', see their power rise (or at least remain unaffected) by the change in the game. [24] Indeed, for every $i \in T'$,

$$\varphi(u_{T'})(i) = \int_0^1 x^{|T'|-1} d\xi(x) \geq \int_0^1 x^{|T|-1} d\xi(x) = \varphi(u_T)(i). \tag{15}$$

For a semivalue φ that is a q-CSPI for some $q \in [0, 1]$, namely, $\varphi = \varphi_q$, any expansion of the set of winning coalitions in the game has a non-negative net effect on the *total power* despite possibly ambiguous individual power variations. Specifically, if $v \in \mathcal{SG}$ is replaced by $w \in \mathcal{SG}$ that satisfies $v \leq w$, the total power cannot go down:

$$\varphi_q(v)(U) \leq \varphi_q(w)(U). \tag{16}$$

This fact is immediate from (12) when $q > 0$, and from (14) when $q = 0$.

We shall state the monotonicity requirement in (16) as an axiom on the behavior of a general power index φ:

Axiom V: Total-power Monotonicity (TP-Mon). If $v, w \in \mathcal{SG}$ and $v \leq w$, then $\varphi(v)(U) \leq \varphi(w)(U)$.

24 A more general (and easily verifiable) version of this property is the following: if $w \in \mathcal{SG}$ is obtained from $v \in \mathcal{SG}$ by adding a single minimal winning coalition T' (that is, $w = \max(v, u_{T'})$), then $\phi_\xi(w)(i) \geq \phi_\xi(v)(i)$ for every $i \in T'$.

As argued in the Introduction, **TP-Mon** seems desirable if there is an intention of using the power index to compute the power of coalitions by additively aggregating the individual power within them. At the same time, **TP-Mon** is sufficiently selective – it is not possessed by all semivalues.[25] For instance, the BPI attains the maximal total power on a given carrier of odd size at the simple majority game, by Theorem 2 in Dubey and Shapley (1979).

4 Results

4.1 Total-power monotonicity of a semivalue

In this section we will characterize the effect of imposing the axiom of **TP-Mon** on the family of semivalues. On a technical level, **TP-Mon** reduces to concavity of the c.d.f. of the representing distribution.

Theorem 1 *A semivalue* $\varphi = \phi_\xi$ *satisfies* ***TP-Mon*** if and only if *the c.d.f.* F_ξ *of the distribution corresponding to* ξ *is concave on* $[0, 1]$.

Proof of Theorem 1 We start with the proof of the "only if" direction of the theorem. Let $\xi \in M([0, 1])$ be such that ϕ_ξ satisfies **TP-Mon**. The main ingredient of the proof will be the following claim. □

Claim Let $0 < a < b < 1$ and $0 < c < d \le 1$ be such that $c - a = d - b > 0$. Then

$$\xi((a, b]) \ge \xi((c, d]). \tag{17}$$

Proof of the claim. We shall first establish (17) under the assumption that

$$\xi(\{a, b, c, d\}) = 0. \tag{18}$$

Fix $\delta > 0$, and let $0 < \varepsilon < \frac{b-a}{2}$ be such that $\xi\left([t^-, t^+]\right) < \delta$ for $t \in \{a, b, c, d\}$, where $t^+ = \min(t + \varepsilon, 1)$, $t^- = \max(t - \varepsilon, 0)$. Also, for any $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\varepsilon}{2}$ and any $x \in [0, 1]$, let Y_x^n be a random variable with the binomial distribution $B(n, x)$. Then

$$\xi((a, b]) \ge \xi\left((a^-, b^+)\right) - 2\delta = \int_{(a^-,b^+)} d\xi(x) - 2\delta$$
$$\ge \int_{(a^-,b^+)} \left(\sum_{k=[an]}^{[bn]} \Pr(Y_x^n = k) \right) d\xi(x) - 2\delta$$
(where $[t]$ stands for the integer part of t)

[25] Notice that manifestations of *individual power* monotonicity expressed by the inequality (15) or, more generally, the statement in Footnote 24, are not selective, in contrast to **TP-Mon**: they are satisfied by *all* semvalues.

$$
\begin{aligned}
&= \sum_{k=[an]}^{[bn]} \left(\int_0^1 \Pr(Y_x^n = k) d\xi\,(x) \right) \\
&\quad - \int_{(a^-,b^+)^c} \left(\sum_{k=[an]}^{[bn]} \Pr(Y_x^n = k) \right) d\xi\,(x) - 2\delta \\
&\geq \sum_{k=[an]}^{[bn]} \left(\int_0^1 \Pr(Y_x^n = k) d\xi\,(x) \right) \\
&\quad - \int_{(a^-,b^+)^c} \Pr\left(\left| \frac{Y_x^n}{n} - x \right| > \frac{\varepsilon}{2} \right) d\xi\,(x) - 2\delta. \qquad (19)
\end{aligned}
$$

By the Chebishev's inequality,

$$
\Pr\left(\left| \frac{Y_x^n}{n} - x \right| > \frac{\varepsilon}{2} \right) \leq \frac{1}{n\varepsilon^2}, \qquad (20)
$$

and hence the expression in (19) is bound from below by

$$
\begin{aligned}
&\sum_{k=[an]}^{[bn]} \left(\int_0^1 \Pr(Y_x^n = k) d\xi\,(x) \right) - \frac{1}{n\varepsilon^2} - 2\delta \\
&\quad = \sum_{k=[an]}^{[bn]} \left(\int_0^1 \binom{n}{k} x^k (1-x)^{n-k} d\xi\,(x) \right) - \frac{1}{n\varepsilon^2} - 2\delta. \qquad (21)
\end{aligned}
$$

For $k = 0, ..., n$, let $w_{n+1,k+1} \in \mathcal{SG}$ be the $k+1$-majority game with carrier $N = \{1, ..., n+1\}$, i.e., $w_{n+1,k+1}(S) = 1$ if and only if $|S \cap N| \geq k+1$. It follows from the definition of ϕ_ξ in (1) and (2) that

$$
\phi_\xi \left(w_{n+1,k+1} \right) (U) = (n+1) \int_0^1 \binom{n}{k} x^k (1-x)^{n-k} d\xi\,(x)\,, \qquad (22)
$$

and so the right-hand side of (21) is equal to

$$
\frac{1}{n+1} \sum_{k=[an]}^{[bn]} \phi_\xi \left(w_{n+1,k+1} \right) (U) - \frac{1}{n\varepsilon^2} - 2\delta.
$$

We have thereby established that

$$
\xi\,((a,b]) \geq \frac{1}{n+1} \sum_{k=[an]}^{[bn]} \phi_\xi \left(w_{n+1,k+1} \right) (U) - \frac{1}{n\varepsilon^2} - 2\delta. \qquad (23)
$$

Since: (i) ϕ_ξ satisfies **TP-Mon;** (ii) $w_{n+1,k+1} \geq w_{n+1,k'+1}$ whenever $k \leq k'$; and (iii) $c - a = d - b > 0$, we obtain

$$\sum_{k=[an]}^{[bn]} \phi_\xi \left(w_{n+1,k+1}\right)(U) \geq \sum_{k=[cn]+1}^{[dn]} \phi_\xi \left(w_{n+1,k+1}\right)(U).$$

From this, (22), and (23) it follows that

$$\begin{aligned}
\xi\left((a,b]\right) &\geq \frac{1}{n+1} \sum_{k=[cn]+1}^{[dn]} \phi_\xi \left(w_{n+1,k+1}\right)(U) - \frac{1}{n\varepsilon^2} - 2\delta \\
&= \sum_{k=[cn]+1}^{[dn]} \int_0^1 \binom{n}{k} x^k (1-x)^{n-k} d\xi\,(x) - \frac{1}{n\varepsilon^2} - 2\delta \\
&\geq \sum_{k=[cn]+1}^{[dn]} \int_{[c^+,d^-]} \binom{n}{k} x^k (1-x)^{n-k} d\xi\,(x) - \frac{1}{n\varepsilon^2} - 2\delta \\
&= \sum_{k=[cn]+1}^{[dn]} \int_{[c^+,d^-]} \Pr(Y_x^n = k) d\xi\,(x) - \frac{1}{n\varepsilon^2} - 2\delta \\
&= \sum_{k=0}^{n} \int_{[c^+,d^-]} \Pr(Y_x^n = k) d\xi\,(x) \\
&\quad - \sum_{k<[cn]+1 \text{ or } k>[dn]} \int_{[c^+,d^-]} \Pr(Y_x^n = k) d\xi\,(x) - \frac{1}{n\varepsilon^2} - 2\delta.
\end{aligned}$$

As

$$\begin{aligned}
\sum_{k=0}^{n} \int_{[c^+,d^-]} \Pr(Y_x^n = k) d\xi\,(x) &= \int_{[c^+,d^-]} \left(\sum_{k=0}^{n} \Pr(Y_x^n = k) \right) d\xi\,(x) \\
&= \xi\left(\left[c^+, d^-\right]\right) \geq \xi((c,d]) - 2\delta,
\end{aligned}$$

we obtain

$$\begin{aligned}
\xi\left((a,b]\right) &\geq \xi\left((c,d]\right) - \sum_{k<[cn]+1 \text{ or } k>[dn]} \int_{[c^+,d^-]} \Pr(Y_x^n = k) d\xi\,(x) - \frac{1}{n\varepsilon^2} - 4\delta \\
&= \xi\left((c,d]\right) - \int_{[c^+,d^-]} \left(\sum_{k<[cn]+1 \text{ or } k>[dn]} \Pr(Y_x^n = k) \right) d\xi\,(x) - \frac{1}{n\varepsilon^2} - 4\delta \\
&\geq \xi\left((c,d]\right) - \int_{[c^+,d^-]} \Pr\left(\left| \frac{Y_x^n}{n} - x \right| > \frac{\varepsilon}{2} \right) d\xi\,(x) - \frac{1}{n\varepsilon^2} - 4\delta.
\end{aligned}$$

By using the Chebishev's inequality (20) again, the last expression is bound from below by $\xi\left((c,d]\right) - \frac{2}{n\varepsilon^2} - 4\delta$. We have thus shown that

$$\xi\left((a,b]\right) \geq \xi\left((c,d]\right) - \frac{2}{n\varepsilon^2} - 4\delta.$$

By letting $n \to \infty$, this turns into $\xi\left((a,b]\right) \geq \xi\left((c,d]\right) - 4\delta$, and since the fixed $\delta > 0$ was arbitrary, the desired inequality (17) is established under Assumption (18).

We will now show that Assumption (18) can be dispensed with. First, notice that when $d = 1$, all the arguments above work without the need to pass from d to d^-. Hence, it is not necessary to assume that $\xi\left(\{d\}\right) = 0$ when $d = 1$ (and, in addition, $\xi\left(\{a, b, c\}\right) = 0$), in order to obtain (17).

Next, for any $0 < x < y \leq 1$, there exists a sequence $\{(a_n, b_n, c_n, d_n)\}_{n=1}^{\infty}$ such that $0 < a_n < x < b_n < 1$, $0 < c_n < y \leq d_n \leq 1$, $c_n - a_n = d_n - b_n > 0$, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x$, $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = y$, $\xi\left(\{a_n, b_n, c_n\}\right) = 0$, and $\xi\left(\{d_n\}\right) = 0$ (unless $d_n = 1$). As (17) holds for such a_n, b_n, c_n, d_n by what has been shown, we have $\xi\left((a_n, b_n]\right) \geq \xi\left((c_n, d_n]\right)$, which translates into $\xi\left(\{x\}\right) \geq \xi\left(\{y\}\right)$ by letting $n \to \infty$. Since the latter inequality holds for all $0 < x < y \leq 1$, ξ cannot have atoms in $(0, 1]$. It follows that (18) always holds, and hence (17) holds for *any* a, b, c, d as in the premise of the claim. □

Proof of Theorem 1 (continued) As has been argued in the last part of the preceding proof, ξ has no atoms in $(0, 1]$. It follows that the c.d.f. F_ξ that corresponds to ξ, given by $F_\xi(x) = \xi\left([0, x]\right)$ for any $x \in [0, 1]$, is continuous on $(0, 1]$. Because a c.d.f. is right-continuous, F_ξ is continuous on the entire closed interval $[0, 1]$. By (17),

$$F_\xi(b) - F_\xi(a) \geq F_\xi(d) - F_\xi(c) \tag{24}$$

for any $0 < a < b < 1$ and $0 < c < d \leq 1$ such that $c - a = d - b > 0$. The continuity of F_ξ on $[0, 1]$ implies that, furthermore, (24) holds even if $a = 0$.

Now, given $0 \leq x < y \leq 1$, consider any rational number $0 < r < 1$, which has the form $r = \frac{m}{n}$ for some $n > m \in \mathbb{N}$. Successive applications of (24) yield

$$\begin{aligned}
&F_\xi\left(\frac{m}{n}x + \frac{n-m}{n}y\right) - F_\xi(x) \\
&= \sum_{k=m+1}^{n}\left(F_\xi\left(\frac{k-1}{n}x + \frac{n-k+1}{n}y\right) - F_\xi\left(\frac{k}{n}x + \frac{n-k}{n}y\right)\right) \\
&\geq (n-m)\left(F_\xi\left(\frac{m}{n}x + \frac{n-m}{n}y\right) - F_\xi\left(\frac{m+1}{n}x + \frac{n-m-1}{n}y\right)\right) \\
&\geq (n-m)\left(F_\xi\left(\frac{m-1}{n}x + \frac{n-m+1}{n}y\right) - F_\xi\left(\frac{m}{n}x + \frac{n-m}{n}y\right)\right) \\
&\geq \frac{n-m}{m}\sum_{k=1}^{m}\left(F_\xi\left(\frac{k-1}{n}x + \frac{n-k+1}{n}y\right) - F_\xi\left(\frac{k}{n}x + \frac{n-k}{n}y\right)\right)
\end{aligned}$$

$$= \frac{n-m}{m}\left(F_\xi(y) - F_\xi\left(\frac{m}{n}x + \frac{n-m}{n}y\right)\right),$$

and hence

$$F_\xi(rx + (1-r)y) \geq rF_\xi(x) + (1-r)F_\xi(y) \tag{25}$$

holds for $r = \frac{m}{n}$. Since F_ξ is continuous on [0, 1], the inequality (25) holds for *any* $0 < r < 1$, which shows that F_ξ is indeed concave on [0, 1]. This establishes the "only if" direction of the theorem.

The proof of the "if" direction of Theorem 1 is deferred to the Appendix, as it is based on the main fact established in the proof of our next Theorem 2. □

4.2 Mixing q-CSPIs: the only way to achieve total-power monotonicity

Theorem 1 contains an implication that is somewhat hidden from sight. The next theorem uncovers it, and points to a tight link between the **TP-Mon** property and the family of generalized CSPIs: a semivalue satisfies **TP-Mon** if and only if it is a *mixture* of q-CSPIs.

Theorem 2 *A semivalue φ satisfies **TP-Mon*** if and only if *there exist a probability measure $\mu \in M([0,1])$, uniquely determined by φ, such that for any $v \in \mathcal{SG}$ and $i \in U$,*

$$\varphi(v)(i) = \int_0^1 \varphi_q(v)(i)\, d\mu(q). \tag{26}$$

Proof The fact that any q-CSPI φ_q satisfies **TP-Mon** has already been noted (see (16)), and it is obvious that any mixture φ of such indices, given by (26), inherits this property. This establishes the "if" direction. To prove the "only if" direction, fix a semivalue φ that satisfies **TP-Mon** . By Theorem 1,[26] $\varphi = \phi_\xi$ for some $\xi \in M([0,1])$ whose c.d.f. F_ξ is continuous[27] and concave on $[0,1]$. The last two properties of F_ξ and its monotonicity as a c.d.f. imply that there exists a nonincreasing function $f_\xi \geq 0$ on (0, 1] such that[28] $F_\xi(t) = F_\xi(0) + \int_0^t f_\xi(x)dx$ for every $t \in (0, 1]$, where $F_\xi(0) = \xi(\{0\})$.

Next, let $g \geq 0$ be any continuous function on $[0,1]$, and assume that $F_\xi(0) < 1$. Notice that

26 Notice that this use of Theorem 1 is legitimate because it relies on the "only if" part of that theorem, which has already been established in the previous section. It is the "if" part that still awaits proof, given in the Appendix.

27 Continuity of F_ξ was established in the proof of Theorem 1, but we did not need to claim both continuity and concavity in the statement of that theorem because concavity of F_ξ on [0, 1] implies its continuity on that interval. Indeed, the only discontinuity of a concave function on [0, 1] might occur at the end-points, but that is impossible because F_ξ is right-continuous and nondecreasing as a c.d.f.

28 One may take f_ξ to be the left-hand derivative of F_ξ on (0, 1]. If $\lim_{x\to 0+} f_\xi(x) = \infty$, then all integrals in the proof that have the form $\int_0^t ...dx$ (for $0 < t \leq 1$), and in which the integrand involves $f_\xi(x)$, should be regarded as improper integrals.

$$\int_0^1 g(x)d\xi(x) = g(0)F_\xi(0) + \int_0^1 g(x)f_\xi(x)dx = g(0)F_\xi(0) + \int_0^1 g(x)\left(\int_0^{f_\xi(x)} ds\right)dx$$
$$= g(0)F_\xi(0) + \int_0^\infty \left(\int_0^1 g(x)I_{s\le f_\xi(x)}dx\right)ds. \tag{27}$$

Denote $a_\xi = \lim_{x\to 0+} f_\xi(x) > 0$,[29] and let h_ξ be a nonincreasing function on $[0, a_\xi)$ defined by $h_\xi(s) = \sup\{x \in [0,1] \mid s \le f_\xi(x)\}$ for every $s \in [0, a_\xi)$; notice that $h_\xi > 0$. The expression in (27) is then equal to

$$g(0)F_\xi(0) + \int_{[0,a_\xi)} \left(\int_0^1 g(x)I_{x\le h_\xi(s)}dx\right)ds$$
$$= g(0)F_\xi(0) + \int_{[0,a_\xi)} h_\xi(s)\left(\int_0^1 g(x)\frac{1}{h_\xi(s)}I_{x\le h_\xi(s)}dx\right)ds.$$

We have thereby shown that

$$\int_0^1 g(x)d\xi(x) = g(0)F_\xi(0) + \int_{[0,a_\xi)} h_\xi(s)\left(\int_0^1 g(x)\frac{1}{h_\xi(s)}I_{x\le h_\xi(s)}dx\right)ds, \tag{28}$$

where $\int_{[0,a_\xi)} h_\xi(s)\,ds = 1 - F_\xi(0)$. Now recall the definition of the probability measure $\xi_q \in M([0,1])$ as the one that is concentrated on $[0,q]$, with $d\xi_q(x) = \frac{1}{q}I_{x\le q}dx$ when $q > 0$. The equality (28) then becomes

$$\int_0^1 g(x)d\xi(x) = \int_0^1 \left(\int_0^1 g(x)d\xi_q(x)\right)d\nu_\xi(q), \tag{29}$$

where $\nu_\xi \in M([0,1])$ is the probability measure determined by the following properties: $\nu_\xi(\{0\}) = F_\xi(0)$, and $\nu_\xi((0,x]) = \int_{[0,a_\xi)} I_{h_\xi(s)\le x}h_\xi(s)\,ds$ for any $x \in (0,1]$.

The measure $\mu = \nu_\xi$ turns out to be the one that is required in (26). Indeed, given $v \in \mathcal{SG}$ and $i \in U$, by using (3) we obtain

$$\varphi(v)(i) = \phi_\xi(v)(i) = \int_0^1 \phi_x(v)(i)\,d\xi(x)$$
$$(\text{by } (29)) = \int_0^1 \left(\int_0^1 \phi_x(v)(i)\,d\xi_q(x)\right)d\nu_\xi(q)$$
$$= \int_0^1 \phi_{\xi_q}(v)(i)d\nu_\xi(q) = \int_0^1 \varphi_q(v)(i)d\nu_\xi(q).$$

Lastly, if $F_\xi(0) = 1$ then ξ is supported on $\{0\}$, i.e., $\xi = \xi_0$, implying that $\varphi = \phi_{\xi_0} = \varphi_0$, and hence (26) holds trivially. Thus, the existence of μ that satisfies (26) has been established for any given TP-monotonic $\varphi = \phi_\xi$.

[29] The limit a_ξ exists because f_ξ is nondecreasing, and its positivity follows from the assumption that $F_\xi(0) < 1$. It may, furthermore, be equal to ∞.

The fact that μ in (26) is determined uniquely by the given φ will be established in the Appendix, based on a useful observation made in the upcoming Remark 2. □

We conclude with two remarks.

Remark 1 *(Generalized Coleman-Shapley* values *and TP-monotonicity of semivalues for all finite games).* Dubey et al. (1981) defined a semivalue on the space of all finite games, $\mathcal{G}$, as a linear projection[30] $\varphi : \mathcal{G} \to \mathcal{AG}$ that satisfies the **Anonymity** and **Non-negativity** axioms of Sect. 2.2 (in the context of general games in $\mathcal{G}$, **Anonymity** needs to be stated for any $v \in \mathcal{G}$, and **Non-negativity** for any monotonic $v \in \mathcal{G}$). Their characterization of semivalues on $\mathcal{G}$ as the family $\left(\phi_\xi\right)_{\xi \in M([0,1])}$ (defined by (1) and (2) for all $v \in \mathcal{G}$) is identical to that in Proposition 1 for simple games. Using this characterization, q-Coleman-Shapley *values* for games in $\mathcal{G}$ can be defined in the same way as q-CSPIs were defined on $\mathcal{SG}$ in Sect. 3.1, and, with **TP-Mon** stated for games in $\mathcal{G}$, all our results (Proposition 2 and Theorems 1, 2) hold for semivalues on $\mathcal{G}$ instead of $\mathcal{SG}$, by identical arguments. □

Remark 2 *(Generalized CSPIs decompose semivalues).* Consider any semivalue φ that satisfies **TP-Mon;** by Theorem 2, it possesses the representation (26) for some $\mu \in M([0,1])$. There turns out to be a strong connection between φ and the semivalue ϕ_μ. Using the terminology of Casajus and Huettner (2018), φ *decomposes* ϕ_μ, that is, for any $v \in \mathcal{SG}$ with a finite carrier N and any $i \in U$,

$$\phi_\mu (v) (i) = \varphi (v) (i) + \sum_{j \in N \setminus i} \left(\varphi (v) (j) - \varphi (v_{-i}) (j)\right), \tag{30}$$

where v_{-i} is the game obtained from v by the removal of player i from its minimal carrier.[31] Thus, if $\varphi (v) (i)$ is interpreted as a measure of i's *direct power*, $\phi_\mu (v) (i)$ may be viewed as a combined measure of i's direct power and his *threat power* (the latter being captured by the total change of the direct power of other players effected by i's exclusion).

Casajus and Huettner (2019) showed that (30) holds when μ is a Dirac measure concentrated on $\frac{1}{2}$ (in which case $\phi_{\frac{1}{2}}$ is the BPI, and $\varphi = \varphi_{\frac{1}{2}}$ is the CSPI). To establish (30) for φ that is given by (26) for a general $\mu \in M([0,1])$, define a game $v_\mu \in \mathcal{G}$ by

$$v_\mu(S) := \int_0^1 v_q (S)\, d\mu (q) \tag{31}$$

(where $v_q (S)$ is given by (10) and (11)) for every $S \in 2^U$. It can be readily seen that

$$\phi_\mu(v)(i) = v_\mu(N) - v_\mu(N \setminus i) \tag{32}$$

[30] I.e., φ acts as the identity map when restricted to $\mathcal{AG}$.

[31] Formally, $v_{-i} (S) := v (S \setminus i)$ for every $S \in 2^U$. Notice that v_{-i} may be the zero game, which is excluded from our definition of simple games. In such a case, $\varphi \left(v_{-i}\right)$ is also taken to be the zero game.

for any $v \in \mathcal{SG}$ with a finite carrier N, and any $i \in N$ (for completeness, this will be stated and proved in Proposition 3 in the Appendix). Also, by integrating both sides of the equation in Proposition 2 over q w.r.t. μ and noticing that the Shapley value operator is interchangeable with integration, another equality is obtained:

$$\varphi(v)(i) = Sh\left(v_\mu\right)(i). \tag{33}$$

By substituting into (30) the expressions obtained for $\phi_\mu(v)$, $\varphi(v)$ in (32) and (33), the equality in (30) follows immediately from the efficiency of the Shapley value.[32]

Since, by Proposition 1, any semivalue has the form ϕ_μ for some $\mu \in M([0,1])$, (30) shows that *any semivalue* can be decomposed by a mixture of generalized CSPIs (specifically, by φ defined in (26) for the corresponding μ). □

A Appendix

A.1 Proof of the "if" direction of Theorem 1

Proof Assume that the c.d.f. F_ξ of the distribution corresponding to ξ is concave on $[0,1]$. The *proof* of the "only if" part of Theorem 2 shows that, in such a case, the semivalue $\varphi = \phi_\xi$ has the representation (26). But then, by the "if" part of Theorem 2, φ satisfies **TP-Mon**. □

A.2 Proof of the uniqueness of a representing measure μ in Theorem 2

Proof Assume that a semivalue φ possesses a representation (26) for some $\mu \in M([0,1])$. Then, as shown in Remark 2, φ decomposes ϕ_μ. But then ϕ_μ is uniquely determined by (30), and μ is in turn uniquely determined by ϕ_μ (due to Proposition 1). □

A.3 Proposition 3

Proposition 3 *Consider $\mu \in M([0,1])$ and $v \in \mathcal{SG}$. For any finite carrier N of v and any $i \in N$,*

$$\phi_\mu(v)(i) = v_\mu(N) - v_\mu(N \backslash i), \tag{34}$$

where $v_\mu \in \mathcal{G}$ is the game given by (31).

Proof It is clear from (1), (2) that, for any $q \in (0,1]$, the q-value ϕ_q is given by

$$\phi_q(v)(i) = E\left[v(\overline{S}_N^q) - v(\overline{S}_N^q \backslash i) \mid i \in \overline{S}_N^q\right],$$

[32] Equalities (32) and (33) extend, respectively, Proposition 1 and Theorem 2 of Casajus and Huettner (2019). Our proof of (30) follows the argument used by these authors in establishing their Corollary 1.

where $\overline{S}_N^q$ is the random coalition of q-active players that satisfies (7). Notice that

$$\begin{aligned}E\left[v(\overline{S}_N^q)-v(\overline{S}_N^q\backslash i)\mid i\in\overline{S}_N^q\right]&=\frac{1}{q}E\left[v(\overline{S}_N^q)-v(\overline{S}_N^q\backslash i)\right]\\&=\frac{1}{q}E[v(N\cap\overline{S}_N^q)]-\frac{1}{q}E[v((N\backslash i)\cap\overline{S}_N^q)]\\&=v_q(N)-v_q(N\backslash i),\end{aligned}$$

and hence

$$\phi_q\,(v)\,(i)=v_q(N)-v_q(N\backslash i). \quad (35)$$

When $q=0$, (35) still holds because $\phi_0\,(v)\,(i)=v(i)$ and $v_0(N)-v_0(N\backslash i)=v(i)$ by (11). By integrating both sides of (35) over q w.r.t. μ and using (3), the desired equality (34) is obtained. □

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