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Weakly differentially monotonic solutions for cooperative games

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Abstract

The principle of differential monotonicity for cooperative games states that the differential of two players' payoffs weakly increases whenever the differential of these players' marginal contributions to coalitions containing neither of them weakly increases. Together with the standard efficiency property and a relaxation of the null player property, differential monotonicity characterizes the egalitarian Shapley values, i.e., the convex mixtures of the Shapley value and the equal division value for games with more than two players. For games that contain more than three players, we show that, cum grano salis, this characterization can be improved by using a substantially weaker property than differential monotonicity. Weak differential monotonicity refers to two players in situations where one player's change of marginal contributions to coalitions containing neither of them is weakly greater than the other player's change of these marginal contributions. If, in such situations, the latter player's payoff weakly/strictly increases, then the former player's payoff also weakly/strictly increases.

Keywords TU game · Shapley value · Differential marginality · Weak differential marginality

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1 Introduction

The Shapley value (Shapley 1953) probably is the most eminent (one-point) solution concept for cooperative games with transferable utility (TU games). Among these solution concepts, the Shapley value can be viewed as *the* measure of the players' *individual* productivity in a game. This view is strongly supported by Young's (1985) characterization via three properties: efficiency, strong monotonicity, and symmetry.[1] Efficiency says that the worth generated by the grand coalition is distributed among the players. Strong monotonicity requires a player's payoff weakly to increase whenever *her* productivity, measured by *her* marginal contributions to coalitions of the other players, weakly increases. Symmetry ensures that equally productive players obtain the same payoff.

Modern societies and institutions, however, distribute their wealth not only based on individual productivity but also on solidarity or egalitarian principles. In order to reflect this fact, alternative solution concepts have been developed. Most notably, Joosten (1996) introduces a particularly appealing class of such solutions, the egalitarian Shapley values, which are the convex mixtures of the Shapley value and the equal division value. That is, an egalitarian Shapley value redistributes the Shapley payoffs as follows: First, the Shapley payoffs are taxed proportionally at a fixed rate. Second, the total tax revenue is distributed equally among all players.[2]

Among the egalitarian Shapley values, the Shapley value is the only value that satisfies strong monotonicity. Two roads have been taken in order to reconcile the marginalism embodied in strong monotonicity with solidarity or egalitarianism.

van den Brink et al. (2013) suggest a relaxation of strong monotonicity called weak monotonicity. Weak monotonicity requires a player's payoff weakly to increase whenever both *her* productivity and the worth generated by the grand coalition weakly increase. Casajus and Huettner (2014b) show that the class of egalitarian Shapley values is characterized by efficiency, symmetry, and weak monotonicity, unless there are just two players.[3]

Casajus (2011) considers a differential version of marginality (see footnote 1) called differential marginality, which demands equal productivity differentials to translate into equal payoff differentials, i.e., whenever the differential of their marginal contributions to coalitions containing neither of them does not change then the differential of their payoffs does not change. Interestingly, differential marginality coincides with the fairness property due to van den Brink (2001) on the full domain of games, for example. Hence, the latter can be replaced with the former in van den Brink's (2001) characterization of the Shapley value, i.e., differential marginality together with efficiency and the null player property characterize the Shapley value.

[1] As already mentioned by Young (1985), strong monotonicity implies and can be relaxed into marginality, i.e., a player's payoff only depends on her own productivity.

[2] Sprumont (1990) suggests another solution with a solidary flavor that later on was characterized by Nowak and Radzik (1994) as the "solidarity value". Casajus and Huettner (2014a) consider a class of generalizations of this value.

[3] Besides efficiency, symmetry and weak monotonicity, the characterizations of the egalitarian Shapley values due to van den Brink et al. (2013) involve a fourth axiom, either linearity or weak covariance. Their characterization using linearity also covers the two-player case.

Later on, Casajus and Huettner (2013) strengthen differential marginality into a differential version of strong monotonicity called strong differential monotonicity. This property requires two players' payoff differential weakly to increase whenever their productivity differential weakly increases. They show that the egalitarian Shapley values are characterized by efficiency, the null player in a productive environment property, and strong differential monotonicity, unless there are just two players. The null player in a productive environment property relaxes the standard null player property by requiring a non-negative payoff for null players only when the worth generated by the grand coalition is non-negative.

Recently, Casajus and Yokote (2017) introduce a substantial relaxation of differential marginality called weak differential marginality. Differential marginality can be rephrased as that equal changes in two players' productivities, i.e., their marginal contributions to coalitions containing neither of them change by the same amount, entails that their payoffs change by the same amount, which obviously implies that both payoffs change in the same direction. Weak differential marginality relaxes differential marginality in this vein. Equal changes in two players' productivities should entail that their payoffs change in the same direction. Using this property, they considerably improve the characterization of the Shapley value by Casajus (2011) . For games with more than two players, the Shapley value can be characterized by efficiency, the null player property, and weak differential marginality.

In this paper, we consider a relaxation of differential monotonicity called weak differential monotonicity, which relaxes differential monotonicity in the same vein as weak differential marginality relaxes differential marginality. Weak differential monotonicity refers to situations where one player's change of marginal contributions is weakly greater than another player's change of marginal contributions. If, in such situations, the latter player's payoff weakly/strictly increases, then the former player's payoff also weakly/strictly increases. First, we show that one cannot replace differential monotonicity with weak differential monotonicity in Casajus and Huettner's (2013) characterization of the egalitarian Shapley values. Second, for games with more than three players, the egalitarian Shapley values can be characterized by efficiency, a relaxation of the dummy player property, and weak differential monotonicity.

The remainder of this paper is organized as follows. In Sect. 2, we give basic definitions and notation. In Sect. 3, we present our main result. Some remarks conclude this paper. An appendix contains the proof of our main result and some complementary findings.

2 Basic definitions and notation

A (**finite TU**) **game** on a non-empty and finite set of players N is given by a **coalition function** $v \in \mathbb{V}(N) := \{f : 2^N \to \mathbb{R} \mid f(\emptyset) = 0\}$, where 2^N denotes the power set of N. Subsets of N are called **coalitions**; $v(S)$ is called the worth of coalition S. Since we deal with a fixed player set N, the latter mostly is dropped as an argument.

Player $i \in N$ is called a **dummy player** in $v \in \mathbb{V}$ if $v(S \cup \{i\}) - v(S) = v(\{i\})$ for all $S \subseteq N \setminus \{i\}$; player $i \in N$ is called a **null player** in $v \in \mathbb{V}$ if $v(S \cup \{i\}) = v(S)$

for all $S \subseteq N \backslash \{i\}$; players $i, j \in N$ are called **symmetric** in $v \in \mathbb{V}$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$.

For $v, w \in \mathbb{V}$ and $\alpha \in \mathbb{R}$, the coalition functions $v + w \in \mathbb{V}$ and $\alpha \cdot v \in \mathbb{V}$ are given by $(v + w)(S) = v(S) + w(S)$ and $(\alpha \cdot v)(S) = \alpha \cdot v(S)$ for all $S \subseteq N$. The game $\mathbf{0} \in \mathbb{V}$ given by $\mathbf{0}(S) = 0$ for all $S \subseteq N$ is called the **null game**. For $T \subseteq N$, $T \neq \emptyset$, the game $u_T \in \mathbb{V}$ given by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise is called a **unanimity game**. Any $v \in \mathbb{V}$ can be uniquely represented by unanimity games. In particular, we have

$$v = \sum_{T \subseteq N : T \neq \emptyset} \lambda_T(v) \cdot u_T, \tag{1}$$

where the coefficients $\lambda_T(v)$ are known as the Harsanyi dividends (Harsanyi 1959) and can be determined recursively by

$$\lambda_T(v) := v(T) - \sum_{S \subsetneq T : S \neq \emptyset} \lambda_S(v) \qquad \text{for all } T \subseteq N,\ T \neq \emptyset. \tag{2}$$

A game $v \in \mathbb{V}$ is called **inessential** if all $i \in N$ are dummy players in v. The set of inessential games is denoted by $\bar{\mathbb{V}}$.

A **solution/value** on N is a mapping $\varphi : \mathbb{V} \to \mathbb{R}^N$. The **Shapley value** (Shapley 1953) , Sh, is given by

$$\mathrm{Sh}_i(v) := \sum_{T \subseteq N : i \in T} |T|^{-1} \cdot \lambda_T(v) \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N. \tag{3}$$

The **equal division value**, ED, is given by

$$\mathrm{ED}_i(v) := \frac{v(N)}{|N|} \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

The **egalitarian Shapley values** (Joosten 1996) , Sh^α, $\alpha \in [0, 1]$ are given by

$$\mathrm{Sh}_i^\alpha(v) = \alpha \cdot \mathrm{Sh}_i(v) + (1 - \alpha) \cdot \mathrm{ED}_i(v) \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N. \tag{4}$$

In the following, we make use of the following standard properties of solutions.

Efficiency, E. For all $v \in \mathbb{V}$, we have $\sum_{\ell \in N} \varphi_\ell(v) = v(N)$.

Null player, N. For all $v \in \mathbb{V}$ and $i \in N$ such that i is a null player in v, we have $\varphi_i(v) = 0$.

Dummy player, D. For all $v \in \mathbb{V}$ and $i \in N$ such that i is a dummy player in v, we have $\varphi_i(v) = v(\{i\})$.

3 Weak differential monotonicity and the egalitarian Shapley values

While the egalitarian Shapley values except the Shapley value itself fail Young's (1985) strong monotonicity property, they satisfy a differential version of this property due to Casajus and Huettner (2013) .

Differential monotonicity, DMo. For all $v, w \in \mathbb{V}$ and $i, j \in N$ such that

$$v(S \cup \{i\}) - v(S \cup \{j\}) \geq w(S \cup \{i\}) - w(S \cup \{j\}) \quad \text{for all } S \subseteq N \backslash \{i, j\},$$

we have

$$\varphi_i(v) - \varphi_j(v) \geq \varphi_i(w) - \varphi_j(w).$$

For games with more than two players, differential monotonicity together with efficiency and the following null player in a productive environment property characterize the egalitarian Shapley values.

Null player in a productive environment, NPE. For all $v \in \mathbb{V}$ and $i \in N$ such that i is a null player in v and $v(N) \geq 0$, we have $\varphi_i(v) \geq 0$.

Theorem 1 (Casajus and Huettner 2013, Theorem 4) *For $|N| > 2$, a value φ satisfies efficiency (**E**), the null player in a productive environment property (**NPE**), and differential monotonicity (**DMo**) if and only if there exists an $\alpha \in [0, 1]$ such that $\varphi = \mathrm{Sh}^{\alpha}$.*

Hypothesis and implication of differential monotonicity can be rewritten as

$$v(S \cup \{i\}) - w(S \cup \{i\}) \geq v(S \cup \{j\}) - w(S \cup \{j\}) \qquad \text{for all } S \subseteq N \backslash \{i, j\} \quad (5)$$

and

$$\varphi_i(v) - \varphi_i(w) \geq \varphi_j(v) - \varphi_j(w), \quad (6)$$

respectively. That is, differential monotonicity can be paraphrased as follows. Whenever one player's change in her productivity is weakly greater than that of another player, the former player's change in her payoff is weakly greater than that of the latter.

In the following, we suggest a considerable relaxation of differential monotonicity called weak differential monotonicity that relaxes differential monotonicity in the same vein as weak differential marginality (Casajus and Yokote 2017) relaxes differential marginality (Casajus 2011) , the latter two properties given below. Note that we rewrite these properties in analogy to (5) and (6). We use the sign function sign : $\mathbb{R} \to \{-1, 0, 1\}$ given by $\mathrm{sign}(x) = 1$ for $x > 0$, $\mathrm{sign}(0) = 0$, and $\mathrm{sign}(x) = -1$ for $x < 0$.

Differential marginality, DM For all $v, w \in \mathbb{V}$ and $i, j \in N$ such that

$$v(S \cup \{i\}) - v(S \cup \{j\}) = w(S \cup \{i\}) - w(S \cup \{j\}) \qquad \text{for all } S \subseteq N \backslash \{i, j\},$$

we have

$$\varphi_i(v) - \varphi_i(w) = \varphi_j(v) - \varphi_j(w).$$

Weak differential marginality, DM$^-$. For all $v, w \in \mathbb{V}$ and $i, j \in N$ such that

$$v(S \cup \{i\}) - w(S \cup \{i\}) = v(S \cup \{j\}) - w(S \cup \{j\}) \qquad \text{for all } S \subseteq N \backslash \{i, j\},$$

we have

$$\operatorname{sign}(\varphi_i(v) - \varphi_i(w)) = \operatorname{sign}\left(\varphi_j(v) - \varphi_j(w)\right).$$

Weak differential marginality can be paraphrased as follows. Whenever two player's productivity changes by the same amount, then their payoffs change in the same direction. For games with more than two players, one can replace the fairness property and differential marginality with weak differential marginality in the characterizations of the Shapley value by van den Brink (2001, Theorem 2.5) and Casajus (2011, Corollary 5), respectively

Theorem 2 (Casajus and Yokote 2017, Theorem 2)*Let $|N| > 2$. The Shapley value is the unique value that satisfies efficiency (**E**), the null player property (**N**), and weak differential marginality (**DM$^-$**).*

We define weak differential monotonicity in analogy to weak differential marginality and differential monotonicity.

Weak differential monotonicity, DMo$^-$ For all $v, w \in \mathbb{V}$ and $i, j \in N$ such that

$$v(S \cup \{i\}) - w(S \cup \{i\}) \geq v(S \cup \{j\}) - w(S \cup \{j\}) \quad \text{for all } S \subseteq N \backslash \{i, j\},$$

we have

$$\operatorname{sign}(\varphi_i(v) - \varphi_i(w)) \geq \operatorname{sign}\left(\varphi_j(v) - \varphi_j(w)\right).$$

One can easily check that the implication of weak differential monotonicity is equivalent to (i) $\varphi_j(v) \geq \varphi_j(w)$ implies $\varphi_i(v) \geq \varphi_i(w)$ and (ii) $\varphi_j(v) > \varphi_j(w)$ implies $\varphi_i(v) > \varphi_i(w)$. That is, weak differential monotonicity can be paraphrased as follows. Whenever one player's change in her productivity is weakly greater than that of another player, then the direction of the change of these players' payoffs does not contradict the changes in their productivities.

Unfortunately, one cannot simply replace differential monotonicity with weak differential monotonicity in Theorem 1. Consider the strictly positively weighted division values (Béal et al. 2016, Theorem 2) . Let

$$\Delta_{++}^N := \left\{ x \in \mathbb{R}^N \mid x_\ell > 0 \text{ for all } \ell \in N \text{ and } \sum_{\ell \in N} x_\ell = 1 \right\}$$

denote the set of all strictly positive weight vectors. For $\omega \in \Delta_{++}^{N}$, the ω-weighted division value WD^{ω} is given by

$$\mathrm{WD}_{i}^{\omega}(v) = \omega_i \cdot v(N) \qquad \text{for all } v \in \mathbb{V} \text{ and } \ell \in N.$$

One can easily check that these values satisfy efficiency, the null player in a productive environment property, and weak differential monotonicity, but fail differential monotonicity and therefore are not egalitarian Shapley values as long as the weights are not uniform.

In order to restore most of the implications of Theorem 1 after replacing differential monotonicity with weak differential monotonicity, we consider a property that is stronger than the null player in a productive environment property but weaker than the dummy player property.

Average dummy player, AD. For all $v \in \mathbb{V}$ and $i \in N$ such that i is a dummy player in v, we have that

(i) $v(\{i\}) \geq v(N)/|N|$ implies $\varphi_i(v) \leq v(\{i\})$,
(ii) $v(\{i\}) \leq v(N)/|N|$ implies $\varphi_i(v) \geq v(\{i\})$.

Whenever a dummy player is weakly more (less) productive than the average of all players, then her payoff is not greater (lower) than her productivity. In the first case, such a player should not be "subsidized" by the other players, in the second, she should not be required to contribute to "subsidizing" other players. Note that average dummy player property is related to the average property used by Yokote and Casajus (2017, Theorem 2) in order to characterize the flat tax and a basic income within a simple framework of the redistribution of income in a society.

Theorem 3 *For $|N| > 3$, a value φ satisfies efficiency (**E**), the average dummy player property (**AD**), and weak differential monotonicity (**DMo**$^-$) if and only if there exists an $\alpha \in [0, 1]$ such that $\varphi = \mathrm{Sh}^{\alpha}$.*

The proof of Theorem 3 can be found in Appendix A. Appendix B contains the counterexample to our characterization for $|N| = 2$. The non-redundancy of our characterization is indicated in Appendix C. It remains an open question whether Theorem 3 holds true for $|N| = 3$ or not.

4 Concluding remarks

As differential marginality, differential monotonicity implicitly assumes that the players' payoff differences are inter-personally comparable. For, differential monotonicity requires that, depending on the games, the difference of one player's payoffs in two games is weakly greater than the difference of some other player's payoffs in these games. In contrast, weak differential marginality and weak differential monotonicity are rather based on intra-personal comparisons. For example, weak differential marginality requires that, depending on the games, one player is better off in one game if and only if some other player also is better off in this game.

Shapley (1988, p. 307) first recognizes that "[i]nterpersonal comparability of utility is generally regarded as an unsound basis on which to erect theories of multipersonal behavior." Even though he then argues that "it enters naturally [...] as a nonbasic, derivative concept playing an important if sometimes hidden role in the theories of bargaining, group decisionmaking, and social welfare", one may not wish to directly impose inter-personal utility comparison. In this sense, Casajus and Yokote (2017, Theorem 2) and our Theorem 3 improve the previous results by van den Brink (2001, Theorem 2.5) , Casajus (2011, Corollary 5), and—cum grano salis— Casajus and Huettner (2013, Theorem 4).

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Appendix A Proof of Theorem 3

Preamble: squeezing water from stone

Proving characterizations for a parametrized classes of solutions with few rather weak axioms is like squeezing water from stone. Their proofs tends to be lengthy and rather technical. Our proof is no exception in this respect. It is by induction on the number of non-vanishing Harsanyi dividends of a game for non-singleton coalitions and can be divided into three major parts. While the first two parts provide the induction basis, the third one the induction step.

The first part (Claim 1) shows the theorem for inessential games, which includes the derivation of the parameter $\alpha \in [0, 1]$ from the solution φ. Its proof largely mimics the proof of Yokote and Casajus (2017, Theorem 2) . Since the latter result involves a related but different average dummy player property and in order to keep our paper self-contained, we present the full proof. The second part (Claims 2, 3, and 4) and the third part extend part one to general games. Their proof is an adaptation of the proof of Casajus and Yokote (2017, Theorem 2) to the egalitarian Shapley values and the use of the average dummy property instead of the null player property.

The proof

It is well-known that any value Sh^{α}, $\alpha \in [0, 1]$ satisfies **E**. Since **DMo** implies **DMo**$^{-}$, Casajus and Huettner (2013, Theorem 4) entails that any Sh^{α} also obeys **DMo**$^{-}$. By (4) and the fact that Sh meets **D**, any Sh^{α} meets **AD**. Let $|N| > 3$ and let the solution φ meet **E**, **AD**, and **DMo**$^{-}$.

For $v \in \mathbb{V}$, set

$$\mathcal{T}_{>1}(v) := \{T \subseteq N \mid |T| > 1 \text{ and } \lambda_T(v) \neq 0\}.$$

We show $\varphi = \mathrm{Sh}^{\alpha}$ for some $\alpha \in [0, 1]$ by induction on $|\mathcal{T}_{>1}(v)|$. For this purpose, we "reduce" $|\mathcal{T}_{>1}(v)|$ without changing $v(N)$ by the following construction: For $T \subseteq N$, $|T| > 1$, let $\bar{u}_T \in \mathbb{V}$ be given by

$$\bar{u}_T := u_T - \sum_{\ell \in T} \frac{u_{\{\ell\}}}{|T|}. \tag{A.1}$$

Note that $\mathrm{Sh}_i^\alpha(\bar{u}_T) = 0$ for all $\alpha \in [0,1]$ and $i \in N$. For $T \in \mathcal{T}_{>1}(v)$, let $v_T \in \mathbb{V}$ be given by

$$v_T := v - \lambda_T(v) \cdot \bar{u}_T. \tag{A.2}$$

By construction, (*) $|\mathcal{T}_{>1}(v_T)| = |\mathcal{T}_{>1}(v)| - 1$ and (**) $v(N) = v_T(N)$.

Induction basis: We show $\varphi(v) = \mathrm{Sh}^\alpha(v)$ for some $\alpha \in [0,1]$ and all $v \in \mathbb{V}$ such that $|\mathcal{T}_{>1}(v)| \leq 1$ by a number of claims and subclaims.

If $|\mathcal{T}_{>1}(v)| = 0$ for $v \in \mathbb{V}$, then $v \in \bar{\mathbb{V}}$, i.e., v is inessential.

Claim 1, C1: There exists some $\alpha \in [0,1]$ such that $\varphi(v) = \mathrm{Sh}^\alpha(v)$ for all $v \in \bar{\mathbb{V}}$.

One can easily check that there is a bijection $\mathbb{R}^N \to \bar{\mathbb{V}}$, $x \mapsto v_x$, where v_x is given by $v_x(S) = \sum_{\ell \in S} x_\ell$ for all $S \subseteq N$. Abusing notation, we identify $\bar{\mathbb{V}}$ with $\mathbb{R}^N$ and write x instead of v_x. By **D**, we have $\mathrm{Sh}(x) = x$ for all $x \in \mathbb{R}^N$ and therefore

$$\mathrm{Sh}_i^\alpha(x) \overset{(4)}{=} \alpha \cdot x_i + (1-\alpha) \cdot |N|^{-1} \cdot \sum_{\ell \in N} x_\ell \quad \text{for all } \alpha \in [0,1], x \in \mathbb{R}^N \text{ and } i \in N. \tag{A.3}$$

Set $n := |N|$. For $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^N$, we define $\lambda \cdot x \in \mathbb{R}^N$ by $(\lambda \cdot x)_\ell = \lambda \cdot x_\ell$ for all $\ell \in N$. Further, for $i, j \in N$, $i \neq j$, we define $e^{ij} \in \mathbb{R}^N$ by $e_i^{ij} = 1$, $e_j^{ij} = -1$, and $e_\ell^{ij} = 0$ for all $\ell \in N \backslash \{i,j\}$. Moreover, for $\mu \in \mathbb{R}$, we define $e^\mu \in \mathbb{R}^N$ by $e_\ell^\mu = \frac{\mu}{n}$ for all $\ell \in N$.

For all $i \in N$, $j \in N \backslash \{i\}$, and $\mu \in \mathbb{R}$, let the mapping $g_{ij}^\mu : \mathbb{R} \to \mathbb{R}$ be given by

$$g_{ij}^\mu(\lambda) := \varphi_i(e^\mu + \lambda \cdot e^{ij}) - \frac{\mu}{n} \quad \text{for all } \lambda \in \mathbb{R}. \tag{A.4}$$

Note that for $\varphi = \mathrm{Sh}^\alpha$, $\alpha \in [0,1]$, we have $g_{ij}^\mu(\lambda) = \alpha \cdot \lambda$ for all $\lambda \in \mathbb{R}$. In the following, we use the mappings g_{ij}^μ in order to derive the parameter α from φ. We proceed by a number of subclaims. First, we show that g_{ij}^μ does not depend on the choice of $j \in N \backslash \{i\}$.

Claim C1a. For $i \in N$ and $\lambda, \mu \in \mathbb{R}$, we have $\varphi_i(e^\mu + \lambda \cdot e^{ij}) = \varphi_i(e^\mu + \lambda \cdot e^{ik})$ for all $j, k \in N \backslash \{i\}$.

For $j = k$, nothing is to show. Let now $j \neq k$ and $\ell \in N \backslash \{i,j,k\}$. Player ℓ is a dummy player in $e^\mu + \lambda \cdot e^{ij} \in \bar{\mathbb{V}}$ and in $e^\mu + \lambda \cdot e^{ik} \in \bar{\mathbb{V}}$ with $\left(e^\mu + \lambda \cdot e^{ij}\right)_\ell = \frac{\mu}{n} = \frac{1}{n} \cdot \left(e^\mu + \lambda \cdot e^{ij}\right)(N)$ and $\left(e^\mu + \lambda \cdot e^{ik}\right)_\ell = \frac{\mu}{n} = \frac{1}{n} \cdot \left(e^\mu + \lambda \cdot e^{ik}\right)(N)$. By **AD**, we have (†) $\varphi_\ell\left(e^\mu + \lambda \cdot e^{ij}\right) = \varphi_\ell\left(e^\mu + \lambda \cdot e^{ik}\right)$. Since i and ℓ are symmetric in $\lambda \cdot e^{ij} - \lambda \cdot e^{ik}$, players i and ℓ, $e^\mu + \lambda \cdot e^{ij}$, and $e^\mu + \lambda \cdot e^{ik}$ satisfy the hypothesis of **DMo**$^-$. Hence, **DMo**$^-$ and (†) imply $\varphi_i\left(e^\mu + \lambda \cdot e^{ij}\right) = \varphi_i\left(e^\mu + \lambda \cdot e^{ik}\right)$. □

For $i \in N$ and $\mu \in \mathbb{R}$, we define $g_i^\mu : \mathbb{R} \to \mathbb{R}$ by

$$g_i^\mu := g_{ij}^\mu \quad \text{for all } \lambda \in \mathbb{R} \text{ and some } j \in N \backslash \{i\}. \tag{A.5}$$

By (A.4) and **C1a**, g_{ij}^{μ} does not depend on the choice of $j \in N \backslash \{i\}$. Hence, g_i^{μ} is well-defined. By **AD**, we have $g_i^{\mu}(0) = 0$. Next, we show that g_i^{μ} does not depend on the choice of $i \in N$.

Claim C1b. For all $i, j \in N$, $i \neq j$ and $\lambda, \mu \in \mathbb{R}$, we have $g_i^{\mu}(\lambda) = g_j^{\mu}(\lambda)$.

For $k \in N \backslash \{i, j\}$, we have

$$g_i^{\mu}(\lambda) + g_k^{\mu}(-\lambda) \overset{(A.4),(A.5)}{=} \varphi_i(e^{\mu} + \lambda \cdot e^{ik}) - \frac{\mu}{n} + \varphi_k(e^{\mu} - \lambda \cdot e^{ki}) - \frac{\mu}{n}$$
$$= \varphi_i(e^{\mu} + \lambda \cdot e^{ik}) - \frac{\mu}{n} + \varphi_k(e^{\mu} + \lambda \cdot e^{ik}) - \frac{\mu}{n}, \quad \text{(A.6)}$$

where the second equation drops from $\lambda \cdot e^{ik} = -\lambda \cdot e^{ki}$. By **AD**, we have $\varphi_{\ell}(e^{\mu} + \lambda \cdot e^{ik}) = \frac{\mu}{n}$ for all $\ell \in N \backslash \{i, k\}$. Hence, **E** entails $\varphi_i(e^{\mu} + \lambda \cdot e^{ik}) + \varphi_k(e^{\mu} + \lambda \cdot e^{ik}) = \frac{2\mu}{n}$. Together with (A.6), we obtain $g_i^{\mu}(\lambda) + g_k^{\mu}(-\lambda) = 0$. Analogously, one shows $g_j^{\mu}(\lambda) + g_k^{\mu}(-\lambda) = 0$, which concludes the proof. □

For $\mu \in \mathbb{R}$, we define $g^{\mu} : \mathbb{R} \to \mathbb{R}$ by $g^{\mu} = g_i^{\mu}$ for some $i \in N$. By **C1b**, g^{μ} is well-defined. In the following, we show certain properties of the mappings g^{μ} and their relation to φ. For later use, we first show that g^{μ} is odd.

Claim C1c. For all $\lambda, \mu \in \mathbb{R}$, we have $g^{\mu}(\lambda) = -g^{\mu}(-\lambda)$.

For $i, j \in N$, $i \neq j$, we have

$$g^{\mu}(\lambda) + g^{\mu}(-\lambda) \overset{(A.4),(A.5)}{=} \varphi_i\left(e^{\mu} + \lambda \cdot e^{ij}\right) - \frac{\mu}{n} + \varphi_j\left(e^{\mu} - \lambda \cdot e^{ji}\right) - \frac{\mu}{n}$$
$$= \varphi_i\left(e^{\mu} + \lambda \cdot e^{ij}\right) - \frac{\mu}{n} + \varphi_j\left(e^{\mu} + \lambda \cdot e^{ij}\right) - \frac{\mu}{n}, \quad \text{(A.7)}$$

where the second equation drops from $\lambda \cdot e^{ij} = -\lambda \cdot e^{ji}$. By **AD**, we have $\varphi_{\ell}\left(e^{\mu} + \lambda \cdot e^{ij}\right) = 0$ for all $\ell \in N \backslash \{i, j\}$. Hence, **E** entails $\varphi_i\left(e^{\mu} + \lambda \cdot e^{ij}\right) + \varphi_j\left(e^{\mu} + \lambda \cdot e^{ij}\right) = \frac{2\mu}{n}$. Together with (A.7), this proves the claim. □

For later use, we show a technical relation between the mappings g^{μ} and φ. Note that in view of **AD**, average players in $x \in \mathbb{R}_{\mu}^{N}$, i.e., players i with $x_i = \frac{\mu}{n}$ are of particular interest. For $\mu \in \mathbb{R}$, set $\mathbb{R}_{\mu}^{N} = \left\{x \in \mathbb{R}^{N} \mid \sum_{\ell \in N} x_{\ell} = \mu\right\}$ and $\bar{\mathbb{R}}_{\mu}^{N} := \left\{x \in \mathbb{R}_{\mu}^{N} \mid \text{there exist some } i \in N \text{ such that } x_i = \frac{\mu}{n}\right\}$. For $x \in \bar{\mathbb{R}}_{\mu}^{N}$, we set $C_{\mu}(x) := \left\{i \in \mathbb{N}_n \mid x_i \neq \frac{\mu}{n}\right\}$.

Claim C1d. For all $\mu \in \mathbb{R}$, $x \in \bar{\mathbb{R}}_{\mu}^{N}$, and $i \in N$, we have $\varphi_i(x) = g^{\mu}\left(x_i - \frac{\mu}{n}\right) + \frac{\mu}{n}$.

We proceed by induction on $\left|C_{\mu}(x)\right|$.

Induction base: For $\left|C_{\mu}(x)\right| = 0$, **AD** entails $\varphi_i(x) = \frac{\mu}{n} = g^{\mu}(0) + \frac{\mu}{n} = g^{\mu}\left(x_i - \frac{\mu}{n}\right) + \frac{\mu}{n}$ for all $i \in N$. Note that $\left|C_{\mu}(x)\right| \neq 1$ for all $x \in \bar{\mathbb{R}}_{\mu}^{N}$. If $\left|C_{\mu}(x)\right| = 2$, then there are $i, j \in N$, $i \neq j$ such that $x = e^{\mu} + \left(x_i - \frac{\mu}{n}\right) \cdot e^{ij} = e^{\mu} + \left(x_j - \frac{\mu}{n}\right) \cdot e^{ji}$. By (A.4) and (A.5), we have $\varphi_i(x) = g^{\mu}\left(x_i - \frac{\mu}{n}\right) + \frac{\mu}{n}$ and $\varphi_j(x) = g^{\mu}\left(x_j - \frac{\mu}{n}\right) + \frac{\mu}{n}$. Moreover, for $k \in N \backslash \{i, j\}$, **AD** implies $\varphi_k(x) = \frac{\mu}{n} = g^{\mu}(0) + \frac{\mu}{n} = g^{\mu}\left(x_k - \frac{\mu}{n}\right) + \frac{\mu}{n}$.

Induction hypothesis: Let the claim hold for all $x \in \bar{\mathbb{R}}^N_\mu$ such that $\left|C_\mu(x)\right| \leq t$, $t \in \mathbb{N}$.

Induction step: Let $x \in \bar{\mathbb{R}}^N_\mu$ be such that $\left|C_\mu(x)\right| = t + 1 > 2$. Suppose $\varphi_i(x) \neq g^\mu\left(x_i - \frac{\mu}{n}\right) + \frac{\mu}{n}$ for some $i \in \mathbb{N}_n$. By **AD**, $i \in C_\mu(x)$. Let $j, k \in C_\mu(x) \setminus \{i\}$, $j \neq k$, and $y = x - \left(x_j - \frac{\mu}{n}\right) \cdot e^{jk}$. Note that $y \in \bar{\mathbb{R}}^N_\mu$, $\left|C_\mu(y)\right| \leq t$, and $\left|C_\mu(y)\right| \neq 1$. By the induction hypothesis, we have $\varphi_i(y) = g^\mu\left(y_i - \frac{\mu}{n}\right) + \frac{\mu}{n} = g^\mu\left(x_i - \frac{\mu}{n}\right) + \frac{\mu}{n}$. By assumption, there exists $\ell \in N \setminus C_\mu(x)$ such that $x_\ell = \frac{\mu}{n}$. Hence, we obtain $\varphi_i(x) - \varphi_i(y) \neq g^\mu\left(x_i - \frac{\mu}{n}\right) + \frac{\mu}{n} - g^\mu\left(x_i - \frac{\mu}{n}\right) - \frac{\mu}{n} = 0$ and $\varphi_\ell(x) - \varphi_\ell(y) = 0$, where the latter drops from **AD.** Since i and ℓ are symmetric in $x - y$, x, y, i, and j satisfy the hypothesis of **DMo**$^-$. Hence, this contradicts **DMo**$^-$. □

For later use, we show crucial properties of the mappings g^μ, where linearity is of particular importance.

Claim C1e. For all $\mu \in \mathbb{R}$, the mapping $g^\mu : \mathbb{R} \to \mathbb{R}$ is linear and monotonic.

We show that the mapping g^μ is additive and monotonic. Then, linearity drops from Aczél (1966, Theorem 1).

Additivity: Let $a, b \in \mathbb{R}$. Let $i, j, k \in N$ and $x \in \mathbb{R}^N_\mu$ be such that $i \neq j$, $j \neq k$, $k \neq i$,

$$x_i = \frac{\mu}{n} + a, \quad x_j = \frac{\mu}{n} + b, \quad x_k = \frac{\mu}{n} - a - b, \quad x_\ell = \frac{\mu}{n} \qquad \text{for all } \ell \in \mathbb{N}_n \setminus \{i, j, k\}.$$

Since $n > 3$, $x \in \bar{\mathbb{R}}^N_\mu$. By **C1d**, we have

$$\varphi_i(x) = g^\mu(a) + \frac{\mu}{n}, \quad \varphi_j(x) = g^\mu(b) + \frac{\mu}{n}, \quad \text{and} \quad \varphi_k(x) = g^\mu(-a-b) + \frac{\mu}{n}.$$

Further, by **AD**, we have $\varphi_\ell(x) = \frac{\mu}{n}$ for all $\ell \in N \setminus \{i, j, k\}$. Hence, we obtain

$$0 \overset{\mathbf{E}}{=} \varphi_i(x) + \varphi_j(x) + \varphi_k(x) - \frac{3\mu}{n} = g^\mu(a) + g^\mu(b) + g^\mu(-a-b) \overset{\mathbf{C1c}}{=} g^\mu(a) + g^\mu(b) - g^\mu(a+b).$$

That is, the mapping g is additive.

Monotonicity: Let $a, b \in \mathbb{R}$ and $i, j, k \in N$ be such that $i \neq j$, $j \neq k$, $k \neq i$, and $a \geq b$. For $x = e^\mu + a \cdot e^{ij}$ and $y = e^\mu + b \cdot e^{ij}$, we have $x_i - y_i = a \geq b = x_k - y_k$. Moreover, by **AD**, $\varphi_k(x) = \varphi_k(y) = \frac{\mu}{n}$. Hence, we obtain

$$g^\mu(a) \overset{(A.4),(A.5)}{=} \varphi_i(x) \overset{\mathbf{DMo}^-}{\geq} \varphi_i(y) \overset{(A.4),(A.5)}{=} g^\mu(b).$$

That is, the mapping g^μ is monotonic. □

For $\mu \in \mathbb{R}$, set $\alpha^\mu := g^\mu(1)$. The next claim already establishes **C1** for all $x \in \mathbb{R}^N_\mu$.

Claim C1f. For all $\mu \in \mathbb{R}$ and $x \in \mathbb{R}^N_\mu$, we have $\varphi(x) = \alpha^\mu \cdot x + (1 - \alpha^\mu) \cdot e^\mu$.

Case (i): For $x \in \bar{\mathbb{R}}^N_\mu$ and $i \in N$, we obtain

$$\varphi_i(x) \overset{\mathbf{C1d}}{=} g_i^\mu\left(x_i - \frac{\mu}{n}\right) + \frac{\mu}{n} \overset{\mathbf{C1e}}{=} \alpha^\mu \cdot \left(x_i - \frac{\mu}{n}\right) + \frac{\mu}{n} = \alpha^\mu \cdot x_i + \left(1 - \alpha^\mu\right) \cdot \frac{\mu}{n}.$$

Case (ii): Let $x \in \mathbb{R}^N_\mu \backslash \bar{\mathbb{R}}^N_\mu$. Suppose $\varphi(x) \neq \alpha^\mu \cdot x + (1 - \alpha^\mu) \cdot e^\mu$. By **E**, we have $\sum_{\ell \in N} \varphi_\ell(x) = \mu = \sum_{\ell \in N}\left[\alpha^\mu \cdot x_\ell + (1 - \alpha^\mu) \cdot \frac{\mu}{n}\right]$. Hence, there are $i, j \in N$ such that $\varphi_i(x) > \alpha^\mu \cdot x_i + (1 - \alpha^\mu) \cdot \frac{\mu}{n}$ and $\varphi_j(x) < \alpha^\mu \cdot x_j + (1 - \alpha^\mu) \cdot \frac{\mu}{n}$. Let $k, \ell \in N\backslash\{i, j\}$, and $y = x - \left(x_k - \frac{\mu}{n}\right) \cdot e^{k\ell}$. Note that $x_i = y_i$ and $x_j = y_j$. Further, note that $y_k = \frac{\mu}{n}$ and therefore $y \in \bar{\mathbb{R}}^N_\mu$. By *Case (i)*, we obtain $\varphi(y) = \alpha^\mu \cdot y + (1 - \alpha^\mu) \cdot e^\mu$. Moreover, we have

$$\varphi_i(x) - \varphi_i(y) > \alpha^\mu \cdot x_i + \left(1 - \alpha^\mu\right) \cdot \frac{\mu}{n} - \left[\alpha^\mu \cdot y_i + \left(1 - \alpha^\mu\right) \cdot \frac{\mu}{n}\right] = 0$$

and

$$\varphi_j(x) - \varphi_j(y) < \alpha^\mu \cdot x_j + \left(1 - \alpha^\mu\right) \cdot \frac{\mu}{n} - \left[\alpha^\mu \cdot y_j + \left(1 - \alpha^\mu\right) \cdot \frac{\mu}{n}\right] = 0.$$

Since i and j are symmetric in $x - y$, x, y, i, and j satisfy the hypothesis of $\mathbf{DMo}^-$. Hence, this contradicts $\mathbf{DMo}^-$. □

Now, we are ready to prove **C1**.

Case (a): Suppose $\alpha^\mu = 0$ for all $\mu \in \mathbb{R}$. By **C1f**, we obtain $\varphi_i(x) = \frac{1}{n} \cdot \sum_{\ell \in N} x_\ell \overset{(A.3)}{=} \mathrm{Sh}^0(x)$ for all $x \in \mathbb{R}^N$.

Case (b): Suppose $\alpha^{\bar{\mu}} \neq 0$ for some $\bar{\mu} \in \mathbb{R}$. By **C1e**, we have $\alpha^\mu > 0$. Set $\alpha := \alpha^{\bar{\mu}}$. We show that

$$\varphi_i(x) = \alpha \cdot x_i + \frac{1 - \alpha}{n} \cdot \sum_{\ell \in N} x_\ell \qquad \text{for all } x \in \mathbb{R}^N \text{ and } i \in N. \tag{A.8}$$

Suppose there exists some $x \in \mathbb{R}^N$ such that (A.8) fails for some $i \in N$. By **C1f**, $x \notin \mathbb{R}^N_{\bar{\mu}}$. By **E** and w.l.o.g., there exists $j \in N\backslash\{i\}$ such that

$$\varphi_i(x) > \alpha \cdot x_i + \frac{1 - \alpha}{n} \cdot \sum_{\ell \in N} x_\ell \tag{A.9}$$

and

$$\varphi_j(x) < \alpha \cdot x_j + \frac{1 - \alpha}{n} \cdot \sum_{\ell \in N} x_\ell. \tag{A.10}$$

Let

$$X := \frac{1 - \alpha}{n} \cdot \left[\left(\sum_{\ell \in N} x_\ell\right) - \bar{\mu}\right]. \tag{A.11}$$

Further, let $k \in N \setminus \{i, j\}$ and let $y \in \mathbb{R}^N$ be given by

$$y_i = x_i + \frac{X}{\alpha}, \quad y_j = x_j + \frac{X}{\alpha}, \quad y_k = -x_i - x_j - \frac{2 \cdot X}{\alpha} + \bar{\mu}, \quad \text{and} \quad y_\ell = 0 \quad (A.12)$$

for all $\ell \in N \setminus \{i, j, k\}$. Since $y \in \mathbb{R}^N_{\bar{\mu}}$, by **C1f**, we have $\varphi(y) = \alpha \cdot y + (1 - \alpha) \cdot e^{\bar{\mu}}$. By (A.9), (A.10), (A.11), and (A.12), we obtain

$$\varphi_i(x) - \varphi_i(y) > \alpha \cdot x_i + \frac{1 - \alpha}{n} \cdot \sum_{\ell \in N} x_\ell - \left(\alpha \cdot x_i + X + \frac{1 - \alpha}{n} \cdot \bar{\mu} \right) = 0$$

and

$$\varphi_j(x) - \varphi_j(y) < \alpha \cdot x_j + \frac{1 - \alpha}{n} \cdot \sum_{\ell \in N} x_\ell - \left(\alpha \cdot x_j + X + \frac{1 - \alpha}{n} \cdot \bar{\mu} \right) = 0.$$

Since i and j are symmetric in $x - y$, x, y, i, and j satisfy the hypothesis of **DMo**$^-$. Therefore, this contradicts **DMo** $^-$. Hence, $\varphi(x) \overset{(A.3)}{=} \mathrm{Sh}^\alpha(x)$ for all $x \in \mathbb{R}^N$.

Finally, we have

$$\alpha \overset{(A.8)}{=} -\varphi_j\left(e^{ij}\right) \overset{\mathbf{AD}}{\leq} -\left(e^{ij}\right)_j = 1 \qquad \text{for } i \in N \text{ and } j \in N \setminus \{i\},$$

which concludes the proof of **C1**. □

If $|\mathcal{T}_{>1}(v)| = 1$ for $v \in \mathbb{V}$, then there are $\delta^v \in \mathbb{R}^N$ and $\beta^v \in \mathbb{R}$, and $T^v \subseteq N$, $|T^v| > 1$ such that $\beta^v \neq 0$ and

$$v = \beta^v \cdot \bar{u}_{T^v} + \sum_{\ell \in N} \delta^v_\ell \cdot u_{\{\ell\}}. \quad (A.13)$$

Set

$$R^v := \left\{ i \in N \mid v(\{i\}) \neq \frac{1}{|N|} \cdot v(N) \right\}.$$

Note that players $i \in N \setminus (R^v \cup T^v)$ are average dummy players, i.e., dummy players with $v(\{i\}) = \frac{1}{|N|} \cdot v(N)$ for which **AD** implies $\varphi(v) = v(\{i\})$.

We now show that $\varphi(v) = \mathrm{Sh}^\alpha(v)$ for all $v \in \mathbb{V}$ with $|\mathcal{T}_{>1}(v)| = 1$ by a number of claims. First, we deal with games in which there exists a average dummy player.

Claim 2, C2: For all $v \in \mathbb{V}$ with $|\mathcal{T}_{>1}(v)| = 1$ and such that $R^v \cup T^v \neq N$, we have $\varphi(v) = \mathrm{Sh}^\alpha(v)$.

By **C1**, we have

$$\varphi\left(v - \beta^v \cdot \bar{u}_{T^v}\right) = \mathrm{Sh}^\alpha\left(v - \beta^v \cdot \bar{u}_{T^v}\right) \overset{(4),(A.1)}{=} \mathrm{Sh}^\alpha(v). \quad (A.14)$$

For $i \in N\backslash(R^v \cup T^v)$, we have

$$\varphi_i(v) \overset{\mathbf{AD}}{=} \frac{v(N)}{|N|} \overset{(4),(A.1)}{=} \mathrm{Sh}_i^\alpha\left(v - \beta^v \cdot \bar{u}_{T^v}\right) \overset{\mathbf{C1}}{=} \varphi_i\left(v - \beta^v \cdot \bar{u}_{T^v}\right). \tag{A.15}$$

Since all players in $N\backslash T^v$ are pairwise symmetric in $-\beta^v \cdot \bar{u}_{T^v}$, v, $v - \beta^v \cdot \bar{u}_{T^v}$, and $i, \ell \in N\backslash T^v$ satisfy the hypothesis of **DMo**$^-$. Hence, we have

$$\varphi_\ell(v) \overset{(A.15),\mathbf{DMo}^-}{=} \varphi_\ell\left(v - \beta^v \cdot \bar{u}_{T^v}\right) \qquad \text{for all } \ell \in N\backslash T^v. \tag{A.16}$$

Since any two players in T^v are pairwise symmetric in $-\beta^v \cdot \bar{u}_{T^v}$, v, $v - \beta^v \cdot \bar{u}_{T^v}$, and $k, \ell \in T^v$ satisfy the hypothesis of **DMo**$^-$, which implies that we have

$$\varphi_k(v) \gtrless \varphi_k\left(v - \beta^v \cdot \bar{u}_{T^v}\right) \quad \text{if and only if} \quad \varphi_\ell(v) \gtrless \varphi_\ell\left(v - \beta^v \cdot \bar{u}_{T^v}\right) \tag{A.17}$$

for all $k, \ell \in T^v$. By **E**, (A.16), and (A.17), we finally have $\varphi(v) = \mathrm{Sh}^\alpha(v)$. □

Next, we handle games in which there are average players but which are not dummy players.

Claim 3, C3: For all $v \in \mathbb{V}$ with $|\mathcal{T}_{>1}(v)| = 1$ such that $R^v \cup T^v = N$ and $|T^v\backslash R^v| \geq 1$, we have $\varphi(v) = \mathrm{Sh}^\alpha(v)$.

Suppose $\varphi(v) \neq \mathrm{Sh}(v)$. By **E**, there are $i, j \in N$ such that

$$\varphi_i(v) > \mathrm{Sh}_i^\alpha(v) \qquad \text{and} \qquad \varphi_j(v) < \mathrm{Sh}_j^\alpha(v). \tag{A.18}$$

Case (i): Suppose $i, j \in R^v\backslash T^v$ or $i, j \in T^v$. By (A.13), we have $v - \beta^v \cdot \bar{u}_{T^v} \in \bar{\mathbb{V}}$. Hence, **C1** implies

$$\varphi\left(v - \beta^v \cdot \bar{u}_{T^v}\right) = \mathrm{Sh}^\alpha\left(v - \beta^v \cdot \bar{u}_{T^v}\right). \tag{A.19}$$

By (A.18) and (A.19), we further have

$$\varphi_i(v) - \varphi_i\left(v - \beta^v \cdot \bar{u}_{T^v}\right) > \mathrm{Sh}_i^\alpha(v) - \mathrm{Sh}_i^\alpha\left(v - \beta^v \cdot \bar{u}_{T^v}\right) = \mathrm{Sh}_i^\alpha\left(\beta^v \cdot \bar{u}_{T^v}\right) \overset{(4),(A.1)}{=} 0$$

and

$$\varphi_j(v) - \varphi_j\left(v - \beta^v \cdot \bar{u}_{T^v}\right) < \mathrm{Sh}_j^\alpha(v) - \mathrm{Sh}_j^\alpha\left(v - \beta^v \cdot \bar{u}_{T^v}\right) = \mathrm{Sh}_j^\alpha\left(\beta^v \cdot \bar{u}_{T^v}\right) \overset{(4),(A.1)}{=} 0.$$

Since i and j are symmetric in $-\beta^v \cdot \bar{u}_{T^v}$, v, $v - \beta^v \cdot \bar{u}_{T^v}$, i, and j satisfy the hypothesis of **DMo**$^-$. Hence, this contradicts **DMo**$^-$.

Case (ii): Suppose, w.l.o.g., $i \in R^v\backslash T^v$ and $j \in T^v$.

Case (ii-a): Suppose $j \in T^v\backslash R^v$. Let $w := v - \beta^v \cdot \bar{u}_{T^v} - \beta^v \cdot \bar{u}_{(T^v\backslash\{j\})\cup\{i\}}$. By **C2**, we have

$$\varphi(w) = \mathrm{Sh}^\alpha(w). \tag{A.20}$$

By (A.18) and (A.20), we further have

$$\varphi_i(v) - \varphi_i(w) > \mathrm{Sh}_i^\alpha(v) - \mathrm{Sh}_i^\alpha(w) = \mathrm{Sh}_i^\alpha\left(\beta^v \cdot \bar{u}_{T^v} + \beta^v \cdot \bar{u}_{(T^v \setminus \{j\}) \cup \{i\}}\right) \overset{(4),(A.1)}{=} 0$$

and

$$\varphi_j(v) - \varphi_j(w) < \mathrm{Sh}_j^\alpha(v) - \mathrm{Sh}_j^\alpha(w) = \mathrm{Sh}_j^\alpha\left(\beta^v \cdot \bar{u}_{T^v} + \beta^v \cdot \bar{u}_{(T^v \setminus \{j\}) \cup \{i\}}\right) \overset{(4),(A.1)}{=} 0.$$

Since i and j are symmetric in $-\beta^v \cdot \bar{u}_{T^v} - \beta^v \cdot \bar{u}_{(T^v \setminus \{j\}) \cup \{i\}}$, v, w, i, and j satisfy the hypothesis of **DMo**$^-$. Hence, this contradicts **DMo**$^-$.

Case (ii-b): Suppose $j \in T^v \cap R^v$. By assumption, there exists $k \in T^v \setminus R^v$ such that $k \neq i$ and $k \neq j$. By **C1**, we have

$$\begin{aligned} &\varphi_j\left(v - \beta^v \cdot \bar{u}_{T^v}\right) = \mathrm{Sh}_j^\alpha\left(v - \beta^v \cdot \bar{u}_{T^v}\right) \quad \text{and} \\ &\varphi_k\left(v - \beta^v \cdot \bar{u}_{T^v}\right) = \mathrm{Sh}_k^\alpha\left(v - \beta^v \cdot \bar{u}_{T^v}\right). \end{aligned} \tag{A.21}$$

By (A.18) and (A.21), we have

$$\varphi_j(v) - \varphi_j\left(v - \beta^v \cdot \bar{u}_{T^v}\right) < \mathrm{Sh}_j^\alpha(v) - \mathrm{Sh}_j^\alpha\left(v - \beta^v \cdot \bar{u}_{T^v}\right) = \mathrm{Sh}_j^\alpha\left(\beta^v \cdot \bar{u}_{T^v}\right) \overset{(4),(A.1)}{=} 0.$$

Since j and k are symmetric in $-\beta^v \cdot \bar{u}_{T^v}$, v, $v - \beta^v \cdot \bar{u}_{T^v}$, i, and k satisfy the hypothesis of **DMo**$^-$. Hence, **DMo**$^-$ entails

$$\varphi_k(v) - \varphi_k(v - \beta^v \cdot \bar{u}_{T^v}) < 0.$$

Since

$$\varphi_k(v - \beta^v \cdot \bar{u}_{T^v}) \overset{(A.21)}{=} \mathrm{Sh}_k^\alpha(v - \beta^v \cdot \bar{u}_{T^v}) \overset{(4),(A.1)}{=} \mathrm{Sh}_k^\alpha(v),$$

we obtain

$$\varphi_k(v) < \mathrm{Sh}_k^\alpha(v). \tag{A.22}$$

Let $z := v - \beta^v \cdot \bar{u}_{T^v} - \beta^v \cdot \bar{u}_{(T^v \setminus \{k\}) \cup \{i\}}$. By (A.18), (A.22), and **C2**, we have

$$\varphi_i(v) - \varphi_i(z) > \mathrm{Sh}_i^\alpha(v) - \mathrm{Sh}_i^\alpha(z) = \mathrm{Sh}_i^\alpha\left(\beta^v \cdot \bar{u}_{T^v} + \beta^v \cdot \bar{u}_{(T^v \setminus \{k\}) \cup \{i\}}\right) \overset{(4),(A.1)}{=} 0 \tag{A.23}$$

and

$$\varphi_k(v) - \varphi_k(z) < \mathrm{Sh}_k^\alpha(v) - \mathrm{Sh}_k^\alpha(z) = \mathrm{Sh}_k^\alpha\left(\beta^v \cdot \bar{u}_{T^v} + \beta^v \cdot \bar{u}_{(T^v \setminus \{k\}) \cup \{i\}}\right) \overset{(4),(A.1)}{=} 0. \tag{A.24}$$

Since $i \in R^v \setminus T^v$ and $k \in T^v \setminus R^v$, i and k are symmetric in $-\beta^v \cdot \bar{u}_{T^v} - \beta^v \cdot \bar{u}_{(T^v \setminus \{k\}) \cup \{i\}}$, v, z, i, and k satisfy the hypothesis of **DMo** $^-$. Hence, (A.23) and (A.24) contradict **DMo** $^-$. □

Finally, we deal with games in which there are no average players.

Claim 4, C4: For all $v \in \mathbb{V}$ with $|\mathcal{T}_{>1}(v)| = 1$ such that $R^v \cup T^v = N$ and $|T^v \backslash R^v| = 0$, we have $\varphi(v) = \mathrm{Sh}^\alpha(v)$.

By assumption, we have $R^v = N$. Suppose $\varphi(v) \neq \mathrm{Sh}(v)$. By **E**, there are $i, j \in N$ such that

$$\varphi_i(v) > \mathrm{Sh}_i^\alpha(v) \qquad \text{and} \qquad \varphi_j(v) < \mathrm{Sh}_j^\alpha(v). \tag{A.25}$$

Let $k \in N \backslash \{i, j\}$, $\ell \in N \backslash \{i, j, k\}$, and $q \in \mathbb{V}$ be given by

$$q := \left(v(\{k\}) - \frac{v(N)}{|N|}\right) \cdot \left(u_{\{k\}} - u_{\{\ell\}}\right). \tag{A.26}$$

By (A.13), we have

$$v - q = \beta^v \cdot \bar{u}_{T^v} + \frac{v(N)}{|N|} \cdot u_{\{k\}} + \left(\delta_\ell^v + \delta_k^v - \frac{v(N)}{|N|}\right) \cdot u_{\{\ell\}} + \sum_{h \in N \backslash \{k\}} \delta_h^v \cdot u_{\{h\}}.$$

Hence, we have $|\mathcal{T}_{>1}(v-q)| = 1$, $T^{v-q} = T^v$, and

$$(v-q)(\{k\}) = \frac{v(N)}{|N|} = \frac{(v-q)(N)}{|N|},$$

where the latter implies $k \notin R^{v-q}$. Note that q is constructed in a way such that k is an average player in $v - q$.

If $k \notin T^v$, then $v - q$ satisfies the hypothesis of **C2** and we obtain

$$\varphi(v-q) = \mathrm{Sh}^\alpha(v-q). \tag{A.27}$$

If $k \in T^v$, then $v - q$ satisfies the hypothesis of **C3** and we also obtain (A.27). By (A.25) and (A.27), we have

$$\varphi_i(v) - \varphi_i(v-q) > \mathrm{Sh}_i^\alpha(v) - \mathrm{Sh}_i^\alpha(v-q) = \mathrm{Sh}_i^\alpha(q) \overset{(4),(A.26)}{=} 0 \tag{A.28}$$

and

$$\varphi_j(v) - \varphi_j(v-q) < \mathrm{Sh}_j^\alpha(v) - \mathrm{Sh}_j^\alpha(v-q) = \mathrm{Sh}_j^\alpha(q) \overset{(4),(A.26)}{=} 0. \tag{A.29}$$

Since i and j are symmetric in $-q$, v, $v-q$, i and j satisfy the hypothesis of **DMo**$^-$. Hence, (A.28) and (A.29) together contradict **DMo**$^-$. □

Note that the induction basis (see page 8) is proved by **C1**, **C2**, **C3**, and **C4**.

Induction hypothesis: Let the claim hold for all $v \in \mathbb{V}$ such that $|\mathcal{T}_{>1}(v)| \leq t$, $t \in \mathbb{N}$, $t \geq 1$.

Induction step: Let now $v \in \mathbb{V}$ be such that $|\mathcal{T}_{>1}(v)| = t + 1$. There exist $S, T \in \mathcal{T}_{>1}(v)$ such that $S \neq T$. By (3), (A.2), (*) (see page 8), and the induction hypothesis, we have

$$\varphi(v_S) = \mathrm{Sh}^\alpha(v_S) = \mathrm{Sh}^\alpha(v) = \mathrm{Sh}^\alpha(v_T) = \varphi(v_T). \tag{A.30}$$

Case (i): $S \cap T \neq \emptyset$. W.l.o.g., $S \backslash T \neq \emptyset$. Let $i \in S \cap T$ and $j \in S \backslash T$. By (A.30) and **DMo**$^-$, we have

$$\begin{array}{llll} \varphi_\ell(v) \gtrless \mathrm{Sh}^\alpha_\ell(v) & \text{if and only if} & \varphi_i(v) \gtrless \mathrm{Sh}^\alpha_i(v) & \text{for all } \ell \in S, \\ \varphi_\ell(v) \gtrless \mathrm{Sh}^\alpha_\ell(v) & \text{if and only if} & \varphi_i(v) \gtrless \mathrm{Sh}^\alpha_i(v) & \text{for all } \ell \in T, \\ \varphi_\ell(v) \gtrless \mathrm{Sh}^\alpha_\ell(v) & \text{if and only if} & \varphi_j(v) \gtrless \mathrm{Sh}^\alpha_j(v) & \text{for all } \ell \in N \backslash T, \end{array}$$

and therefore

$$\varphi_\ell(v) \gtrless \mathrm{Sh}^\alpha_\ell(v) \quad \text{if and only if} \quad \varphi_i(v) \gtrless \mathrm{Sh}^\alpha_i(v) \qquad \text{for all } \ell \in N. \tag{A.31}$$

Case (ii): $S \cup T \neq N$. W.l.o.g., $S \backslash T \neq \emptyset$. Let $i \in N \backslash (S \cup T)$ and $j \in S \backslash T$. By (A.30) and **DMo**$^-$, we have

$$\begin{array}{llll} \varphi_\ell(v) \gtrless \mathrm{Sh}^\alpha_\ell(v) & \text{if and only if} & \varphi_i(v) \gtrless \mathrm{Sh}^\alpha_i(v) & \text{for all } \ell \in N \backslash S, \\ \varphi_\ell(v) \gtrless \mathrm{Sh}^\alpha_\ell(v) & \text{if and only if} & \varphi_i(v) \gtrless \mathrm{Sh}^\alpha_i(v) & \text{for all } \ell \in N \backslash T, \\ \varphi_\ell(v) \gtrless \mathrm{Sh}^\alpha_\ell(v) & \text{if and only if} & \varphi_j(v) \gtrless \mathrm{Sh}^\alpha_j(v) & \text{for all } \ell \in S, \end{array}$$

and therefore

$$\varphi_\ell(v) \gtrless \mathrm{Sh}^\alpha_\ell(v) \quad \text{if and only if} \quad \varphi_i(v) \gtrless \mathrm{Sh}^\alpha_i(v) \qquad \text{for all } \ell \in N. \tag{A.32}$$

Case (iii): $S \cap T = \emptyset$ and $S \cup T = N$. Hence, $\mathcal{T}_{>1}(v) = \{S, T\}$. Let $i \in S$, $j \in T$, and $w \in \mathbb{V}$ be given by

$$w = v_S - \lambda_S(v) \cdot u_{(S \backslash \{i\}) \cup \{j\}} + \frac{\lambda_S(v)}{|S|} \cdot \sum_{\ell \in (S \backslash \{i\}) \cup \{j\}} u_{\{\ell\}}. \tag{A.33}$$

By construction, we have $\mathcal{T}_{>1}(w) = \{(S \backslash \{i\}) \cup \{j\}, T\}$ and (****) $v(N) = w(N)$. In view of *Case (i)*, we have (*****) $\varphi(w) = \mathrm{Sh}(w)$.

Since i and j are symmetric in $v - w$, v, w, i, and j satisfy the hypothesis of **DMo**$^-$. Hence, by **DMo**$^-$ and (A.30), we have

$$\begin{array}{llll} \varphi_j(v) \gtrless \varphi_j(w) \overset{(*****)}{=} \mathrm{Sh}^\alpha_j(v) & \text{if and only if} & \varphi_i(v) \gtrless \varphi_i(w) \overset{(*****)}{=} \mathrm{Sh}^\alpha_i(v), & \\ \varphi_\ell(v) \gtrless \mathrm{Sh}^\alpha_\ell(v) & \text{if and only if} & \varphi_i(v) \gtrless \mathrm{Sh}^\alpha_i(v) & \text{for all } \ell \in S, \\ \varphi_\ell(v) \gtrless \mathrm{Sh}^\alpha_\ell(v) & \text{if and only if} & \varphi_j(v) \gtrless \mathrm{Sh}^\alpha_j(v) & \text{for all } \ell \in T, \end{array}$$

and therefore

$$\varphi_\ell(v) \gtrless \mathrm{Sh}^\alpha_\ell(v) \quad \text{if and only if} \quad \varphi_i(v) \gtrless \mathrm{Sh}^\alpha_i(v) \qquad \text{for all } \ell \in N. \tag{A.34}$$

Finally, (A.31), (A.32), (A.34), and **E** imply $\varphi(v) = \mathrm{Sh}^\alpha(v)$. □

Appendix B Counterexample to Theorem 3 for $|N| = 2$

Theorem 3 fails for $|N| = 2$. Let $N = \{1, 2\}$. Consider the solution $\varphi^{\heartsuit} : \mathbb{V} \to \mathbb{R}^2$ given by

$$\left(\varphi_1^{\heartsuit}(v), \varphi_2^{\heartsuit}(v)\right) = \begin{cases} (\mathrm{Sh}_1(v), \mathrm{Sh}_2(v)), & \mathrm{Sh}_1(v) \geq 0,\ \mathrm{Sh}_2(v) \geq 0, \\ \left(\mathrm{Sh}_1(v) + \frac{\mathrm{Sh}_2(v)}{2}, \frac{\mathrm{Sh}_2(v)}{2}\right), & \mathrm{Sh}_1(v) > 0,\ \mathrm{Sh}_2(v) < 0,\ v(N) \geq 0, \\ \left(\frac{\mathrm{Sh}_1(v)}{2}, \mathrm{Sh}_2(v) + \frac{\mathrm{Sh}_1(v)}{2}\right), & \mathrm{Sh}_1(v) > 0,\ \mathrm{Sh}_2(v) < 0,\ v(N) < 0, \\ (\mathrm{Sh}_1(v), \mathrm{Sh}_2(v)), & \mathrm{Sh}_1(v) \leq 0\ \mathrm{Sh}_2(v) \leq 0, \\ \left(\mathrm{Sh}_1(v) + \frac{\mathrm{Sh}_2(v)}{2}, \frac{\mathrm{Sh}_2(v)}{2}\right), & \mathrm{Sh}_1(v) < 0, \mathrm{Sh}_2(v) > 0,\ v(N) \geq 0, \\ \left(\frac{\mathrm{Sh}_1(v)}{2}, \mathrm{Sh}_2(v) + \frac{\mathrm{Sh}_1(v)}{2}\right), & \mathrm{Sh}_1(v) < 0, \mathrm{Sh}_2(v) > 0,\ v(N) < 0 \end{cases}$$

for all $v \in \mathbb{V}$. One can easily check that $\varphi^{\heartsuit} \neq \mathrm{Sh}^{\alpha}$ for all $\alpha \in [0, 1]$ and that $\varphi^{\heartsuit}$ inherits **E**, **AD**, and **DMo** $^-$ from Sh.

Appendix C Non-redundancy of Theorem 3 for $|N| > 3$

Our characterization is non-redundant for $|N| > 3$. The value $\varphi^{\mathbf{E}}$ given by $\varphi_i^{\mathbf{E}}(v) = v(\{i\})$ for all $v \in \mathbb{V}$ and $i \in N$ satisfies **AD** and **DMo**$^-$ but not **E**. The strictly positively weighted division values (Béal et al. 2016, Theorem 2) with non-uniform weights satisfy **E** and **DMo**$^-$ but not **AD**. For $v \in \mathbb{V}$, let $D_0(v) := \{i \in N \mid i \text{ is a dummy player in } v\}$. The value $\varphi^{\mathbf{DMo}^-}$ given by

$$\varphi_i^{\mathbf{DMo}^-}(v) = \begin{cases} \frac{v(N) - \sum_{\ell \in D_0(v)} v(\{\ell\})}{|N \backslash D_0(v)|}, & i \in N \backslash D_0(v), \\ v(\{i\}), & D_0(v) \end{cases} \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N$$

satisfies **E** and **AD** but not **DMo**$^-$.

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