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ORIGINAL PAPER

Multiplayer games as extension of misère games

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Abstract

We introduce i-misère play to multiplayer impartial games, which is an extension of Li's rank-based play and, at the same time, which contains the normal play and the misère play as special cases when the number of players is two. We characterize losing positions of such plays for both NIM and Moore's game. i-misère play is defined as a special case of a more general notion of preference-based play, and we also study properties of some other preference-based plays.

Keywords Combinatorial game theory · Multiplayer game · Misère play · NIM · Moore's game

1 Introduction

Multiplayer combinatorial games are difficult to analyze, because of the possibility of coalitions in multiplayer games. For example, consider a NIM position $(1, 2)$ in three-player NIM. If the current player moves to $(1, 0)$ or $(0, 2)$, then the second player wins. However if the current player moves to $(1, 1)$ then the second player moves to $(1, 0)$ and the third player wins. So, the current player has no winning strategy but she can choose whether the second player becomes the winner or the third player does.

With this observation, we introduce the notion of a preference-based play to multiplayer games. We consider that each player has her preference order, which is a total ordering of all the players, and the objective of each player is to let the last moving player of the game be the most preferred player with respect to her order. Of course, it is difficult to determine which move will lead to the most preferred result. However, we show that, under some assumptions, the most preferable move of each player is determined and thus the outcomes of each game position is determined. In this paper, we analyze such outcome of plays with preference order.

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We are particularly interested in the case that the preference order of each player is equal to the play order starting with the i-th player, which we call the i-misère play. 0-misère game is equivalent to Li's theory of multiplayer combinatorial games. In addition, for the two-player game, 0-misère play is equivalent to normal play and 1-misère play is equivalent to misère play. Therefore, it is a generalization of Li's theory (1978) as well as the generalization of both normal and misère games to multiplayer games.

We introduce early results and define i-misère play in Sect. 2, study i-misère NIM in Sect. 3 and study i-misère Moore's game in Sect. 4. In Sect. 5, we study another symmetric preference order and in Sect. 6, we study some asymmetric preference orders.

2 Normal, misère and multiplayer NIM

Among the early results of combinatorial game theory is a winning strategy for NIM by Bouton (1902). NIM is a two-player game with some heaps of stones, and the current player chooses one of the heaps and takes out some stones. The winner of NIM is the player who takes out the last stone. NIM is an impartial game in that both player have the same options in every game position.

We say that a game position is an N-position or a P-position if the next player or the previous player has a winning strategy, respectively. The following is one of the most important facts for a two-player impartial game.

Theorem 1 *A game position of an impartial two-player game is an N-position or a P-position.*

We can analyze whether a NIM position is an N-position or a P-position in a simple way, by calculating modulo-2 sum without carry which is denoted by $\oplus$ (NIM sum).

Theorem 2 (Bouton 1902) *A NIM position $(n_1, n_2, \ldots, n_k)$ is a P-position if and only if $n_1^{\langle 2\rangle} \oplus n_2^{\langle 2\rangle} \oplus \cdots \oplus n_k^{\langle 2\rangle} = 0$. Here, $n^{\langle 2\rangle}$ is the binary notation of n.*

In contrast to normal play, a game is called in misère play if the last player to move is the loser. We can also analyze misère NIM game in the following way.

Theorem 3 (Bouton 1902) *In misère NIM game, a position $(n_1, n_2, \ldots, n_k)$ is a P-position if and only if*

$$\begin{cases} n_1^{\langle 2\rangle} \oplus n_2^{\langle 2\rangle} \oplus \ldots \oplus n_k^{\langle 2\rangle} = 0 & (\exists j.\, n_j > 1) \\ n_1^{\langle 2\rangle} \oplus n_2^{\langle 2\rangle} \oplus \ldots \oplus n_k^{\langle 2\rangle} = 1 & (\forall j.\, n_j \leq 1). \end{cases}$$

2.1 Multiplayer game

In this paper we consider multiplayer games. We assume that there are m players $P_0, P_1, \ldots, P_{m-1}$ and they play in this order. For simplicity, any arithmetic in the subscript (e.g. P_{i+k}) is done modulo m.

In two-player impartial games, as Theorem 1, each game position is determined whether it is a P-position or an N-position. In contrast, as we mentioned above, multiplayer combinatorial games are difficult to analyze because of its possibility of coalitions. Therefore, people usually study multiplayer games by adding some assumptions to determine the game result. Robert Li (1978) defined a rank system. Krawec (2012; 2015) introduced alliance matrix and Liu and Duan (2017), Liu and Wang (2017a, b), Liu and Yang (2018); Liu and Zhou (2018); Liu and Wu (2019) studied some multiplayer games with Krawec's definitions.

In this paper, we introduce the notion of a preference-based play to multiplayer games. We assume that each player has her own preference order, which is a total ordering of all the players. We study the situation where each player knows the content of each other's preference, and it is common knowledge that every player knows the preferences of the other players, that is, each player knows that other players know the preferences of other players, and so on. We call such a play a preference-based play. In this article, we study the case that each player behaves optimally so that her most preferred player will move last, and if she cannot, then she behaves so that her second preferred player will move last,..., and so on.

We write the preference order of a player as $P_{i(0)} > P_{i(1)} > \cdots > P_{i(m-1)}$ if she wants player $P_{i(0)}$ to be the last player to move, and if that is impossible, she wants player $P_{i(1)}$ to play last, ..., and it is the worst result that player $P_{i(m-1)}$ becomes the last player to move.

Definition 1 (*Preference matrix*) Following Krawec (2012), we introduce an $m \times m$ matrix notation to express the preference orders of all the players.

$$\begin{bmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,m-1} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,m-1} \\ \vdots & \vdots & & \vdots \\ A_{m-1,0} & A_{m-1,1} & \cdots & A_{m-1,m-1} \end{bmatrix}$$

Here, $A_{j,k}$ is the index of the k-th preferred player of P_j relative to j with the most preferred player called the 0-th preferred player. That is, the preference order of P_j is $P_{j+A_{j,0}} > P_{j+A_{j,1}} > \cdots > P_{j+A_{j,m-1}}$.

Example 1 If the preference order of P_0, P_1 and P_2 are $P_0 > P_1 > P_2$, $P_2 > P_0 > P_1$ and $P_1 > P_2 > P_0$, respectively, then the preference matrix is

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Example 2 If the preference order of P_0, P_1, P_2 and P_3 are $P_1 > P_2 > P_3 > P_0$, $P_2 > P_3 > P_0 > P_1$, $P_3 > P_0 > P_1 > P_2$ and $P_0 > P_1 > P_2 > P_3$, respectively, then the preference matrix is

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 3 & 0 \end{bmatrix}.$$

Our notion of preference-based play is similar to the notion of alliance by Krawec (2012). The difference is that the objective of alliance-based play is to let the specified player have no moves, where the objective of our play is to let the specified player play last. The two notions are convertible and we can express our results with the notion of alliance-based play. However, then the constructions in the next section becomes complicated and we cannot state our result (Theorem 6) as a generalization of Li's result. The following notion of a last moving player is a reformulation of Krawec's notion of game value and Theorem 4 is his result expressed with the position of the last moving player.

Definition 2 For a game position G, we write $opt(G)$ as the set of all options at G. That is, for every $G' \in opt(G)$, one can move from G to G'.

Definition 3 We define the function l to be

$$l(G,t) = \begin{cases} m-1 & \text{if } G \text{ is an end position} \\ A_{t,q} & \text{otherwise.} \end{cases}$$

for $q = min\{j \in \mathbb{N} \mid l(G', t+1) + 1 = A_{t,j} \ with \ G' \in opt(G)\}$.

We call l the last moving player function because the following theorem holds.

Theorem 4 (Krawec) *If every player plays optimally, then $P_{t+l(G,t)}$ moves last in the game that starts with position G and player P_t. If the starting position G is an end position, then we consider P_{t-1} to have moved last.*

Proof If G is an end position then $l(G,t) = m-1$ by Definition 3 and player P_{t+m-1} moves last.

Otherwise, by induction hypothesis, for each $G' \in opt(G)$, $P_{t+1+l(G',t+1)}$ is the player who moves last in the game starts with the position G' and player P_{t+1}. Since we assume that every player plays optimally, P_t chooses G' which minimizes the number j such that $l(G', t+1) + 1 = A_{t,j}$.

By using this theorem, it is guaranteed that for multiplayer games in preference-based play, the result of each game is uniquely determined as using Theorem 1 for two-player games.

Definition 4 We say a multiplayer game is *preference-impartial* if the game is impartial and its preference matrix satisfies, $A_{i,k} = A_{j,k}$ for every $i, j, k < m$. In this case, we abbreviate the preference matrix as:

$$\begin{bmatrix} A_0 & A_1 & \cdots & A_{m-1} \end{bmatrix}$$

Definition 5 In a preference-impartial game, $l(X, t) = l(X, t')$ for any t and t'. Therefore we simply describe it as a unary function $l(X)$.

$$l(G) = \begin{cases} m-1 & \text{if } G \text{ is an end position} \\ A_q & \text{otherwise} \end{cases}$$

for $q = min\{j \in \mathbb{N} \mid l(G') + 1 = A_j \ with \ G' \in opt(G)\}$.

For $G = (n_1, n_2, \ldots, n_k)$, we will abbreviate $l(G)$ as $l(n_1, n_2, \ldots, n_k)$ when they will be no confusion.

Definition 6 If a game is a preference-impartial game and its preference matrix is

$$\begin{bmatrix} i & (i+1) & \cdots & (m-1) & 0 & 1 & \cdots & (i-1) \end{bmatrix},$$

then we say that it is in i-*misère play*.

In i-misère play, each player wants the i-th player to be the last moving player. The reason why we call it misère is that we obtain two-player misère play when $m = 2$ and $i = 1$. In a preference impartial game, according to Theorem 4, if every player plays optimally, then $P_{l(G)}$ plays last in the game G starting with player P_0. Therefore, the person whose preference order starts (resp. ends) with $P_{l(G)}$ obtains the most pleasant (resp. unpleasant) result. We call them the winner and the loser of the game, and say that a position is a winning (resp. losing) position if the starting player is the winner (resp. loser). Note that an N-position is a winning position and a P-position is a losing position for two-player games. In an i-misère game, a position is a winning position if $l(G) = i$ and is a losing position if $l(G) = i - 1$.

Of course, we can define other positions by using the values of $l(G)$. However, it seems to be difficult to characterize the positions.

2.2 Li's theory

In 1978, Li considered multiplayer NIM with rank system. He defined that the winner of the game is the person who moves last like two-player normal play. In addition, players are assigned a rank, ranging from bottom to top in the order of P_{k+1}, P_{k+2}, …, P_{m-1}, P_0, P_1, …, P_{k-1}, P_k when P_k is the winner and each player adopts an optimal strategy toward her own highest possible rank. That is, by definitions above, Li's theory is in 0-misère play, under the following preference matrix:

$$\begin{bmatrix} 0 \ 1 \cdots m-1 \end{bmatrix}$$

and with our terminology, the highest ranked player is the winner and the lowest ranked player is the loser.

In order to describe Li's result, we define a notion of modulo-m NIM sum.

Definition 7 For $k \geq 2$, let SEQ_k be the set of sequences of $\{0, 1, \ldots, k-1\}$ that do not start with 0. For simplicity, we write $0 \in SEQ_k$ for the empty sequence. Note that

$SEQ_k \subseteq SEQ_m$ if $k \leq m$. For a non-negative integer x, we write $x^{\langle k \rangle} \in SEQ_k$ for the k-ary notation of x. That is, $x^{\langle k \rangle} = (x_t, x_{t-1}, \ldots, x_1, x_0)$ if $x = \Sigma_{0 \leq s \leq t} x_s k^s$.

Definition 8 (*Generalized NIM sum*) On SEQ_m, we define the component-wise modulo-m addition operation $\oplus_m$ as follows. For $x, y \in SEQ_m$, if x and y have different length, then we prepend 0 s to the shorter sequence to adjust their length and then do modulo-m addition without carry on each component and then remove the leading 0 s from the result so that $x \oplus_m y$ do not start with 0. We simply write $\oplus$ for $\oplus_2$.

Example 3 $39^{\langle 2 \rangle} \oplus_3 17^{\langle 2 \rangle} = 100111 \oplus_3 010001 = 110112$

Example 4 $5^{\langle 2 \rangle} \oplus_3 7^{\langle 2 \rangle} \oplus_3 9^{\langle 2 \rangle} \oplus_3 15^{\langle 2 \rangle} = 0101 \oplus_3 0111 \oplus_3 1001 \oplus_3 1111 = 2021$

By using these definitions and notations, Li's result is described in the following theorem.

Theorem 5 (Robert Li 1978) *In 0-misère NIM,* $(n_1, n_2, \ldots, n_k)$ *is a losing position if and only if* $n_1^{\langle 2 \rangle} \oplus_m n_2^{\langle 2 \rangle} \oplus_m \cdots \oplus_m n_k^{\langle 2 \rangle} = 0$.

For the case $m = 2$, we can obtain Theorem 2 from this theorem. Therefore, this theorem is a generalization of Theorem 2 for multiplayer case. In the next section, we show a more generalized theorem.

3 Multiplayer misère NIM

In this section, we show a generalization of Li's theorem to i-misère play.

Theorem 6 *In i-misère NIM,* $(n_1, n_2, \ldots, n_k)$ *is a losing position if and only if*

$$\begin{cases} n_1^{\langle 2 \rangle} \oplus_m n_2^{\langle 2 \rangle} \oplus_m \cdots \oplus_m n_k^{\langle 2 \rangle} = 0 \;\; (\exists j.\, n_j > 1) \\ n_1^{\langle 2 \rangle} \oplus_m n_2^{\langle 2 \rangle} \oplus_m \cdots \oplus_m n_k^{\langle 2 \rangle} = i \;\; (\forall j.\, n_j \leq 1). \end{cases}$$

This theorem also means that we can know whether a given game position G is a winning position by checking whether G has an option G' which satisfies this condition.

For the case $(m, i) = (2, 0)$ we can obtain Theorem 2, for the case $(m, i) = (2, 1)$, we can obtain Theorem 3 and for the case $(m, i) = (m, 0)$, we can obtain Theorem 5 from this theorem. That is, this theorem reveal a hidden connection between misère NIM and multiplayer NIM.

In order to prove this theorem, we prepare some definitions and lemmas. In following Lemmas and Definitions, we consider i-misère play of m-player game in general. Recall that in i-misère play, G is a losing position iff $l(G) = (i - 1) \bmod m$.

Lemma 1 *In i-misère play,*

$$l(G) = \begin{cases} m - 1 & \text{if } G \text{ is an end position} \\ (min\{(l(G') + 1 - i) \bmod m \mid G' \in opt(G)\} + i) \bmod m & \text{otherwise.} \end{cases}$$

Proof In i-misère play, $A_j = (j + i) \bmod m$. Therefore, by Definition 5,

$$l(G) = \begin{cases} m - 1 & \text{if } G \text{ is an end position} \\ (q + i) \bmod m & \text{otherwise} \end{cases}$$

for $q = min\{j \in \mathbb{N} \mid l(G') + 1 = (i + j) \bmod m \ with\ G' \in opt(G)\}$

Definition 9 For $r \geq 1$, we define $M^r(G)$ as the set of game positions which are reached in no more than r moves from G. That is, $M^1(G) = opt(G)$ and $M^r(G) = opt(M^{r-1}(G) \cup G)$ for $r > 1$. In addition, we define $M^r(G) = \phi$ for $r < 1$.

Definition 10 We define $Level(G)$ as follows:

$$Level(G) = \begin{cases} m - i & \text{if } G \text{ is an end position} \\ min(\{r(G), m - i + e(G)\}) & \text{otherwise} \end{cases}$$

where $r(G)$ is the least number such that $M^{r(G)}(G)$ contains a losing position and $e(G)$ is the least number such that $M^{e(G)}(G)$ contains an end position. If such an $r(G)$ does not exist, then we set $r(G) = \infty$.

Lemma 2 *$Level(G) \leq m$.*

Proof Without loss of generality, assume that the current player is P_0. Let $x = l(G)$ and

$$u = \begin{cases} (x - i) \bmod m & (x \neq i) \\ m & (x = i). \end{cases}$$

If the game ends before P_u's turn, then $x \leq i$ and $M^x(G)$ contains an end position.

Otherwise, since we assumed that each player moves optimally, P_u is given a losing position after u moves from the first move by P_0. Therefore, $M^u(G)$ contains a losing position.

Lemma 3 *$l(G) = (Level(G) + i - 1) \bmod m$.*

Proof If G is an end position, then clearly $l(G) = (Level(G) + i - 1) \bmod m$. For the rest of the proof, we assume that G is not an end position.

If $Level(G) = 1$ then $min(\{r(G), m - i + e(G)\}) = 1$. $m - i + e(G) \neq 1$ because $m \geq i + 1$ and $e(G) > 0$. Therefore, we have $r(G) = 1$ and it means that there is an option $G' \in opt(G)$ such that G' is a losing position. Since G' is a losing position, $l(G') = (i - 1) \bmod m$. Therefore,

$$\begin{aligned} l(G) &= (min\{(l(Y) + 1 - i) \bmod m \mid Y \in opt(G)\} + i) \bmod m \\ &= i \\ &= (Level(G) + i - 1) \bmod m. \end{aligned}$$

Otherwise, since $Level(G) = min(\{r(G), m - i + e(G)\})$, there is an option $G' \in opt(G)$ such that $Level(G') = min(\{r(G) - 1, m - i + e(G) - 1\})$. Note that this holds even if $e(G) = 1$ because there exists $G' \in opt(G)$ such that G' is an end position and by Definition 10, $Level(G') = m - i$. In addition, there is no option G' which satisfies $r(G') \leq r(G) - 2$ nor $m - i + e(G') \leq m - i + e(G) - 2$.

Therefore, there is an option $G' \in opt(G)$ such that $Level(G') = Level(G) - 1$ and there is no option $G' \in opt(G)$ such that $1 \leq Level(G') \leq Level(G) - 2$. By induction hypothesis, there is an option $G' \in opt(G)$ such that $l(G') = (i - 1 + Level(G) - 1) \bmod m$ and there is no option $G' \in opt(G)$ such that $l(G') = (i - 1 + Level(G) - 2) \bmod m$ nor $l(G') = (i - 1 + Level(G) - 3) \bmod m$ nor …nor $l(G') = i \bmod m$. Therefore,

$$\begin{aligned} l(G) &= (min\{(l(G') + 1 - i) \bmod m \mid G' \in opt(G)\} + i) \bmod m \\ &= (((Level(G) - 1) \bmod m) + i) \bmod m \\ &= (Level(G) + i - 1) \bmod m. \end{aligned}$$

Lemma 4 *Let S be a set of game positions of i-misère play. S is the set of losing positions if and only if*

(i) $\forall G \in S\ \forall G' \in E.\ G' \notin M^{i-1}(G)$
(ii) $\forall G, G' \in S.\ G' \notin M^{m-1}(G)$
(iii) $\forall G \notin S.\ (\exists G' \in S, G' \in M^{m-1}(G)) \vee (\exists G'' \in E, G'' \in M^{i-1}(G))$.

Here, E is the set of end positions.

Proof Note that if G is a losing position, then $l(G) = (i - 1) \bmod m$. Assume that S is the set of losing positions. Then, by Lemma 3, $\forall G \in S.\ Level(G) = m$ and by Definition 10, $min(\{r(G), m - i + e(G)\}) = m$. Therefore, (i) and (ii) hold. Next, by Lemma 3, $\forall G \notin S$, $Level(G) < m$ and therefore (iii) holds.

On the other hand, assume that (i), (ii), and (iii) hold. From (i) and (ii), $\forall G \in S.\ Level(G) = m$ and from (iii), $\forall G \notin S.\ Level(G) < m$. Therefore, S is the set of game position G such that $l(G) = (i - 1) \bmod m$.

Lemma 5 *Let m be a positive integer. Assume that non-negative integers $n_1, n_2, \ldots, n_k$ satisfy $n_1^{\langle 2\rangle} \oplus_m n_2^{\langle 2\rangle} \oplus_m \cdots \oplus_m n_k^{\langle 2\rangle} \neq 0$. Then there exist $n'_1, n'_2, \ldots, n'_k (0 \leq n'_j \leq n_j)$ such that $n_1'^{\langle 2\rangle} \oplus_m n_2'^{\langle 2\rangle} \oplus_m \cdots \oplus_m n_k'^{\langle 2\rangle} = 0$ and $n_j = n'_j$ for all but at most $m - 1$ values of j.*

Proof Let u be the maximal length of binary notations of n_j. Suppose that $n_j^{\langle 2\rangle} = (n_{j,u}, n_{j,u-1}, \cdots, n_{j,1})$ and if the length of $n_j^{\langle 2\rangle}$ is shorter than u, then we prepend 0's to adjust its length. We calculate with the following algorithm a subset α of $\{1, \ldots, k\}$ specifying the indices of the n_j that will be reduced. First, we set $v = u$ and $\alpha = \phi$ and start at step $s = 0$. At step s, let $V = \{j | j \notin \alpha, n_{j,v} = 1\}$. Let $p = |V|\ mod\ m$. If $p \leq m - |\alpha| - 1$, set $n_{j,v} = 0$ for $j \in \alpha$, choose a subset $J \subseteq V$ of p elements, and set $n_{j,v} = 0$ for $j \in J$ and add $j \in J$ to α. If $p \geq m - |\alpha|$, choose a subset $J' \subseteq \alpha$ of $m - p$ elements, set $n_{j,v} = 1$ for $j \in J'$ and set $n_{j,v}$=0 for $j \in \alpha \setminus J'$. Then we set $v = v - 1$, and if $v \geq 1$, proceed to step $s + 1$.

Lemma 6 *Let m be a positive integer. Assume that non-negative integers $n_1, n_2, \ldots, n_k$ satisfy $n_1^{\langle 2\rangle} \oplus_m n_2^{\langle 2\rangle} \oplus_m \cdots \oplus_m n_k^{\langle 2\rangle} = 0$. Then there does not exist $n'_1, n'_2, \ldots, n'_k (0 \leq n'_j \leq n_j)$ such that $n_1'^{\langle 2\rangle} \oplus_m n_2'^{\langle 2\rangle} \oplus_m \cdots \oplus_m n_k'^{\langle 2\rangle} = 0$ and $n_j = n'_j$ for all but at most $m-1$ values of j. However, one can obtain $n_1'^{\langle 2\rangle} \oplus_m n_2'^{\langle 2\rangle} \oplus_m \cdots \oplus_m n_k'^{\langle 2\rangle} = 0$ by reducing m of them unless $n_1 = n_2 = \cdots = n_k = 0$.*

Proof We define $f(n_1, n_2, \ldots, n_k)$ as the left-most non-zero component of $n_1^{\langle 2\rangle} \oplus_m n_2^{\langle 2\rangle} \oplus_m \cdots \oplus_m n_k^{\langle 2\rangle}$ and if $n_1^{\langle 2\rangle} \oplus_m n_2^{\langle 2\rangle} \oplus_m \cdots \oplus_m n_k^{\langle 2\rangle} = 0$, then we define $f(n_1, n_2, \ldots, n_k) = 0$. If $n_1^{\langle 2\rangle} \oplus_m n_2^{\langle 2\rangle} \oplus_m \cdots \oplus_m n_k^{\langle 2\rangle} = 0$, then reducing any $n_j > 0$ to $n'_j(n'_j < n_j)$, results in $f(n_1, n_2, \ldots, n_{j-1}, n'_j, n_{j+1}, \ldots, n_k) = m-1$. And if $f = f(n_1, n_2, \ldots, n_{j-1}, n_j, n_{j+1}, \ldots, n_k) > 0$, then reducing any $n_j > 0$ to $n'_j(n'_j < n_j)$, yields $f(n_1, n_2, \ldots, n_{j-1}, n'_j, n_{j+1}, \ldots, n_k) \in \{f, f-1, m-1\}$. Therefore one needs to reduce at least f members of $\{n_1, n_2, \ldots, n_k\}$ in order to make their generalized NIM sum 0. In addition, from Lemma 5, one can do so by reducing at most $m-1$ more members.

Lemma 7 *Assume that $k \geq m-1$ and non-negative integers $n_1, n_2, \ldots, n_k$ satisfy $n_1^{\langle 2\rangle} \oplus_m n_2^{\langle 2\rangle} \oplus_m \cdots \oplus_m n_k^{\langle 2\rangle} = 0$ and $\forall j. n_j \leq 1$. Then for any $i (1 \leq i \leq m-1)$, and for any subset $J \subseteq \{1, \ldots, k\}$ of cardinality $m-1$, there exist $n'_1, n'_2, \ldots, n'_k (\forall j. n'_j \leq 1)$ such that $n_1'^{\langle 2\rangle} \oplus_m n_2'^{\langle 2\rangle} \oplus_m \cdots \oplus_m n_k'^{\langle 2\rangle} = i$ and $n_j = n'_j$ for $j \notin J$.*

Proof Let $p = n_{j_1}^{\langle 2\rangle} \oplus_m n_{j_2}^{\langle 2\rangle} \oplus_m \cdots \oplus_m n_{j_{m-1}}^{\langle 2\rangle}$ for $J = \{j_1, j_2, \ldots, j_{m-1}\}$. If $i \leq m-1-p$, then there are i elements whose values are 0 in $n_{j_1}, n_{j_2}, \ldots, n_{j_{m-1}}$ and otherwise, $p \geq m-i$ and it means that there are $m-i$ elements whose values are 1. In the former case, by adding 1 to i elements whose values are 0, we can obtain $n'_1, n'_2, \ldots, n'_k (n'_j \leq 1)$ such that $n_1'^{\langle 2\rangle} \oplus_m n_2'^{\langle 2\rangle} \oplus_m \cdots \oplus_m n_k'^{\langle 2\rangle} = i$ and $n_j = n'_j$ if $j \notin J$, and in the latter case, by reducing 1 from $m-i$ elements whose values are 1, we can also obtain such numbers.

Proof of Theorem 6 Let $S_1 = \{(n_1, n_2, \ldots, n_k) \mid n_1^{\langle 2\rangle} \oplus_m n_2^{\langle 2\rangle} \oplus_m \cdots \oplus_m n_k^{\langle 2\rangle} = 0 \ (\exists j. n_j > 1)\}$, $S_2 = \{(n_1, n_2, \ldots, n_k) \mid n_1^{\langle 2\rangle} \oplus_m n_2^{\langle 2\rangle} \oplus_m \cdots \oplus_m n_k^{\langle 2\rangle} = i \ (\forall j. n_j \leq 1)\}$, and $S = S_1 \cup S_2$.

Assume that $G = (g_1, g_2, \ldots, g_k) \in S$ and G' is the end position. There are two cases:

(i)-1 $G \in S_1$: By Lemma 6, $G' \notin M^{i-1}(G)$
(i)-2 $G \in S_2$: It is clear that $G' \notin M^{i-1}(G)$

Next, assume that $G = (g_1, g_2, \ldots, g_k) \in S$ and $G' = (g'_1, g'_2, \ldots, g'_k) \in S$. There are three cases:

(ii)-1 $G, G' \in S_1$: By Lemma 6, $G' \notin M^{m-1}(G)$.
(ii)-2 $G, G' \in S_2$: It is clear that $G' \notin M^{m-1}(G)$.
(ii)-3 $G \in S_1$ and $G' \in S_2$: Since $g_1^{\langle 2\rangle} \oplus_m g_2^{\langle 2\rangle} \oplus_m \cdots \oplus_m g_k^{\langle 2\rangle} = 0$, G has at least m elements which are larger than 1 and thus, $G' \notin M^{m-1}(G)$.

Therefore, if $G, G' \in S$, then $G' \notin M^{m-1}(G)$.

Next, assume that $G = (g_1, g_2, \ldots, g_k) \notin S$. There are two cases:

(iii)-1 There are less than i elements which are larger than 0 in G: Clearly $\exists G'' \in E.\ G'' \in M^{i-1}(G)$.

(iii)-2 There are more than or equal to i elements which are larger than 0 in G: By Lemma 5, there exist $g'_1, g'_2, \ldots, g'_k (0 \leq g'_j \leq g_j)$ such that $g_1'^{\langle 2\rangle} \oplus_m g_2'^{\langle 2\rangle} \oplus_m \cdots \oplus_m g_k'^{\langle 2\rangle} = 0$ and $g_j = g'_j$ for all but at most $m-1$ values of j. If there exist $g'_j > 1$ then $\exists G' = (g'_1, g'_2, \ldots, g'_k)$ such that $G' \in S_1$ and $G' \in M^{m-1}(G)$.

On the other hand, if $g'_j \leq 1$ for all $j \leq k$, then there are two cases:

(iii)-2a There are less than m elements which are larger than 0 in G: Since there are i or more elements which are larger than 0 in G, one can reduce or remain them so that i elements are 1 and the others are 0. Therefore, $\exists G' = (g'_1, g'_2, \ldots, g'_k)$ such that $G' \in S_2$ and $G' \in M^{m-1}(G)$.

(iii)-2b There are m or more elements which are larger than 0 in G. Since $g_1'^{\langle 2\rangle} \oplus_m g_2'^{\langle 2\rangle} \oplus_m \cdots \oplus_m g_k'^{\langle 2\rangle} = 0$, there are m or more elements which are 1. Then by Lemma 7, there also exists $G'' = (g''_1, g''_2, \ldots, g''_k)(g''_j \leq 1)$ such that $g_1''^{\langle 2\rangle} \oplus_m g_2''^{\langle 2\rangle} \oplus_m \cdots \oplus_m g_k''^{\langle 2\rangle} = i$ and $g_j = g''_j$ for all but at most $m-1$ values of j. Therefore, $\exists G'' = (g''_1, g''_2, \ldots, g''_k)$ such that $G'' \in S_2$ and $G'' \in M^{m-1}(G)$.

Therefore, by Lemma 4, S is the set of losing positions of i-misère NIM.

4 *m*-Player misère Moore's game

Moore's game, or NIM_t, is a game in which players can choose up to t heaps and take any numbers of stones from them (Moore 1909) . Therefore, NIM_1 is the original NIM.

Theorem 7 (Moore 1909) *A game position $(n_1, n_2, \ldots, n_k)$ of NIM_t is a P-position if and only if $n_1^{\langle 2\rangle} \oplus_{t+1} n_2^{\langle 2\rangle} \oplus_{t+1} \cdots \oplus_{t+1} n_k^{\langle 2\rangle} = 0$.*

Robert Li (1978) showed the following theorem for multiplayer NIM_t.

Theorem 8 (Robert Li 1978) *In 0-misère play of m-player NIM_t, $l(n_1, n_2, \ldots, n_k) = 0$ if and only if $n_1^{\langle 2\rangle} \oplus_v n_2^{\langle 2\rangle} \oplus_v \cdots \oplus_v n_k^{\langle 2\rangle} = 0$ where $v = t(m-1)+1$.*

We can also extend this theorem to i-misère play.

Theorem 9 *In i-misère play of m-player NIM_t, $(n_1, n_2, \ldots, n_k)$ is a losing position if and only if,*

$$\begin{cases} n_1^{\langle 2\rangle} \oplus_v n_2^{\langle 2\rangle} \oplus_v \cdots \oplus_v n_k^{\langle 2\rangle} = 0 \ \ (\exists j.\, n_j > 1) \\ n_1^{\langle 2\rangle} \oplus_v n_2^{\langle 2\rangle} \oplus_v \cdots \oplus_v n_k^{\langle 2\rangle} = u \ \ (\forall j.\, n_j \leq 1) \end{cases}$$

where $v = t(m-1)+1$ and

$$\begin{cases} u = 0 \ \ (i = 0) \\ u = t(i-1)+1 \ \ (1 \leq i \leq m-1). \end{cases}$$

.

Proof Let $S_1 = \{(n_1, n_2, \ldots, n_k) \mid n_1^{\langle 2\rangle} \oplus_v n_2^{\langle 2\rangle} \oplus_v \cdots \oplus_v n_k^{\langle 2\rangle} = 0 \ \ (\exists j.\, n_j > 1)\}$ and $S_2 = \{(n_1, n_2, \ldots, n_k) \mid n_1^{\langle 2\rangle} \oplus_v n_2^{\langle 2\rangle} \oplus_v \cdots \oplus_v n_k^{\langle 2\rangle} = u \ \ (\forall j.\, n_j \leq 1)\}$, and $S = S_1 \cup S_2$.

Assume that $G = (g_1, g_2, \ldots, g_k) \in S$ and G' is the end position. There are two cases:

(i)-1 $G \in S_1$: By Lemma 6, $G' \notin M^{i-1}(G)$
(i)-2 $G \in S_2$: It is clear that $G' \notin M^{i-1}(G)$

Next, assume that $G = (g_1, g_2, \ldots, g_k) \in S$ and $G' = (g'_1, g'_2, \ldots, g'_k) \in S$. There are three cases:

(ii)-1 $G, G' \in S_1$: By Lemma 6, $G' \notin M^{m-1}(G)$.
(ii)-2 $G, G' \in S_2$: It is clear that $G' \notin M^{m-1}(G)$.
(ii)-3 $G \in S_1$ and $G' \in S_2$: Since $g_1^{\langle 2\rangle} \oplus_v g_2^{\langle 2\rangle} \oplus_v \cdots \oplus_v g_k^{\langle 2\rangle} = 0$, G has at least v elements which are larger than 1 and thus, $G' \notin M^{m-1}(G)$.

Therefore, if $G, G' \in S$, then $G' \notin M^{m-1}(G)$.
Next, assume that $G = (g_1, g_2, \ldots, g_k) \notin S$. There are two cases:

(iii)-1 There are less than u elements which are larger than 0 in G: Clearly $\exists G'' \in E.\, G'' \in M^{i-1}(G)$.
(iii)-2 There are more than or equal to u elements which are larger than 0 in G: By Lemma 5, there exist $g'_1, g'_2, \ldots, g'_k (0 \leq g'_j \leq g_j)$ such that $g_1'^{\langle 2\rangle} \oplus_v g_2'^{\langle 2\rangle} \oplus_v \cdots \oplus_v g_k'^{\langle 2\rangle} = 0$ and $g_j = g'_j$ for all but at most $v-1$ values of j. If there exist $g'_j > 1$ then $\exists G' = (g'_1, g'_2, \ldots, g'_k)$ such that $G' \in S_1$ and $G' \in M^{m-1}(G)$.

On the other hand, if $g'_j \leq 1$ for all $j \leq k$, then there are two cases:

(iii)-2a There are less than v elements which are larger than 0 in G: Since there are u or more elements which are larger than 0 in G, one can reduce or remain them so that u elements are 1 and the others are 0. Therefore, $\exists G' = (g'_1, g'_2, \ldots, g'_k)$ such that $G' \in S_2$ and $G' \in M^{m-1}(G)$.
(iii)-2b There are v or more elements which are larger than 0 in G. Since $g_1'^{\langle 2\rangle} \oplus_v g_2'^{\langle 2\rangle} \oplus_v \cdots \oplus_v g_k'^{\langle 2\rangle} = 0$, there are v or more elements which are 1. Then by Lemma 7, there also exists $G'' = (g''_1, g''_2, \ldots, g''_k)(g''_j \leq 1)$ such that $g_1''^{\langle 2\rangle} \oplus_v g_2''^{\langle 2\rangle} \oplus_v \cdots \oplus_v g_k''^{\langle 2\rangle} = u$ and $g_j = g''_j$ for all but at most $v-1$ values of j. Therefore, $\exists G'' = (g''_1, g''_2, \ldots, g''_k)$ such that $G'' \in S_2$ and $G'' \in M^{m-1}(G)$.

Therefore, by Lemma 4, S is the set of losing positions of i-misère Moore's game.

5 Reverse form

In Sects. 3 and 4, we considered i-misère play where each player's preference order is the same as the play order. In this section, we study the situation that for each player P_i, her preference order is reverse to the play order.

Definition 11 For $0 \le i \le m-1$, we say that preference-based play of an m-player game is an *i-reverse play* if it is a preference-impartial game and the preference matrix is the following:

$$\left[i \ (i-1) \cdots 1 \ 0 \ (m-1) \cdots (i+1) \right]$$

In i-misère play, if $l(G) = (i-1) \bmod m$, then G is a losing position. On the other hand, in i-reverse play, if $l(G) = (i-1) \bmod m$ then the secondly preferred player of the current player is going to be the last moving player of G.

In two-play normal and misère NIM. for all $j (1 \le j \le k)$ and for all non-negative integers $n_1, n_2, \ldots, n_{j-1}, n_{j+1}, \ldots, n_k$, there is exactly one non-negative integer n_j such that $(n_1, n_2, \ldots, n_{j-1}, n_j, n_{j+1}, \ldots, n_k)$ is a P-position. In i-misère play with $m > 2$, there is no such uniqueness. However, we show that there exists such a uniqueness in i-reverse play. This result suggests that i-reverse play is also a natural extension of two-player normal and misère play.

Theorem 10 *In i-reverse play, for all $j(1 \le j \le k)$ and for all non-negative integers $n_1, n_2, \ldots, n_{j-1}, n_{j+1}, \ldots, n_k$, there is exactly one non-negative integer n_j such that $l(n_1, n_2, \ldots, n_{j-1}, n_j, n_{j+1}, \ldots, n_k) = (i-1) \bmod m$.*

Proof Without loss of generality, we assume $j = k$ and we prove there is exactly one n_k such that $l(n_1, n_2, \ldots, n_{k-1}, n_k) = (i-1) \bmod m$ for all non-negative integers $n_1, n_2, \ldots, n_{k-1}$. First, we show that there is at most one n_k such that $l(n_1, n_2, \ldots, n_{k-1}, n_k) = (i-1) \bmod m$. Assume for a contradiction that $l(G) = l(G') = (i-1) \bmod m$ where $G = (n_1, n_2, \ldots, n_{k-1}, n_k)$ and $G' = (n_1, n_2, \ldots, n_{k-1}, n'_k)(n_k < n'_k)$. Then, the current player can move from G' to G. Since $l(G) = (i-1) \bmod m$, $l(G') = i$, which is a contradiction.

Next, we show that there is at least one n_k such that $l(n_1, n_2, \ldots, n_{k-1}, n_k) = (i-1) \bmod m$. Assume for a contradiction that there exist $(n_1, n_2, \ldots, n_{k-1})$ such that for all non-negative integer n_k, $l(n_1, n_2, \ldots, n_{k-1}, n_k) \ne (i-1) \bmod m$. Similarly to the case of $i-1$, for each s $(s \not\equiv i (mod\ m))$, there is at most one n_k such that $l(n_1, n_2, \ldots, n_{k-1}, n_k) = s$. Therefore, there are infinitely many n_k such that $l(n_1, n_2, \ldots, n_{k-1}, n_k) = i$. Then, the current player has a move to G' such that $l(G') = (i-1) \bmod m$. It can not be a move to take some stones from n_k, because we assumed for all non-negative integer n'_k, $l(n_1, n_2, \ldots, n_{k-1}, n'_k) \ne (i-1) \bmod m$. From the pigeonhole principle, there exist n'_k, n''_k $(n'_k < n''_k)$ and $n'_1(n'_1 < n_1)$ such that $l(G_1) = l(n'_1, n_2, \ldots, n_{k-1}, n'_k) = (i-1) \bmod m$ and $l(G_2) = l(n'_1, n_2, \ldots, n_{k-1}, n''_k) = (i-1) \bmod m$. On the other hand, the current player can move from G_2 to G_1. Since $l(G_1) = (i-1) \bmod m$, $l(G_2) = i$, which is a contradiction.

6 Asymmetric forms in three-player NIM

Finally, we study the case that each player has a different preference order for the case of three-player NIM. We have already studied the cases named 0-misère, 1-misère, and 2-misère play;

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 1 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

and the cases named 0-reverse, 1-reverse, and 2-reverse play;

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}.$$

We showed some properties of $l(G, t)$ though its characterization is an open problem for some of the cases. In this section, we study the following preference orders which are not symmetric.

Semi-normal form Each player prefers herself first. Two players secondly prefer the same player and the other player secondly prefers her next player. There are three possibilities of preference orders which are essentially the same.

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

Semi-reverse form Each player prefers herself first. Two players secondly prefer the same player and the other player secondly prefers her previous player. There are three possibilities of preference orders which are essentially the same.

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

Without loss of generality, we consider semi-normal form

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

and semi-reverse form

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Theorem 11 *The overall result for the semi-normal form is listed in Table 1 and for the semi-reverse form is listed in Table 2. Here,* $(\alpha; \beta; \gamma)$ *means that* $l(G, 0) = \alpha, l(G, 1) = \beta$*, and* $l(G, 2) = \gamma$*.*

Table 1 Semi-normal form

(I) $3n$ heaps	
(I-1) each heap has 1 stone; $(1, 1, \dots, 1)$	(2; 2; 2)
(I-2) $(3n - 1)$ heaps have 1 stone and the other heap has $x(\geq 2)$ stones; $(1, 1, \dots, 1, x)$	(0; 0; 0)
(I-3) otherwise	(0; 2; 1)
(II) $(3n + 1)$ heaps	
(II-1) $3n$ heaps have 1 stone and the other heap has $x(\geq 1)$ stones; $(1, 1, \dots, 1, x)$	(0; 0; 0)
(II-2) otherwise	(0; 2; 1)
(III) $(3n + 2)$ heaps	
(III-1) each heap has 1 stone; $(1, 1, \dots, 1)$	(1; 1; 1)
(III-2) $(3n + 1)$ heaps have 1 stone and the other heap has 2 stones; $(1, 1, \dots, 1, 2)$	(1; 2; 1)
(III-3) otherwise	(0; 2; 1)

Table 2 Semi-reverse form

(I) $3n$ heaps	
(I-1) each heap has 1 stone; $(1, 1, \dots, 1)$	(2; 2; 2)
(I-2) $(3n - 1)$ heaps have 1 stone and the other heap has $x(\geq 2)$ stones; $(1, 1, \dots, 1, x)$	(0; 0; 0)
(I-3) $(3n - 2)$ heaps have 1 stone, one heap has 2 stones and the other heap has $x(\geq 2)$ stones; $(1, 1, \dots, 1, 2, x)$	(0; 2; 0)
(I-4) if $n = 1$ and each heap has 2 stones; $(2, 2, 2)$	(0; 1; 1)
(I-5) otherwise	(0; 2; 1)
(II) $(3n + 1)$ heaps	
(II-1) $3n$ heaps have 1 stone and the other heap has $x(\geq 1)$ stones; $(1, 1, \dots, 1, x)$	(0; 0; 0)
(II-2) $(3n - 1)$ heaps have 1 stone and the other two heaps have 2 stones; $(1, 1, \dots, 1, 2, 2)$	(0; 1; 1)
(II-3) otherwise	(0; 2; 1)
(III) $(3n + 2)$ heaps	
(III-1) each heap has 1 stone; $(1, 1, \dots, 1)$	(1; 1; 1)
(III-2) $(3n + 1)$ heaps have 1 stone and the other heap has 2 stones; $(1, 1, \dots, 1, 2)$	(2; 2; 1)
(III-3) $(3n + 1)$ heaps have 1 stone and the other heap has $x(\geq 3)$ stones; $(1, 1, \dots, 1, x)$	(0; 2; 0)
(III-4) $3n$ heaps have 1 stone, one heap has 2 stones and the other heap has $x(\geq 2)$ stones; $(1, 1, \dots, 1, 2, x)$	(0; 2; 0)
(III-5) if $n = 0$ and each heap has 3 stones; $(3, 3)$	(0; 1; 1)
(III-6) otherwise	(0; 2; 1)

Proof We prove the semi-normal case by induction.

- (I-1): If each heap has 1 stone, then every player has only one choice to take one stone from a heap and the result is (2; 2; 2).
- (I-2): The current player can be the last moving player by taking all but one stone from the maximal heap. From (I-1), she becomes the last moving player.
- (II-1): The current player can be the last moving player by taking all stones from the maximal heap. From (I-1), she becomes the last moving player.
- (III-1): If each heap has 1 stone, then every player has only one choice to take one stone from a heap and the result is (1; 1; 1).
- (III-2): The current player has no way to be the last moving player. From (II), she can make her next player the last moving player by taking a stone from a heap which has 1 stone. From (III-1), she can make her previous player the last moving player by taking a stone from a heap which has 2 stones. Since this is semi-normal form, the result is (1; 2; 1).
- (I-3),(II-2),(III-3): If there are $3n + 2$ heaps and one heap has 3 or more stones and the other heaps have one stone, then player P_1 and player P_2 have no strategies to be the last moving player because they cannot make the game position (I-1). On the other hand, they can make P_0 the last moving player by taking all stones from the maximal heap. In addition, player P_0 can be the last moving player by taking all but two stones from the maximal heap. Therefore, the result is (0; 2; 1). In other cases, there are two heaps which have two or more stones. Assume that player P_0 is the first player. If there are just two heaps and they have just two stones, then player P_0 can be the last moving player by taking a stone from one heap. Otherwise, by induction, player P_0 becomes the last moving player because she can always move so that the two heaps have two or more stones.
  If player P_1 is the first player, then she cannot bring the game to position (I-1), so she cannot be the last moving player. However, if there are just two heaps which have just two stones, then she can take a stone from one heap and make P_0 the last moving player. Otherwise, she can make P_0 the last moving player by induction.
  Finally, if player P_2 is the first player, then she cannot bring the game to position (I-1), so she cannot be the last moving player. However, if there are just two heaps which have just two stones, then she can take all stones from one heap and make P_0 the last moving player. Otherwise, she can make P_0 the last moving player by induction.

We can prove the semi-reverse case by induction as the semi-normal case, but it is more complex, so we omit it.

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