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Non-manipulability of uniform price auctions with a large number of objects

Tomoya Tajika[1] · Tomoya Kazumura[2]

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Abstract
When agents (bidders) have multi-demand preferences, uniform price auctions are generally not immune to agents' strategic manipulation, and they may achieve an inefficient allocation. We consider economies in which a large number of identical objects have to be allocated. Agents have quasi-linear preferences with non-increasing incremental valuations. We explore the incentives of agents in uniform price auctions. An important assumption on preferences is proposed, called "no monopoly." It requires that preferences should be correlated in such a way that no agent's incremental valuation for an additional object when he receives sufficiently many objects is higher than those of the other agents. We show that under no monopoly and other mild assumptions on preferences, as the number of objects goes to infinity, the payment in any uniform price auction converges to that in a Vickrey auction. We deduce that when there are sufficiently many objects, truth-telling is an approximate Bayesian Nash equilibrium in each uniform price auction.

Keywords Uniform price auction · No monopoly · Large market · ϵ-Bayesian Nash equilibrium

JEL Classification D44 · D71 · D61 · D82

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✉ Tomoya Tajika
tomoyatajika@gmail.com

Tomoya Kazumura
pge003kt@gmail.com

1 Institute of Economic Research, Hitotsubashi University, 2-1 Naka, Kunitachi, Tokyo 186-8603, Japan

2 Department of Industrial Engineering and Economics, School of Engineering, Tokyo Institute of Technology, W9-74, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8552, Japan

1 Introduction

Auctions are frequently conducted in the world to allocate scarce resources. Examples include auctions of spectrum licenses, government debts, and public assets. The *uniform price auction* is one of the most frequently used auction formats in practice. It satisfies several desirable properties in theory. If agents (bidders) truthfully report their bids, uniform price auctions achieve efficiency, one of the most important goals in conducting an auction. In addition, a uniform price auction generates a fair allocation as each agent faces the same price scheme.[1] In addition, a simple ascending auction calculates allocations selected by a uniform price auction (Gul and Stacchetti 2000). However, uniform price auctions are generally not immune to strategic manipulation by agents when agents have multi-demand preferences. In a uniform price auction, agents have an incentive to underreport their valuations (Ausubel et al. 2014).[2,3] This behavior, which is called "demand reduction," results in uniform price auctions failing to achieve an efficient and fair allocation.

In reality, many auctions have a large number of agents. Several authors focused on this case and explored the possibility of avoiding strategic manipulation in uniform price auctions. Under various assumptions on preferences, agents' incentives to misreport their valuations in uniform price auctions were shown to vanish as the number of agents goes to infinity (Swinkels 2001; Jackson and Kremer 2006; Bodoh-Creed 2013; Azevedo and Budish 2018).

However, there are also large auctions in which a large number of objects are allocated and the number of agents is relatively small. An example is treasury security auctions. In Canada, the average and minimum numbers of participants at 3-month treasury bill auctions between 1998 and 2003 are 17 and 11 (Hortaçsu and Kastl 2012). The existing literature focusing on economies with large populations does not cover such cases.

In contrast to the existing literature, we explore the incentives of agents in uniform price auctions when there are many objects. Specifically, we consider a model in which multiple identical objects are to be sold and agents have quasi-linear preferences of which incremental valuations are non-increasing. We show that under several assumptions on the domain of admissible preferences and beliefs about agents' preferences, when there are sufficiently many objects, truth-telling is an approximate Bayesian Nash equilibrium in any uniform price auction. This result holds even if agents have different prior beliefs.

An important assumption to establish our result is called *no monopoly*. Suppose there are some threshold quantity x and an agent such that for each $y \geq x$, the incremental valuation to him of the $(y+1)$st unit exceeds that of the other agents. Then, as the number of objects in the economy goes to infinity, at an efficient allocation, only this agent's assigned objects goes to infinity. *No monopoly* requires that in the support of each agent's prior belief, there should not be a preference profile where

[1] Moreover, a uniform price auction assigns an allocation where each agent finds his assignment at least as desirable as the others' assignments. This property is called *no-envy* (Foley 1967).

[2] Baisa (2016a) shows a parallel result when preferences are allowed to be non-quasi-linear.

[3] If agents have quasi-linear and unit-demand preferences, a uniform price auction coincides with some Vickrey auction. Thus, agents have no incentive to misreport their valuations.

there is such an agent. Under no monopoly, agents' preferences should be correlated in a way that if an agent has high incremental valuations for an additional object when he receives sufficiently many objects, then there should be another agent whose incremental valuations for an additional object when he receives sufficiently many objects are at least as high as his. However, no monopoly allows for the existence of an agent whose valuations for objects are higher than those of any other agent.

Our main result does not hold without no monopoly. We provide an example to illustrate this point (Example 2). Furthermore, we fail to obtain the same result without other assumptions. However, we can show alternative results without them. In our main result, we vary the number of objects, while fixing the number of agents. We also discuss the case in which both the numbers of agents and objects increase, and obtain a partial result. Specifically, we consider replica economies. We show that truth-telling is a Nash equilibrium in a particular uniform price auction if we replicate an economy sufficiently many times.

An application of our main result is treasury securities auctions. Uniform price auctions are used in several countries, such as the US and the UK, to sell treasury securities. Each securities type is auctioned separately, and has a large number of copies of it. Thus, our assumption that a large number of identical objects is allocated is satisfied. Furthermore, there is a limited number of agents.[4] If no monopoly is satisfied, then our main result supports the use of uniform price auctions in treasury security auctions.

1.1 Related literature

Several papers in auction theory study large auctions. Many of them focus on some specific auction format(s). For example, Bodoh-Creed (2013) considers uniform price auctions, and Swinkels (1999) considers discriminatory auctions.[5] Swinkels (2001) and Jackson and Kremer (2006) focus on both uniform price auctions and discriminatory auctions. Cripps and Swinkels (2006) and Fudenberg et al. (2007) consider double auctions. Azevedo and Budish (2018) exceptionally consider a general large market model including a variety of models, such as auction model and matching model. The authors introduce a notion called "strategy-proofness in the large." This notion requires truth-telling to be approximately optimal against any distribution of the other agents' preferences when the market is sufficiently large. Azevedo and Budish's (2018) result implies that uniform price auctions are strategy-proof in the large.

Our work differs from Swinkels (2001), Jackson and Kremer (2006), Bodoh-Creed (2013), and Azevedo and Budish (2018) in the following respects. First, our study and these four studies focus on different types of large economies. In their models, there are sufficiently many agents. On the other hand, in our model, there are sufficiently many objects, whereas the number of agents can be small.

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[4] In our study, we further assume that agents have private value for the objects. Hortaçsu and Kastl (2012) provide an empirical evidence that agents can have private value for securities.

[5] The *discriminatory auction* is an auction such that the object allocation is determined in order that the sum of valuations is maximized, and each agent pays the valuation for the objects he obtains.

Second, these four studies assume that each agent demands up to some fixed number of objects. We do not make this assumption. In our model, this assumption would imply that when agents receive more than some units of the object, the incremental valuation for an additional object is zero. Thus, if we made such an assumption and there are sufficiently many objects, then the price given by a uniform price auction would be zero. This implies that no agent would benefit from misreporting their preferences. Hence, under this assumption, our result would be obvious.

Some studies also focus on large markets in a variety of models, such as the classical exchange economy (Roberts and Postlewaite 1976; Otani and Sicilian 1982; Otani 1990; Jackson and Manelli 1997) and matching model (Immorlica and Mahdian 2005; Kojima and Pathak 2009; Che and Kojima 2010; Che et al. 2018; Che and Tercieux 2018; Lee 2017). Those studies also consider models with a large number of agents. Thus, our study and those studies focus on different types of large markets. An exception is Kojima and Manea (2010) who, like us, consider a situation in which there are a large number of objects in an object assignment model without money.

Efficiency in auctions has been investigated in the literature. *Vickrey auctions* are a well-known efficient auctions that satisfy *strategy-proofness*, that is, each agent has an incentive to report his true preferences. Furthermore, Vickrey auctions are the unique efficient and strategy-proof auctions at which each agent who receives no object pay nothing (Holmström 1979; Chew and Serizawa 2007).

However, Vickrey auctions are seldom used in practice for several reasons. Ausubel and Milgrom (2006) point out several of their drawbacks, such as low revenue, non-monotonicity of the auctioneer's revenue with respect to the number of agents, and vulnerability to collusion by agents. These problems do not occur in our setting in which valuation functions have non-increasing incremental valuations. However, there are drawbacks of Vickrey auctions even in our setting. For example, Vickrey auctions are not fair in the sense that agents pay different prices even when they receive the same objects. Hence, it is worth exploring non-Vickrey auctions.

Our analysis has an interesting implication for Vickrey auctions. In general, a uniform price auction require agents to pay more than a Vickrey auction does. In the proof of our main result, we show that as the number of objects in the economy becomes large, the difference in payment between uniform price and Vickrey auctions becomes close to zero. Then, Vickrey auctions are almost equivalent to uniform price auctions. This observation implies that in the large economies that we study, Vickrey auctions no longer have the drawback of unfair pricing, since uniform price auctions offer agents fair pricing.

The rest of this article is organized as follows. In Sect. 2, we introduce the model and definitions. We state the main result in Sect. 3. In Sect. 4, we discuss assumptions of our main result in detail. In Sect. 5, we conclude. All the proofs appear in the Appendix.

2 Preliminaries and definitions

There are n agents and $\overline{x}$ copies of an object. The set of agents is denoted by $N := \{1, \ldots, n\}$. Let $X := \{0, 1, \ldots, \overline{x}\}$. For each $i \in N$, let $N_{-i} := N \backslash \{i\}$. A typical (*consumption*) *bundle* is a pair $(x, t) \in X \times \mathbb{R}$, where x is the number of copies of the object and t is a payment. Thus, the *consumption set* is $X \times \mathbb{R}$.

Agents have preferences over $\mathbb{Z}_+ \times \mathbb{R}$, where $\mathbb{Z}_+$ is the set of non-negative integers. Preferences are assumed to be quasi-linear. Thus, for each preference relation, there is a *valuation function* $v : \mathbb{Z}_+ \to \mathbb{R}_+$ such that $v(0) = 0$, and the preference relation is represented by the function $u(x, t; v) := v(x) - t$ for each $(x, t) \in \mathbb{Z}_+ \times \mathbb{R}$.

For each valuation function v and $x \in \mathbb{Z}_+$, the *incremental valuation of* v *at* x is $v(x+1) - v(x)$. Each valuation function v satisfies the following conditions.

MONOTONICITY: For each pair $x, x' \in \mathbb{Z}_+$, if $x < x'$, $v(x) \leq v(x')$.

NON- INCREASING INCREMENTAL VALUATIONS: For each pair $x, x' \in \mathbb{Z}_+$, if $x < x'$, $v(x+1) - v(x) \geq v(x'+1) - v(x')$.

Let $\mathcal{V}^*$ be the set of valuation functions satisfying these properties. These two properties imply that for each $v \in \mathcal{V}^*$, $\lim_{x\to\infty}(v(x+1) - v(x))$ exists. For each $v \in \mathcal{V}^*$, let $v^\infty := \lim_{x\to\infty}(v(x+1) - v(x))$. Let $\mathcal{V} \subset \mathcal{V}^*$ be the set of admissible valuation functions. A valuation profile is an n-tuple $v := (v_1, \ldots, v_n) \in \mathcal{V}^N$. For each $i \in N$, each $N' \subseteq N$, and each $v \in \mathcal{V}^N$, let $v_{-i} := (v_j)_{j \in N \setminus \{i\}}$, $v_{N'} := (v_i)_{i \in N'}$, and $v_{-N'} := (v_i)_{i \in N \setminus N'}$.

Each agent has a belief about the other agents' preferences. For each agent $i \in N$, his prior belief is denoted by $\Phi_i : 2^{\mathcal{V}^N} \to [0, 1]$. Given $v_i \in \mathcal{V}$, his posterior belief is given by $\Phi_i(\cdot|v_i)$. Let $\varphi_i(\cdot|v_i)$ be the corresponding probability density function.

An object allocation is an n-tuple $(x_1, \ldots, x_n) \in X^N$ such that $\sum_{i \in N} x_i \leq \overline{x}$. Denote the set of object allocations by A. An allocation is an n-tuple $((x_1, t_1), \ldots, (x_n, t_n)) \in (X \times \mathbb{R})^N$ such that $(x_1, \ldots, x_n) \in A$. We denote the set of allocations by Z.

A (*direct*) *mechanism* is a mapping $f : \mathcal{V}^N \to Z$. For each $v \in \mathcal{V}^N$ and each $i \in N$, let $x_i^f(v)$ and $t_i^f(v)$ be the objects assigned to agent i and his payment under f at v, respectively, and we write $f_i(v) := (x_i^f(v), t_i^f(v))$. For each $v \in \mathcal{V}^N$ we also write $x^f(v) := (x_1^f(v), \ldots, x_n^f(v))$ and $t^f(v) := (t_1^f(v), \ldots, t_n^f(v))$. A strategy of agent $i \in N$ in a mechanism f is a mapping $\sigma_i : \mathcal{V} \to \mathcal{V}$. We denote a strategy profile by $\sigma := (\sigma_1, \ldots, \sigma_n)$, and given $v \in \mathcal{V}^N$ and $i \in N$, we write $\sigma(v) := (\sigma_j(v_j))_{j \in N}$ and $\sigma_{-i}(v_{-i}) = (\sigma_j(v_j))_{j \in N_{-i}}$. A strategy profile σ is *truth-telling* if for each $i \in N$ and each $v_i \in \mathcal{V}$, $\sigma_i(v_i) = v_i$.

The following is an approximate version of the notion of the Bayesian Nash equilibrium.

Definition 1 Given a mechanism f and $\epsilon \in \mathbb{R}_{++}$, a strategy profile $\sigma := (\sigma_1, \ldots, \sigma_n)$ is an ϵ-*Bayesian Nash equilibrium in* f if for each $i \in N$, each strategy σ_i', and each $v_i \in \mathcal{V}$,

$$\int_{v_{-i} \in \mathcal{V}^{N_{-i}}} u(f_i(\sigma(v_i, v_{-i})); v_i)\varphi_i(v_{-i}|v_i)dv_{-i}$$
$$\geq \int_{v_{-i} \in \mathcal{V}^{N_{-i}}} u(f_i(\sigma_i'(v_i), \sigma_{-i}(v_{-i})); v_i)\varphi_i(v_{-i}|v_i)dv_{-i} - \epsilon.$$

Given a mechanism f and $v \in \mathcal{V}^N$, let (f, v) be the static game with complete information in which N is the set of players, $\mathcal{V}$ is the action set for each player, f is the outcome function, and v represents the players' preferences.

Now we define uniform price auctions. Given $v \in \mathcal{V}^N$, let $V_{\overline{x}}(v)$ and $V_{\overline{x}+1}(v)$ be the $\overline{x}$th and $(\overline{x}+1)$st highest incremental valuations at v, respectively.

Definition 2 A mechanism f is a *uniform price auction* if there is a *price scheme* $p : \mathcal{V}^N \to \mathbb{R}$ such that for each $v \in \mathcal{V}^N$, $p(v) \in [V_{\overline{x}+1}(v), V_{\overline{x}}(v)]$ and for each $i \in N$,

$$x_i^f(v) \geq |\{x \in X \backslash \{\overline{x}\} : v_i(x+1) - v_i(x) > p(v)\}|,$$
$$x_i^f(v) \leq |\{x \in X \backslash \{\overline{x}\} : v_i(x+1) - v_i(x) \geq p(v)\}|, \text{ and},$$
$$t_i^f(v) = p(v) \cdot x_i^f(v).$$

A uniform price auction with a price scheme p is a *minimum uniform price auction* if for each $v \in \mathcal{V}^N$, $p(v) = V_{\overline{x}+1}(v)$. A uniform price auction with a price scheme p is a *maximum uniform price auction* if for each $v \in \mathcal{V}^N$, $p(v) = V_{\overline{x}}(v)$.

3 Main result

We study the incentive properties of uniform price auctions. It is already known that in a uniform price auction, truth-telling is not a Bayesian Nash equilibrium. For this reason, we focus on particular economies: Economies with many objects. In addition, we weaken the equilibrium concept to ϵ-Bayesian Nash equilibrium.

Before stating our main result, we introduce three assumptions.

Assumption 1 (*Rapid convergence*). For each $v \in \mathcal{V}$, $\lim_{x\to\infty} x \cdot (v(x+1) - v(x) - v^{\infty}) = 0$.

Although this assumption is not common, it is implied by standard assumptions. For example, it is natural to assume that valuation functions take only integer values. Assumption 1 is satisfied by this integer value assumption. In addition, Assumption 1 is satisfied if valuation functions take only discrete values, that is, there is $\delta \in \mathbb{R}_{++}$ such that for each $v \in \mathcal{V}$ and each $x \in X$, $v(x) = a \cdot \delta$ for some $a \in \mathbb{Z}_+$.

The second assumption states that the set of admissible valuation functions is finite.

Assumption 2 $\mathcal{V}$ is finite.

Given $i \in N$, let $\text{supp}_{\Phi_i}(\mathcal{V}^N)$ be the support of Φ_i. That is,

$$\underset{\Phi_i}{\text{supp}}(\mathcal{V}^N) = \bigcap_{V \subset \mathcal{V}^N : \Phi_i(V)=1} V.$$

The last assumption states that each agent believes that no agent's incremental valuation for an additional object when he receives sufficiently many objects is higher than those of the other agents.

Assumption 3 (*No monopoly*). For each $i \in N$ and each $v \in \text{supp}_{\Phi_i}(\mathcal{V}^N)$, there are no $j \in N$ and $\hat{x} \in \mathbb{Z}_+$ such that for each $k \in N \setminus \{j\}$ and each $x \in \mathbb{Z}_+$ with $x \geq \hat{x}$, $v_j(x+1) - v_j(x) > v_k(x+1) - v_k(x)$.

Note that no monopoly requires that agents' preferences should be correlated in the following way. Suppose that an agent, say agent i, has high incremental valuations when he receives many objects. Then, there should be another agent whose incremental valuations when he receives sufficiently many objects are as high as those of agent i. In other words, no monopoly is satisfied if there are at least two agents whose incremental valuation converges to the same point and the limit is greater than the others'.[6] However, no monopoly does not imply that every agent has the same incremental valuation in the limit.

Our main result states that under these three assumptions, if there are sufficiently many objects, truth-telling is almost optimal for every agent in any uniform price auction.

Theorem 1 *Under Assumptions 1, 2, and 3, for each $\epsilon \in \mathbb{R}_{++}$, there is $\hat{x} \in \mathbb{Z}_+$ such that if $\overline{x} \geq \hat{x}$, truth-telling is an ϵ-Bayesian Nash equilibrium in any uniform price auction.*

The key to prove Theorem 1 is the relation between uniform price auctions and Vickrey auctions. Formally, a *Vickrey auction* is a mechanism f such that for each $v \in \mathcal{V}^N$,

$$x^f(v) \in \underset{(x_i)_{i\in N} \in A}{\arg\max} \sum_{i \in N} v_i(x_i),$$

and for each $i \in N$,

$$t_i^f(v) = \max_{(x_i)_{i\in N} \in A} \sum_{j \in N \setminus \{i\}} v_j(x_j) - \sum_{j \in N \setminus \{i\}} v_j(x_j^f(v)).$$

In general, a uniform price auction does not coincide with a Vickrey auction. However, we show that under our assumptions, as $\overline{x}$ goes to infinity, at each valuation profile in the support, a uniform price auction gets sufficiently close to a Vickrey auction.

For the intuition, consider the situation where there are two agents, that is $N = \{1, 2\}$. Let f be a uniform price auction and p be the price scheme associated with f. Let $v \in \mathcal{V}^N$. For simplicity, suppose v_1 and v_2 are strictly monotone, that is, for each $i \in N$ and each pair $x, x' \in \mathbb{Z}_+$ with $x > x'$, $v_i(x) > v_i(x')$. We assume that all the objects are assigned to the agents at $f(v)$, that is, $x_1^f(v) + x_2^f(v) = \overline{x}$.[7] As we have seen, an implication of no monopoly is that there are at least two agents whose incremental valuation in the limit is at least as large as the others. Thus, $v_1^\infty = v_2^\infty$.

By the definition of uniform price auctions and the strict monotonicity of the valuation functions, as $\overline{x}$ goes to infinity, both $x_1^f(v)$ and $x_2^f(v)$ go to infinity. To see why,

[6] We give the formal proof in the proof of Theorem 1.

[7] In the formal proof of Theorem 1, we explain that we can assume the condition without loss of generality.

suppose by contradiction that as $\overline{x}$ goes to infinity, only the assignment of one of the agents, say agent 1, goes to infinity. If $\overline{x}$ is large enough (and hence $x_1^f(v)$ is large enough), $v_1(x_1^f(v)) - v_1(x_1^f(v) - 1)$ is sufficiently close to v_1^∞. On the other hand, even though $\overline{x}$ goes to infinity, $v_2(x_2^f(v)+1) - v_2(x_2^f(v))$ does not get close to v_2^∞. By $v_1^\infty = v_2^\infty$, the strict monotonicity of the valuation functions, and these observations, if $\overline{x}$ is large enough, we have $v_1(x_1^f(v)) - v_1(x_1^f(v)-1) < v_2(x_2^f(v)+1) - v_2(x_2^f(v))$. This contradicts the definition of uniform price auctions.

Without loss of generality, focus on agent 1. By the definition of f, $p(v) \in [V_{\overline{x}+1}(v), V_{\overline{x}}(v)]$. In this example, $V_{\overline{x}}(v) = \min\{v_1(x_1^f(v)) - v_1(x_1^f(v)-1), v_2(x_2^f(v)) - v_2(x_2^f(v) - 1)\}$ and $V_{\overline{x}+1}(v) = \max\{v_1(x_1^f(v) + 1) - v_1(x_1^f(v)), v_2(x_2^f(v) + 1) - v_2(x_2^f(v))\}$. Since both $x_1^f(v)$ and $x_2^f(v)$ go to infinity as $\overline{x}$ goes to infinity, all of these incremental valuations converge to $v_1^\infty = v_2^\infty$. Thus, as $\overline{x}$ goes to infinity, $t_1^f(v)$ becomes sufficiently close to $x_1^f(v) \cdot v_1^\infty$.

Now let g be a Vickrey auction such that $x^g = x^f$. By the definition of Vickrey auctions,

$$t_1^g(v) = \max_{x \in X} v_2(x) - v_2(x_2^g(v)) = v_2(\overline{x}) - v_2(x_2^f(v)) = \sum_{x=x_2^f(v)}^{\overline{x}-1} v_2(x+1) - v_2(x).$$

Note that $\overline{x} - x_2^f(v) = x_1^f(v)$. Since $x_2^f(v)$ goes to infinity as $\overline{x}$ goes to infinity, $\sum_{x=x_2^f(v)}^{\overline{x}-1} v_2(x+1) - v_2(x)$ gets close to $x_1^f(v) \cdot v_2^\infty$. Thus, by $v_2^\infty = v_1^\infty$, as $\overline{x}$ goes to infinity, $t_1^g(v)$ also becomes sufficiently close to $x_1^f(v) \cdot v_1^\infty$.

Vickrey auctions are known to be *strategy-proof*, that is, no agent has an incentive to misreport preferences.[8] The strategy-proofness of Vickrey auctions, the above claim, and our assumptions lead to the desired result.[9]

In the above discussion, no monopoly (and thus, the condition $v_1^\infty = v_2^\infty$) plays an important role. Indeed, the above discussion is not applicable to the case that $v_1^\infty > v_2^\infty$. In this case, under a uniform price auction, agent 1 can reduce the price by reporting a smaller valuation. Although it may also reduce the number of objects allocated to agent 1, this loss is exceeded by the gain from the price reduction in some environment. Hence, when $v_1^\infty > v_2^\infty$, truth-telling is not necessarily an ϵ-Bayesian Nash equilibrium in uniform price auctions for some $\epsilon \in \mathbb{R}_{++}$. We provide a concrete example of this argument in Sect. 4.4.

4 Discussions

In this section, we discuss the assumptions of Theorem 1.

[8] We state the formal definition of strategy-proofness in Appendix A.

[9] We give a formal proof in Appendix B.

4.1 Reserve price

An auctioneer sometimes wishes to sell an object only if the price of the object is at least as large as a given reserve price. When there is a reserve price, the notion of uniform price auctions is modified as follows.

Definition 3 Given a reserve price $r \in \mathbb{R}_+$, a mechanism f is a *uniform price auction with reserve price* r if there is a price scheme $p : \mathcal{V}^N \to \mathbb{R}$ such that for each $v \in \mathcal{V}^N$, $p(v) \in [\max\{V_{\overline{x}+1}(v), r\}, \max\{V_{\overline{x}}(v), r\}]$ and for each $i \in N$,

$$x_i^f(v) \geq |\{x \in X \backslash \{\overline{x}\} : v_i(x+1) - v_i(x) > p(v)\}|,$$
$$x_i^f(v) \leq |\{x \in X \backslash \{\overline{x}\} : v_i(x+1) - v_i(x) \geq p(v)\}|, \text{ and,}$$
$$t_i^f(v) = p(v) \cdot x_i^f(v).$$

Corollary 1 *Under Assumptions 1, 2, and 3, for each $r \in \mathbb{R}_+$ and each $\epsilon \in \mathbb{R}_{++}$, there is $\hat{x} \in \mathbb{Z}_+$ such that if $\overline{x} \geq \hat{x}$, truth-telling is an ϵ-Bayesian Nash equilibrium in any uniform price auction with reserve price r.*

The idea of the proof of Corollary 1 is the following. For each uniform price auction with reserve price $r \in \mathbb{R}_+$, there is a new model with an augmented set of agents in which Assumptions 1, 2, and 3 are satisfied, and for each valuation profile in the support, a uniform price auction (without a reserve price) assigns each agent except for the added agents a bundle that is indifferent to the bundle given by the uniform price auction with reserve price r.[10] By Theorem 1, when there are many objects, truth-telling is an approximate Bayesian Nash equilibrium in the uniform price auction. Thus, the same conclusion holds for the uniform price auction with reserve price r.

4.2 Rapid convergence

As we have noted in Sect. 3, Assumption 1 is satisfied if the valuation functions take only discrete values. If we replace Assumption 1 with this condition, we can strengthen the equilibrium concept.

Proposition 1 *Assume that each $v \in \mathcal{V}$ takes only discrete values. Then, under Assumptions 2 and 3, there is $\hat{x} \in \mathbb{Z}_+$ such that if $\overline{x} \geq \hat{x}$, then truth-telling is a Bayesian Nash equilibrium in any uniform price auction.*

Theorem 1 does not hold without Assumption 1. The following example shows that no matter how many objects there are, truth-telling is not even an ϵ-Bayesian Nash equilibrium for some $\epsilon \in \mathbb{R}_{++}$ in a uniform price auction.

Example 1 Let $N := \{1, 2\}$ and $\overline{x} = 2k$, $k \in \mathbb{N}$. Let f be a uniform price auction. Let $v := (v_1, v_2) \in \mathcal{V}^N$ be such that for each $x \in \mathbb{Z}_+ \backslash \{0\}$,

$$v_1(x) = v_2(x) = \sum_{\ell=1}^{x} \frac{1}{\ell}.$$

[10] We give a formal proof in Appendix C.

Assume that for each $v_1' \in \mathcal{V}$, $\Phi_1(\{v_2\}|v_1') = 1$. Clearly, this valuation function violates Assumption 1. Since f is a uniform price auction,

$$x_1^f(v) = x_2^f(v) = k.$$

Note that $V_{\overline{x}}(v) = \frac{1}{k}$ and $V_{\overline{x}+1}(v) = \frac{1}{k+1}$. Thus, $t_1^f(v) \to 1$ as $k \to \infty$. Let $v_1' \in \mathcal{V}$ be such that for each $x \in \mathbb{Z}_+ \backslash \{0\}$,

$$v_1'(x) = \sum_{\ell=1}^{x} \frac{1}{\alpha \cdot \ell}, \quad \alpha > 1.$$

Then, since f is a uniform price auction, we have $x_1^f(v_1', v_2) + x_2^f(v_1', v_2) = 2k$, $\frac{1}{\alpha \cdot x_1^f(v_1', v_2)} \geq \frac{1}{x_2^f(v_1', v_2)+1}$ and $\frac{1}{x_2^f(v_1', v_2)} \geq \frac{1}{\alpha \cdot (x_1^f(v_1', v_2)+1)}$. This implies

$$x_1^f(v_1', v_2) \approx \frac{2k}{1+\alpha}, \quad V_{\overline{x}}(v_1', v_2) \approx \frac{1}{\alpha \cdot x_1^f(v_1', v_2)}.$$

Then,

$$u(f_1(v); v_1) \approx \sum_{\ell=1}^{k} \frac{1}{\ell} - 1,$$
$$u(f_1(v_1', v_2); v_1) \approx \sum_{\ell=1}^{\frac{2k}{1+\alpha}} \frac{1}{\ell} - \frac{1}{\alpha}.$$

When $\alpha = 1.66$, the difference converges to

$$\lim_{k \to \infty} u(f_1(v); v_1) - u(f_i(v_1', v_2); v_1) \approx -0.11 < 0.$$

This implies that for some $\epsilon \in \mathbb{R}_{++}$, no matter how large $\overline{x}$ is, truth-telling is not an ϵ-Bayesian Nash equilibrium in f. □

4.3 Finiteness

Assumption 2 is necessary to ensure that each potential incremental valuation uniformly converges sufficiently rapidly. Under complete information, however, we obtain the following result.

Proposition 2 *Suppose Assumptions 1 and 3 hold. Let f be a uniform price auction. For each $i \in N$, each $v \in \text{supp}_{\Phi_i}(\mathcal{V}^N)$, and each $\epsilon \in \mathbb{R}_{++}$, there is $\hat{x} \in \mathbb{N}$ such that if $\overline{x} \geq \hat{x}$, truth-telling is an ϵ-Nash equilibrium in (f, v).*

4.4 No monopoly

Theorem 1 does not hold if no monopoly is violated. The following example shows that no matter how many objects there are, truth-telling is not an ϵ-Bayesian Nash equilibrium for some $\epsilon \in \mathbb{R}_{++}$ in a uniform price auction.

Example 2 Let $N := \{1, 2\}$ and $\overline{x} \geq 3$. Let f be a uniform price auction. Let $v := (v_1, v_2) \in \mathcal{V}^N$ be such that for each $x \in \mathbb{Z}_+ \backslash \{0\}$,

$$v_1(x) = 10x + 10 \text{ and } v_2(x) = 5x + 20.$$

Assume that for each $v_1' \in \mathcal{V}$, $\Phi_1(\{v_2\}|v_1') = 1$. Note that for each $x \in \mathbb{Z}_+ \backslash \{0\}$, $v_1(x+1) - v_1(x) = 10 > 5 = v_2(x+1) - v_2(x)$. Thus, no monopoly is violated.

Since f is a uniform price auction,

$$x_1^f(v) = \overline{x} - 1 \text{ and } x_2^f(v) = 1.$$

Note that $V_{\overline{x}}(v) = V_{\overline{x}+1}(v) = 10$. Thus, $t_1^f(v) = 10(\overline{x} - 1)$.

Let $v_1' \in \mathcal{V}$ be such that for each $x \in \mathbb{Z}_+ \backslash \{0\}$, $v_1'(x) = 8x + 12$. Since f is a uniform price auction,

$$x_1^f(v_1', v_2) = \overline{x} - 1 \text{ and } x_2^f(v_1', v_2) = 1.$$

Note that $V_{\overline{x}}(v_1', v_2) = V_{\overline{x}+1}(v_1', v_2) = 8$. Thus, $t_1^f(v_1', v_2) = t_1^f(v_1', v_2) = 8(\overline{x} - 1)$. Therefore,

$$\begin{aligned}
&\int_{v_2' \in \mathcal{V}} u(f_1(v_1, v_2'); v_1)\varphi_1(v_2'|v_1)dv_2' - \int_{v_2' \in \mathcal{V}} u(f_1(v_1', v_2'); v_1)\varphi_1(v_2'|v_1)dv_2' \\
&= u_1(f_1(v); v_1) - u_1(f_1(v_1', v_2); v_1) \\
&= v_1(\overline{x} - 1) - 10(\overline{x} - 1) - (v_1(\overline{x} - 1) - 8(\overline{x} - 1)) \\
&= -2(\overline{x} - 1) < 0.
\end{aligned}$$

Hence, for each $\epsilon \in \mathbb{R}_{++}$ with $\epsilon < 2(\overline{x} - 1)$, truth-telling is not an ϵ-Bayesian Nash equilibrium in f. □

4.5 Economies with many agents: replica economies

In Theorem 1, the number of agents is fixed and we investigate how the number of objects affects agents' incentives. In this subsection, we focus on cases in which there are sufficiently many objects and agents. Precisely, we consider replica economies and investigate the incentive properties of uniform price auctions. This section assumes complete information, that is, agents' preferences are publicly known.

An *economy* is a tuple $(N, v, \overline{x})$ where $v \in \mathcal{V}^N$. A *subeconomy* of an economy $(N, v, \overline{x})$ is a tuple $(N', v', \overline{x}')$ where $N' \subseteq N$, $v' \in \mathcal{V}^{N'}$ with the property that for each $i \in N'$, $v'_i = v_i$, and $\overline{x}' \in X$. Given $N' \subseteq N$, $v \in \mathcal{V}^{N'}$, and $v' \in \mathcal{V}$, let $N(v', N', v) := \{i \in N' : v_i = v'\}$.

Definition 4 Given $K \in \mathbb{N}$, an economy $(N, v, \overline{x})$ is a *K-replica* of a subeconomy $(N', v', \overline{x}')$ if (i) $|N| = K \cdot |N'|$, (ii) $|N(v'_i, N, v)| = K \cdot |N(v'_i, N', v')|$ for each $i \in N'$, and (iii) $\overline{x} = K \cdot \overline{x}'$.

The following theorem states that if an economy is generated by replicating a subeconomy of itself sufficiently many times, truth-telling is a Nash equilibrium in minimum uniform price auctions.

Theorem 2 *Let $\mathcal{V} := \mathcal{V}^*$. Let f be a minimum uniform price auction. Let $v \in \mathcal{V}^N$ and $(N', v', \overline{x}')$ be a subeconomy of $(N, v, \overline{x})$. Suppose that $(N, v, \overline{x})$ is a K-replica of $(N', v', \overline{x}')$ for some $K \in \mathbb{N}$ with $K > \overline{x}'$. Then, truth-telling is a Nash equilibrium in (f, v).*

We make some remarks about Theorem 2. First, note that in Theorem 2, we do not make Assumptions 1, 2, and 3. In other words, truth-telling is a Nash equilibrium for each valuation profile in $(\mathcal{V}^*)^N$.

Second, as is the case for Theorem 1, the key for the proof of Theorem 2 is the relation between minimum uniform price auctions and Vickrey auctions. In the proof, we show that if an economy is a K-replica of a subeconomy for a sufficiently large $K \in \mathbb{N}$, a minimum uniform price auction and a Vickrey auction assign the same allocation at the economy. The strategy-proofness of Vickrey auctions and the equivalence lead to the desired result.

Gul and Stacchetti (1999) show the same equivalence result in a model in which there can be several different types of objects. However, the authors assume that each agent can receive at most one object for each type. Since we do not make this assumption, the result by Gul and Stacchetti (1999) does not imply Theorem 2.

The last remark is that Theorem 2 does not necessarily hold for other uniform price auctions. The following example shows that there is a uniform price auction in which even if an economy is generated by replicating a subeconomy sufficiently many times, truth-telling is not a Nash equilibrium.

Example 3 Let $\mathcal{V} := \mathcal{V}^*$ and f be a maximum uniform price auction. Let $K \in \mathbb{N}$. Let $(N, v, \overline{x})$ be a K-replica of the following subeconomy $(N', v', \overline{x}')$: $N' = \{1, 2\}$, $\overline{x}' = 2$, and for each $x \in \mathbb{Z}_+$,

$$v'_1(x) = \begin{cases} x & \text{if } x \leq 2, \\ 2 & \text{otherwise,} \end{cases} \quad \text{and } v'_2(x) = \begin{cases} 2x & \text{if } x \leq 2, \\ 4 & \text{otherwise.} \end{cases}$$

Note that for each $i \in N$, either $v_i = v'_1$ or $v_i = v'_2$. Since f is a uniform price auction, for each $i \in N$,

$$x_i^f(v) = \begin{cases} 0 & \text{if } v_i = v'_1, \\ 2 & \text{if } v_i = v'_2. \end{cases}$$

Note that $V_{\overline{x}}(v) = 2$. Thus, for each $i \in N$ with $v_i = v'_2$, $t_i^f(v) = 4$.

Let $i \in N$ be such that $v_i = v'_2$ and let $v''_i \in \mathcal{V}$ be such that for each $x \in \mathbb{Z}_+$,

$$v''_i(x) = \begin{cases} 1.5x & \text{if } x \leq 2, \\ 3 & \text{otherwise.} \end{cases}$$

Then, for each $j \in N$,

$$x_j^f(v''_i, v_{-i}) = \begin{cases} 0 & \text{if } v_j = v'_1, \\ 2 & \text{otherwise.} \end{cases}$$

Thus, $x_i^f(v''_i, v_{-i}) = 2$. Note that $V_{\overline{x}}(v''_i, v_{-i}) = 1.5$. Thus, $t_i^f(v''_i, v_{-i}) = 3$. Therefore,

$$v_i(x_i^f(v)) - t_i^f(v) - [v_i(x_i^f(v''_i, v_{-i})) - t_i^f(v''_i, v_{-i})] = -1.$$

This implies that truth-telling is not a Nash equilibrium in (f, v). □

5 Concluding remarks

We have investigated the incentive properties of uniform price auctions when there are sufficiently many identical objects. We showed that under several assumptions on preferences and agents' beliefs, if there are sufficiently many objects, truth-telling is an approximate Bayesian Nash equilibrium in any uniform price auction. The key assumption for the result is that of no monopoly. Without this assumption, truth-telling is no longer an approximate Bayesian Nash equilibrium in a uniform price auction even if there are many objects.

There are several directions for future research. One concerns the case in which there are several different objects. Another is to allow preferences to be non-quasi-linear. Quasi-linearity is plausible only when agents' payments are sufficiently small compared with their income levels: a large payment in an auction affects his future consumption plan, which affects the valuations for the objects in the auction. However, in many applications of auction theory, such as auctions of spectrum licenses and treasury securities, agents' payments are large. Several studies show that when preferences are allowed to be non-quasi-linear, typically no allocation rule satisfies efficiency, strategy-proofness, and other mild conditions (Baisa 2016b; Kazumura and Serizawa 2017). Hence, it is of interest to find efficient auctions that are immune to strategic manipulation by agents in large economies when preferences can be non-quasi-linear.

Appendix

Appendix A: Preliminaries

Given $N' \subseteq N$ and $v \in \mathcal{V}^{N'}$, an object allocation $(x_i)_{i \in N} \in A$ is *efficient* for v if $\sum_{i \in N \setminus N'} v_i(x_i) = \max_{(y_i)_{i \in N} \in A} \sum_{i \in N'} v_i(y_i)$.[11] For each $N' \subseteq N$ and each $v \in \mathcal{V}^{N'}$, let $P(v)$ be the set of efficient object allocations for v.

Remark 1 Let $N' \subseteq N$, $v \in \mathcal{V}^{N'}$ and $(x_i)_{i \in N} \in P(v)$. For each pair $i, j \in N'$ with $i \neq j$, if $x_j > 0$, then $v_i(x_i + 1) - v_i(x_i) \leq v_j(x_j) - v_j(x_j - 1)$.

Remark 2 In a uniform price auction f, for each $v \in \mathcal{V}^N$, $x^f(v) \in P(v)$.

A uniform price auction requires agents to pay more money than a Vickrey auction if they have the same object allocation rule (Gul and Stacchetti 1999).

Fact 1 (Gul and Stacchetti 1999) *Let f and g be a uniform price auction and a Vickrey auction, respectively. If $x^f = x^g$, then for each $v \in \mathcal{V}^N$ and each $i \in N$, $t_i^f(v) \geq t_i^g(v)$.*

Now we state the formal definition of strategy-proofness.

Definition 5 A mechanism f is *strategy-proof* if for each $v \in \mathcal{V}^N$, each $i \in N$, and each $v_i' \in \mathcal{V}$, $u(f_i(v); v_i) \geq u(f_i(v_i', v_{-i}); v_i)$.

It is known that Vickrey auctions are strategy-proof. Using Fact 1 and the strategy-proofness of Vickrey auctions, we obtain the following sufficient condition for truth-telling to be an ϵ-Bayesian Nash equilibrium in a uniform price auction.

Proposition 3 *Let f and g be a uniform price auction and a Vickrey auction such that $x^f = x^g$. Let $\epsilon \in \mathbb{R}_+$. Suppose that for each $i \in N$ and each $v \in \text{supp}_{\Phi_i}(\mathcal{V}^N)$, $t_i^f(v) - t_i^g(v) \leq \epsilon$. Then, truth-telling is an ϵ-Baysian Nash equilibrium in f.*

Proof Let $i \in N$ and $v_i \in \mathcal{V}$. For each $(v_i, v_{-i}) \in \text{supp}_{\Phi_i}(\mathcal{V}^N)$, $t_i^f(v_i, v_{-i}) \leq t_i^g(v_i, v_{-i}) + \epsilon$. Thus, for each $v_i' \in \mathcal{V}$,

$$\begin{aligned}
&\int_{v_{-i} \in \mathcal{V}^{N-i}} [v_i(x_i^f(v_i, v_{-i})) - t_i^f(v_i, v_{-i})]\varphi_i(v_{-i}|v_i)dv_{-i} \\
&\geq \int_{v_{-i} \in \mathcal{V}^{N-i}} [v_i(x_i^g(v_i, v_{-i})) - t_i^g(v_i, v_{-i})]\varphi_i(v_{-i}|v_i)dv_{-i} - \epsilon \\
&\geq \int_{v_{-i} \in \mathcal{V}^{N-i}} [v_i(x_i^g(v_i', v_{-i})) - t_i^g(v_i', v_{-i})]\varphi_i(v_{-i}|v_i)dv_{-i} - \epsilon \\
&\geq \int_{v_{-i} \in \mathcal{V}^{N-i}} [v_i(x_i^f(v_i', v_{-i})) - t_i^f(v_i', v_{-i})]\varphi_i(v_{-i}|v_i)dv_{-i} - \epsilon,
\end{aligned}$$

where the second inequality follows from the strategy-proofness of Vickrey auctions, and the last inequality follows from Fact 1. Hence, truth-telling is an ϵ-Bayesian Nash equilibrium. □

[11] In the literature, this notion is sometimes called *decision efficiency*.

The following is an analogue of Proposition 3.

Proposition 4 *Let f and g be a uniform price auction and a Vickrey auction such that $x^f = x^g$. Let $\epsilon \in \mathbb{R}_+$ and $v \in \mathcal{V}^N$ be such that $t^f(v) - t^g(v) \leq \epsilon$ for each $i \in N$. Then, truth-telling is an ϵ-Nash equilibrium in (f, v).*

We omit the proof since it is similar to the proof of Proposition 3.

Appendix B: Proof of Theorem 1

Let $\epsilon \in \mathbb{R}_{++}$. By Assumptions 1 and 2, there is $x^*(\epsilon) \in \mathbb{Z}_+$ such that for each $v_i \in \mathcal{V}$ and each $x \in \mathbb{Z}_+$ with $x \geq x^*(\epsilon)$,

$$x \cdot |v_i(x+1) - v_i(x) - v_i^\infty| < \epsilon. \tag{1}$$

Suppose $\overline{x} \geq |N| \cdot x^*(\epsilon)$.

Let f be a uniform price auction. Let $p : \mathcal{V}^N \to \mathbb{R}$ be the price scheme associated with f. Note that there can be several uniform price auctions that have the same price scheme. However, they always assign each agent bundles that he finds indifferent. Thus, if truth-telling is an ϵ-Bayesian Nash equilibrium in a uniform price auction, then the same conclusion holds for any other uniform price auction that has the same price scheme. Hence, without loss of generality, we can assume that f always assigns all the objects, that is, for each $v \in \mathcal{V}^N$, $\sum_{i \in N} x_i^f(v) = \overline{x}$.

Let g be a Vickrey auction such that $x^g = x^f$. By Proposition 3, all we need to prove Theorem 1 is that for each $i \in N$ and each $v \in \text{supp}_{\Phi_i}(\mathcal{V}^N)$, $t_i^f(v) - t_i^g(v) < \epsilon$. Let $i \in N$ and $v \in \text{supp}_{\Phi_i}(\mathcal{V}^N)$. Denote $v^\infty := \max_{j \in N} v_j^\infty$ and let $N^* := \{j \in N : v_j^\infty = v^\infty\}$.

Step 1 $|N^*| \geq 2$.

Proof Since N is finite, there is $j \in N$ such that $v_j^\infty = v^\infty$. Thus, $|N^*| \geq 1$. Suppose, by contradiction, that $|N^*| = 1$. Then, for each $k \in N \backslash \{j\}$, $v_j^\infty > v_k^\infty$. Let $\delta = v_j^\infty - \max_{k \in N \backslash \{j\}} v_k^\infty$. Since N is finite, $\delta > 0$.

Note that there is $x' \in \mathbb{Z}_+$ such that for each $x \in \mathbb{Z}_+$ with $x \geq x'$ and each $k \in N$, $v_k(x+1) - v_k(x) \leq v_k^\infty + \delta/2$. Then, for each $x \in \mathbb{Z}_+$ with $x \geq x'$ and each $k \in N \backslash \{j\}$,

$$v_j(x+1) - v_j(x) \geq v_j^\infty > \delta/2 + v_k^\infty \geq v_k(x+1) - v_k(x).$$

However, since $v \in \text{supp}_{\Phi_i}(\mathcal{V}^N)$, this inequality contradicts Assumption 3. □

Step 2 $t_i^f(v) < x_i^f(v) \cdot v^\infty + \epsilon$.

Proof Let $j \in \arg\max_{k \in N} x_k^f(v)$. Since $\overline{x} \geq |N| \cdot x^*(\epsilon)$ and $\sum_{k \in N} x_k^f(v) = \overline{x}$, $x_j^f(v) \geq x^*(\epsilon)$. Then, by (1), $x_j^f(v) \cdot (v_j(x_j^f(v)) - v_j(x_j^f(v) - 1) - v_i^\infty) < \epsilon$. Thus, by the definition of $p(v)$,

$$p(v) \leq v_j(x_j^f(v)) - v_j(x_j^f(v) - 1) < v_j^\infty + \frac{\epsilon}{x_j^f(v)} \leq v^\infty + \frac{\epsilon}{x_j^f(v)}.$$

Hence,

$$t_i^f(v) = p(v) \cdot x_i^f(v) < x_i^f(v) \cdot v^\infty + \frac{\epsilon \cdot x_i^f(v)}{x_j^f(v)} \leq x_i^f(v) \cdot v^\infty + \epsilon.$$

□

Step 3 *There is* $(x_j)_{j \in N} \in P(v_{-i})$ *such that for each* $j \in N_{-i}$, $x_j \geq x_j^f(v)$.

Proof Suppose by contradiction that for each $(x_j)_{j \in N} \in P(v_{-i})$, $x_j < x_j^f(v)$ for some $j \in N_{-i}$. Let

$$P^* := \mathop{\arg\min}_{(x_j)_{j \in N} \in P(v_{-i})} |\{j \in N_{-i} : x_j < x_j^f(v)\}|.$$

Since N is finite, $P^* \neq \emptyset$. Let $j \in N_{-i}$ be such that $x_j < x_j^f(v)$ for some $(x_j)_{j \in N} \in P^*$, and let $P^*(j) := \{(x_k)_{k \in N} \in P^* : x_j < x_j^f(v)\}$. Let

$$(x_k)_{k \in N} \in \mathop{\arg\max}_{(y_k)_{k \in N} \in P^*(j)} y_j.$$

Claim 1 *There is* $k \in N \backslash \{i, j\}$ *such that* $x_k > x_k^f(v)$.

Proof Suppose by contradiction that for each $k \in N \backslash \{i, j\}$, $x_k \leq x_k^f(v)$. Then, by $x_j < x_j^f(v)$, $\sum_{k \in N_{-i}} x_k < \sum_{k \in N_{-i}} x_k^f(v) \leq \overline{x}$. Let $(y_k)_{k \in N} \in A$ be such that $y_i = 0$, $y_j = x_j + 1$, and for each $k \in N \backslash \{i, j\}$, $y_k = x_k$. Note that $(y_k)_{k \in N}$ is feasible because $\sum_{k \in N} y_k = 1 + \sum_{k \in N_{-i}} x_k \leq \overline{x}$. Moreover, by $v_j(y_j) = v_j(x_j + 1) \geq v_j(x_j)$, $\sum_{k \in N_{-i}} v_k(y_k) \geq \sum_{k \in N_{-i}} v_k(x_k)$. Thus, $(x_k)_{k \in N} \in P(v_{-i})$ implies $(y_k)_{k \in N} \in P(v_{-i})$.

By $x_j < x_j^f(v)$, we have either $y_j = x_j^f(v)$ or $y_j < x_j^f(v)$. If $y_j = x_j^f(v)$, then

$$|\{k \in N_{-i} : y_k < x_k^f(v)\}| < |\{k \in N_{-i} : x_k < x_k^f(v)\}|,$$

which contradicts $(x_k)_{k \in N} \in P^*$. Thus, $y_j < x_j^f(v)$. This implies $(y_k)_{k \in N} \in P^*(j)$. By $y_j > x_j$, however, this also contradicts the definition of $(x_k)_{k \in N}$. □

Claim 2 $v_j(x_j + 1) - v_j(x_j) < v_k(x_k) - v_k(x_k - 1)$.

Proof Suppose by contradiction that $v_j(x_j+1)-v_j(x_j)\geq v_k(x_k)-v_k(x_k-1)$. Let $(y_\ell)_{\ell\in N}\in A$ be such that $y_i=0$, $y_j=x_j+1$, $y_k=x_k-1$, and for each $\ell\in N\backslash\{i,j,k\}$, $y_\ell=x_\ell$. Note that $(y_\ell)_{\ell\in N}$ is feasible because $\sum_{\ell\in N}y_\ell=\sum_{\ell\in N_{-i}}x_\ell\leq\overline{x}$. Moreover,

$$\sum_{\ell\in N_{-i}}v_\ell(y_\ell)=v_j(x_j+1)+v_k(x_k-1)+\sum_{\ell\in N\backslash\{i,j,k\}}v_\ell(x_\ell)\geq\sum_{\ell\in N_{-i}}v_\ell(x_\ell).$$

By $(x_\ell)_{\ell\in N}\in P(v_{-i})$, we have $(y_\ell)_{\ell\in N}\in P(v_{-i})$.

By $x_j<x_j^f(v)$, we have either $y_j=x_j^f(v)$ or $y_j<x_j^f(v)$. If $y_j=x_j^f(v)$, then

$$|\{\ell\in N_{-i}:y_\ell<x_\ell^f(v)\}|<|\{\ell\in N_{-i}:x_\ell<x_\ell^f(v)\}|,$$

which contradicts $(x_\ell)_{\ell\in N}\in P^*$. Thus, $y_j<x_j^f(v)$. This implies $(y_k)_{k\in N}\in P^*(j)$. By $y_j>x_j$, however, this also contradicts the definition of $(x_\ell)_{\ell\in N}$. □

By $x_j<x_j^f(v)$, non-increasing incremental valuations of v_j, Claims 1 and 2,

$$\begin{aligned}v_j(x_j^f(v))-v_j(x_j^f(v)-1)&\leq v_j(x_j+1)-v_j(x_j)\\&<v_k(x_k)-v_k(x_k-1)\\&\leq v_k(x_k^f(v)+1)-v_k(x_k^f(v)).\end{aligned}$$

This contradicts Remark 1. □

Let

$$P^*(v_{-i}):=\{(x_j)_{j\in N}\in P(v_{-i}):\text{ for each } j\in N_{-i}, x_j\geq x_j^f(v)\}.$$

By Step 3, $P^*(v_{-i})\neq\emptyset$.

Step 4 *Let $(x_j)_{j\in N}\in P^*(v_{-i})$ and $j\in N_{-i}$. If $x_j>x_j^f(v)$, $v_j(x_j)-v_j(x_j-1)\geq v^\infty$.*

Proof Suppose, by contradiction, that $x_j>x_j^f(v)$ and $v_j(x_j)-v_j(x_j-1)<v^\infty$. By Step 1, $N^*\backslash\{i\}\neq\emptyset$. Let $k\in N^*\backslash\{i\}$. Then, $v_k(x_k+1)-v_k(x_k)\geq v^\infty$, and thus, $k\neq j$. Thus, $v_k(x_k+1)-v_k(x_k)\geq v^\infty>v_j(x_j)-v_j(x_j-1)$. By $(x_\ell)_{\ell\in N}\in P^*(v_{-i})$, this inequality contradicts Remark 1. □

Step 5 *Completing the proof.*

Let $(x_j)_{j\in N}\in P^*(v_{-i})$. Without loss of generality, we assume that $\sum_{j\in N_{-i}}x_j=\overline{x}$. Note that $t_i^g(v)=\sum_{j\in N_{-i}}v_j(x_j)-\sum_{j\in N_{-i}}v_j(x_j^g(v))=\sum_{j\in N_{-i}}v_j(x_j)-\sum_{j\in N_{-i}}v_j(x_j^f(v))$. Thus,

$$\begin{aligned}
t_i^f(v) - t_i^g(v) &< x_i^f(v)\cdot v^\infty + \epsilon - \Bigg(\sum_{j\in N_{-i}} v_j(x_j) - \sum_{j\in N_{-i}} v_j(x_j^f(v))\Bigg) \\
&= x_i^f(v)\cdot v^\infty + \epsilon \\
&\quad - \sum_{j\in N_{-i},\, x_j > x_j^f(v)} (v_j(x_j) - v_j(x_j - 1) + v_j(x_j - 1) - v_k(x_j - 2) \\
&\quad + \cdots + v_j(x_j^f(v)+1) - v_j(x_j^f(v))) \\
&\le x_i^f(v)\cdot v^\infty + \epsilon - \sum_{j\in N_{-i},\, x_j > x_j^f(v)} (v_j(x_j) - v_j(x_j - 1))\cdot (x_j - x_j^f(v)) \\
&\le x_i^f(v)\cdot v^\infty + \epsilon - \sum_{j\in N_{-i},\, x_j > x_j^f(v)} v^\infty \cdot (x_j - x_j^f(v)),
\end{aligned}$$

where the first inequality follows from Step 2, the second inequality from non-increasing incremental valuations, and the last inequality from Step 4. Note that

$$\begin{aligned}
\sum_{j\in N_{-i},\, x_j > x_j^f(v)} (x_j - x_j^f(v)) &= \sum_{j\in N_{-i},\, x_j > x_j^f(v)} (x_j - x_j^f(v)) \\
&\quad + \sum_{j\in N_{-i},\, x_j = x_j^f(v)} (x_j - x_j^f(v)) \\
&= \overline{x} - \sum_{j\in N_{-i}} x_j^f(v) \\
&= x_i^f(v).
\end{aligned}$$

Hence, $t_i^f(v) - t_i^g(v) < \epsilon$. □

Appendix C: Proof of Corollary 1

Let $\epsilon \in \mathbb{R}_{++}$. Let $r \in \mathbb{R}_+$ and f be a uniform price auction with reserve price r. Let $p : \mathcal{V}^N \to \mathbb{R}$ be the price scheme associated with f. Note that there can be several uniform price auctions with reserve price r that have the same price scheme. However, they always assign each agent bundles that he finds indifferent. Thus, if truth-telling is an ϵ-Bayesian Nash equilibrium in a uniform price auction with reserve price r, then the same conclusion holds for any other uniform price auction with reserve price r that has the same price scheme. Hence, without loss of generality, we can assume that f always assigns as many objects as possible, that is, for each $v \in \mathcal{V}^N$,

$$\sum_{i \in N} x_i^f(v) = \min \left\{ \overline{x}, \sum_{i \in N} |\{x \in X \backslash \{\overline{x}\} : v_i(x+1) - v_i(x) \geq p(v)\}| \right\}. \quad (2)$$

Now, we construct a new model where there are two additional agents. Let $N' := \{1 \ldots, n, n+1, n+2\}$, and $v^* \in \mathcal{V}^*$ be such that for each $x \in \mathbb{Z}_+$, $v^*(x) = x \cdot r$. Let $\mathcal{V}' := \mathcal{V} \cup \{v^*\}$. For each $i \in N'$, we denote i's prior belief in the new setting by Φ'_i and its probability density function by φ'_i. For each $\mathcal{V}'' \subseteq \mathcal{V}^N$, let $(\mathcal{V}'', v^*) := \{v \in (\mathcal{V}')^{N'} : v_N \in (\mathcal{V}'')^N$ and $v_{n+1} = v_{n+2} = v^*\}$. For each $i \in N$ and each $\mathcal{V}'' \subseteq \mathcal{V}^N$, we assume $\Phi'_i(\mathcal{V}'', v^*) = \Phi_i(\mathcal{V}'')$. Note that for each $i \in N$ and each $v \in (\mathcal{V}')^{N'}$, $v \in \text{supp}_{\Phi'_i}((\mathcal{V}')^{N'})$ if and only if $v_{n+1} = v_{n+2} = v^*$ and $v_N \in \text{supp}_{\Phi_i}(\mathcal{V}^N)$. For each $i \in \{n+1, n+2\}$, we assume $\Phi'_i = \Phi'_1$. Note that in the new model, Assumptions 1, 2, and 3 are satisfied.

We define a mechanism g in the new model as follows. Let $v \in (\mathcal{V}')^{N'}$. If $v_{n+1} = v_{n+2} = v^*$, then

for each $i \in N$, $x_i^g(v) = x_i^f(v_N)$, $x_{n+1}^g(v) = \overline{x} - \sum_{i \in N} x_i^g(v)$, and $x_{n+2}^g(v) = 0$, and

for each $i \in N'$, $t_i^g(v) = x_i^g(v) \cdot p(v_N)$.

Otherwise, $g(v)$ is an allocation given by a uniform price auction (without a reserve price).

Claim 3 *g is a uniform price auction in the new model.*

Proof Let $v \in (\mathcal{V}')^{N'}$. If $v_{n+1} \neq v^*$ or $v_{n+2} \neq v^*$, then by the definition of g, $g(v)$ is an allocation given by a uniform price auction. Thus, suppose $v_{n+1} = v_{n+2} = v^*$. Denote $p := p(v_N)$. We show that $p \in [V_{\overline{x}+1}(v), V_{\overline{x}}(v)]$ and for each $i \in N'$,

$$|\{x \in X \backslash \{\overline{x}\} : v_i(x+1) - v_i(x) > p\}| \leq x_i^g(v) \\ \leq |\{x \in X \backslash \{\overline{x}\} : v_i(x+1) - v_i(x) \geq p\}|.$$

Note that the latter claim is done for each $i \in N$ because $x_i^g(v) = x_i^f(v_N)$. We have two cases.

Case 1: $p > r$. By $p \leq \max\{r, V_{\overline{x}}(v_N)\}$ and $p > r$, $p \leq V_{\overline{x}}(v_N)$. By $N \subseteq N'$, $V_{\overline{x}}(v) \geq V_{\overline{x}}(v_N)$. Thus, $p \leq V_{\overline{x}}(v)$. By $v_{n+1} = v_{n+2} = v^*$ and the definition of v^*, if $V_{\overline{x}+1}(v_N) > r$, then $V_{\overline{x}+1}(v) = V_{\overline{x}+1}(v_N) \leq p$. If $V_{\overline{x}+1}(v_N) \leq r$, then $V_{\overline{x}+1}(v) = r < p$. Hence, $p \in [V_{\overline{x}+1}(v), V_{\overline{x}}(v)]$.

By $p \leq V_{\overline{x}}(v_N)$, $\sum_{i \in N} |\{x \in X \backslash \{\overline{x}\} : v_i(x+1) - v_i(x) \geq p\}| \geq \overline{x}$. Thus, by (2), $\sum_{i \in N} x_i^f(v_N) = \overline{x}$. This implies that $x_{n+1}^g(v) = 0$. By the definition of v^* and $p > r$, $\{x \in X \backslash \{\overline{x}\} : v^*(x+1) - v^*(x) > p\} = \{x \in X \backslash \{\overline{x}\} : v^*(x+1) - v^*(x) \geq p\} = \emptyset$. Therefore, for each $i \in \{n+1, n+2\}$, by $v_i = v^*$ and $x_i^g(v) = 0$, $|\{x \in X \backslash \{\overline{x}\} : v_i(x+1) - v_i(x) > p\}| \leq x_i^g(v) \leq |\{x \in X \backslash \{\overline{x}\} : v_i(x+1) - v_i(x) \geq p\}|$.

Case 2. $p = r$. By $v_{n+1} = v^*$ and the definition of v^*, $V_{\overline{x}}(v) \geq r = p$. By $r = p \geq V_{\overline{x}+1}(v_N)$ and $v_{n+1} = v^*$, $V_{\overline{x}+1}(v) = r$. Hence, $p \in [V_{\overline{x}+1}(v), V_{\overline{x}}(v)]$.

By the definition of v^*, $\{x \in X\backslash\{\overline{x}\} : v^*(x+1) - v^*(x) > r\} = \emptyset$ and $\{x \in X\backslash\{\overline{x}\} : v^*(x+1) - v^*(x) \geq r\} = X\backslash\{\overline{x}\}$. Hence, for each $i \in \{n+1, n+2\}$, by $p = r$ and $v_i = v^*$, $|\{x \in X\backslash\{\overline{x}\} : v_i(x+1) - v_i(x) > p\}| \leq x_i^g(v) \leq |\{x \in X\backslash\{\overline{x}\} : v_i(x+1) - v_i(x) \geq p\}|$. □

By Claim 3 and Theorem 1, there is $\hat{x} \in \mathbb{Z}_+$ such that if $\overline{x} \geq \hat{x}$, truth-telling is an ϵ-Bayesian Nash equilibrium in g.

Suppose $\overline{x} \geq \hat{x}$. Let $i \in N$ and $v_i, v_i' \in \mathcal{V}$. By the definition of Φ_i' and g,

$$\begin{aligned}
&\int_{v_{-i} \in \mathcal{V}^{N\backslash\{i\}}} [v_i(x_i^f(v_i, v_{-i})) - t_i^f(v_i, v_{-i})]\varphi_i(v_{-i}|v_i)dv_{-i} \\
&= \int_{v'_{-i} \in (\mathcal{V}')^{N'\backslash\{i\}}} [v_i(x_i^g(v_i, v'_{-i})) - t_i^g(v_i, v'_{-i})]\varphi_i'(v'_{-i}|v_i)dv'_{-i} \\
&\geq \int_{v'_{-i} \in (\mathcal{V}')^{N'\backslash\{i\}}} [v_i(x_i^g(v_i', v'_{-i})) - t_i^g(v_i', v'_{-i})]\varphi_i'(v'_{-i}|v_i)dv'_{-i} - \epsilon \\
&= \int_{v_{-i} \in \mathcal{V}^{N\backslash\{i\}}} [v_i(x_i^f(v_i', v_{-i})) - t_i^f(v_i', v_{-i})]\varphi_i(v_{-i}|v_i)dv_{-i} - \epsilon.
\end{aligned}$$

Hence, truth-telling is an ϵ-Bayesian Nash equilibrium in f. □

Appendix D: Proofs of Propositions 1 and 2

Proof of Proposition 1 By Assumption 2 and the assumption that each valuation function takes discrete values, there is $x^* \in \mathbb{Z}_+$ such that for each $i \in N$, each $v \in \text{supp}_{\Phi_i}(\mathcal{V}^N)$, each $j \in N$, and each $x \in \mathbb{Z}_+$ with $x > x^*$,

$$v_j(x+1) - v_j(x) = v_j^\infty.$$

Let $\overline{x} > |N| \cdot x^*$. Let f be a uniform price auction. As we explained in the proof of Theorem 1, without loss of generality, we assume that for each $v \in \mathcal{V}^N$, $\sum_{i \in N} x_i^f(v) = \overline{x}$. Let g be a Vickrey auction such that $x^g = x^f$. Then, by following the proof of Theorem 1, we can show that for each $i \in N$, each $v \in \text{supp}_{\Phi_i}(\mathcal{V}^N)$, and each $j \in N$, $t_j^f(v) = t_j^g(v)$. Thus, by Proposition 3, we obtain the desired result. □

Proof of Proposition 2 As we explained in the proof of Theorem 1, without loss of generality, we assume that for each $v' \in \mathcal{V}^N$, $\sum_{i \in N} x_i^f(v') = \overline{x}$. Let $i \in N$ and $v \in \text{supp}_{\Phi_i}(\mathcal{V}^N)$. Let $\epsilon \in \mathbb{R}_{++}$. By Assumption 1, for each $j \in N$, there is $x_j^*(\epsilon) \in \mathbb{Z}_+$ such that for each $x \in \mathbb{Z}_+$ with $x \geq x_j^*(\epsilon)$,

$$x \cdot |v_j(x+1) - v_j(x) - v_j^\infty| < \epsilon.$$

Let $\overline{x} > |N| \cdot \max_{j \in N} x_j^*(\epsilon)$. Let g be a Vickrey auction such that $x^g = x^f$. Then, by following the proof of Theorem 1, we can show that for each $j \in N$, $t_j^f(v) - t_j^g(v) < \epsilon$. Therefore, by Proposition 4, we obtain the desired result. □

Appendix E: Proof of Theorem 2

As we explained in the proof of Theorem 1, without loss of generality, we assume that for each $v \in \mathcal{V}^N$, $\sum_{i \in N} x_i^f(v) = \overline{x}$. Since $(N, v, \overline{x})$ is a K-replica economy of $(N', v', \overline{x}')$, there is $(N_i)_{i \in N'} \in (2^N)^{N'}$ such that (i) $\bigcup_{i \in N'} N_i = N$, (ii) for each pair $i, j \in N'$ with $i \neq j$, $N_i \cap N_j = \emptyset$, (iii) for each $i \in N'$, $|N_i| = K$, and (iv) for each $i \in N'$ and each $j \in N_i$, $v_j = v'_i$. Let $(x'_i)_{i \in N'}$ be an efficient object allocation in the subeconomy $(N', v', \overline{x}')$, i.e.,

$$(x'_i)_{i \in N'} \in \arg\max \left\{ \sum_{i \in N'} v'_i(x_i) : (x_i)_{i \in N'} \in \{0, \ldots, \overline{x}'\}^{N'} \text{ and } \sum_{i \in N'} x_i \leq \overline{x}' \right\}.$$

Without loss of generality, we assume $\sum_{i \in N'} x'_i = \overline{x}'$. Let $(x_i)_{i \in N} \in A$ be such that for each $i \in N$, $x_i = x'_j$ where $j \in N'$ and $i \in N_j$.

Step 1 $(x_i)_{i \in N} \in P(v)$.

Proof Suppose by contradiction that $(x_i)_{i \in N} \notin P(v)$. Let

$$(y_i^*)_{i \in N} \in \mathop{\arg\min}_{(y_i)_{i \in N} \in P(v)} \sum_{i \in N} |y_i - x_i|.$$

By $(x_i)_{i \in N} \notin P(v)$ and $(y_i^*)_{i \in N} \in P(v)$, $\sum_{i \in N} v_i(y_i^*) > \sum_{i \in N} v_i(x_i)$. Thus, there is $i \in N$ such that $v_i(y_i^*) > v_i(x_i)$. By monotonicity, $y_i^* > x_i$. By the definition of $(x_j)_{j \in N}$,

$$\sum_{j \in N} x_j = \sum_{j \in N'} K \cdot x'_j = K \cdot \overline{x}' = \overline{x} \geq \sum_{j \in N} y_j^*.$$

Thus, by $y_i^* > x_i$, there is $j \in N \backslash \{i\}$ such that $y_j^* < x_j$. □

Claim 4 $v_i(y_i^*) - v_i(y_i^* - 1) = v_j(y_j^* + 1) - v_j(y_j^*)$

Proof By the definition of $(x_k)_{k \in N}$, there is $k \in N'$ such that $i \in N_k$ and $x'_k = x_i$. For the same reason, there is $\ell \in N'$ such that $j \in N_\ell$ and $x'_\ell = x_j$. Note that $x'_\ell = x_j > y_j^* \geq 0$. Thus, by the definition of $(x'_{i'})_{i' \in N'}$ and Remark 1, $v'_k(x'_k + 1) - v'_k(x'_k) \leq v'_\ell(x'_\ell) - v'_\ell(x'_\ell - 1)$. Therefore,

$$\begin{aligned} v_i(x_i + 1) - v_i(x_i) &= v'_k(x'_k + 1) - v'_k(x'_k) \\ &\leq v'_\ell(x'_\ell) - v'_\ell(x'_\ell - 1) \\ &= v_j(x_j) - v_j(x_j - 1). \end{aligned} \tag{3}$$

By $(y^*_{i'})_{i' \in N} \in P(v)$ and Remark 1, $v_i(y^*_i) - v_i(y^*_i - 1) \geq v_j(y^*_j + 1) - v_j(y^*_j)$. Thus, by $y^*_j < x_j$, (3), $y^*_i > x_i$, and non-increasing incremental valuations,

$$\begin{aligned} v_i(y^*_i) - v_i(y^*_i - 1) &\geq v_j(y^*_j + 1) - v_j(y^*_j) \\ &\geq v_j(x_j) - v_j(x_j - 1) \\ &\geq v_i(x_i + 1) - v_i(x_i) \\ &\geq v_i(y^*_i) - v_i(y^*_i - 1). \end{aligned}$$

Therefore, $v_i(y^*_i) - v_i(y^*_i - 1) = v_j(y^*_j + 1) - v_j(y^*_j)$. □

Let $(y_k)_{k \in N} \in A$ be such that for each $k \in N$,

$$y_k = \begin{cases} y^*_i - 1 & \text{if } k = i, \\ y^*_j + 1 & \text{if } k = j, \\ y^*_k & \text{otherwise.} \end{cases}$$

By Claim 4, $v_i(y^*_i - 1) + v_j(y^*_j + 1) = v_i(y^*_i) + v_j(y^*_j)$. Thus,

$$\sum_{k \in N} v_k(y_k) = v_i(y^*_i - 1) + v_j(y^*_j + 1) + \sum_{k \in N \setminus \{i,j\}} v_k(y^*_k) = \sum_{k \in N} v_k(y^*_k).$$

Thus, by $(y^*_k)_{k \in N} \in P(v)$, we have $(y_k)_{k \in N} \in P(v)$. Moreover, by $y^*_i > x_i$ and $y^*_j < x_j$,

$$\begin{aligned} \sum_{k \in N} |y_k - x_k| &= |y^*_i - 1 - x_i| + |y^*_j + 1 - x_j| + \sum_{k \in N \setminus \{i,j\}} |y^*_k - x_k| \\ &= -2 + \sum_{k \in N} |y^*_k - x_k| \\ &< \sum_{k \in N} |y^*_k - x_k|. \end{aligned}$$

This contradicts the definition of $(y^*_k)_{k \in N}$. □

Let

$$N^* := \{i \in N : v_i(x_i + 1) - v_i(x_i) = V_{\overline{x}+1}(v)\}.$$

Step 2 $N^* \neq \emptyset$.

Proof By Step 1, Remark 1, and non-increasing incremental valuations, for each $i \in N$, each $j \in N$, and each $x \in X \setminus \{0\}$ with $x \leq x_i$,

$$v_i(x) - v_i(x - 1) \geq v_i(x_i) - v_j(x_i - 1) \geq v_j(x_j + 1) - v_j(x_j)$$

Hence, by $\sum_{i \in N} x_i = \overline{x}$ and non-increasing incremental valuations, $\max_{i \in N}\{v_i(x_i + 1) - v_i(x_i)\} = V_{\overline{x}+1}(v)$, which completes the proof. □

Note that for each $i \in N^*$, there are at least $K - 1$ other agents who have the same valuation function and object assignment at $(x_j)_{j \in N}$ as agent i. Thus, $|N^*| \geq K$. Fix $i \in N$ and let

$$\begin{aligned} N_1 &:= \{j \in N \backslash \{i\} : x_j^f(v) = x_j\}, \\ N_2 &:= \{j \in N \backslash \{i\} : x_j^f(v) > x_j\}, \text{ and} \\ N_3 &:= \{j \in N \backslash \{i\} : x_j^f(v) < x_j\}. \end{aligned}$$

Let $N_j^* := N_j \cap N^*$ for each $j = 1, 2, 3$.

Step 3 $x_i^f(v) \leq |N_1^*| + \sum_{j \in N_3^*}(x_j - x_j^f(v) + 1) + \sum_{j \in N_3 \backslash N_3^*}(x_j - x_j^f(v))$.

Proof Note that $N^* \backslash \{i\} = N_1^* \cup N_2^* \cup N_3^*$. Thus, by $|N^*| \geq K$ and $K > \overline{x}' \geq x_i$,

$$|N_1^*| + |N_2^*| + |N_3^*| = |N^* \backslash \{i\}| \geq x_i. \tag{4}$$

By $N_2 \supseteq N_2^*$ and the definition of N_2,

$$|N_2^*| \leq |N_2| \leq \sum_{j \in N_2} (x_j^f(v) - x_j).$$

By $\sum_{j \in N} x_j^f(v) = \overline{x} = \sum_{j \in N} x_j$,

$$\begin{aligned} x_i^f(v) + \sum_{j \in N_1} x_j^f(v) + \sum_{j \in N_2} x_j^f(v) + \sum_{j \in N_3} x_j^f(v) = \sum_{j \in N} x_j^f(v) = \sum_{j \in N} x_j = x_i \\ + \sum_{j \in N_1} x_j + \sum_{j \in N_2} x_j + \sum_{j \in N_3} x_j. \end{aligned}$$

By $\sum_{j \in N_1} x_j^f(v) = \sum_{j \in N_1} x_j$ and $|N_2^*| \leq \sum_{j \in N_2}(x_j^f(v) - x_j)$,

$$x_i^f(v) - x_i = \sum_{j \in N_3} (x_j - x_j^f(v)) - \sum_{j \in N_2} (x_j^f(v) - x_j) \leq \sum_{j \in N_3} (x_j - x_j^f(v)) - |N_2^*|. \tag{5}$$

Therefore, by (4) and (5),

$$\begin{aligned} x_i^f(v) &= x_i + (x_i^f(v) - x_i) \\ &\leq |N_1^*| + |N_2^*| + |N_3^*| + \sum_{j \in N_3} (x_j - x_j^f(v)) - |N_2^*| \\ &\leq |N_1^*| + |N_3^*| + \sum_{j \in N_3} (x_j - x_j^f(v)) \\ &\leq |N_1^*| + \sum_{j \in N_3^*} (x_j - x_j^f(v) + 1) + \sum_{j \in N_3 \setminus N_3^*} (x_j - x_j^f(v)). \end{aligned}$$

□

By Step 3, there is $(y_j^*)_{j \in N_1^* \cup N_3} \in X^{|N_1^* \cup N_3|}$ such that $\sum_{j \in N_1^* \cup N_3} y_j^* = x_i^f(v)$ and for each $j \in N_1^* \cup N_3$,

$$y_j^* \leq \begin{cases} 1 & \text{if } j \in N_1^*, \\ x_j - x_j^f(v) + 1 & \text{if } j \in N_3^*, \\ x_j - x_j^f(v) & \text{if } j \in N_3 \setminus N_3^*. \end{cases}$$

Let $(y_j)_{j \in N} \in X^N$ be such that for each $j \in N$,

$$y_j = \begin{cases} 0 & \text{if } j = i, \\ x_j^f(v) + y_j^* & \text{if } j \in N_1^* \cup N_3, \\ x_j^f(v) & \text{otherwise.} \end{cases}$$

Note that

$$\sum_{j \in N} y_j = \sum_{j \in N_1^* \cup N_3} y_j^* + \sum_{j \in N_{-i}} x_j^f(v) = x_i^f(v) + \sum_{j \in N_{-i}} x_j^f(v) = \overline{x}.$$

Thus, $(y_j)_{j \in N} \in A$. Moreover, for each $j \in N_{-i}$, $y_j \geq x_j^f(v)$.

Step 4 *Let $j \in N_{-i}$ be such that $y_j > x_j^f(v)$. For each $x \in \{x_j^f(v) + 1, \ldots, y_j\}$, $v_j(x) - v_j(x-1) = V_{\overline{x}+1}(v)$.*

Proof By the definition of f and non-increasing incremental valuations,

$$V_{\overline{x}+1}(v) \geq v_j(x_j^f(v) + 1) - v_j(x_j^f(v)) \geq v_j(y_j) - v_j(y_j - 1).$$

Thus, we complete the proof if we show $V_{\overline{x}+1}(v) \leq v_j(y_j) - v_j(y_j - 1)$. Note that by $y_j > x_j^f(v)$, we have $j \in N_1^* \cup N_3$.

Case 1: $j \in N_1^*$. By the definition of y^*, we have $y_j = x_j^f(v) + y_j^* \leq x_j + 1$. By $y_j > x_j^f(v) = x_j$, we have $y_j = x_j + 1$. Thus, by $j \in N^*$,

$$V_{\overline{x}+1}(v) = v_j(x_j + 1) - v_j(x_j) = v_j(y_j) - v_j(y_j - 1).$$

Case 2: $j \in N_3^*$. By the definition of y^*, we have $y_j = x_j^f(v) + y_j^* \leq x_j + 1$. Thus, by $j \in N^*$ and non-increasing incremental valuations,

$$V_{\overline{x}+1}(v) = v_j(x_j + 1) - v_j(x_j) \leq v_j(y_j) - v_j(y_j - 1).$$

Case 3: $j \in N_3 \backslash N_3^*$. By $N^* \neq \emptyset$ and $j \in N_3 \backslash N_3^*$, there is $k \in N^*$ such that $k \neq j$. Note that by $j \in N_3$, $x_j > x_j^f(v) \geq 0$. Thus, by $k \in N^*$, $(x_\ell)_{\ell \in N} \in P(v)$ and Remark 1,

$$V_{\overline{x}+1}(v) = v_k(x_k + 1) - v_k(x_k) \leq v_j(x_j) - v_j(x_j - 1).$$

By the definition of y^*, $y_j = x_j^f(v) + y_j^* \leq x_j$. Thus, by non-increasing incremental valuations,

$$V_{\overline{x}+1}(v) \leq v_j(x_j) - v_j(x_j - 1) \leq v_j(y_j) - v_j(y_j - 1).$$

□

Step 5 *Completing the proof.*

Let g be a Vickrey auction such that $x^g = x^f$. By Step 4, for each $j \in N_{-i}$ with $y_j > x_j^f(v)$,

$$\begin{aligned} v_j(y_j) - v_j(x_j^f(v)) &= v_j(y_j) - v_j(y_j - 1) + (v_j(y_j - 1) - v_j(y_j - 2)) \\ &\quad + \cdots + (v_j(x_j^f(v) + 1) - v_j(x_j^f(v))) \\ &= V_{\overline{x}+1}(v) \cdot (y_j - x_j^f(v)). \end{aligned}$$

By the definition of y,

$$\sum_{j \in N_{-i}} (y_j - x_j^f(v)) = \sum_{j \in N_1^* \cup N_3} y_j^* = x_i^f(v).$$

Thus, we have

$$\begin{aligned} t_i^g(v) &= \max_{(x''_j)_{j\in N}\in A} \sum_{j\in N_{-i}} v_j(x''_j) - \sum_{j\in N_{-i}} v_j(x_j^g(v)) \\ &\geq \sum_{j\in N_{-i}} v_j(y_j) - \sum_{j\in N_{-i}} v_j(x_j^f(v)) \\ &= \sum_{j\in N_{-i}} (v_j(y_j) - v_j(x_j^f(v))) \\ &= V_{\overline{x}+1}(v) \cdot \sum_{j\in N_{-i}} (y_j - x_j^f(v)) \\ &= V_{\overline{x}+1}(v) \cdot x_i^f(v) \\ &= t_i^f(v). \end{aligned}$$

Hence, by Proposition 4, we obtain the desired result. □

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