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A q-player impartial avoidance game for generating finite groups

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Abstract We study a q-player variation of the impartial avoidance game introduced by Anderson and Harary, where q is a prime. The game is played by the q players taking turns selecting previously-unselected elements of a finite group. The losing player is the one who selects an element that causes the set of jointly-selected elements to be a generating set for the group, with the previous player winning. We introduce a ranking system for the other players to prevent coalitions. We describe the winning strategy for these games on cyclic, nilpotent, dihedral, and dicyclic groups.

Keywords Group theory · Game theory · Impartial game · Maximal subgroup

Mathematics Subject Classification 91A46 · 20D30

1 Introduction

The game *Do Not Generate* was introduced by Anderson and Harary Anderson and Harary (1987). In this game, two players take turns selecting previously unselected elements of a finite group until the group is generated by the jointly-selected elements. The losing player is the first player who selects an element that causes the jointly-selected elements to generate the entire group. The strategies were classified in Barnes

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(1988), and the strategies and nim-numbers for these games were classified in Benesh et al. (2016a) and Benesh et al. (2016b). *Do Not Generate* cannot be played on the trivial group, as in this case the empty set generates the entire group implying there are no legal moves for either player. Thus, we will assume that all groups are nontrivial. We modify this game to include q players for a prime q, and we use the notation $\mathrm{DNG}_q(G)$ to denote the game on a nontrivial finite group G with q players.

If n is an integer, we define $[n]_q$ to be the unique integer i in $\{1, 2, 3, \ldots, q\}$ such that $n \equiv i \mod q$. We will simply write $[n]$ for $[n]_q$ if there is no risk of confusion, and we will write $[H]$ instead of $[|H|]$ if H is a group. Additionally, let $P_1, P_2, \ldots, P_q$ denote the q players of $\mathrm{DNG}_q(G)$ for a nontrivial finite group G, where P_i makes the ith move of the game.

We give two results on $\mathrm{DNG}_q(G)$ on finite nilpotent groups G, which are finite groups that can be written as a direct product of their Sylow subgroups. The first is a special case of the second, and $d(G)$ denotes the minimum size of a generating set of a group G.

Theorem 7 *Let G be a nontrivial finite cyclic group of order n. Let p be a prime divisor of n such that $[n/p]$ is minimal. Then $P_{[n/p]}$ has a winning strategy in* $\mathrm{DNG}_q(G)$.

Theorem 12 *Let G be a nontrivial finite nilpotent group. If q is a prime that divides $|G|$, then the winner of* $\mathrm{DNG}_q(G)$ *is*

1. P_j *if* $|G| \equiv jq \mod q^2$ *for some* $j \in \{1, 2, \ldots, q-1\}$ *and* $d(G) \le j$
2. P_q *if* $|G| \equiv jq \mod q^2$ *for some* $j \in \{1, 2, \ldots, q-1\}$ *and* $d(G) > j$
3. P_q *if* $|G| \equiv 0 \mod q^2$.

If we are considering $\mathrm{DNG}_3(G)$, we can be more specific than the previous result.

Corollary 13 *Let G be a nontrivial finite nilpotent group, and let H be the direct product of Sylow r-groups of G such that $r \equiv 1 \mod 3$ and K be the direct product of Sylow t-groups of G such that $t \equiv 2 \mod 3$. Then for* $\mathrm{DNG}_3(G)$,

1. P_1 *has a winning strategy in the following cases.*
   (a) $|G| \equiv 1 \mod 3$ *and* $2d(H) \ge d(K) + 1$
   (b) $|G| \equiv 2 \mod 3$ *and* $2d(K) \ge d(H) + 1$
   (c) $|G| \equiv 3 \mod 9$ *and* $d(G) = 1$
2. P_2 *has a winning strategy in the following cases.*
   (a) $|G| \equiv 1 \mod 3$ *and* $2d(H) < d(K) + 1$
   (b) $|G| \equiv 2 \mod 3$ *and* $2d(K) < d(H) + 1$
   (c) $|G| \equiv 6 \mod 9$ *and* $d(G) \le 2$
3. P_3 *has a winning strategy in the following cases.*
   (a) $|G| \equiv 0 \mod 9$
   (b) $|G| \equiv 3 \mod 9$ *and* $d(G) \ge 2$
   (c) $|G| \equiv 6 \mod 9$ *and* $d(G) \ge 3$

Our final result is about dihedral and dicyclic groups, where dicyclic groups are generalizations of the quaternion group of order 8.

Theorem 15 *Let G be a dihedral group or dicyclic group and define*

$$W = \{[|G|/p] : p \mid |G|, p \text{ odd}\}$$
$$X = \{[|G|/p] : p \mid |G|, p \text{ odd}, [|G|/p] \ne 1\}$$

We let $m = \min W$ *if* $W \neq \emptyset$ *and* $m' = \min X$ *if* $X \neq \emptyset$. *Then for* $\mathrm{DNG}_q(G)$,

1. $P_{[|G|/2]}$ *has a winning strategy if one of the following is true.*
   (a) $|G| = 2^k$ *for some* k.
   (b) $[|G|/2] \leq m$
   (c) $1 = m < [|G|/2]$, $X = \emptyset$, *and* $|G|/2$ *is even*
   (d) $1 = m < [|G|/2] \leq m'$, $X \neq \emptyset$
2. P_m *has a winning strategy if one of the following is true.*
   (a) $1 < m < [|G|/2]$
   (b) $m = 1$, $X = \emptyset$, *and* $|G|/2$ *is odd*
3. $P_{m'}$ *has a winning strategy if* $1 = m < m' < [|G|/2]$ *and* $X \neq \emptyset$.

2 Preliminaries

We are generalizing a 2-player game to a q-player game for a prime $q \geq 2$. This introduces the possibility of the players forming coalitions, which greatly adds to the complexity of the game. Such games were studied in Li (1978), Propp (2000), and Straffin (1985), with various strategies for reducing the complexity. We adopt the convention from Li (1978), which is described next.

We will call the players $P_1, \ldots, P_q$, where P_i makes the nth move if and only if $n \equiv i \mod q$. If P_m makes the last legal move, we say that P_m wins, and we rank each player in the following order:

$$P_m, P_{m+1}, P_{m+2}, \ldots, P_q, P_1, P_2, \ldots, P_{m-1}.$$

Thus, P_m wins, P_{m+1} is runner-up, P_{m+2} is the second runner-up, and so on. Thus, if P_m wins, then P_i ends up in $[i - m + 1]$th place. We will assume that each P_i plays optimally to optimize the P_i's rank at the end of the game by minimizing $[i - m + 1]$.

The intuition for the game $\mathrm{DNG}_q(G)$ on a nontrivial finite group G follows. Once the game is over, the q players will realize that they simply took turns selecting elements from a single maximal subgroup M, although they may not realize what M is early in the game. To see this, consider that if the elements $X := \{x_1, \ldots, x_n\}$ have been selected after the nth turn of the game, then one considers the subgroup $H := \langle X \rangle$ to determine whether the game is over. If $H < G$, play continues. Because G is a finite group, H is contained in a maximal subgroup M. The $[n + 1]$st player will be able to avoid losing on the $(n + 1)$st turn exactly when $X \neq M$ by selecting any $g \in M \backslash X$. If $X = M$, then $\langle X \cup \{h\} \rangle = G$ for any $h \in G \backslash X$.

Thus, for every $\mathrm{DNG}_q(G)$ game, there will be a unique maximal subgroup M such that every element of M will be selected if the game is played optimally; let $\overline{M}$ denote this maximal subgroup. We can now determine the outcome of the game according to the value of $[\overline{M}]$.

Lemma 1 *For any nontrivial finite group* G, $P_{[\overline{M}]}$ *wins* $\mathrm{DNG}_q(G)$.

Proof We can use the Division Algorithm to write $|\overline{M}| = nq + r$ for some $0 \leq r \leq q - 1$. Then we see that each of the q players contributes n elements to the pool of

Fig. 1 This is an optimally played game for $\mathrm{DNG}_3(G)$ on $G := D_8 \times \mathbb{Z}_3$ with $D_8 = \langle r, f\rangle$ of order 8 and $\mathbb{Z}_3 = \{0, 1, 2\}$. Notice that the game is determined with the selection of $(\mathbf{f}, \mathbf{0})$, since that guarantees that $|\overline{M}|$ will be $8 \equiv 2 \bmod 3$, guaranteeing a victory for P_2 by Lemma 1

Player	Element selected	Subgroup generated
1	$(r, 0)$	$\mathbb{Z}_4$
2	$(\mathbf{f}, \mathbf{0})$	D_8
3	$(e, 0)$	D_8
1	$(r^2, 0)$	D_8
2	$(rf, 0)$	D_8
3	$(r^3 f, 0)$	D_8
1	$(r^3, 0)$	D_8
2	$(r^2 f, 0)$	D_8
3	$g \in G \setminus D_8 \times \{0\}$	G

Fig. 2 This is an optimally played game for $\mathrm{DNG}_3(G)$ on $G := \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ with $\mathbb{Z}_2 = \{0, 1\}$ and $\mathbb{Z}_3 = \{0, 1, 2\}$. Notice that the game is determined with the selection of $(\mathbf{0}, \mathbf{0}, \mathbf{1})$, since that guarantees that $|\overline{M}|$ will be divisible by 3, guaranteeing a victory for P_3 by Lemma 2

Player	Element selected	Subgroup generated
1	$(1, 0, 0)$	$\mathbb{Z}_2$
2	$(\mathbf{0}, \mathbf{0}, \mathbf{1})$	$\mathbb{Z}_2 \times \mathbb{Z}_3$
3	$(0, 0, 2)$	$\mathbb{Z}_2 \times \mathbb{Z}_3$
1	$(1, 0, 1)$	$\mathbb{Z}_2 \times \mathbb{Z}_3$
2	$(0, 0, 0)$	$\mathbb{Z}_2 \times \mathbb{Z}_3$
3	$(1, 0, 2)$	$\mathbb{Z}_2 \times \mathbb{Z}_3$
1	$g \in G \setminus \mathbb{Z}_2 \times \{0\} \times \mathbb{Z}_3$	G

selected elements, and the first r players get to select an additional element of $\overline{M}$. Therefore, P_{r+1} must select an element outside of $\overline{M}$, and hence generates the group and loses. Because $r = [\overline{M}]$ if $r > 0$ and P_q wins if $r = 0$, we see that $P_{[\overline{M}]}$ wins $\mathrm{DNG}_q(G)$.

The following lemma describes P_q's advantage, noting that Cauchy's Theorem guarantees that an element of order q exists.

Lemma 2 *Let G be a nontrivial finite group G and $X = \{x_1, \ldots, x_j\}$ denote the set of elements selected by $P_1, \ldots, P_j$. If $\langle X\rangle < G$ and X contains an element of order q, then P_q wins $DNG_q(G)$.*

Proof Since $X \subseteq \overline{M}$, $\overline{M}$ will contain an element of order q. By Lagrange's Theorem, q divides $|\overline{M}|$. By Lemma 1, $P_{[\overline{M}]} = P_q$ wins.

Let D_8 denote the dihedral group of order 8. Figure 1 shows an optimally played game of $\mathrm{DNG}_3(G)$ for the nilpotent group $G := D_8 \times \mathbb{Z}_3$, where P_2 wins with the help of P_1 by Lemma 1. Figure 2 shows an optimally played game where P_3 wins on $\mathrm{DNG}_3(G)$ on $G := \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ by Lemma 2, with help from P_2.

Let $d(G)$ denote the minimum size of a generating set of a group G. We end with a couple of general results.

Proposition 3 *If G is a finite group such that q divides $|G|$ and $d(G) \geq q + 1$, then P_q has a winning strategy in $DNG_q(G)$.*

Proof Suppose that P_i selects x_i for $1 \leq i \leq q - 1$. If there is a i such that x_i has order q, then P_q wins by Lemma 2. So suppose that none of the x_i have order q. By Cauchy's Theorem, there is an element t of order q in G, and $\langle x_1, \ldots, x_{q-1}, t\rangle < G$ since $d(G) \geq q + 1$. Then P_q wins by Lemma 2.

We now discuss some qualities of groups where Lemma 2 applies, which will allow us to partially generalize the results about $\mathrm{DNG}_2(G)$ from Benesh et al. (2016b). Before stating the results for $\mathrm{DNG}_2(G)$, we state a definition.

Definition 4 Let G be a noncyclic finite group, and denote the set of maximal subgroups of G by $\mathcal{M}_G$. We say that a subset $\mathcal{X}$ of $\mathcal{M}_G$ *n-covers* G if for every n elements $g_1, \ldots, g_n \in G$, there is a maximal subgroup $M \in \mathcal{X}$ such that $g_1, \ldots, g_n \in M$.

One of the main results of Benesh et al. (2016b) about $\mathrm{DNG}_2(G)$ is below.

Corollary 5 [Benesh et al. (2016b), Corollary 5.5] *If G is a noncyclic group of even order, then the second player has a winning strategy if and only if the set of maximal subgroups of even order 1-covers G.*

We let $\mathcal{M}_G^q$ be the set of maximal subgroups of G with orders divisible by q. The next proposition is a first approximation for a $\mathrm{DNG}_q(G)$ version of the Corollary 5.

Proposition 6 *Let G be a finite noncyclic group. If $\mathcal{M}_G^q$ j-covers G, then P_i does not have a winning strategy for $1 \leq i \leq j$ for $\mathrm{DNG}_q(G)$. In particular, P_q has a winning strategy if $\mathcal{M}_G^q$ $(q-1)$-covers G. Additionally, if $\mathcal{M}_G^q$ does not 1-cover G, then P_q does not have a winning strategy for $\mathrm{DNG}_q(G)$.*

Proof Suppose that $\mathcal{M}_G^q$ j-covers G and P_i initially selects an element x_i of G for $1 \leq i \leq j$. Because $\mathcal{M}_G^q$ j-covers G, there is a maximal subgroup $M \in \mathcal{M}_G^q$ such that $x_1, \ldots, x_j \in M$. Since P_{j+1} prefers P_q to win over P_i for $1 \leq i \leq j$, P_{j+1} could prevent such P_i from winning by selecting an element t of M of order q, which would result in a win for P_q by Lemma 2. Since P_{j+1} may have a better strategy than selecting t, we conclude that P_k will win for some $j+1 \leq k \leq q$. So if $j = q-1$, then P_q wins.

Now assume that $\mathcal{M}_G^q$ does not 1-cover G. Recall that P_1 prefers any player to win over P_q. Since $\mathcal{M}_G^q$ does not 1-cover G, P_1 can choose an element not contained in $\cup\mathcal{M}_G^q$. Then $\overline{M} \notin \mathcal{M}_G^q$, so P_q does not win by Lemma 1. □

3 Cyclic groups

We start by considering a cyclic group G with generator g and order n. In the case of cyclic groups, P_1 can determine $\overline{M}$ by selecting g^p on the first move to generate a maximal subgroup of order n/p for some prime p. This allows us to conclude the following.

Theorem 7 *Let G be a nontrivial finite cyclic group of order n. Let p be a prime divisor of n such that $[n/p]$ is minimal. Then $P_{[n/p]}$ has a winning strategy in $\mathrm{DNG}_q(G)$.*

Proof By Lemma 1, it suffices to consider the maximal subgroups of G. It is well-known that every maximal subgroup M of G has order n/t for some prime t if G is cyclic, so we conclude that the winner will be $P_{[n/t]}$ for some t. Because G is cyclic, every subgroup is also cyclic, and P_1 can optimize its ranking by selecting a generator of a maximal subgroup of order n/p such that $[n/p]$ is minimal. Thus, $P_{[n/p]}$ wins by Lemma 1.

Corollary 8 *Let G be a nontrivial finite cyclic group of order n, and suppose that the game is $DNG_3(G)$.*

1. *If $n \equiv 1 \bmod 3$ and there is a prime number p dividing n such that $p \equiv 1 \bmod 3$, then P_1 has a winning strategy.*
2. *If $n \equiv 1 \bmod 3$ and every prime number p dividing n is such that $p \equiv 2 \bmod 3$, then P_2 has a winning strategy.*
3. *If $n \equiv 2 \bmod 3$, then P_1 has a winning strategy.*
4. *If $n \equiv 0 \bmod 9$, then P_3 has a winning strategy.*
5. *If $n \equiv 3 \bmod 9$, then P_1 has a winning strategy.*
6. *If $n \equiv 6 \bmod 9$, then P_2 has a winning strategy.*

Proof If $n \equiv 1 \bmod 3$, then $[n/p] = 1$ if $p \equiv 1 \bmod 3$ and $[n/p] = 2$ otherwise. Thus, P_1 wins if there is a prime p such that $[n/p] = 1$ and P_2 wins otherwise when $n \equiv 1 \bmod 3$. If $n \equiv 2 \bmod 3$, then there must be a prime p such that $p \equiv 2 \bmod 3$. Then $[n/p] = 1$ and P_1 wins.

So assume that $n \equiv 0 \bmod 3$, in which case $[n/p] = 3$ for all primes $p \neq 3$. If $n \equiv 0 \bmod 9$, then $[n/3] = 3$ as well, and hence P_3 wins regardless of strategy. If $n \equiv 3 \bmod 9$, then $[n/3] = 1$ and P_1 wins. If $n \equiv 6 \bmod 9$, then $[n/3] = 2$, and P_2 wins because P_1 prefers P_2 over P_3.

4 Nilpotent groups

Recall that nilpotent groups are a generalization of abelian groups and have the following properties.

Theorem 9 [Isaacs (2009), Theorem 8.19] *Let G be a finite group. Then the following are equivalent.*

1. *G is nilpotent.*
2. *Every maximal subgroup of G is normal in G.*
3. *G is isomorphic to a direct product of its Sylow subgroups.*

Proposition 10 [Isaacs (2009), Problem 8.11] *If M is a maximal subgroup of a finite nilpotent group G, then $|M| = |G|/p$ for some prime divisor p of $|G|$.*

Corollary 11 *If p is a prime and G is a nontrivial finite p-group of order p^n, then $P_{[p^{n-1}]}$ wins $DNG_q(G)$.*

Proof Every maximal subgroup has order $|G|/p = p^{n-1}$, so $P_{[p^{n-1}]}$ wins by Lemma 1.

We will generalize Theorem 7 in the theorem below by determining the outcomes when $d(G) \geq 2$ in the case where q divides $|G|$. By Lemma 1, the orders of the maximal subgroups are key to proving the next result. Thus, we will first use Proposition 10 to determine the orders of the maximal subgroups, and then we will determine which of those orders can be the order of $\overline{M}$.

Theorem 12 *Let G be a nontrivial finite nilpotent group. If q is a prime that divides $|G|$, then the winner of $DNG_q(G)$ is*

1. P_j *if* $|G| \equiv jq \bmod q^2$ *for some* $j \in \{1, 2, \ldots, q-1\}$ *and* $d(G) \leq j$
2. P_q *if* $|G| \equiv jq \bmod q^2$ *for some* $j \in \{1, 2, \ldots, q-1\}$ *and* $d(G) > j$
3. P_q *if* $|G| \equiv 0 \bmod q^2$.

Proof If $d(G) \geq 1$ and $|G| \equiv 0 \bmod q^2$, then $|G|/p \equiv 0 \bmod q$ for all such p dividing $|G|$, including $p = q$. Therefore, P_q wins in this case, regardless of strategy.

So suppose $|G| \equiv jq \bmod q^2$ for some $j \in \{1, \ldots, q-1\}$. Then for every maximal subgroup M, $[M] = [|G|/p] = j$ if $p = q$ and $[M] = q$ otherwise. Thus, P_j and P_q are the only two players who can win, with P_k preferring P_j to win for $k \in \{1, \ldots, j\}$ and P_l preferring P_q to win for $l \in \{j+1, j+2, \ldots, q\}$.

We can write $G = Q \times H$ for some subgroup Q of order q and subgroup H such that q does not divide $|H|$. Note that Q is cyclic, so $d(G) = d(H)$ since $G = Q \times H$ and $\gcd(|Q|, |H|) = 1$. If $d(H) = d(G) \leq j$, then P_1 to $P_{d(H)}$ will select generators of H with their first moves, with H a maximal subgroup of order $|G|/q$. Thus, P_j will win.

If $d(H) = d(G) > j$, then suppose that P_1 to P_j select elements $x_1, \ldots, x_j \in G$ with $X := \langle x_1, \ldots, x_j \rangle$ with their first moves. If q divides $|X|$, then P_q will win by Lemma 1. If q does not divide $|X|$, then X is a subgroup of H. But $d(H) > j$, so X is a proper subgroup of H. Since P_{j+1} prefers P_q to win, P_{j+1} can select an element $t \in G$ of order q, yielding a proper subgroup isomorphic to $Q \times X < Q \times H = G$. Thus, q divides $|\langle x_1, \ldots, x_j, t \rangle|$ and P_q wins by Lemma 2.

Note that Theorem 12 agrees with Theorem 7 for cyclic groups.

Corollary 13 *Let G be a nontrivial finite nilpotent group, and let H be the direct product of all Sylow r_i-groups of G such that* $r_i \equiv 1 \bmod 3$ *and K be the direct product of Sylow t_j-groups of G such that* $t_j \equiv 2 \bmod 3$. *Then the following is true for $DNG_3(G)$.*

1. P_1 *has a winning strategy in the following cases.*
   (a) $|G| \equiv 1 \bmod 3$ *and* $2d(H) \geq d(K) + 1$
   (b) $|G| \equiv 2 \bmod 3$ *and* $2d(K) \geq d(H) + 1$
   (c) $|G| \equiv 3 \bmod 9$ *and* $d(G) = 1$
2. P_2 *has a winning strategy in the following cases.*
   (a) $|G| \equiv 1 \bmod 3$ *and* $2d(H) < d(K) + 1$
   (b) $|G| \equiv 2 \bmod 3$ *and* $2d(K) < d(H) + 1$
   (c) $|G| \equiv 6 \bmod 9$ *and* $d(G) \leq 2$
3. P_3 *has a winning strategy in the following cases.*
   (a) $|G| \equiv 0 \bmod 9$
   (b) $|G| \equiv 3 \bmod 9$ *and* $d(G) \geq 2$
   (c) $|G| \equiv 6 \bmod 9$ *and* $d(G) \geq 3$

Proof The results follow from Theorem 12 if 3 divides $|G|$, so we may assume that 3 does not divide $|G|$. Thus, we may write $G = H \times K$, and P_3 cannot win because no maximal subgroup has order divisible by 3. Then P_3 will help P_1 and, after the first two elements are selected, P_1 effectively selects two elements for every element P_2 selects.

Suppose first that $|G| \equiv 1 \bmod 3$. Note that since $|H|$ and $|K|$ are coprime, the maximal subgroups of G have the form $L \times K$ and $H \times J$ for maximal subgroups L of H and J of K by Thévenaz (1997, Lemma 1.3). Since P_1 wants $[\overline{M}] = 1$, P_1 wants $|\overline{M}| = |G|/r_i$ for some i. In other words, P_1 wants $\{e\} \times K \leq \overline{M}$. Therefore, P_1 and P_3 should select elements of the form $(e, k) \in H \times K$ where e is the identity of H and k is an element of a generating set of minimal size of K. Similarly, P_2 will select elements of the form (h, e') where e' is the identity of K and h is an element of a generating set of H of minimal size. Because P_1 and P_3 each get to choose a generator of K for every generator of H that P_2 chooses, we see after some simple algebra that P_1 (with P_3's help) will be able to generate K before P_2 can generate H if and only if $2d(H) \geq d(K) + 1$. A similar argument shows that P_1 wins when $|G| \equiv 2 \bmod 3$ if and only if $2d(K) \geq d(H) + 1$.

5 Dihedral and dicyclic groups

Let D_{2n} denote a dihedral group of order $2n$ and Q_{4n} denote the dicyclic group of order $4n$, which has the presentation $\langle a, x \mid a^{2n} = 1, x^2 = a^n, x^{-1}ax = a^{-1}\rangle$ where 1 is the identity of Q_{4n}. Note that Q_8 is isomorphic to the quaternion group. Additionally, $D_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong Q_4$ if $n = 1$, and D_2 is not usually considered a dihedral group and Q_4 is not usually considered a dicyclic group. However, the results below hold for $n = 1$, so we will allow for $n = 1$.

Notice that both D_{2n} and Q_{4n} have a cyclic subgroup C of index 2 such that every element not in C acts on C by inversion. We start with a statement about the maximal subgroups of certain metacyclic groups that are similar to D_{2n} and Q_{4n}.

Proposition 14 *Let G be finite group with a cyclic subgroup C of index 2 such that every $g \in G \backslash C$ acts on C via inversion. If M is a maximal subgroup of G, then either $|M| = |G|/2$ or $|M| = |G|/p$ for some prime p dividing $|C|$. Moreover, there is a maximal subgroup L of each such order in G, where $d(L) = 1$ if $L = C$ and $d(L) = 2$ otherwise.*

Proof Let M be a maximal subgroup of G. If $M \leq C$, then $M = C$ with $|G : M| = 2$. Then C is a maximal subgroup of order $|G|/2$ with $d(C) = 1$.

So assume that M is not contained in C, let $H = M \cap C$, and let t be any element of order 2 in $M \backslash C$. The element t acts by inversion on all elements of C, so t normalizes H and hence $H\langle t\rangle$ is a subgroup of M. We will show that $M = H\langle t\rangle$. It is clear that $H\langle t\rangle \leq M$, so it suffices to show that M is contained in $H\langle t\rangle$. Let $x \in M$. If $x \in M \cap C = H$, then $x \in H\langle t\rangle$. So assume that x is not in C. Since $G = C\langle t\rangle$, we see that $x = ct$ for some $c \in C$. Then $c = (ct)t = xt \in M$, so $c \in M \cap C = H$ and $x = ct \in H\langle t\rangle$. Therefore, $M \leq H\langle t\rangle$, and we conclude that $M = H\langle t\rangle$.

If H is not maximal in C, then there is proper subgroup K of C properly containing H. Because t acts by inversion on all of C, we have $M = H\langle t\rangle < K\langle t\rangle < C\langle t\rangle = G$, which contradicts the maximality of M. Therefore, H is maximal in C and has order

$|C|/p$ for some prime divisor p of $|C|$. Then

$$|M| = |H\langle t\rangle| = |H||\langle t\rangle|/|H \cap \langle t\rangle| = |H||\langle t\rangle| = (|C|/p)(2) = |G|/p.$$

Finally, let p be an odd prime divisor of $|G|$. Then p divides $|C|$. Because C is cyclic, it has a maximal subgroup D of order $|C|/p$. Then $L := D\langle t\rangle$ is a maximal subgroup of G of order $|G|/p$ for any $t \in G\backslash C$. Since $L = \langle D, t\rangle$ and D is cyclic, $d(L) = 2$.

We can now again determine the outcomes of the avoidance game on dihedral and dicyclic groups by considering the orders of maximal subgroups.

Theorem 15 *Let G be a dihedral group or dicyclic group and define*

$$W = \{[|G|/p] : p \text{ divides } |G|,\ p \text{ odd}\}$$
$$X = \{[|G|/p] : p \text{ divides } |G|,\ p \text{ odd},\ [|G|/p] \neq 1\}$$

We let $m = \min W$ *if* $W \neq \emptyset$ *and* $m' = \min X$ *if* $X \neq \emptyset$. *Then for* $DNG_q(G)$,

1. $P_{[|G|/2]}$ *has a winning strategy if one of the following is true.*
   (a) $|G| = 2^k$ *for some* k.
   (b) $[|G|/2] \leq m$
   (c) $1 = m < [|G|/2]$, $X = \emptyset$, *and* $|G|/2$ *is even*
   (d) $1 = m < [|G|/2] \leq m'$, $X \neq \emptyset$
2. P_m *has a winning strategy if one of the following is true.*
   (a) $1 < m < [|G|/2]$
   (b) $m = 1$, $X = \emptyset$, *and* $|G|/2$ *is odd*
3. $P_{m'}$ *has a winning strategy if* $1 = m < m' < [|G|/2]$ *and* $X \neq \emptyset$.

Proof If $|G| = 2^k$ for some k, then every maximal subgroup of G has order 2^{k-1} by Proposition 14. Thus, $P_{[|G|/2]}$ will win. So suppose that $|G|$ is not a power of 2. Then an odd prime divides $|G|$, so W is nonempty and m is defined.

Let C denote the cyclic subgroup of order $|G|/2$. By Proposition 14, P_k will win for some $k = [|G|/p]$ for a prime p that divides $|G|$, and P_1 will prefer that k is $[|G|/2]$ if $|G|/2 \leq m$ and m otherwise. Because C has order $|G|/2$, P_1 will select a generator of C on the first move if $[|G|/2] \leq m$, so $P_{[|G|/2]}$ will win in this case.

So suppose that $m < [|G|/2]$. This means that P_1 wants P_m to win, and thus will help generate a maximal subgroup of order $|G|/p$ for some odd p dividing G. Let p be an odd prime dividing $|G|$. Then there is a maximal subgroup M_p of order $|G|/p$ with $d(M_p) = 2$ by Proposition 14. Since $d(M_p) = 2$, P_1 cannot unilaterally determine $\overline{M}$ unless $\overline{M} = C$, which P_1 does not want.

If $1 < m < [|G|/2]$, then P_2 also prefers P_m to win. Since all maximal subgroups are 2-generated by Proposition 14, P_1, and P_2 can select elements that generate a maximal subgroup of order $|G|/p$ such that $[|G|/p] = m$. Thus, P_m wins.

If $1 = m < [|G|/2]$, however, then P_2 does not prefer $P_m = P_1$ to win. If $X = \emptyset$, then $[|G|/p] = 1$ for all odd primes p that divide $|G|$ and P_2 prefers that $P_{[|G|/2]}$

wins. If $|C| = |G|/2$ is odd, then P_1 can pick any $t \in G\backslash C$, which implies $\overline{M}$ will have order $|G|/p$ for some odd prime p dividing $|C|$. Then $[\overline{M}] = 1$ and P_1 wins.

If $X = \emptyset$ and n is even, then $\overline{M}$ will have order $|G|/2$ or $|G|/p$ for some p dividing $|C|$ with $[|G|/p] = 1$ if p is odd. In this case, P_2 prefers $|\overline{M}| = |G|/2$. If P_1 picks an element of C, then P_2 selects a generator of C to ensure that $\overline{M} = C$. If P_1 picks a $t \in G\backslash C$, then P_2 selects an element $c \in C$ such that $|\langle c\rangle| = |C|/2$. Then $|\langle t, c\rangle| = |G|/2$. In either case, $P_{[|G|/2]}$ wins.

Suppose now that $1 = m < [|G|/2]$ and $X \neq \emptyset$. In this case, P_1 could potentially win, but P_2 still prefers that anyone but P_1 wins. Because $X \neq \emptyset$, there is a $y \in C$ such that $|C : \langle y\rangle| = p_{m'}$ for some prime $p_{m'}$ with $[|G|/p_{m'}] = m'$.

1. If P_1 selects some $x \in C$ on the first move, then because C is cyclic, P_2 can select any $y \in C$ such that $\langle x, y\rangle = C$. Then $\overline{M} = C$ and $P_{[C]} = P_{[|G|/2]}$ wins.
2. If P_1 selects any $t \in G\backslash C$ on the first move, then P_2 selects y so that $P_{m'}$ wins.

Therefore, P_1 cannot win. If $[|G|/2] \leq m'$, P_1's second preference is for $P_{[|G|/2]}$ to win. Then P_1 can select a generator of C on the first move so that $P_{[|G|/2]}$ wins. If $m' < |G|/2$, then P_1 and P_2 both want $P_{m'}$ to win, and will select t and y as above.

Corollary 16 *Let D_{2n} be a dihedral group. Then for $DNG_3(D_{2n})$,*

1. *P_1 has a winning strategy if and only if $n \equiv 1 \bmod 3$.*
2. *P_2 has a winning strategy if and only if $n \equiv 2 \bmod 3$ or $n \equiv 3 \bmod 9$.*
3. *P_3 has a winning strategy if and only if $n \equiv 0 \bmod 9$ or $n \equiv 6 \bmod 9$.*

Proof We will use the notation from Theorem 15. If $n \equiv 1 \bmod 3$, then $[|G|/2] = [n] = 1 \leq m$, so P_1 wins. If $n \equiv 0 \bmod 9$, then $[|G|/2] = 3 = m$, so P_3 wins. If $n \equiv 3 \bmod 9$, then $[|G|/2] = 3$ and $[|G|/p] = [2n/p]$ is 2 if $p = 3$ and 3 otherwise, so $m = 2$. Thus, P_2 wins if $n \equiv 3 \bmod 9$ since $1 < 2 = m < 3 = [|G|/2]$.

If $n \equiv 6 \bmod 9$, then $[|G|/3] = 1$ and $[|G|/p] = 3$ if $p \neq 3$. This implies that $m = 1$. Since $n \equiv 6 \bmod 9$, either $n = 2^k \cdot 3$ for some $k \geq 1$ or there is an odd prime $p \neq 3$ that divides n. If $n = 2^k \cdot 3$, then $X = \emptyset$ and $|G|/2 = 2^k \cdot 3$ is even, so $P_{[|G|/2]} = P_3$ wins by Item 1c of Theorem 15. If such an odd p exists, then $[|G|/p] = 3 \neq 1$ and $X \neq \emptyset$. This time, P_3 wins by Item 1d of Theorem 15. In either case, P_3 wins.

Now suppose that $n \equiv 2 \bmod 3$, in which case n must have a prime divisor u such that $u \equiv 2 \bmod 3$. Additionally, 3 will not divide $|\overline{M}|$, so P_3 cannot win. However, we see that there is no case in Theorem 15 where P_1 wins since $[|G|/2] = 2$ is not odd and $1 \notin \{[|G|/2], m'\}$. Because P_3 cannot win, we conclude that P_2 wins.

The proof of the next corollary exactly mirrors that of Corollary 16, and is thus omitted. The only difference worth noting is that the cyclic subgroup of index 2 in a dicyclic group has order $2n$ rather than n.

Corollary 17 *Let Q_{4n} be a dicyclic group. Then for $DNG_3(Q_{4n})$,*

1. *P_1 has a winning strategy if and only if $2n \equiv 1 \bmod 3$.*
2. *P_2 has a winning strategy if and only if $2n \equiv 2 \bmod 3$ or $2n \equiv 3 \bmod 9$.*
3. *P_3 has a winning strategy if and only if $2n \equiv 0 \bmod 9$ or $2n \equiv 6 \bmod 9$.*

6 Further questions

We close with some open questions.

1. What are the strategies and outcomes for groups other than cyclic, dihedral, and nilpotent groups? Perhaps supersolvable or simple groups could be classified—supersolvable groups are a next logical generalization of nilpotent groups, and a lot is known about the maximal subgroups of finite simple groups.
2. Can we generalize these results for the analogous game involving n players when n is not prime? In this case, Lemma 2 may not apply because Cauchy's Theorem does not guarantee an element of order n.
3. What are the winning strategies in the analogous achievement game where the player who generates the group wins, rather than loses?
4. Can we extend the result in Proposition 6 by determining exactly who wins if $\mathcal{M}_G^q$ j-covers G? This has been done in Benesh et al. (2016b) for $\text{DNG}_2(G)$, as stated in Corollary 5.

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