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ORIGINAL PAPER

Warm-glow giving in networks with multiple public goods

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Abstract This paper explores a voluntary contribution game in the presence of warm-glow effects. There are many public goods and each public good benefits a different group of players. The structure of the game induces a bipartite network structure, where players are listed on one side and the public good groups they form are listed on the other side. The main result of the paper shows the existence and uniqueness of a Nash equilibrium. The unique Nash equilibrium is also shown to be asymptotically stable. Then the paper provides some comparative statics analysis regarding pure redistribution, taxation and subsidies. It appears that small redistributions of wealth may sometimes be neutral, but generally, the effects of redistributive policies depend on how public good groups are related in the contribution network structure.

Keywords Multiple public goods · Warm-glow effects · Bipartite contribution structure · Nash equilibrium · Comparative statics

JEL Classification C72 · D64 · H40

1 Introduction

In many real life situations, people are organized in social groups with a common goal whose achievement has the characteristics of a public good (Olson 1965; Cornes

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and Sandler 1986). When individual actions are unobservable, a joint work by a team of co-workers can be regarded as such (Holmstrom 1982). Colleagues working on a joint project, students working on a group report, neighbors creating a good social atmosphere or friends planning a party are only a few examples of social groups providing their members with a public good.[1] As a result, people's well-being is often dependent on the private provision of many public goods. Securing the sustainability of these goods, for which generally no market mechanism exists, is therefore a problem of considerable practical importance.

On the academic side, theoretical work with multiple public goods has mainly concerned models in which voluntary contributions are driven by " pure altruism ".[2] In other words, people are supposed to be indifferent to the means by which the public goods are provided, and to only care for the total supply of each public good (Kemp 1984; Bergstrom et al. 1986; Cornes and Schweinberger 1996; Cornes and Itaya 2010). Controlled laboratory experiments, however, contradict this assumption. In practice, for moral, emotional or even social reasons, people enjoy a private benefit, commonly called and henceforth referred to as " warm-glow ", from the act of contributing, independently of the utility they gain from the aggregate amounts of contributions (Andreoni 1993, 1995; Palfrey and Prisbrey 1996, 1997; Andreoni and Miller 2002; Eckel et al. 2005; Gronberg et al. 2012; Ottoni-Wilhelm et al. 2014).

Although a great deal is known about the effect of warm-glow on the provision of a single public good (see, e.g., Andreoni 1990), there exists no theoretic analysis of voluntary contributions to multiple public goods in the presence of warm-glow. Further analysis is then required since the extension to many public goods may be related to different types of strategic behavior (see, e.g., Cornes and Itaya 2010). This problem is addressed here by focusing on multiple public goods for which people's preferences are not separable. The set of voluntary contributions is modelled as a directed bipartite network or graph (henceforth, graph) in which contributions flow through links that connect a set of agents to a set of public goods.[3] For example in graph g_0 of Fig. 1, where a_1, a_2, a_3 are the agents and p_1, p_2, p_3 are the public goods, the presence of a link from a_1 to p_1 captures the fact that a_1 belongs to the group providing p_1. This means that a_1 can contribute to and benefit from the provision of p_1. The absence of a link from a_1 to p_3, by contrast, means that a_1 does not belong to the group providing p_3, i.e., a_1 cannot contribute to and benefit from the provision of p_3. Hence, the bipartite graph reflects existing membership structure; links represent membership ties between people and social groups.

Agents are initially endowed with a fixed amount of a private good and decide on their contributions to the various public goods they are connected to. Two key assumptions underlie this analysis. First, the warm-glow part of preferences is separable in

[1] See, e.g., Brekke et al. (2007) for more stylized examples.

[2] See Becker (1974) for an early analysis of altruism and voluntary contributions.

[3] Bipartite graphs have previously been used, for example, to model economic exchange when buyers have relationships with sellers (Kranton and Minehart 2001), and water extraction when users draw on resource from multiple sources (Ilkiliç 2011).

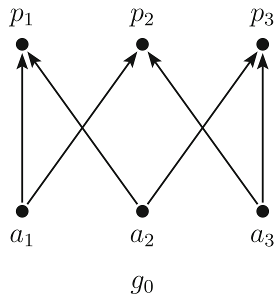


Fig. 1 A bipartite graph with 3 agents and 3 public goods

each public good. This assumption is consistent with experimental findings that indicate an imperfect substitution between the various contributions made by individuals (Reinstein 2011). People enjoy warm-glow over contributions to individual public goods, rather than over their total contribution. Agents are therefore distinguishable in terms of substitution patterns between public goods. Second, the marginal warm-glow of a contribution decreases in the size of the contribution. This assumption is consistent with observed behavior of individuals who generally prefer to make smaller contributions to more public goods (Null 2011).

The purpose of this paper is to analyze voluntary contributions to several public goods under warm-glow preferences. The main result establishes the existence and uniqueness of a Nash equilibrium. Using a continuous adjustment process, the unique Nash equilibrium is also shown to be asymptotically stable. Further assuming that every agent contributes to every public good (as, e.g., in Kemp 1984), the paper extends existing results regarding the effects of pure redistribution, taxation and subsidies.[4] Specifically, it is shown that redistributive policies often yield both desirable and undesirable effects whose intensity depends on two main factors: the topology of the contribution graph structure and the altruism coefficients of all agents. Hence, a significant contribution of this work lies in the introduction of warm-glow in the literature on multiple public goods.[5] This work also enriches the analysis of public good games played on fixed networks by considering multidimensional strategies and non-linear best-response functions.[6]

The message emerging from this analysis is that while Andreoni (1990)'s existence, uniqueness and stability results go through with multiple public goods, the conditions

[4] Furthermore, the comparative statics results involving corner solutions carry over exactly from the pure altruism case with many public goods (see Cornes and Itaya 2010).

[5] Previous results in this literature are restricted to purely altruistic agents. See Kemp (1984), Bergstrom et al. (1986) and Cornes and Itaya (2010) for neutrality and other comparative statics results. For the design of efficient mechanisms, see Cornes and Schweinberger (1996) and Mutuswami and Winter (2004). For the characterization of strategy-proof social choice functions, see Barberà et al. (1991) and Reffgen and Svensson (2012).

[6] Much of this literature is concerned with games in which agents decide how much to contribute to a single public good (i.e., strategies are unidimensional). See Bramoullé and Kranton (2007), Bloch and Zenginobuz (2007) and Bramoullé et al. (2014) for the case of linear best-responses. For the non-linear case, see Bramoullé et al. (2014), Rébillé and Richefort (2014) and Allouch (2015).

required for the usual comparative statics results to hold are more restrictive in the presence of many related public goods and separable warm-glow effects. In the next section, the model of warm-glow giving with multiple public goods is presented. In Sect. 3, the existence of a unique and stable equilibrium is established. Sect. 4 solves for the sufficient conditions for neutrality of wealth redistribution to hold. Sect. 5 examines the implications of government tax policies. A discussion of the main contributions and limitations concludes the paper.

2 A model of impure altruism with multiple public goods

There are n agents $a_1, \ldots, a_n$, m public goods $p_1, \ldots, p_m$ and one private good. Each agent a_i consumes an amount q_i of the private good and participates in the provision of one or more public goods. The set of possible contributions is called the *contribution structure*, which is represented as a directed bipartite graph g.

To this end, the contribution structure is formalized as a triplet $g = (A, P, L)$, where $A = \{a_1, ..., a_n\}$ and $P = \{p_1, ..., p_m\}$ are two disjoint sets of nodes formed by agents and public goods, and L is a set of directed links, each link going from an agent to a public good. A link from agent a_i to public good p_j is denoted as ij. Agent a_i is a member of the group providing p_j if and only if ij is a link in L. In this case, agent a_i is said to be a *potential contributor* to public good p_j. It is assumed, without loss of generality, that the corresponding undirected bipartite graph of g, obtained by removing the direction of the links, is connected.[7] Let $r(g)$ be the number of links in L.

Example 1 Figure 2 presents the directed bipartite graphs of two simple contribution structures g_1 and g_2. The corresponding undirected graph of g_1 belongs to the class of complete bipartite graphs. Connected graphs of this class contain $m \times n$ links. The corresponding undirected graph of g_2 belongs to the class of acyclic bipartite graphs. Connected graphs of this class contain $m + n - 1$ links. A large number of contribution structures lies between these two polar cases.

Given a contribution structure g, let $N_g(a_i)$ be the set of public goods to which a_i can potentially contribute, i.e.,

$$N_g(a_i) = \left\{p_j \in P \text{ such that } ij \in L\right\},$$

and similarly, $N_g(p_j)$ is the group of potential contributors to public good p_j. The number of public goods in $N_g(a_i)$ and the number of agents in $N_g(p_j)$ are respectively denoted $r_g(a_i)$ and $r_g(p_j)$. It is assumed, without restricting the generality of the model, that each agent belongs to at least one public good group, i.e., $r_g(a_i) \geq 1$ for all $a_i \in A$, and each public good group is composed of at least two agents, i.e., $r_g(p_j) \geq 2$ for all $p_j \in P$.

Let $x_{ij} \geq 0$ be the contribution by agent a_i to public good p_j. Agent a_i is endowed with wealth w_i which he allocates between the private good q_i and his total contribution

[7] An undirected bipartite graph is connected if any two nodes are connected by a path.

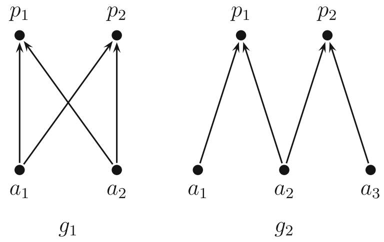


Fig. 2 Two different contribution structures for the provision of two public goods

$X_i = \sum_{p_j \in N_g(a_i)} x_{ij}$. For convenience, it is assumed that each public good can be produced from the private good with a unit-linear technology.[8] It is also assumed that agents are *impurely altruistic*, i.e., an agent a_i involved in the provision of a public good p_j cares about both p_j's total supply, given by $G_j = \sum_{a_i \in N_g(p_j)} x_{ij}$, and his own contribution to p_j.[9]

The utility function $U_i : \mathbb{R}_+^{r(g)} \to \mathbb{R}_+$ of agent a_i is given by

$$U_i = \sum_{p_j \in N_g(a_i)} \left\{ b_j \left(G_j \right) + \delta_{ij} \left(x_{ij} \right) \right\} + c_i(q_i),$$

where $b_j : \mathbb{R}_+ \to \mathbb{R}_+$ is the collective benefit from p_j's total supply, $\delta_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$ is the warm-glow from own contribution to p_j, and $c_i : \mathbb{R}_+ \to \mathbb{R}_+$ is the personal benefit from private consumption.[10] Hence, a contribution x_{ij} enters the utility function of a_i three times: once as part of G_j, once alone like a private good, and once as part of $q_i = w_i - X_i$. Accordingly, the utility function of agent a_i is not separable with respect to each public good. The marginal utility with respect to x_{ij} does depend on the contributions by a_i to public goods other than p_j.

Warm-glow effects vary from public good to public good, as well as from agent to agent. Thus, agents can be distinguished by their marginal rates of substitution, as in Kemp (1984), Bergstrom et al. (1986), Cornes and Schweinberger (1996) and Cornes and Itaya (2010). This specification is also consistent with recent empirical findings by Null (2011) and Reinstein (2011), who show that contributions to multiple public goods are imperfectly substitutable. Moreover, for the rest of the paper, the value functions will satisfy the following properties.

8 This assumption is almost innocuous. See, e.g., Bergstrom et al. (1986, p. 31) for a discussion.

9 There exist at least three alternative approaches to model impure altruism: one in which people care about the well-being of others (Margolis 1982; Bourlès et al. 2017), another one in which voluntary contributions are subject to a principle of reciprocity (Sudgen 1984), and a third one in which public goods are jointly produced with private goods (Cornes and Sandler 1984).

10 When $P = \{p_1\}$, the utility function of agent a_i reduces to

$$U_i = b_1(G_1) + \delta_{i1}(x_{i1}) + c_i(q_i).$$

This specification complies with the assumptions of the usual impure altruism model with a single public good (Andreoni 1990). It is also a special case of the joint production model by Cornes and Sandler (1984). This further indicates that the model developped in this paper is not a direct extension of Bramoullé and Kranton (2007)'s network public good game.

Assumption 1 For each link $ij \in L$, b_j, δ_{ij} and c_i are increasing, twice continuously differentiable functions, with b_j concave, δ_{ij} strongly concave and c_i concave.

Increasing value functions yield to Samuelson's efficiency condition like in the pure altruism model (see, e.g., Cornes and Itaya 2010). The rest of the above assumption reflects the convexity of preferences with respect to each individual contribution. Hence, Assumption 1 is consistent with empirical findings by Null (2011), who show that agents prefer to distribute their total contribution between many public goods rather than giving all to a single public good.[11] For simplicity, assume further that the private good is essential and consider the following multiple public goods game. Given a contribution structure g, each agent $a_i \in A$ faces the optimization problem

$$\max_{\{x_{ij} \text{ s.t. } p_j \in N_g(a_i)\}, q_i} \sum_{p_j \in N_g(a_i)} \left\{ b_j\left(G_j\right) + \delta_{ij}\left(x_{ij}\right) \right\} + c_i\left(q_i\right)$$
$$\begin{aligned} \text{s.t. } \quad & q_i + X_i = w_i, \\ & X_i = \textstyle\sum_{p_j \in N_g(a_i)} x_{ij}, \\ & G_j = \textstyle\sum_{a_i \in N_g(p_j)} x_{ij}, \\ & x_{ij} \geq 0, \quad \text{for all } p_j \in N_g(a_i). \end{aligned}$$

Pure strategy Nash equilibria under simultaneous decision-making are investigated.

3 Existence, uniqueness and stability of the Nash equilibrium

First, the existence and uniqueness of a Nash equilibrium is established. By substituting the budget constraint into the utility function, and in turn by using the specifications for X_i and G_j, the maximization problem of agent a_i is equivalent to

$$\max_{\{x_{ij} \text{ s.t. } p_j \in N_g(a_i)\}}$$
$$\sum_{p_j \in N_g(a_i)} \left\{ b_j\left(\sum_{a_i \in N_g(p_j)} x_{ij} \right) + \delta_{ij}\left(x_{ij}\right) \right\} + c_i\left(w_i - \sum_{p_j \in N_g(a_i)} x_{ij} \right)$$
$$\text{s.t. } x_{ij} \geq 0, \quad \text{for all } p_j \in N_g(a_i).$$

The problem of agent a_i is to choose $r_g(a_i)$ nonnegative numbers. His strategy space is therefore a subset of the $r_g(a_i)$-dimensional Euclidean space, and the multiple public goods game belongs to the class of the "concave N-person games" studied by Rosen (1965). Using Rosen's analysis, the following result is obtained.

Theorem 1 *Let Assumption 1 be satisfied. Then, the multiple public goods game admits a unique Nash equilibrium.*

[11] Assumption 1, though, does not prevent the model from exhibiting free-riding effects.

Proof The proof of Theorem 1, together with all of the other proofs, appears in the Appendix. □

Three comments on Theorem 1 are in order. First, this result extends the existence and uniqueness result of Andreoni (1990) to the more general setting of multiple public goods with additive separable utility functions. To get an intuition for this result, it is interesting to note that a_i's maximization problem could be replaced by

$$\max_{\{x_{ij} \text{ s.t. } p_j \in N_g(a_i)\}} \Psi = \sum_{p_j \in P} \left\{ b_j \left(\sum_{a_i \in N_g(p_j)} x_{ij} \right) \right\} + \sum_{ij \in L} \left\{ \delta_{ij} \left(x_{ij} \right) \right\} + \sum_{a_i \in A} \left\{ c_i \left(w_i - \sum_{p_j \in N_g(a_i)} x_{ij} \right) \right\}$$
$$\text{s.t. } x_{ij} \geq 0, \quad \text{for all } p_j \in N_g(a_i),$$

without changing the first-order conditions. In other words, Ψ is a best-response potential of the multiple public goods game (Voorneveld 2000). Under Assumption 1, Ψ is strongly concave with respect to x_{ij} for all $ij \in L$. Following Bourlès et al. (2017), this potential property might be used to construct an alternative proof of Theorem 1. Therefore, clearly, it appears that the key to the uniqueness result in the private provision of many public goods under warm-glow preferences is the assumption of separable and strongly concave warm-glow functions.[12]

Second, Theorem 1 extends the uniqueness result of Ilkiliç (2011) to non-linear best-response functions. To see this, consider the first-order condition of a_i's maximization problem with respect to x_{ij}, i.e.,

$$b_j' \left(G_j \right) + \delta_{ij}' \left(x_{ij} \right) - c_i' \left(w_i - X_i \right) + \mu_{ij} = 0,$$

with

$$\mu_{ij} x_{ij} = 0, \qquad \mu_{ij} \geq 0,$$

where μ_{ij} is the Karush–Kuhn–Tucker multiplier associated with the constraint $x_{ij} \geq 0$. Ilkiliç (2011) studies a game with linear quadratic utility functions where a player's first-order condition would become here

$$\alpha - \beta G_j - \beta x_{ij} - \gamma X_i + \mu_{ij} = 0,$$

with

$$\mu_{ij} x_{ij} = 0, \qquad \mu_{ij} \geq 0,$$

where $\alpha, \beta, \gamma > 0$. Hence, it is clear that the first-order conditions coincide when b_j, δ_{ij} and c_i are some specific concave down quadratic functions. In this case, Theorem 3

[12] This claim is confirmed by a close inspection of the actual proof of Theorem 1.

of Ilkiliç (2011), which expresses the equilibrium as a function of a network centrality measure (i.e., a modified Bonacich centrality measure), applies to the model presented in this paper.[13]

Third, it is worth checking whether Theorem 1 carries over heterogeneous benefit functions or not. Suppose, for instance, that a_i's utility function is given by

$$U_i = \sum_{p_j \in N_g(a_i)} \left\{ b_{ij}\left(G_j\right) + \delta_{ij}\left(x_{ij}\right) \right\} + c_i\left(q_i\right),$$

where $b_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$ is a_i's benefit from p_j's total supply. Following the same lines as in the proofs of Lemma 1 and Theorem 2 in Rébillé and Richefort (2015), a sufficient condition for the uniqueness of a Nash equilibrium is that the Jacobian matrix of marginal utilities be a strictly row diagonally dominant matrix.[14] Here, this condition is equivalent to

$$\delta''_{ij} < \left[r_g(p_j) - 2\right] b''_{ij} + \left[r_g(a_i) - 2\right] c''_i,$$

for all $ij \in L$. When each agent belongs to at most two public good groups and each public good group is composed of exactly two agents (like for example in graphs g_0, g_1 and g_2), the above uniqueness condition is satisfied, thanks to the strong concavity of warm-glow functions. Otherwise, additional conditions on the concavity of all three value functions are necessary.

The dynamic stability of the unique Nash equilibrium is now explored. For this purpose, the best-response functions at each link of the contribution structure are considered. The best-response functions specify the optimal contribution at each link for each fixed contribution level at the other links. Let $G_{-i,j} = G_j - x_{ij}$ denote the sum of all contributions to public good p_j by agents other than a_i and $X_{i,-j} = X_i - x_{ij}$ denote the sum of all contributions by agent a_i to public goods other than p_j. Under the Nash assumption, $G_{-i,j}$ and $X_{i,-j}$ are treated exogenously. Hence, solving the first-order condition with respect to x_{ij} yields the best-response

$$x_{ij} = \max\left\{0, \phi_{ij}\left(G_{-i,j}, w_i - X_{i,-j}\right)\right\},$$

where ϕ_{ij} is a non-linear function defined on $\mathbb{R}$. By definition, the solution of the system of best-response functions is the unique Nash equilibrium of the multiple public goods game.

The following autonomous dynamic system, adapted from the Cournot literature on multiproduct firms (see, e.g., Zhang and Zhang 1996), is specified:

$$\dot{x}_{ij} = \frac{\mathrm{d}x_{ij}}{\mathrm{d}t} = \max\left\{0, \phi_{ij}\left(G_{-i,j}, w_i - X_{i,-j}\right)\right\} - x_{ij}, \quad \text{for all } ij \in L.$$

[13] This would establish that a contribution increases (resp. decreases) with the number of even (resp. odd) length paths that start from it in the (corresponding undirected) contribution structure.

[14] In particular, it can be shown that all Nash equilibria admitted by the multiple public goods game are solutions to a non-linear complementarity problem (Rébillé and Richefort 2015). See, e.g., Karamardian (1969) for fundamental results in the field.

This system assumes that agents continuously adjust their contributions at each link by choosing the best-response to the contributions at the other links. Hence, by construction, a stationary state of this system is a Nash equilibrium. Moreover, a Nash equilibrium is said to be *asymptotically stable* if this system converges back to the Nash equilibrium following any small enough perturbation.[15] Let G_j^* denote the total equilibrium supply of public good p_j and X_i^* denote the total equilibrium contribution by agent a_i. Following Bramoullé et al. (2014) and Allouch (2015), the links are partitioned into three sets: the set of clearly active links

$$B = \left\{ij \in L \text{ s.t. } b_j'\left(0 + G_{-i,j}^*\right) + \delta_{ij}'(0) - c_i'\left(w_i - 0 - X_{i,-j}^*\right) > 0\right\}$$

formed by links that would still be active even after a small pertubation; the set of inactive links being just at the margin of becoming active

$$C = \left\{ij \in L \text{ s.t. } b_j'\left(0 + G_{-i,j}^*\right) + \delta_{ij}'(0) - c_i'\left(w_i - 0 - X_{i,-j}^*\right) = 0\right\}$$

formed by links that might become active after a small pertubation; and the set of clearly inactive links

$$D = \left\{ij \in L \text{ s.t. } b_j'\left(0 + G_{-i,j}^*\right) + \delta_{ij}'(0) - c_i'\left(w_i - 0 - X_{i,-j}^*\right) < 0\right\}$$

formed by links that would still be inactive even after a small pertubation. The stability analysis will be restricted to set $B \cup D$, i.e., contribution structures in which all links are either clearly active or clearly inactive.

Assumption 2 $C = \emptyset$.

This assumption, also used in Allouch (2015), is very reasonable. In fact, a Nash equilibrium in which one or more inactive links would be just at the margin of becoming active is quite unlikely to occur (see, e.g., footnote 16 and its proof in Bramoullé et al. 2014). Using this assumption, the asymptotic stability of the Nash equilibrium is established.

Theorem 2 *Let Assumptions 1 and 2 be satisfied. Then, the Nash equilibrium of the multiple public goods game is asymptotically stable.*

Theorem 2 extends the stability result of Andreoni (1990) to multidimensional strategy spaces. A different way to see this is to solve the first-order conditions of clearly active links with respect to the total supply of public goods. The first-order condition of a clearly active link ij may be written

$$b_j'\left(G_j\right) + \delta_{ij}'\left(G_j - G_{-i,j}\right) - c_i'\left(G_{-i,j} - G_j + w_i - X_{i,-j}\right) = 0.$$

15 See, e.g., Definition 4.1 in Khalil (2002).

Totally differentiating this expression and rearranging yields

$$dG_j = \frac{\delta''_{ij}}{b''_j + \delta''_{ij} + c''_i} dG_{-i,j} + \frac{c''_i}{b''_j + \delta''_{ij} + c''_i} \left(dG_{-i,j} + dw_i - dX_{i,-j} \right),$$

where the term $\delta''_{ij}/(b''_j + \delta''_{ij} + c''_i)$ comes from the warm-glow component of a_i's preferences and denote a_i's marginal willingness to contribute to public good p_j for egoistic reasons. Furthermore, the second term $c''_i/(b''_j + \delta''_{ij} + c''_i)$ comes from the altruistic component of a_i's preferences and denote a_i's marginal willingness to contribute to public good p_j for altruistic reasons. Under Assumption 1, both terms are between zero and one, meaning that all warm-glow effects, all public goods and the private good are supposed to be normal, just like in the single public good case.

4 Neutral redistributions of wealth

The inefficiency of the Nash equilibrium is a famous outcome of voluntary contribution models (see, e.g., Cornes and Sandler 1986). Public goods are under-produced because contributions are strategic substitutes and generate positive externalities. Hence, agents have incentives to contribute less than the optimal level. To minimize this inefficiency, it is important to have a better understanding of individual reactions to various public policies, as well as welfare effects of these policies. This section examines the effects of wealth transfers between agents. For this purpose, a slightly stronger assumption about the convexity of individual preferences is stated.

Assumption 1' For each link $ij \in L$, b_j, δ_{ij} and c_i are increasing, twice continuously differentiable functions, with b_j concave, δ_{ij} strongly concave and c_i strongly concave.

Moreover, it is also assumed that all links are active and that the set of active links remains unchanged after the (small enough) redistribution.

Assumption 2' $C = D = \emptyset$.

There are two main justifications for this assumption. First, interior equilibria are more likely to emerge under warm-glow preferences than under pure altruism.[16] Second, the comparative statics involving corner solutions with purely altruistic agents is now well-established (see, e.g., Bergstrom et al. 1986; Cornes and Itaya 2010). According to Andreoni (1990, p. 466), the results obtained in the pure altruism case shall extend to warm-glow preferences. Hence, considering corner equilibria here will not add to the insights of Bergstrom et al. (1986) and Cornes and Itaya (2010).[17]

Following Andreoni (1990), agents are identified by their altruism with respect to the different public goods they are connected to. The altruism of agent a_i with respect to public good p_j is given by

[16] See, e.g., Cornes and Itaya (2010, p. 364) for a discussion.

[17] Another possible justification for Assumption 2' may be that agents must be active, even very slightly, to secure their memberships in public good groups. The interiority of the equilibrium would then be the result of group formation processes, not studied in this paper and well worth exploring in future research. See, e.g., Brekke et al. (2007) for the analysis of a group formation game in which group membership is only available to active agents.

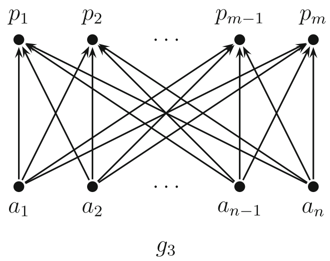


Fig. 3 Contribution structure with n agents and m public goods, candidate for neutral redistributions of wealth

$$\alpha_{ij} = \frac{c''_i}{\delta''_{ij} + c''_i} \in (0, 1).$$

If a_i has high altruism with respect to p_j, δ''_{ij} will be close to zero, so α_{ij} will be close to one. If a_i has low altruism with respect to p_j, δ''_{ij} will be far distant from zero, so α_{ij} will be close to zero. More generally, the closer δ''_{ij} is to zero, the nearer α_{ij} is to one, hence the more agent a_i can be thought of as having high altruism with respect to public good p_j. The following partial neutrality result is then obtained.

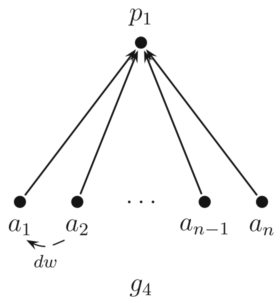
Proposition 1 *Let Assumptions 1' and 2' be satisfied. Then, a wealth transfer between any agents such that $\sum_{a_i \in A} dw_i = 0$ will not change the total supply of each public good whenever agents have identical altruism with respect to each public good, i.e., $\alpha_{ij} = \alpha_j$ for all $ij \in L$, and the contribution structure g is complete.*

A few comments on Proposition 1 are useful. First, a contribution structure is said to be *complete* whenever each agent is involved in the provision of all public goods, in other words, whenever each agent is a member of each public good group and can therefore potentially contribute to and benefit from the provision of each public good. Such a membership structure is depicted in Fig. 3. Along with Assumption 2', this means that every agent contributes positively to every public good.[18] This is a fairly strong assumption. Thus, consistent with empirical findings (see, e.g., Hochman and Rodgers 1973; Reinstein 2011), the above result shows first of all that redistributions of wealth will generally not be neutral.

However, when every agent contributes to every public good, Proposition 1 shows that pure altruism is indeed sufficient for neutrality: if α_{ij} tends to one for all $ij \in L$, then dG_j tends to zero for all $p_j \in P$, as in Kemp (1984) and to a lesser extent as in Cornes and Itaya (2010), although in this case, the equilibrium may not be unique and stable (see, e.g., Rébillé and Richefort 2015). Proposition 1 also shows that pure altruism is only one of the cases in which small redistributions of wealth are neutral.

[18] An example of such a situation is given in Kemp (1984), in which agents are countries and public goods are international pure public consumption goods or global-level common-pool resources. In this case, warm-glow can be thought of as being a local, country-specific benefit derived from own contribution. For instance, national policy measures to protect the environment provide benefits which are both local (i.e., private) and global (i.e., collective). See, e.g., Kaul et al. (1999) for more details and examples.

Fig. 4 Wealth transfer from agent a_2 to agent a_1 in presence of n agents and a single public good



In fact, this property holds whenever agents are equally altruistic with respect to each public good, as long as the contribution structure is complete and all links are active.[19]

Regardless of the contribution structure, the proof of Proposition 1 shows that a transfer between any two agents, say agents a_1 and a_2, such that $dw_1 = -dw_2 = dw > 0$, has an effect on the supply of each public good such that

$$dG_j = k_j \left(\alpha_{1j} - \alpha_{2j}\right) dw - k_j \sum_{a_i \in N_g(p_j)} \alpha_{ij} dX_{i,-j}, \quad \text{for all } p_j \in P,$$

where $k_j \in (0, 1]$. Three simple cases are now discussed in more details.

- In presence of a single public good, the above result reduces to the same expression obtained by Andreoni (1990), i.e.,

$$dG_1 = k_1 \left(\alpha_{11} - \alpha_{21}\right) dw,$$

  where $k_1 \in (0, 1]$. The transfer does not change G_1 if and only if $\alpha_{11} = \alpha_{21}$. It has the desired effect on G_1 if and only if $\alpha_{11} > \alpha_{21}$. In this case, the only possible contribution structure is the complete $n \times 1$ bipartite graph, depicted in Fig. 4.
- When there are two agents and two public goods, the contribution structure is also necessarily complete (see the 2×2 bipartite graph g_1). In this case, a transfer from a_2 to a_1 such that $dw_1 = -dw_2 = dw > 0$ yields

$$dG_1 = k_1 \left[\alpha_{11} \left(dw - dx_{12}\right) - \alpha_{21} \left(dw + dx_{22}\right)\right]$$

  and

$$dG_2 = k_2 \left[\alpha_{12} \left(dw - dx_{11}\right) - \alpha_{22} \left(dw + dx_{21}\right)\right],$$

  where $k_1, k_2 \in (0, 1]$. If $\alpha_{1j} = \alpha_{2j} = \alpha_j$ for a given public good p_j, the transfer does not change G_1 if and only if it does not change G_2. Accordingly, if $\alpha_{1j} = \alpha_{2j}$

[19] For example, quadratic value functions such that

$$\delta_{ij}\left(x_{ij}\right) = \sigma_{ij} x_{ij} - \frac{\theta_j}{2} x_{ij}^2 \quad \text{and} \quad c_i\left(q_i\right) = \xi_i q_i - \frac{\psi}{2} q_i^2$$

for all $ij \in L$, where $\sigma_{ij}, \xi_i > 0$, $\theta_j \in (0, \sigma_{ij}/w_i)$ and $\psi \in (0, \xi_i/w_i)$, fulfil the neutrality condition over the altruism coefficients.

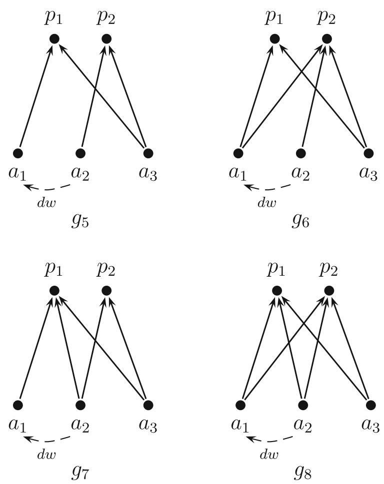


Fig. 5 Wealth transfer from agent a_2 to agent a_1 in presence of three agents and two public goods

for all p_j, the transfer does not change G_1 and G_2 simultaneously. Furthermore, if $\alpha_{1j} > \alpha_{2j}$ for a given public good p_j, the transfer increases G_1 if it decreases G_2, and vice versa. Hence, if $\alpha_{i1} > \alpha_{k1}$ and $\alpha_{i2} > \alpha_{k2}$ for a given agent a_i, where a_k is the other agent, the transfer might increase or decrease G_1 and G_2 simultaneously.

- When there are three agents and two public goods, the contribution structure may not be complete. If the third agent is connected to both public goods, four contribution structures, depicted in Fig. 5, are possible. In the complete graph g_8, a transfer of wealth from a_2 to a_1 yields

$$dG_1 = k_1 \left[\alpha_{11} (dw - dx_{12}) - \alpha_{21} (dw + dx_{22}) - \alpha_{31} dx_{32}\right]$$

and

$$dG_2 = k_2 \left[\alpha_{12} (dw - dx_{11}) - \alpha_{22} (dw + dx_{21}) - \alpha_{32} dx_{31}\right],$$

where $k_1, k_2 \in (0, 1]$. Thus, it is easy to show that the above conclusions from the 2×2 bipartite graph still hold. Suppose now that some links are removed, as in graphs g_5, g_6 and g_7. The contribution structure is therefore no longer complete.[20] In these graphs, the transfer might increase or decrease G_1 and G_2, simultaneously or not, depending on the altruism coefficients of the three agents, as well as their position in the contribution structure.

Lastly, Proposition 1 can also be expressed as follows.

Proposition 2 *Let Assumptions 1' and 2' be satisfied, and let the contribution structure g be complete. Then, the total supply of each public good is independent of the distribution of wealth if and only if each best-response function can be written in the form*

[20] Intuitively, considering incomplete contribution structures almost amounts to relaxing the assumption that $D = \emptyset$, since clearly inactive links could be practically treated as missing links (see, e.g., Ilkiliç 2011).

$$x_{ij} = \phi^*_{ij}\left(G_{-i,j}\right) + \alpha_j\left(w_i - X_{i,-j}\right),$$

where $\alpha_j \in (0, 1)$*,* ϕ^*_{ij} *is a decreasing function for all* $ij \in L$*, and* α_j *is identical across all agents for any* $p_j \in P$*.*

For complete contribution structures, the class of best-response functions specified in Proposition 2 will be sufficient for each public good to be independent of redistributions of wealth. However, the quasi-linear form of these functions, along with all other sufficient conditions, highlights the fact that neutrality, even partial, would only hold for very particular game structures. An interesting way to think of this result may be to consider the case of complete neutrality.

Indeed, if both the set of public goods and the consumption of the private good are required to be independent of wealth redistributions, it can be shown that Proposition 2 gives rise to an impossibility result. Totally differentiating the best-response functions in Proposition 2 yields

$$dx_{ij} = \phi^{*\prime}_{ij} dG_{-i,j} + \alpha_j\left(dw_i - dX_{i,-j}\right).$$

Assuming $dG_j = 0$ and rearranging, it appears that

$$dw_i = dX_{i,-j} + \frac{1 + \phi^{*\prime}_{ij}}{\alpha_j} dx_{ij}.$$

It follows that ϕ^*_{ij} must be an affine function since $dq_i = 0$ if and only if $dw_i = dX_i$ if and only if $\phi^{*\prime}_{ij} = \alpha_j - 1$. Thus, $\phi^*_{ij}(G_{-i,j}) = \zeta_{ij} + \left(\alpha_j - 1\right) G_{-i,j}$, where ζ_{ij} is a constant. The best-response functions in Proposition 2 may then be written

$$x_{ij} = \zeta_{ij} + \left(\alpha_j - 1\right) G_{-i,j} + \alpha_j\left(w_i - X_{i,-j}\right),$$

or equivalently,

$$\zeta_{ij} + \left(\alpha_j - 1\right) G_j + \alpha_j\left(w_i - X_i\right) = 0.$$

Hence, complete neutrality requires that warm-glow functions be linear functions, which contradicts Assumption 1'.

5 Subsidies and direct grants

In this section, it is assumed that public goods may be provided both publicly and privately.[21] Suppose that each individual contribution x_{ij} is subsidized at a rate $s_{ij} \in (0, 1)$ by the government and suppose that these subsidies are financed through lump

[21] The effects of government intervention on the private provision of public goods has a long tradition in economics. The main question is to which extent public provision crowds out private contributions. See, e.g., Abrams and Schmitz (1984), Andreoni (1993), Eckel et al. (2005), Gronberg et al. (2012) and Ottoni-Wilhelm et al. (2014) for empirical studies on this issue.

sum taxes $\tau_{ij} > 0$. All net tax receipts are dedicated to the provision of public goods, either through subsidies towards individual contributions, or through direct grants.

For all $p_j \in P$, let $T_j = \sum_{a_i \in N_g(p_j)} \{\tau_{ij} - s_{ij} x_{ij}\}$ be the government's net tax receipts with respect to public good p_j, and let $\tilde{G}_j = G_j + T_j$ be the joint supply of public good p_j. The utility function of agent a_i is now given by

$$U_i = \sum_{p_j \in N_g(a_i)} \left\{ b_j\left(\tilde{G}_j\right) + \delta_{ij}\left(x_{ij}\right) \right\} + c_i(q_i).$$

Let $\tilde{x}_{ij} = x_{ij}(1 - s_{ij}) + \tau_{ij}$ represents a_i's contribution to public good p_j. Then, a_i's budget constraint becomes $w_i = q_i + \tilde{X}_i$, where $\tilde{X}_i = \sum_{p_j \in N_g(a_i)} \tilde{x}_{ij}$. It follows that a_i's maximization problem may be written

$$\max_{\{\tilde{x}_{ij} \text{ s.t. } p_j \in N_g(a_i)\}} \sum_{p_j \in N_g(a_i)} \left\{ b_j\left(\sum_{a_i \in N_g(p_j)} \tilde{x}_{ij} \right) + \delta_{ij}\left(\frac{\tilde{x}_{ij} - \tau_{ij}}{1 - s_{ij}} \right) \right\} c_i\left(w_i - \sum_{p_j \in N_g(a_i)} \tilde{x}_{ij} \right)$$
$$\text{s.t. } \tilde{x}_{ij} - \tau_{ij} \geq 0, \quad \text{for all } p_j \in N_g(a_i).$$

Similarly to the neutrality analysis, it is assumed that all links are active and that the set of active links remains unchanged after a (small enough) change in lump sum taxes and/or subsidies (Assumption 2'). Hence, substituting $\tilde{X}_i = \tilde{x}_{ij} + \tilde{X}_{i,-j}$ and $\tilde{G}_j = \tilde{x}_{ij} + \tilde{G}_{-i,j}$ into the first-order condition of a_i's maximization problem with respect to $\tilde{x}_{ij}$ yields

$$b'_j\left(\tilde{x}_{ij} + \tilde{G}_{-i,j}\right) + \frac{1}{1 - s_{ij}} \delta'_{ij}\left(\frac{\tilde{x}_{ij} - \tau_{ij}}{1 - s_{ij}} \right) - c'_i\left(w_i - \tilde{x}_{ij} - \tilde{X}_{i,-j} \right) = 0.$$

Solving this with respect to $\tilde{x}_{ij}$ yields the best-response

$$\tilde{x}_{ij} = \phi_{ij}\left(\tilde{G}_{-i,j}, s_{ij}, \frac{\tau_{ij}}{1 - s_{ij}}, w_i - \tilde{X}_{i,-j} \right).$$

The second argument, s_{ij}, appears because of the expression multiplying a_i's marginal warm-glow function in the first-order condition. The third argument comes from the warm-glow component of a_i's utility function. The altruism coefficient is now given by

$$\tilde{\alpha}_{ij} = \frac{c''_i}{\frac{\delta''_{ij}}{(1 - s_{ij})^2} + c''_i} \in (0, 1).$$

The effects of changing lump sum taxes are first analyzed.

Proposition 3 *Let Assumptions 1' and 2' be satisfied, let the contribution structure g be complete, and let $\tilde{\alpha}_{ij} = \tilde{\alpha}_j$ for all $ij \in L$. Then, any increase (resp. decrease) in the lump sum taxes with respect to a given public good, say public good p_1, will:*

(i) increase (resp. decrease) the total supply of p_1,
(ii) decrease (resp. increase) the total supply of any other public good,
(iii) increase (resp. decrease) the total amount of contributions.

The above proposition establishes that direct grants financed by lump sum taxation will incompletely crowd out private contributions. In fact, regardless of the contribution structure, the proof of Proposition 3 shows that changing lump sum taxes affects the total supply of each public good such that

$$d\tilde{G}_j = \tilde{k}_j \sum_{a_i \in N_g(p_j)} \{\left(1 - \tilde{\alpha}_{ij}\right) d\tau_{ij} - \tilde{\alpha}_{ij} d\tilde{X}_{i,-j}\}, \quad \text{for all } p_j \in P,$$

where $\tilde{k}_j \in (0, 1]$ and $d\tau_{ij}$ is the change in a_i's tax rate with respect to p_j. In presence of a single public good, the above result reduces to the same expression obtained by Andreoni (1990), just like in the previous section. In this case, any change in the lump sum taxes has the desired effect on the total supply of the single public good, and since agents are impurely altruistic, the crowding out effect is incomplete because agents always prefer the bundle with the highest warm-glow.

In a complete contribution structure composed of equally altruistic agents with respect to each public good, changing lump sum taxes with respect to a given public good, say p_1, yields

$$d\tilde{G}_1 = \tilde{k}_1 \left(1 - \tilde{\alpha}_1\right) d\tau_1 - \tilde{k}_1 \tilde{\alpha}_1 \sum_{p_j \in P \setminus \{p_1\}} d\tilde{G}_j$$

and

$$d\tilde{G}_l = -\tilde{k}_l \tilde{\alpha}_l \sum_{p_j \in P \setminus \{p_l\}} d\tilde{G}_j, \quad \text{for all } p_l \in P \setminus \{p_1\},$$

where $\tilde{k}_j \in (0, 1]$ for all $p_j \in P$ and $d\tau_1$ is the variation in p_1's total tax revenue, i.e., $d\tau_1 = \sum_{a_i \in N_g(p_1)} d\tau_{i1}$. Hence, any change in τ_1 produces desired effects on the total supply of p_1 and undesired effects on the total supply of any other public good p_l. Moreover, these effects depend on the altruism of all agents with respect to each public good:

- The more altruistic the agents are with respect to p_1, the lower the change in G_1;
- The more altruistic the agents are with respect to any other public good p_l, the higher the change in G_l.

This result is therefore consistent with empirical findings by Feldstein and Taylor (1976) and Reece (1979), who show that different public goods, thus inducing different warm glow effects, exhibit different responses to tax policy changes.

A similar result is now established with subsidies.

Proposition 4 *Let Assumptions 1' and 2' be satisfied, let the contribution structure g be complete, and let* $\tilde{\alpha}_{ij} = \tilde{\alpha}_j$ *for all* $ij \in L$. *Then, any increase (resp. decrease) in the subsidy rates with respect to a given public good, say public good* p_1, *will:*

(i) increase (resp. decrease) the total supply of p_1,
(ii) decrease (resp. increase) the total supply of any other public good,
(iii) increase (resp. decrease) the total amount of contributions.

In presence of a single public good, subsidies are always more desirable than direct grants because impurely altruistic agents prefer to contribute directly rather than indirectly (Andreoni 1990). To check the robustness of this fact when there are multiple public goods, suppose that the government raises the subsidy rates with respect to public good p_1 and finances this by raising lump sum taxes with respect to p_1. Totally differentiating the best-response functions and rearranging as in the proofs yields

$$d\tilde{G}_1 = \tilde{k}_1 \sum_{a_i \in N_g(p_1)} \left\{ (1-\tilde{\alpha}_{i1})\, d\tau_{i1} + \left(\tilde{\alpha}_{i1}\kappa_{i1} + (1-\tilde{\alpha}_{i1}) \frac{\tau_{i1}}{1-s_{i1}} \right) ds_{i1} - \tilde{\alpha}_{i1} d\tilde{X}_{i,-1} \right\},$$

where $\kappa_{i1} > 0$. In a complete contribution structure composed of equally altruistic agents with respect to each public good, it holds that

$$\begin{aligned} d\tilde{G}_1 &= \tilde{k}_1 (1-\tilde{\alpha}_1)\, d\tau_1 - \tilde{k}_1\tilde{\alpha}_1 \sum_{a_i \in A} d\tilde{X}_{i,-1} + \tilde{k}_1 \sum_{a_i \in A} \left\{ \left(\tilde{\alpha}_1\kappa_{i1} + (1-\tilde{\alpha}_1) \frac{\tau_{i1}}{1-s_{i1}} \right) ds_{i1} \right\} \\ &= d\tilde{G}_1\Big|_{\text{grants}} + \tilde{k}_1 \sum_{a_i \in A} \left\{ \left(\tilde{\alpha}_1\kappa_{i1} + (1-\tilde{\alpha}_1) \frac{\tau_{i1}}{1-s_{i1}} \right) ds_{i1} \right\} \\ &> d\tilde{G}_1\Big|_{\text{grants}} \quad > \quad 0, \end{aligned}$$

and since $d\tilde{G}_l$ is a linear decreasing function of $d\tilde{G}_1$,

$$d\tilde{G}_l < d\tilde{G}_l\Big|_{\text{grants}} < 0, \quad \text{for all } p_l \in P \backslash \{p_1\}.$$

Hence, lump sum taxes with respect to p_1 spent on subsidizing contributions yield greater effects than lump sum taxes with respect to p_1 spent on direct grants. First, they have a greater desired effect on the total supply of p_1, just like in the single public good case. Second, they have a greater undesired effect on the total supply of any other public good.

It is therefore interesting to check whether subsidies or direct grants Pareto-dominate. Suppose that direct grants dedicated to the provision of public good p_1 are increased by $d\tau_{i1}$. Totally differentiating a_i's utility function yields

$$dU_i|_{\text{grants}} = K_i - \frac{\delta'_{i1}}{1-s_{i1}} d\tau_{i1},$$

where $K_i = \sum_{p_j \in N_g(a_i)} \{b'_j d\tilde{G}_j + \delta'_{ij} d\tilde{x}_{ij}/(1-s_{ij})\} - c'_i d\tilde{X}_i$. Now, suppose that direct grants dedicated to the provision of p_1 and subsidies with respect to p_1 are increased simultaneously by $(d\hat{\tau}_{i1}, ds_{i1})$, so that the same change in the equilibrium supply of each public good and in a_i's equilibrium contributions occurs. Totally differentiating a_i's utility function yields

$$dU_i|_{\text{subsidies}} = K_i - \frac{\delta'_{i1}}{1-s_{i1}} \left(d\hat{\tau}_{i1} - x_{i1} ds_{i1}\right),$$

where $x_{i1} = (\tilde{x}_{i1} - \tau_{i1})/(1-s_{i1}) \geq 0$. From the above, it is known that $d\hat{\tau}_{i1} \leq d\tau_{i1}$. Hence, in a complete contribution structure composed of equally altruistic agents with respect to each public good,

$$d\hat{\tau}_{i1} - x_{i1} ds_{i1} \leq d\tau_{i1} \iff dU_i|_{\text{subsidies}} \geq dU_i|_{\text{grants}}.$$

Consequently, an increase in the subsidy rates will increase utility more than an equivalent increase in direct grants.

6 Conclusion

This paper explores a voluntary contribution game with m public goods in which players enjoy warm-glow for their contributions. Each public good benefits a different group of players. Players are initially endowed with a fixed amount of a private good and decide on their contributions to the various public good groups they are affiliated to. Under this framework, the contribution structure forms a bipartite graph between the players and the public goods. The main result of the paper is to show the existence and uniqueness of a Nash equilibrium. The asymptotic stability of the unique equilibrium is also established. These findings confirm the crucial role played by warm-glow effects in the equilibrium analysis, whether there is one public good, like in Andreoni (1990)'s analysis, or many related public goods.

Then the paper provides some comparative statics analysis regarding pure redistribution and public provision. When applied to the case of $m = 1$ public good, the results presented in this paper provide the same conditions as those obtained in the existing literature. However, the comparative statics of the single public good case cannot be extended to the more general setting of multiple public goods: in general, the neutrality conditions for m public goods in isolation are not generalizable to m related public goods. Moreover, the multiple public goods game leads to an intuitive conclusion that cannot be achieved when only one public good is considered: the impact of direct grants and subsidies is highly dependent on how public good groups are related in the contribution graph structure.

These results suggest three important reasons for examining the private provision of multiple public goods in a bipartite graph context. First, it makes it possible to show that Ilkiliç (2011)'s results on equilibrium characterization apply to the multiple public goods game with quadratic preferences. Second, it serves to illustrate a significant feature of the usual neutrality result: under pure altruism, neutrality requires that the agents involved in the redistribution be embodied in a path of clearly active links (Cornes and Itaya 2010), while under warm-glow preferences, the agents need to be embodied in a complete structure of clearly active links. Third, it helps to demonstrate that a complete contribution structure is necessary for Andreoni (1990)'s comparative statics results.

Still, it is likely that the comparative statics results presented in this paper can be extended further by relaxing the requirement on the completeness of the contribution structure. In fact, the comparative statics analysis will not be over until conditions on the contribution structure will be found which are both necessary and sufficient. This could probably be achieved by considering some specific, tractable utility functions. Furthermore, the results on the existence, uniqueness and stability of the Nash equilibrium do not impose any structural requirements. They are based on properties of individual preferences, and may eventually be extended to the general class of network games of strategic substitutes with multidimensional strategy spaces and non-linear best-response functions.

Appendix

Given a contribution structure g, let $\mathbf{x}_g$ stand for the column vector of contributions: $\mathbf{x}_g$ is the *link by link profile of contributions* and has size $r(g)$. The links in $\mathbf{x}_g$ are sorted in lexicographic order: the contribution x_{ij} is listed above the contribution x_{kl} when $i < k$ or when $i = k$ and $j < l$. For the contribution structures g_1 and g_2 given in Fig. 2,

$$\mathbf{x}_{g_1} = \begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_{g_2} = \begin{pmatrix} x_{11} \\ x_{21} \\ x_{22} \\ x_{32} \end{pmatrix}.$$

The Nash equilibrium of the multiple public goods game is noted $\mathbf{x}_g^*$.

Proof of Theorem 1 Because of the budget constraints, the allowed contributions are limited by the requirement that $\mathbf{x}_g$ be selected from a convex and compact set S such that

$$S = \prod_{ij \in L} [0, w_i] \subset \mathbb{R}_+^{r(g)}.$$

Then, the existence of a Nash equilibrium follows from fixed point arguments (such as Kakutani fixed point theorem) as in Theorem 1 of Rosen (1965).

To prove the uniqueness of the Nash equilibrium, Theorems 2 and 6 of Rosen (1965) are applied, which entail that the Nash equilibrium of the multiple public goods game is unique whenever the $r(g) \times r(g)$ Jacobian matrix of marginal utilities $\mathbf{J}(\mathbf{x}_g)$ is a symmetric negative definite matrix for all $\mathbf{x}_g \in S$. Observe that, for all $ij \in L$,

$$\frac{\partial^2 U_i}{\partial x_{kl} \partial x_{ij}} (\mathbf{x}_g) = \begin{cases} b_j'' \left(G_j\right) + \delta_{ij}'' \left(x_{ij}\right) + c_i'' \left(w_i - X_i\right), & \text{for } kl \in L \text{ s.t. } kl = ij; \\ c_i'' \left(w_i - X_i\right), & \text{for } kl \in L \text{ s.t. } k = i \text{ and } l \neq j; \\ b_j'' \left(G_j\right), & \text{for } kl \in L \text{ s.t. } k \neq i \text{ and } l = j; \\ 0, & \text{for } kl \in L \text{ s.t. } k \neq i \text{ and } l \neq j, \end{cases}$$

so $\mathbf{J}(\mathbf{x}_g)$ is a symmetric matrix which can be decomposed as

$$\mathbf{J}\left(\mathbf{x}_g\right) = \mathbf{B}\left(\mathbf{x}_g\right) + \Delta\left(\mathbf{x}_g\right) + \mathbf{C}\left(\mathbf{x}_g\right),$$

where $\mathbf{B}(\mathbf{x}_g)$ is the Jacobian matrix of marginal collective benefits, $\Delta(\mathbf{x}_g)$ is the Jacobian matrix of marginal warm-glow, and $\mathbf{C}(\mathbf{x}_g)$ is the Jacobian matrix of marginal private consumption. Both $\mathbf{B}(\mathbf{x}_g)$, $\Delta(\mathbf{x}_g)$ and $\mathbf{C}(\mathbf{x}_g)$ are symmetric matrices. Moreover, $\Delta(\mathbf{x}_g)$ is a diagonal matrix with all diagonal elements negative since under Assumption 1, $\delta_{ij}''(.) < 0$ for all $ij \in L$. Then, $\Delta(\mathbf{x}_g)$ is negative definite for all $\mathbf{x}_g \in S$. In the following lemmas, it is shown that both $\mathbf{B}(\mathbf{x}_g)$ and $\mathbf{C}(\mathbf{x}_g)$ are negative semidefinite for all $\mathbf{x}_g \in S$, so $\mathbf{J}(\mathbf{x}_g)$ is a sum of a symmetric negative definite matrix and two symmetric negative semidefinite matrices. Hence, $\mathbf{J}(\mathbf{x}_g)$ is symmetric negative definite for all $\mathbf{x}_g \in S$, and uniqueness is established. □

Lemma 1 $\mathbf{B}(\mathbf{x}_g)$ *is negative semidefinite for all* $\mathbf{x}_g \in S$.

Proof To show that $\mathbf{B}(\mathbf{x}_g)$ is negative semidefinite for all $\mathbf{x}_g \in S$, it is proved that there exists a matrix $\mathbf{R}_g$, with possibly dependent columns, such that $-\mathbf{B}(\mathbf{x}_g) = {\mathbf{R}_g}^{\mathsf{T}} \mathbf{R}_g$ (see Strang 1988, p. 333). Observe that, for all $ij \in L$,

$$-\frac{\partial^2 b_j}{\partial x_{kl} \partial x_{ij}} \left(\mathbf{x}_g\right) = \begin{cases} -b_j'' \left(G_j\right), & \text{for } kl \in L \text{ s.t. } l = j; \\ 0, & \text{for } kl \in L \text{ s.t. } l \neq j, \end{cases}$$

so $-\mathbf{B}(\mathbf{x}_g)$ is a symmetric matrix. For $s \in \{1, \ldots, m\}$, let $\mathbf{v}^s \in \mathbb{R}_+^{r(g)}$ be such that

$$v_{ij}^s = \begin{cases} \sqrt{-b_j''(G_j)}, & \text{for } ij \in L \text{ s.t. } j = s; \\ 0, & \text{for } ij \in L \text{ s.t. } j \neq s. \end{cases}$$

Define $\mathbf{R}_g$ as a partitioned matrix such that

$${\mathbf{R}_g}^{\mathsf{T}} = \left(\mathbf{v}^1 \ \ldots \ \mathbf{v}^m\right)_{r(g) \times m}.$$

It is straightforward to check that $-\mathbf{B}(\mathbf{x}_g) = {\mathbf{R}_g}^{\mathsf{T}} \mathbf{R}_g$, so $\mathbf{B}(\mathbf{x}_g)$ is negative semidefinite for all $\mathbf{x}_g \in S$. □

Lemma 2 $\mathbf{C}(\mathbf{x}_g)$ *is negative semidefinite for all* $\mathbf{x}_g \in S$.

Proof Let's prove that there exists a matrix $\mathbf{R}_g$ such that $-\mathbf{C}(\mathbf{x}_g) = \mathbf{R}_g{}^{\top}\mathbf{R}_g$. Observe that, for all $ij \in L$,

$$-\frac{\partial^2 c_i}{\partial x_{kl}\partial x_{ij}}\left(\mathbf{x}_g\right) = \begin{cases} -c_i''\left(w_i - X_i\right), & \text{for } kl \in L \text{ s.t. } k = i; \\ 0, & \text{for } kl \in L \text{ s.t. } k \neq i, \end{cases}$$

so $-\mathbf{C}(\mathbf{x}_g)$ is a symmetric matrix. For $t \in \{1, \ldots, n\}$, let $\mathbf{w}^t \in \mathbb{R}_+^{r(g)}$ be such that

$$w_{ij}^t = \begin{cases} \sqrt{-c_i''(w_i - X_i)}, & \text{for } ij \in L \text{ s.t. } i = t; \\ 0, & \text{for } ij \in L \text{ s.t. } i \neq t. \end{cases}$$

Define $\mathbf{R}_g$ as a partitioned matrix such that

$$\mathbf{R}_g{}^{\top} = \left(\mathbf{w}^1 \ \ldots \ \mathbf{w}^n\right)_{r(g)\times n}.$$

It is straightforward to check that $-\mathbf{C}(\mathbf{x}_g) = \mathbf{R}_g{}^{\top}\mathbf{R}_g$, so $\mathbf{C}(\mathbf{x}_g)$ is negative semidefinite for all $\mathbf{x}_g \in S$. □

Proof of Theorem 2 Since asymptotic stability is a local property, it can be assumed that clearly inactive links remain inactive following a small change of contributions at the other links (see, e.g.,Bramoullé et al. 2014; Allouch 2015). Hence, under Assumption 2, the dynamic system reduces to

$$\dot{x}_{ij} = \phi_{ij}\left(G_{-i,j}, w_i - X_{i,-j}\right) - x_{ij}, \quad \text{for all } ij \in B.$$

Let $\mathbf{Z}(\mathbf{x}_B)$ be the $r(B) \times r(B)$ Jacobian matrix of the function $z_{ij}(\mathbf{x}_B) = \phi_{ij}(G_{-i,j}, w_i - X_{i,-j}) - x_{ij}$ for all $ij \in B$. To prove the asymptotic stability of the Nash equilibrium, Lyapunov's indirect method is applied. This entails that the Nash equilibrium of the multiple public goods game is asymptotically stable whenever the real part of each eigenvalue of $\mathbf{Z}(\mathbf{x}_B^*)$ is negative.[22]

For all $ij \in B$, observe that

$$\frac{\partial z_{ij}}{\partial x_{kl}}(\mathbf{x}_B) = \begin{cases} -1, & \text{for } kl \in B \text{ s.t. } kl = ij; \\ \frac{-c_i''(w_i - X_i)}{b_j''(G_j) + \delta_{ij}''(x_{ij}) + c_i''(w_i - X_i)}, & \text{for } kl \in B \text{ s.t. } k = i \text{ and } l \neq j; \\ \frac{-b_j''(G_j)}{b_j''(G_j) + \delta_{ij}''(x_{ij}) + c_i''(w_i - X_i)}, & \text{for } kl \in B \text{ s.t. } k \neq i \text{ and } l = j; \\ 0, & \text{for } kl \in B \text{ s.t. } k \neq i \text{ and } l \neq j, \end{cases}$$

so $\mathbf{Z}(\mathbf{x}_B)$ is an asymmetric matrix which can be decomposed as

$$\mathbf{Z}\left(\mathbf{x}_B\right) = \mathbf{Y}\left(\mathbf{x}_B\right)\mathbf{J}(\mathbf{x}_B),$$

[22] See, e.g., Theorem 4.7 in Khalil (2002).

where $\mathbf{J}(\mathbf{x}_B)$ is the Jacobian matrix of marginal utilities and $\mathbf{Y}(\mathbf{x}_B)$ is a diagonal matrix with all diagonal elements positive, i.e.,

$$[\mathbf{Y}(\mathbf{x}_B)]_{ij,ij} = -\frac{1}{b_j''\left(G_j\right) + \delta_{ij}''\left(x_{ij}\right) + c_i''\left(w_i - X_i\right)} > 0, \quad \text{for all } ij \in B.$$

Then, $\mathbf{Y}(\mathbf{x}_B)$ is a symmetric positive definite matrix for all $\mathbf{x}_B \in S_B = \prod_{ij\in B}[0, w_i]$. It has been shown in the proof of Theorem 1 that under Assumption 1, $\mathbf{J}(\mathbf{x}_g)$ is a symmetric negative definite matrix for all $\mathbf{x}_g \in S$. Given that any principal submatrix of a symmetric negative definite matrix is symmetric negative definite, $\mathbf{J}(\mathbf{x}_B)$ is a symmetric negative definite matrix for all $\mathbf{x}_B \in S_B$. It follows that $-\mathbf{Z}(\mathbf{x}_B)$ is the product of two symmetric positive definite matrices, $\mathbf{Y}(\mathbf{x}_B)$ and $-\mathbf{J}(\mathbf{x}_B)$. By Theorem 2 in Ballantine (1968), all the eigenvalues of $-\mathbf{Z}(\mathbf{x}_B)$ are real and positive for all $\mathbf{x}_B \in S_B$. Thus, all the eigenvalues of $\mathbf{Z}(\mathbf{x}_B^*)$ are real and negative, and asymptotic stability of the Nash equilibrium is established. □

Proof of Proposition 1 Totally differentiating the best-response functions at each link $ij \in B$ yields

$$dx_{ij} = \frac{\partial \phi_{ij}}{\partial G_{-i,j}} dG_{-i,j} + \frac{\partial \phi_{ij}}{\partial (w_i - X_{i,-j})}\left(dw_i - dX_{i,-j}\right).$$

It follows that

$$dx_{ij} = -\frac{b_j''}{b_j'' + \delta_{ij}'' + c_i''} dG_{-i,j} + \frac{c_i''}{b_j'' + \delta_{ij}'' + c_i''}\left(dw_i - dX_{i,-j}\right),$$

or equivalently, since $dG_{-i,j} = dG_j - dx_{ij}$,

$$dx_{ij} = -\frac{b_j''}{\delta_{ij}'' + c_i''} dG_j + \alpha_{ij}\left(dw_i - dX_{i,-j}\right).$$

Since $C = D = \emptyset$, all links are clearly active, i.e., $L = B$. Hence, summing across all $a_i \in N_g(p_j)$ and solving for dG_j yields

$$dG_j = k_j \sum_{a_i \in N_g(p_j)} \left\{\alpha_{ij}\left(dw_i - dX_{i,-j}\right)\right\}, \quad \text{for all } p_j \in P, \tag{1}$$

where

$$k_j = \left(1 + \sum_{a_i \in N_g(p_j)} \frac{b_j''}{\delta_{ij}'' + c_i''}\right)^{-1} \in (0, 1].$$

Since $\alpha_{ij} = \alpha_j$ for all $ij \in L$, Eq. (1) becomes

$$dG_j = k_j \alpha_j \sum_{a_i \in N_g(p_j)} \{dw_i - dX_{i,-j}\}, \quad \text{for all } p_j \in P.$$

Moreover, since g is a complete bipartite graph, it holds that $N_g(a_i) = P$ for all $a_i \in A$, and equivalently $N_g(p_j) = A$ for all $p_j \in P$. Hence,

$$\sum_{a_i \in N_g(p_j)} dw_i = \sum_{a_i \in A} dw_i = 0$$

and

$$\sum_{a_i \in N_g(p_j)} dX_{i,-j} = \sum_{a_i \in A} dX_{i,-j} = \sum_{p_l \in P \setminus \{p_j\}} dG_l.$$

It follows that, for all $p_j \in P$,

$$dG_j = -k_j \alpha_j \sum_{p_l \in P \setminus \{p_j\}} dG_l.$$

From this last equation, it appears that

$$\sum_{p_l \in P} dG_l = \left(1 - \frac{1}{k_1 \alpha_1}\right) dG_1 = \ldots = \left(1 - \frac{1}{k_m \alpha_m}\right) dG_m,$$

so it holds that

$$\text{sign}\,(dG_1) = \ldots = \text{sign}\,(dG_m)\,.$$

Then, for all $p_j \in P$,

$$\begin{aligned}\text{sign}\left(dG_j\right) &= \text{sign}\left(\sum_{p_l \in P \setminus \{p_j\}} dG_l\right)\\ &= \text{sign}\left(k_j \alpha_j \sum_{p_l \in P \setminus \{p_j\}} dG_l\right)\\ &= \text{sign}\left(-dG_j\right)\end{aligned}$$

if and only if $dG_j = 0$. □

Proof of Proposition 2 When the contribution structure is complete, a best-response function of the form given is sufficient since identical values of the altruism coefficient among all agents with respect to each public good is sufficient. The remainder of the proof is therefore devoted to the necessary condition.

Since $C = D = \emptyset$, $x_{ij} = \phi_{ij}(G_{-i,j}, w_i - X_{i,-j})$ holds for all $ij \in L$. Moreover, since $dG_j = 0$ for all $p_j \in P$, the total differential of the best-response functions given in the proof of Proposition 1 yields

$$dx_{ij} = \alpha_j \left(dw_i - dX_{i,-j}\right), \quad \text{for all } ij \in L,$$

where $\alpha_j = \alpha_j(\mathbf{x}_g^*)$. This implies that $\phi_{ij}(G_{-i,j}, w_i - X_{i,-j})$ is linear in $w_i - X_{i,-j}$. Then, it holds that

$$x_{ij} = \phi_{ij}\left(G_{-i,j}, w_i - X_{i,-j}\right) = \phi_{ij}^*\left(G_{-i,j}\right) + \alpha_j\left(w_i - X_{i,-j}\right), \quad \text{for all } ij \in L,$$

where ϕ_{ij}^* is decreasing since $\partial\phi_{ij}/\partial G_{-i,j} = -b_j''/(b_j'' + \delta_{ij}'' + c_i'') \leq 0$. □

Proof of Proposition 3 Totally differentiating the best-response functions at each link $ij \in B$ while keeping $ds_{ij} = dw_i = 0$ yields

$$d\tilde{x}_{ij} = \frac{\partial\phi_{ij}}{\partial G_{-i,j}} dG_{-i,j} + \frac{\partial\phi_{ij}}{\partial\left(\frac{\tau_{ij}}{1-s_{ij}}\right)} \times \frac{1}{1-s_{ij}} d\tau_{ij} - \frac{\partial\phi_{ij}}{\partial(w_i - X_{i,-j})} dX_{i,-j},$$

or equivalently,

$$d\tilde{x}_{ij} = -\frac{b_j''}{b_j'' + \frac{\delta_{ij}''}{(1-s_{ij})^2} + c_i''} dG_{-i,j} + \frac{\frac{\delta_{ij}''}{(1-s_{ij})^2}}{b_j'' + \frac{\delta_{ij}''}{(1-s_{ij})^2} + c_i''} d\tau_{ij} - \frac{c_i''}{b_j'' + \frac{\delta_{ij}''}{(1-s_{ij})^2} + c_i''} dX_{i,-j}.$$

Since $C = D = \emptyset$, all links are clearly active, i.e., $L = B$. Hence, rearranging as in the proof of Proposition 1 yields

$$d\tilde{G}_j = \tilde{k}_j \sum_{a_i \in N_g(p_j)} \left\{\left(1 - \tilde{\alpha}_{ij}\right) d\tau_{ij} - \tilde{\alpha}_{ij} d\tilde{X}_{i,-j}\right\}, \quad \text{for all } p_j \in P, \tag{2}$$

where

$$\tilde{k}_j = \left(1 + \sum_{a_i \in N_g(p_j)} \frac{b_j''}{\frac{\delta_{ij}''}{(1-s_{ij})^2} + c_i''}\right)^{-1} \in (0, 1].$$

Let $\tau_j = \sum_{a_i \in N_g(p_j)} \tau_{ij}$ denote the total lump sum taxes with respect to public good p_j. Since the contribution structure is complete and $\tilde{\alpha}_{ij} = \tilde{\alpha}_j$ for all $ij \in L$, Equation (2) can be rearranged as

$$d\tilde{G}_j = \tilde{k}_j\left(1 - \tilde{\alpha}_j\right) d\tau_j - \tilde{k}_j \tilde{\alpha}_j \sum_{p_l \in P \setminus \{p_j\}} d\tilde{G}_l, \quad \text{for all } p_j \in P.$$

Hence, assuming that $d\tau_1 \neq 0$ and $d\tau_l = 0$ for all $p_l \in P\backslash\{p_1\}$ yields

$$d\tilde{G}_1 = \tilde{k}_1 (1 - \tilde{\alpha}_1) d\tau_1 - \tilde{k}_1\tilde{\alpha}_1 \sum_{p_l \in P\backslash\{p_1\}} d\tilde{G}_l$$

and

$$d\tilde{G}_l = -\tilde{k}_l\tilde{\alpha}_l \sum_{p_j \in P\backslash\{p_l\}} d\tilde{G}_j, \quad \text{for all } \; p_l \in P\backslash\{p_1\}.$$

From this last equation, it appears that

$$\sum_{p_j \in P} d\tilde{G}_j = \left(1 - \frac{1}{\tilde{k}_2\tilde{\alpha}_2}\right) d\tilde{G}_2 = ... = \left(1 - \frac{1}{\tilde{k}_m\tilde{\alpha}_m}\right) d\tilde{G}_m. \tag{3}$$

Hence, it holds that

$$d\tilde{G}_l = \beta_l d\tilde{G}_1, \quad \text{for all} \, p_l \in P\backslash\{p_1\}, \tag{4}$$

where

$$\beta_l = \left(-\frac{1}{\tilde{k}_l\tilde{\alpha}_l} - \sum_{p_j \in P\backslash\{p_1, p_l\}} \left\{ \frac{1 - \frac{1}{\tilde{k}_l\tilde{\alpha}_l}}{1 - \frac{1}{\tilde{k}_j\tilde{\alpha}_j}} \right\} \right)^{-1} \in (-1, 0).$$

Now, let $d\tau_1 > 0$ and suppose that $d\tilde{G}_1 \leq 0$. Then, from Equation (4), $d\tilde{G}_l \geq 0$ for all $p_l \in P\backslash\{p_1\}$, and therefore, from Equation (3), $\sum_{p_j \in P} d\tilde{G}_j \leq 0$. Hence,

$$-d\tilde{G}_1 \geq \sum_{p_l \in P\backslash\{p_1\}} d\tilde{G}_l \geq 0.$$

It follows that

$$\begin{aligned} d\tilde{G}_1 &= \tilde{k}_1 (1 - \tilde{\alpha}_1) d\tau_1 - \tilde{k}_1\tilde{\alpha}_1 \sum_{p_l \in P\backslash\{p_1\}} d\tilde{G}_l \\ &\geq \tilde{k}_1 (1 - \tilde{\alpha}_1) d\tau_1 - \tilde{k}_1\tilde{\alpha}_1 \left(-d\tilde{G}_1\right) \\ &= \tilde{k}_1 (1 - \tilde{\alpha}_1) d\tau_1 + \tilde{k}_1\tilde{\alpha}_1 d\tilde{G}_1. \end{aligned}$$

Then, it appears that

$$d\tilde{G}_1 \left(1 - \tilde{k}_1\tilde{\alpha}_1\right) \geq \tilde{k}_1 (1 - \tilde{\alpha}_1) d\tau_1 \iff d\tilde{G}_1 \geq \frac{\tilde{k}_1(1 - \tilde{\alpha}_1)}{1 - \tilde{k}_1\tilde{\alpha}_1} d\tau_1 > 0,$$

a contradiction. The same contradiction can easily be obtained under the assumption that $d\tilde{G}_1 \geq 0$ when $d\tau_1 < 0$. Hence, sign($d\tau_1$) = sign($d\tilde{G}_1$) = sign($-d\tilde{G}_l$) for all $p_l \in P\backslash\{p_1\}$ = sign($\sum_{p_j \in P} d\tilde{G}_j$). □

Proof of Proposition 4 Totally differentiating the best-response functions at each link $ij \in B$ while keeping $d\tau_{ij} = dw_i = 0$ yields

$$d\tilde{x}_{ij} = \frac{\partial \phi_{ij}}{\partial G_{-i,j}} dG_{-i,j} + \frac{\partial \phi_{ij}}{\partial s_{ij}} ds_{ij} + \frac{\partial \phi_{ij}}{\partial (\frac{\tau_{ij}}{1-s_{ij}})} \times \frac{\tau_{ij}}{(1-s_{ij})^2} ds_{ij} - \frac{\partial \phi_{ij}}{\partial (w_i - X_{i,-j})} dX_{i,-j},$$

or equivalently,

$$d\tilde{x}_{ij} = -\frac{b''_j}{b''_j + \frac{\delta''_{ij}}{(1-s_{ij})^2} + c''_i} dG_{-i,j} - \frac{\frac{\delta'_{ij}}{(1-s_{ij})^2} - \frac{\delta''_{ij}\tau_{ij}}{(1-s_{ij})^3}}{b''_j + \frac{\delta''_{ij}}{(1-s_{ij})^2} + c''_i} ds_{ij} - \frac{c''_i}{b''_j + \frac{\delta''_{ij}}{(1-s_{ij})^2} + c''_i} dX_{i,-j}.$$

Since $C = D = \emptyset$, all links are clearly active, i.e., $L = B$. Hence, rearranging as in the proof of Proposition 1 yields

$$d\tilde{G}_j = \tilde{k}_j \sum_{a_i \in N_g(p_j)} \left\{ \left(\tilde{\alpha}_{ij}\kappa_{ij} + (1 - \tilde{\alpha}_{ij}) \frac{\tau_{ij}}{1 - s_{ij}} \right) ds_{ij} - \tilde{\alpha}_{ij} d\tilde{X}_{i,-j} \right\}, \quad \text{for all } p_j \in P, \tag{5}$$

where

$$\kappa_{ij} = \frac{\frac{\partial \phi_{ij}}{\partial s_{ij}}}{\frac{\partial \phi_{ij}}{\partial (w_i - \tilde{X}_{i,-j})}} = \frac{-\frac{\delta'_{ij}}{(1-s_{ij})^2}}{c''_i} > 0,$$

and $\tilde{k}_j \in (0, 1]$ as in the proof of Proposition 3. Since the contribution structure is complete and $\tilde{\alpha}_{ij} = \tilde{\alpha}_j$ for all $ij \in L$, Equation (5) can be rearranged as

$$d\tilde{G}_j = \tilde{k}_j \sum_{a_i \in A} \left\{ \left(\tilde{\alpha}_j \kappa_{ij} + (1 - \tilde{\alpha}_j) \frac{\tau_{ij}}{1 - s_{ij}} \right) ds_{ij} \right\} - \tilde{k}_j \tilde{\alpha}_j \sum_{p_l \in P \setminus \{p_j\}} d\tilde{G}_l, \quad \text{for all } p_j \in P.$$

Hence, assuming that $ds_{i1} \neq 0$ for at least one agent $a_i \in N_g(p_1)$ and $ds_{il} = 0$ for all $a_i \in N_g(p_l)$ for all $p_l \in P \setminus \{p_1\}$ yields

$$d\tilde{G}_1 = \tilde{k}_1 \sum_{a_i \in A} \left\{ \left(\tilde{\alpha}_1 \kappa_{i1} + (1 - \tilde{\alpha}_1) \frac{\tau_{i1}}{1 - s_{i1}} \right) ds_{i1} \right\} - \tilde{k}_1 \tilde{\alpha}_1 \sum_{p_l \in P \setminus \{p_1\}} d\tilde{G}_l$$

and

$$d\tilde{G}_l = -\tilde{k}_l \tilde{\alpha}_l \sum_{p_j \in P \setminus \{p_l\}} d\tilde{G}_j, \quad \text{for all } p_l \in P \setminus \{p_1\}.$$

From this last equation, observe that Equations (3) and (4) hold, and since

$$\tilde{\alpha}_1 \kappa_{i1} + (1 - \tilde{\alpha}_1) \frac{\tau_{i1}}{1 - s_{i1}} > 0, \quad \text{for all } a_i \in A,$$

the same contradiction as in the proof of Proposition 3 can easily be obtained. □

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