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Optimal deterrence of cooperation

Stéphane Gonzalez[1] · **Aymeric Lardon**[2]

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Abstract We introduce axiomatically a new solution concept for cooperative games with transferable utility inspired by the core. While core solution concepts have investigated the sustainability of cooperation among players, our solution concept, called contraction core, focuses on the deterrence of cooperation. The main interest of the contraction core is to provide a monetary measure of the robustness of cooperation in the grand coalition. We motivate this concept by providing optimal fine imposed by competition authorities for the dismantling of cartels in oligopolistic markets. We characterize the contraction core on the set of balanced cooperative games with transferable utility by four axioms: the two classic axioms of non-emptiness and individual rationality, a superadditivity principle and a weak version of a new axiom of consistency.

Keywords TU-game · Contraction core · Optimal fine · Cournot oligopoly · Axiomatization

JEL Classification C71 · D43

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✉ Aymeric Lardon
aymeric.lardon@unice.fr

Stéphane Gonzalez
stephane.gonzalez@univ-st-etienne.fr

1 Univ Lyon, UJM Saint-Etienne, CNRS, GATE L-SE UMR 5824, 42023 Saint-Étienne, France

2 Université Côte d'Azur, GREDEG, CNRS, 250 rue Albert Einstein, 06560 Valbonne, France

1 Introduction

One of the main issues in cooperative game theory concerns the possibility for players to cooperate all together. A well-known solution concept for cooperative games with transferable utility (henceforth TU-games) dealing with the existence of stable cooperative agreements is the core (Gillies 1953). The classic Bondareva–Shapley theorem establishes that the non-emptiness of the core is characterized by the balancedness property as proved independently in Bondareva (1963) and Shapley (1967). A possible interpretation of the balancedness property that interests us is the following: each player must distribute one unit of time among all the coalitions of which she is a member; the balancedness property stipulates that an optimal time allocation for players is to devote all their unit of time to the grand coalition, i.e., the whole set of players.

Even in the case where the core is empty the literature has investigated the possibility to enforce a stable cooperative agreement by introducing other core solution concepts: the strong and the weak ε-cores (Shapley and Shubik 1966), the (weak) least core (Maschler et al. 1979; Young et al. 1982), the aspiration core (Albers 1979; Cross 1967; Bennett 1983), the extended core (Bejan and Gómez 2009), the negotiation set (Gonzalez and Grabisch 2015b) and the d-multicoalitional core (Gonzalez and Grabisch 2016). Some of these variants of the core are non-empty when applied to non-balanced TU-games and coincide with the core on the set of balanced TU-games.

Until now, solution concepts inspired by the core have restricted attention to the sustainability of cooperation. Nevertheless, in many competitive environments, cooperation is not socially desirable, and players must be discouraged to work all together. For example, horizontal agreements on prices between firms are punished by competition authorities. Similarly, drug cartels are reproved to protect population. In the same vein, the dismantling of terrorist groups appears to be of primary importance for national security. Furthermore, to the best of our knowledge, even when cooperation is efficient, the robustness of stable cooperative agreements has not been studied yet. For example, how sensible is the cohesion of collaborative activities on research and development to discovery values? Or does the stability of trade agreements depend crucially on transportation costs? To meet these challenges head on, a general solution concept spanning several fields of economics (industrial organization, innovation, international trade, criminology…) appears fundamental in order to provide insight into the deterrence of cooperation.

In this article, we investigate the deterrence of cooperation among players for balanced cooperative TU-games by imposing monetary penalty on the grand coalition. Precisely, we are interested in finding the minimal amount of fine, called the optimal fine, under which cooperation can no longer be sustained. This leads us to consider a new solution concept, called contraction core, which contains all stable cooperative agreements for which any fine increase makes these agreements unstable. In this sense, the contraction core contains all the "weakest" stable cooperative agreements further to the optimal fine imposed on the grand coalition. This fine can be interpreted as a measure of the robustness of cooperation in the grand coalition. In terms of time allocation, this means that authority deters the formation of the grand coalition and that players must devote fractions of their unit of time to any other coalition as a second best time allocation. This notion will be used for the definition of feasibility and efficiency conditions related to our solution concept. Unlike some of the core solution concepts mentioned above, the contraction core does not contain the core, and so it is not a core extension. Moreover, the contraction core has the advantage of being a singleton on the set of balanced and symmetric TU-games.

Following in the footsteps of previous works (Trockel 2005; Moulin 2014) which deal with microeconomics by using cooperative concepts, we propose an illustrative example of oligopolistic markets in order to motivate our solution concept. In economic welfare analysis, it is a well-established and old idea that monopoly power often negatively affects social welfare. Although cooperation on research and development activities may have beneficial welfare effects (D'Aspremont and Jacquemin 1988), most of horizontal agreements on sales prices are considered as harmful to social welfare. The cooperative approach of oligopoly games is of great interest in order to analyze the stability of cartels which are one of the main preoccupations of competition authorities.[1] We point out that our analysis does not pay attention to the welfare effects of trade restriction as advocated by the rule of reason in antitrust law, but focuses on the deterrence of monopoly power which leads, a priori, to welfare losses. Thus, the contraction core constitutes an effective tool to prevent the formation of cartels. Precisely, we consider the set of Cournot oligopoly TU-games in γ-characteristic function form (Hart and Kurz 1983; Chander and Tulkens 1997) which is appropriate in the context of oligopoly industries. Under this approach, the worth of any coalition (the cartel profit) is enforced by a competition setting in which any cartel faces external firms acting individually. We assume that the inverse demand function is affine and firms operate at constant and identical marginal costs. These assumptions ensure that the balancedness property holds on this set of Cournot oligopoly TU-games as shown by Lardon (2012), and so the contraction core is well-defined. After having determined the worth of any coalition, we compute the contraction core and we provide an expression of the optimal fine imposed by competition authorities in order to deter the formation of the grand coalition which corresponds, in the present case, to the cartel comprising all the firms. Surprisingly, this expression differs depending on the number of firms and leads to distinguish markets of small size (less than five firms) and those of medium and large size (more than six firms).

Beyond this economic application, in order to get a better grasp of the contraction core, we provide an axiomatic characterization of this new solution concept on the set of balanced TU-games. We invoke the two classic axioms of non-emptiness and individual rationality as well as a superadditivity principle and a weak version of a new axiom of consistency. The original superadditivity and consistency properties (Peleg 1986) used to characterize the core, implicitly depend on grand coalition feasibility. We replace them with similar properties based on a new definition of feasibility derived from non-trivial coalition formation which relies on second best time allocation for players.[2] We impose this feasibility requirement on our superadditivity principle. Our consistency principle is based on an appropriated reduced games property. Traditional

[1] The developing theory of oligopoly TU-games comprises many contributions such as Zhao (1999), Norde et al. (2002), Driessen and Meinhardt (2005), Lardon (2012) and Lekeas and Stamatopoulos (2014) among others.

[2] Bejan and Gómez (2012) use a more relaxed feasibility condition based on first best time allocation.

reduced games (Davis and Maschler 1965) used by Peleg (1986) make an exception to the grand coalition in order to ensure grand coalition feasibility. Bejan and Gómez (2012) use a more general version (Moldovanu and Winter 1994) that treats all coalitions in the same way. Our axiom of consistency is based on a new modified version of reduced games which makes again an exception to the grand coalition. Precisely, unlike any other coalition, the grand coalition of any reduced game is not allowed to cooperate with the complementary coalition. Moreover, given any second best time allocation in the original TU-game, under certain conditions, we provide a formula to compute the corresponding second best time allocation in any of its reduced games. Our axioms of superadditivity and consistency do not coincide with those of Peleg (1986) and their generalized versions in Bejan and Gómez (2012) on the set of balanced TU-games.

The article is organized as follows. Section 2 presents the contraction core as well as some of its properties. Section 3 gives an illustrative example of oligopolistic markets for the deterrence of monopoly power. In Sect. 4, we provide an axiomatic characterization of the contraction core. Section 5 deals with a natural extension of the contraction core on the set of all TU-games.

2 Cooperatives games and the contraction core

2.1 Cooperatives games with transferable utility

A cooperative **TU-game** is an ordered pair (N, v) consisting of a finite set of players N and a **characteristic function** $v : 2^N \longrightarrow \mathbb{R}$ such that $v(\emptyset) = 0$ where 2^N denotes the power set of N. Subsets of N are called **coalitions**, and we call $v(S)$ the worth of coalition S. The **size** of coalition S is denoted by $s = |S|$. Let Γ denote the **set of TU-games**.

Later in the paper, we will use both simple and symmetric TU-games. A TU-game (N, v) is **simple** if for any coalition $S \in 2^N\backslash\{\emptyset, N\}$, we have $v(S) \in \{0, 1\}$ and $v(N) = 1$. A coalition S such that $v(S) = 1$ is called a **winning coalition**. A player $i \in N$ is called a **veto player** if she belongs to any winning coalition. A TU-game (N, v) is **symmetric** if there exists a mapping $f : \mathbb{N} \longrightarrow \mathbb{R}$ such that for any coalition $S \in 2^N\backslash\{\emptyset\}$, we have $v(S) = f(s)$.

Let $\mathcal{B} \subseteq 2^N\backslash\{\emptyset\}$ be a collection of coalitions. Then $\mathcal{B}$ is said to be a **balanced collection of coalitions** if for every $S \in \mathcal{B}$ there exists a balancing weight $\delta_S \in \mathbb{R}_+$ such that $\sum_{S\in\mathcal{B}:i\in S} \delta_S = 1$ for every $i \in N$. Consider δ_S as an amount of time allocated to coalition S by any of its members. When each player has one unit of time, the requirement that $\sum_{S\in\mathcal{B}:i\in S} \delta_S = 1$ is then a time feasibility condition. We denote by $\Lambda(N)$ the **set of balanced collections** and $\Lambda^*(N)$ the **set of balanced collections not containing the grand coalition** when $n \geq 2$. By convention, $\Lambda^*(N) = \Lambda(N)$ when $n = 1$. A TU-game (N, v) is **balanced** if for every balanced collection $\mathcal{B} \in \Lambda(N)$ it holds that $\sum_{S\in\mathcal{B}} \delta_S v(S) \leq v(N)$. Let Γ_c denote the **subset of balanced TU-games**. On the set Γ_c a best time allocation for players is to devote all their unit of time to the grand coalition.

2.2 Feasibility as second best time allocation

We now introduce the appropriate notion of feasibility which will be useful for the definition of the contraction core. In a TU-game $(N, v) \in \Gamma$, every player $i \in N$ may receive a **payoff** $x_i \in \mathbb{R}$. A vector $x \in \mathbb{R}^N$ is a **payoff vector**. For any coalition $S \in 2^N \backslash \{\emptyset\}$ and any payoff vector $x \in \mathbb{R}^N$, we define $x(S) = \sum_{i \in S} x_i$ and we denote by $x^S \in \mathbb{R}^S$ the vector such that $x_i^S = x_i$ for all $i \in S$.

Generally speaking, feasibility is a restriction on players' payoffs and can be interpreted in terms of time allocation. The classic feasibility condition, called the grand coalition feasibility, is defined as the set of payoff vectors, denoted by $X(N, v)$, that are feasible when players allocate their unit of time to the grand coalition, i.e.:

$$X(N, v) = \left\{x \in \mathbb{R}^N : x(N) \leq v(N)\right\}.$$

A more relaxed feasibility condition which considers non-trivial coalition formation, is defined as the set of payoff vectors, denoted by $X_\Lambda(N, v)$, that are feasible when players can devote fractions of their time to any coalition, not just the grand coalition, i.e.:

$$X_\Lambda(N, v) = \left\{x \in \mathbb{R}^N : x(N) \leq \sum_{S \in \mathcal{B}} \delta_S v(S) \text{ for some } \mathcal{B} \in \Lambda(N)\right\}.$$

On the set Γ_c, both conditions of feasibility are equivalent since a best time allocation for players is to form the grand coalition. Now, suppose authority prevents the formation of the grand coalition. The new feasibility condition that interests us becomes any possible arrangement between players who distribute a fraction of their time to any coalition except grand coalition.

Definition 2.1 For any TU-game $(N, v) \in \Gamma_c$, the **set of feasible payoff vectors** of (N, v), denoted by $X_{\Lambda^*}(N, v)$, is defined as:

$$X_{\Lambda^*}(N, v) = \left\{x \in \mathbb{R}^N : x(N) \leq \sum_{S \in \mathcal{B}} \delta_S v(S) \text{ for some } \mathcal{B} \in \Lambda^*(N)\right\}.$$

On the set Γ_c, this feasibility condition relies on coalition formation of players as second best time allocation. This leads to define the associated efficiency condition.

Definition 2.2 For any TU-game $(N, v) \in \Gamma_c$, the **set of efficient payoff vectors** of (N, v), denoted by $X^*_{\Lambda^*}(N, v)$, is defined as:

$$X^*_{\Lambda^*}(N, v) = \arg\max \left\{x(N) : x \in X_{\Lambda^*}(N, v)\right\}.$$

On the set Γ_c, any efficient payoff vector is exactly achieved by a second best time allocation for players on $X_\Lambda(N, v)$.[3]

[3] Precisely, the second best time allocation is also a first best one in the non-generical case where $X_{\Lambda^*}(N, v) = X_\Lambda(N, v)$.

2.3 Contraction core

The new feasibility and efficient conditions related to second best time allocation permit to define the main object of our study on the set Γ_c, namely the contraction core.

Definition 2.3 For any TU-game $(N, v) \in \Gamma_c$, the **contraction core**, denoted by $CC(N, v)$, is defined as:

$$CC(N, v) = \left\{x \in X^*_{\Lambda^*}(N, v) : \forall S \subset N, x(S) \geq v(S)\right\}.$$

The contraction core contains all efficient payoff vectors[4] achieved by any second best time allocation that satisfy a relaxed coalitional stability condition for which the grand coalition is not taken into account.

The following are the definitions of the core, the aspiration core and the weak least core for which we will make comparisons with the contraction core.[5] The core (Gillies 1953) of a TU-game $(N, v) \in \Gamma$, denoted by $C(N, v)$, is defined as:

$$C(N, v) = \{x \in X(N, v) : \forall S \subseteq N, x(S) \geq v(S)\}.$$

Bondareva (1963) and Shapley (1967) showed that any TU-game $(N, v) \in \Gamma$ is balanced if and only if $C(N, v) \neq \emptyset$.
The aspiration core (Albers 1979; Cross 1967; Bennett 1983) of a TU-game $(N, v) \in \Gamma$, denoted by $AC(N, v)$, is defined as:

$$AC(N, v) = \{x \in X_\Lambda(N, v) : \forall S \subseteq N, x(S) \geq v(S)\}.$$

Both the core and the aspiration core contain all feasible payoff vectors (with the understanding that we consider grand coalition feasibility for the former and feasibility as first best time allocation for the latter) that satisfy the classic coalitional stability condition.
We now introduce the concept of the weak ε-core (Shapley and Shubik 1966) that will be useful for the definition of the weak least core. Given any $\varepsilon \in \mathbb{R}$, the weak ε-core (or the per-capita ε-core) of a TU-game $(N, v) \in \Gamma$, denoted by $C_\varepsilon(N, v)$, is defined as:[6]

$$C_\varepsilon(N, v) = \left\{x \in \mathbb{R}^N : x(N) = v(N) \text{ and } \forall S \subset N, x(S) \geq v(S) - s\varepsilon\right\}.$$

[4] We need to use the set of efficient payoff vectors in the definition of the contraction core in order to deal with the one-player case.

[5] While the contraction core is defined on the set Γ_c, the core, the aspiration core and the weak least core are defined on the set Γ.

[6] Shapley and Shubik (1966) also define another generalization of the core called the strong ε-core. In this case, every coalition faces to the same cost ε regardless of its cardinality.

The weak least core (Young et al. 1982) of a TU-game $(N, v) \in \Gamma$, denoted by $LC_w(N, v)$, is defined as the intersection of all the non-empty weak ε-cores, i.e., $LC_w(N, v) = C_{\underline{\varepsilon}}(N, v)$ where $\underline{\varepsilon}$ is the smallest ε such that $C_\varepsilon(N, v) \neq \emptyset$.

2.4 Deterrence of cooperation

We now show that the contraction core is relevant in order to deal with the deterrence of cooperation. Given any TU-game $(N, v) \in \Gamma_c$ and any $t \in \mathbb{R}_+$, its **t-contraction** is the TU-game, denoted by (N, v^t), such that $v^t(S) = v(S)$ for any $S \subset N$, and $v^t(N) = v(N) - t$. In particular, we assign real number $\underline{\mathrm{t}}(N, v)$, called the **optimal fine**, to any TU-game $(N, v) \in \Gamma_c$, which is defined as:

$$\underline{\mathrm{t}}(N, v) = \begin{cases} \inf\{t \in \mathbb{R} : \forall k > t, (N, v^k) \text{ is not balanced}\} & \text{if } n \geq 2; \\ 0 & \text{if } n = 1. \end{cases}$$

The $\underline{\mathrm{t}}(N, v)$-contraction corresponds to the original TU-game (N, v) for which the grand coalition must pay optimal fine $\underline{\mathrm{t}}(N, v)$. This optimal fine gives the minimal amount for which any fine increase makes cooperation in the grand coalition unstable. It can be considered as a measure of the robustness of stable cooperative agreements.[7] An alternative formula of the optimal fine easier to compute is the following:

$$\underline{\mathrm{t}}(N, v) = v(N) - \max_{\mathcal{B} \in \Lambda^*(N)} \sum_{S \in \mathcal{B}} \delta_S v(S).$$

Whenever the core is non-empty, the optimal fine $\underline{\mathrm{t}}(N, v)$ is equal to the difference between $v(N)$ and the minimum no-blocking payoff defined by Zhao (2001). The minimum no-blocking payoff also coincides with the worth of the grand coalition of the root game introduced by Calleja et al. (2009). We show that the contraction core of any TU-game $(N, v) \in \Gamma_c$ is equal to the core of its $\underline{\mathrm{t}}(N, v)$-contraction.

Proposition 2.4 *For any TU-game $(N, v) \in \Gamma_c$, it holds that $CC(N, v) = C(N, v^{\underline{\mathrm{t}}(N,v)})$.*

Proof First, we prove that $C(N, v^{\underline{\mathrm{t}}(N,v)}) \subseteq CC(N, v)$. Take any $x \in C(N, v^{\underline{\mathrm{t}}(N,v)})$. Then, it holds that:

$$\begin{aligned} x(N) &= v^{\underline{\mathrm{t}}(N,v)}(N) \\ &= v(N) - \underline{\mathrm{t}}(N, v) \\ &= \max_{\mathcal{B} \in \Lambda^*(N)} \sum_{S \in \mathcal{B}} \delta_S v(S). \end{aligned}$$

Hence $x \in X^*_{\Lambda^*}(N, v)$. Moreover, for any $S \in 2^N \backslash \{\emptyset, N\}$ we have:

$$\begin{aligned} x(S) &\geq v^{\underline{\mathrm{t}}(N,v)}(S) \\ &= v(S), \end{aligned}$$

[7] Observe that $\underline{\mathrm{t}}(N, v) = 0$ for the one-player case since no cooperation occurs.

which proves that $x \in CC(N, v)$.
Second, we prove that $CC(N, v) \subseteq C(N, v^{\underline{t}(N,v)})$. Take any $x \in CC(N, v)$. Since $x \in X^*_{\Lambda^*}(N, v)$ the above equalities imply that $x(N) = v^{\underline{t}(N,v)}(N)$. Moreover, it holds that for any $S \in 2^N \backslash \{\emptyset, N\}$, $x(S) \geq v(S) = v^{\underline{t}(N,v)}(S)$. Hence $x \in C(N, v^{\underline{t}(N,v)})$. □

The contraction core contains all the "weakest" stable cooperative agreements further to the optimal fine imposed on the grand coalition. This means that authority deters the formation of the grand coalition which compels players to find another almost unstable agreement in the contraction core. Furthermore, given any TU-game $(N, v) \in \Gamma_c$ and any vector of "taxes" $t \in \mathbb{R}^N_+$ such that $t(N) = \underline{t}(N, v)$, if $x \in CC(N, v)$ then $y = x + t \in C(N, v)$.[8] However, the converse does not always hold as showed in the following example.

Example 2.5 Consider the TU-game $(N, v) \in \Gamma_c$ such that $N = \{1, 2\}$, $v(\{1\}) = 2$, $v(\{2\}) = 7$ and $v(\{1, 2\}) = 10$. It holds that $C(N, v) = \text{convex hull}\{(3, 7); (2, 8)\}$, $\underline{t}(N, v) = 1$ and $CC(N, v) = \{(2, 7)\}$. We have $x = (2, 7) \in CC(N, v)$ and $y = x + (1/2) \times e = (2, 5; 7, 5) \in C(N, v)$ where $e = (1, 1)$. However, it holds that $y = (3, 7) \in C(N, v)$ but $x = y - (1/2) \times e = (2, 5; 6, 5) \notin CC(N, v)$.

Proposition 2.4 holds for any core extension which coincides with the core on the set Γ_c such that the aspiration core (Albers 1979; Cross 1967; Bennett 1983), the extended core (Bejan and Gómez 2009) and the negotiation set (Gonzalez and Grabisch 2015b). Moreover, although the contraction core is defined on the set Γ_c, we argue that it is also closely related to the aspiration core applied to the set $\Gamma \backslash \Gamma_c$. Given any TU-game $(N, v) \in \Gamma_c$, we consider the associated TU-game $(N, \underline{v}) \in \Gamma \backslash \Gamma_c$ such that $\underline{v}(S) = v(S)$ for any $S \subset N$ and $\underline{v}(N) = v(N) - M$ where $M \in \mathbb{R}_{++}$ is sufficiently large to discourage the formation of the grand coalition. Then, it is straightforward to verify that $CC(N, v) = AC(N, \underline{v})$. This result implies that authority does not have to worry about the calculation of the optimal fine $\underline{t}(N, v)$. Even if it decides to impose an extremely harsh fine M on the grand coalition, considering the aspiration core of $(N, \underline{v})$ permits to return to the contraction core of (N, v). We now show that there is a bijection between the contraction core and the weak least core.

Proposition 2.6 *For any TU-game $(N, v) \in \Gamma_c$, it holds that $LC_w(N, v) = C_{\underline{\varepsilon}}(N, v)$ where $\underline{\varepsilon} = -\underline{t}(N, v)/n$. Additionally, $x \in CC(N, v)$ if and only if $y = x - \underline{\varepsilon} \times e \in LC_w(N, v)$ where $e = (1, \ldots, 1)$.*

Proof First, we prove that $\underline{\varepsilon} = -\underline{t}(N, v)/n$ is the smallest ε such that $C_\varepsilon(N, v) \neq \emptyset$. Take any $\varepsilon < \underline{\varepsilon}$ and assume by contradiction that there exists $y \in C_\varepsilon(N, v)$. We define payoff vector $x \in \mathbb{R}^N$ such that $x = y + \varepsilon \times e$. Then, it follows from $y(N) = v(N)$ that:

$$\begin{aligned} x(N) &= y(N) + n\varepsilon \\ &= v(N) + n\varepsilon \\ &= v^{-n\varepsilon}(N). \end{aligned}$$

[8] We refer to Bejan and Gómez (2009) for a detailed discussion on tax rules.

Moreover, for any $S \in 2^N \setminus \{\emptyset, N\}$ we have:

$$\begin{aligned} x(S) &= y(S) + s\varepsilon \\ &\geq v(S), \end{aligned}$$

which proves that $x \in C(N, v^{-n\varepsilon})$. Hence we conclude that $\varepsilon \geq -\underline{t}(N, v)/n = \underline{\varepsilon}$, a contradiction.
Second, we prove that $x \in CC(N, v)$ if and only if $y = x - \underline{\varepsilon} \times e \in LC_w(N, v)$. Take any $x \in CC(N, v)$. Then, it holds that:

$$\begin{aligned} y(N) &= x(N) - n\underline{\varepsilon} \\ &= x(N) + \underline{t}(N, v) \\ &= v(N). \end{aligned}$$

Moreover, for any $S \in 2^N \setminus \{\emptyset, N\}$ we have:

$$\begin{aligned} y(S) &= x(S) - s\underline{\varepsilon} \\ &\geq v(S) - s\underline{\varepsilon}, \end{aligned}$$

which proves that $y \in C_{\underline{\varepsilon}}(N, v) = LC_w(N, v)$. Then, take any $y \in LC_w(N, v)$. Since $y(N) = v(N)$ the above equalities imply that $x(N) = v^{\underline{t}(N,v)}(N)$. Moreover, it holds that for any $S \in 2^N \setminus \{\emptyset, N\}$, $y(S) \geq v(S) - s\underline{\varepsilon}$ which is equivalent to $x(S) \geq v(S)$. Hence, it holds that $x \in C(N, v^{\underline{t}(N,v)})$ and, by Proposition 2.4, $x \in CC(N, v)$. □

2.5 Properties of the contraction core

One of the main advantages of the contraction core is to be a singleton on the set of balanced and symmetric TU-games. In order to prove this result, we first introduce the concept of autonomous coalition (Gonzalez and Grabisch 2015a). Given any TU-game $(N, v) \in \Gamma_c$, a coalition $S \in 2^N \setminus \{\emptyset\}$ is **autonomous** for (N, v) if for any payoff vector $x \in C(N, v)$, it holds that $x(S) = v(S)$.

Proposition 2.7 (Gonzalez and Grabisch 2015a) *For any TU-game $(N, v) \in \Gamma_c$, the following statements are equivalent:*

1. *There exists a coalition $S \in 2^N \setminus \{\emptyset, N\}$ which is autonomous for (N, v).*
2. *For all $t > 0$, it holds that $C(N, v^t) = \emptyset$.*

Furthermore, Gonzalez and Grabisch (2015a) prove that the set of autonomous coalitions is a balanced collection.

Proposition 2.8 *For any symmetric TU-game $(N, v) \in \Gamma_c$, the contraction core $CC(N, v)$ is a singleton.*

Proof It follows from the symmetry of (N, v) that its $\underline{t}(N, v)$-contraction $(N, v^{\underline{t}(N,v)})$ is also symmetric. By Proposition 2.4, it holds that $CC(N, v) = C(N, v^{\underline{t}(N,v)})$. It is well-known that payoff vector $x \in \mathbb{R}^N$ such that $x_i = v^{\underline{t}(N,v)}(N)/n$ for all $i \in N$ is a core element of any symmetric and balanced TU-game. Moreover, it follows from Proposition 2.7 that there exists an autonomous coalition $K \subset N$ of size $k < n$. The symmetry of (N, v) implies that any coalition S of size k is also autonomous. The collection of all coalitions of size k, denoted by $\mathcal{B}$, is a balanced collection with weight $\delta_S = \binom{n-1}{k-1}$ for any $S \in \mathcal{B}$. We define payoff vector $x' \in \mathbb{R}^N$ such that $x'_i = v(K)/k$ for all $i \in N$. Hence, it holds that:

$$\begin{aligned} x'(N) &= \sum_{S \in \mathcal{B}} \binom{n-1}{k-1} x'(S) \\ &= \sum_{S \in \mathcal{B}} \binom{n-1}{k-1} v(S) \\ &= v^{\underline{t}(N,v)}(N), \end{aligned}$$

where the last equality follows from Proposition 2.7. We conclude that $x_i = x'_i$ for all $i \in N$, and so $x' \in CC(N, v)$.

It remains to show that $x' \in CC(N, v)$ is the unique element of the contraction core. Suppose by contradiction that there exists $y \in CC(N, v)$ such that $y \neq x'$. Then, there exists a player $j \in N$ such that $y_j > v(K)/k$ and a player $i \in N$ such that $y_i < v(K)/k$. Since $k < n$, there exists an autonomous coalition T of size k such that $j \in T$ and $i \notin T$. Hence, we deduce that $\sum_{r \in T \setminus \{j\}} y_r < (v(K)/k) \times (k-1)$, and so $\sum_{r \in (T \cup \{i\}) \setminus \{j\}} y_r < v(K)$, a contradiction with the fact that $(T \cup \{i\}) \setminus \{j\}$ is also an autonomous coalition. □

Next, we provide a subset of simple TU-games in which the contraction core is not a singleton.

Proposition 2.9 *For any simple TU-game $(N, v) \in \Gamma_c$ with at least two veto players and at least two winning coalitions, the contraction core is not a singleton.*

Proof It is known that the core of any simple TU-game contains any payoff vector that distributes all the gains of the grand coalition among veto players. Take any simple TU-game (N, v) with at least two veto players and at least two winning coalitions. It holds that $v(N) = 1$ and $v(S) = 1$ for some $S \in 2^N \setminus \{\emptyset, N\}$ which implies that $CC(N, v) = C(N, v)$. Moreover, since there are at least two veto players, the above mentioned result on the core permits to conclude that the contraction core is not a singleton. □

The following example shows that the contraction core may not be a singleton even on the set Γ_c.

Example 2.10 Consider the TU-game $(N, v) \in \Gamma_c$ such that $N = \{1, 2, 3\}$, $v(\{1\}) = v(\{2\}) = 0$, $v(\{3\}) = 3$, $v(\{1, 2\}) = 6$, $v(\{1, 3\}) = v(\{2, 3\}) = 0$, and $v(\{1, 2, 3\}) = 15$. Then, it holds that $\underline{t}(N, v) = 6$ and $CC(N, v) = \text{convex hull}\{(6, 0, 3); (0, 6, 3)\}$.

3 Illustrative example

Usually, oligopolistic markets are modeled by means of non-cooperative games in which every profit-maximizing firm pursues Nash strategies. However, in other oligopoly situations firms do not always behave non-cooperatively and if sufficient communication is feasible it may be possible for firms to sign collusive agreements. In this section, we consider a fully cooperative approach by converting a normal form Cournot oligopoly game into a Cournot oligopoly TU-game in which firms can form cartels acting as a single player. While the core is an appropriate solution concept in order to study the existence of stable collusive agreements, we apply the contraction core to oligopolistic markets in order to compute the optimal fine imposed by competition authorities for cartel deterrence. We analyze a quantity competition between n firms. Every firm $i \in N$ produces **quantity** $q_i \in \mathbb{R}_+$ of a homogeneous good. Furthermore, we consider the following affine **inverse demand function**:

$$p(Q) = a - bQ,$$

where a is the intercept of demand, b is the slope of p and $Q = \sum_{j \in N} q_j$ is the total output of the market. Each firm produces at constant **average and marginal cost** $c \in \mathbb{R}_+$. **Profits** for the ith producer in terms of quantities, π_i, are expressed as:

$$\pi_i((q_j)_{j \in N}) = (p(Q) - c)q_i.$$

Without loss of generality, we assume that $c = 0$.

Following Hart and Kurz (1983) and Chander and Tulkens (1997), we consider a situation in which some firms form a cartel (coalition) S while the remaining players in $N \backslash S$ continue to act independently. Cartel members are assumed to act as a single firm maximizing their joint profit by correlating their strategies. This leads to consider the set of Cournot oligopoly TU-games in γ-characteristic function form defined as:

$$v_\gamma(S) = \sum_{i \in S} \pi_i(q_i^*, \tilde{q}_j),$$

where $(q_i^*, \tilde{q}_j)$ is the Cournot-Nash equilibrium between S and the other players with the understanding that each firm $i \in S$ produces identical quantity q_i^* and each outsider $j \in N \backslash S$ chooses the same quantity $\tilde{q}_j$.[9] Under these considerations, we can compute the worth of any coalition as established in the following proposition.

Proposition 3.1 *Let* $(N, v_\gamma) \in \Gamma$ *be an oligopoly TU-game in* γ*-characteristic function form. Then for any coalition* $S \in 2^N \backslash \{\emptyset\}$*, it holds that:*

$$v_\gamma(S) = \frac{1}{b}\left(\frac{a}{n - s + 2}\right)^2.$$

[9] This is a consequence of the symmetric cost assumption.

Proof Take any $S \in 2^N \backslash \{\emptyset\}$. Cartel members' and outsiders' optimal quantities are characterized by the first order conditions:

$$\forall i \in S, \quad \frac{\partial}{\partial q_i} \sum_{k \in S} \pi_k(q) = 0 \Longleftrightarrow 2b \sum_{k \in S} q_k^* = a - b \sum_{j \in N \backslash S} q_j,$$

and

$$\forall j \in N \backslash S, \quad \frac{\partial}{\partial q_j} \pi_j(q) = 0 \Longleftrightarrow 2b\tilde{q}_j = a - b \sum_{k \in N \backslash \{j\}} q_k,$$

respectively. Since the inverse demand function is affine and firms operate at the same marginal cost, any Cournot–Nash equilibrium implies that identical parties must choose identical strategies (quantities), i.e., for any $i, k \in S$, $q_i^* = q_k^*$ and for any $j, l \in N \backslash S$, $\tilde{q}_j = \tilde{q}_l$. From this remark, the intersection of the two above reaction functions yields:

$$q_i^* = \frac{a}{sb(n - s + 2)} \quad \text{and} \quad \tilde{q}_j = \frac{a}{b(n - s + 2)},$$

which permits to compute the worth of coalition S as:

$$\begin{aligned} v_\gamma(S) &= \sum_{i \in S} \pi_i(q_i^*, \tilde{q}_j) \\ &= \frac{1}{b}\left(\frac{a}{n - s + 2}\right)^2. \end{aligned}$$

This concludes the proof. □

Proposition 3.1 shows that any Cournot oligopoly TU-game (N, v_γ) is symmetric. The worth $v_\gamma(S)$ of any coalition S is increasing with the intercept of demand a and the size s of coalition S. Moreover, it is decreasing with the slope b and the number of outsiders $n - s$. Furthermore, Lardon (2012) has proved that $(N, v_\gamma) \in \Gamma_c$.

It is now possible to provide the optimal fine imposed by competition authorities in order to deter the grand coalition.

Proposition 3.2 *For any Cournot oligopoly TU-game $(N, v_\gamma) \in \Gamma_c$, it holds that:*

$$\underline{t}(N, v_\gamma) = \begin{cases} \dfrac{1}{b}\left(\dfrac{a(n-1)}{2(n+1)}\right)^2 & \text{if } n \le 5; \\[2ex] \dfrac{a^2}{b}\left(\dfrac{5n - 9}{36(n-1)}\right) & \text{if } n \ge 5. \end{cases}$$

Proof Since the Cournot oligopoly TU-game (N, v_γ) is symmetric, the non-emptiness of the core is characterized by the following condition:

$$\forall S \in 2^N \backslash \{\emptyset\}, \quad \frac{v_\gamma(S)}{s} \le \frac{v_\gamma(N)}{n}.$$

It follows that the optimal penalty $\underline{t}(N, v_\gamma)$ can be computed as:

$$\begin{aligned}\underline{t}(N, v_\gamma) &= v_\gamma(N) - n \max_{S \subset N} \frac{v_\gamma(S)}{s} \\ &= \frac{a^2}{4b} - n \max_{s \in \{1,\ldots,n-1\}} \frac{a^2}{sb(n-s+2)^2}.\end{aligned}$$

It remains to find the size s which minimizes the function $f(s) = s(n-s+2)^2$ defined on $[1; n-1]$. We deduce from $f'(s) = (n-s+2)(n-3s+2)$ and $f''(s) = -4n+6s-8$ that f:

- attains its maximum at point $s^* = (n+2)/3$ where $1 < s^* < n-1$ for any $n \geq 3$;
- is strictly increasing on $[1; s^*]$ and strictly decreasing on $[s^*; n-1]$.

Hence it holds that $\arg\min_{s \in [1,\ldots,n-1]} f(s) \subseteq \{1; n-1\}$. We distinguish two cases:

- if $n = 2$ it trivially holds that f attains its minimum at $s = 1$.
- assume that $n \geq 3$. It follows from $f(1) = (n+1)^2$ and $f(n-1) = 9(n-1)$ that:

$$\arg \min_{s \in [1,\ldots,n-1]} f(s) = \begin{cases} \{1\} & \text{if } 3 < n < 5; \\ \{1; n-1\} & \text{if } n = 5; \\ \{n-1\} & \text{if } n > 5. \end{cases}$$

Thus, when $2 \leq n \leq 5$ it holds that:

$$\begin{aligned}\underline{t}(N, v_\gamma) &= \frac{a^2}{4b} - n \frac{a^2}{b(n+1)^2} \\ &= \frac{1}{b}\left(\frac{a(n-1)}{2(n+1)}\right)^2.\end{aligned}$$

Moreover, when $n \geq 5$ it holds that:

$$\begin{aligned}\underline{t}(N, v_\gamma) &= \frac{a^2}{4b} - n \frac{a^2}{9b(n-1)} \\ &= \frac{a^2}{b}\left(\frac{5n-9}{36(n-1)}\right),\end{aligned}$$

which concludes the proof. □

Proposition 3.2 shows that the optimal fine imposed by competition authorities is increasing with the intercept of demand a and the number of firms n. Moreover, it is decreasing with the slope b. Surprisingly, the expression of the optimal fine leads to distinguish markets of small size ($n \leq 5$) and those of medium and large size ($n \geq 6$) for the deterrence of monopoly power.

We know by Propositions 2.8 and 3.1 that the contraction core of any Cournot oligopoly TU-game $(N, v_\gamma) \in \Gamma_c$ is a singleton. Proposition 3.2 permits to go further by providing an expression of the contraction core.

Corollary 3.3 *For any Cournot oligopoly TU-game $(N, v_\gamma) \in \Gamma_c$, the contraction core is expressed as:*

$$CC(N, v_\gamma) = \begin{cases} \left\{ \frac{1}{b}\left(\frac{a}{n+1}\right)^2 \times e \right\} & \text{if } n \leq 5; \\ \left\{ \left(\frac{a^2}{9b(n-1)}\right) \times e \right\} & \text{if } n \geq 5; \end{cases}$$

where $e = (1, \ldots, 1)$.

This result shows that, regardless of the number of firms, each individual payoff in the contraction core is increasing with the intercept of demand a and decreasing with the slope b and the number of firms n.

4 Axiomatization of the contraction core

In this section, we provide an axiomatic characterization of the contraction core on the set Γ_c.

Let Γ_0 be any arbitrary subset of Γ. A **solution** on Γ_0 is a mapping σ that assigns a (possibly empty) subset $\sigma(N, v) \subseteq X_{\Lambda^*}(N, v)$ to any TU-game (N, v).

4.1 Axioms

We now present the axioms relevant to our analysis. The first two are classic in the literature on core axiomatizations.

Definition 4.1 Non-emptiness (NE) A solution σ on Γ_0 satisfies *NE* if for any $(N, v) \in \Gamma_0$, $\sigma(N, v) \neq \emptyset$.

Definition 4.2 Individual rationality (IR) A solution σ on Γ_0 satisfies *IR* if for any $(N, v) \in \Gamma_0$, every $x \in \sigma(N, v)$, and every $i \in N$, $x_i \geq v(\{i\})$.

Both of these axioms are satisfied by all core extensions discussed in the introduction, and so are useful in characterizing them.

Next, we introduce three versions of reduced games and their corresponding consistency axioms in order to make core comparisons. The first reduced game type makes a special treatment to the grand coalition and permits to characterize the core (Peleg 1986). The **DM-reduced game** (Davis and Maschler 1965) of $(N, v) \in \Gamma$ with respect to $S \subseteq N$ and $x \in \mathbb{R}^N$ is the game $(S, v_{S,x}) \in \Gamma$ defined for any $T \in 2^S$ as:

$$v_{S,x}(T) = \begin{cases} 0 & \text{if } T = \emptyset; \\ v(N) - x(N \backslash S) & \text{if } T = S; \\ \max\{v(T \cup Q) - x(Q) : Q \subseteq N \backslash S\} & \text{otherwise.} \end{cases}$$

Definition 4.3 DM-consistency (DM-CON) A solution σ on Γ_0 satisfies *DM-CON* if for any $(N, v) \in \Gamma_0$, every $S \in 2^N \backslash \{\emptyset\}$ and every $x \in \sigma(N, v)$, then $(S, v_{S,x}) \in \Gamma_0$ and $x^S \in \sigma(S, v_{S,x})$.

The second version is more general and treats all coalitions in the same way and permits to characterize the aspiration core. The **modified DM-reduced game** (Bejan and Gómez 2012) of $(N, v) \in \Gamma$ with respect to $S \subseteq N$ and $x \in \mathbb{R}^N$ is the game $(S, v_*^{S,x}) \in \Gamma$ defined for any $T \in 2^S$ as:

$$v_*^{S,x}(T) = \begin{cases} 0 & \text{if } T = \emptyset; \\ \max\{v(T \cup Q) - x(Q) : Q \subseteq N \backslash S\} & \text{otherwise.} \end{cases}$$

Definition 4.4 MDM-consistency (MDM-CON) A solution σ on Γ_0 satisfies *MDM-CON* if for any $(N, v) \in \Gamma_0$, every $S \in 2^N \backslash \{\emptyset\}$ and every $x \in \sigma(N, v)$, then $(S, v_*^{S,x}) \in \Gamma_0$ and $x^S \in \sigma(S, v_*^{S,x})$.

We can verify that the contraction core does not satisfies *MDM-CON* on Γ_c.

Example 4.5 Consider the TU-game $(N, v) \in \Gamma_c$ such that $N = \{1, 2, 3\}$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1, 2\}) = 4$, $v(\{1, 3\}) = v(\{2, 3\}) = 2$, and $v(\{1, 2, 3\}) = 10$. It holds that $\underline{t}(N, v) = 6$ and $CC(N, v) = C(N, v^{\underline{t}(N,v)}) = \{(2, 2, 0)\}$. When $S = \{1\}$ and $x = (2, 2, 0)$, the modified DM-reduced game is given by $v_*^{\{1\},x}(\{1\}) = v(\{1, 2, 3\}) - 2 - 0 = 8$. Thus, $2 \notin CC(\{1\}, v_*^{\{1\},x}) = \{8\}$ so that the contraction core does not satisfied *MDM-CON*.

The third version which is relevant for our results makes again a special treatment to the grand coalition of any reduced game which is not allowed to cooperate with the complementary coalition. This permits to satisfy the feasibility condition related to second best time allocation. The **new modified DM-reduced game** of $(N, v) \in \Gamma$ with respect to $S \subset N$ and $x \in \mathbb{R}^N$ is the game $(S, v_{S,x}^*) \in \Gamma$ defined for any $T \in 2^S$ as:

$$v_{S,x}^*(T) = \begin{cases} 0 & \text{if } T = \emptyset; \\ \max\{v(T \cup Q) - x(Q) : Q \subset N \backslash S\} & \text{if } T = S; \\ \max\{v(T \cup Q) - x(Q) : Q \subseteq N \backslash S\} & \text{otherwise.} \end{cases}$$

Definition 4.6 NMDM-consistency (NMDM-CON) A solution σ on Γ_0 satisfies *NMDM-CON* if for any $(N, v) \in \Gamma_0$, every $S \in 2^N \backslash \{\emptyset, N\}$ and every $x \in \sigma(N, v)$, then $(S, v_{S,x}^*) \in \Gamma_0$ and $x^S \in \sigma(S, v_{S,x}^*)$.

Observe that the three axioms of consistency defined above satisfy the following logical equality: *DM-CON* $\vee$ *NMDM-CON* = *MDM-CON*. The following weak version of the axiom of consistency will be useful for the axiomatic characterization of the contraction core.

Definition 4.7 Weak-NMDM-consistency (W-NMDM-CON) A solution σ on Γ_0 satisfies *W-NMDM-CON* if for any $(N, v) \in \Gamma_0$, every $S \in 2^N \backslash \{\emptyset, N\}$ such that $s = 1$, and every $x \in \sigma(N, v)$, then $(S, v_{S,x}^*) \in \Gamma_0$ and $x^S \in \sigma(S, v_{S,x}^*)$.

The last axiom differs from the classic superadditivity axiom on the feasibility requirement.

Definition 4.8 Conditional Superadditivity (C-SUPA) A solution σ on Γ_0 satisfies *C-SUPA* if for any (N, v_A),$(N, v_B) \in \Gamma_0$, every $x_A \in \sigma(N, v_A)$ and every $x_B \in \sigma(N, v_B)$, then $x_A + x_B \in \sigma(N, v_A + v_B)$ whenever $(N, v_A + v_B) \in \Gamma_0$ and $x_A + x_B$ is feasible for $(N, v_A + v_B)$, i.e., $x_A + x_B \in X_{\Lambda^*}(N, v_A + v_B)$.

While the feasibility requirement related to first best time allocation is redundant on the set Γ_c, ours is not trivially satisfied since the grand coalition is deterred.

4.2 Axiomatization

Before characterizing the contraction core, we first provide the following lemma.

Lemma 4.9 *Take any $\mathcal{B} \in \Lambda^*(N)$ where $n \geq 2$ with balanced weights $(\delta_H)_{H \in \mathcal{B}}$. For any $S \in 2^N \backslash \{\emptyset, N\}$ where $s \geq 2$ such that there exists $H \in \mathcal{B}$ satisfying $H \cap S \neq \emptyset$ and $H \cap S \subset S$, we define:*

$$\mathcal{B}^S = \{T \subset S : T = H \cap S \neq \emptyset \text{ for some } H \in \mathcal{B}\},$$

and for every $T \in \mathcal{B}^S$:

$$\hat{\delta}_T = \left(1 - \sum_{\substack{H \in \mathcal{B}: \\ H \cap S = S}} \delta_H\right)^{-1} \sum_{\substack{H \in \mathcal{B}: \\ T = H \cap S}} \delta_H.$$

Then, $\mathcal{B}^S \in \Lambda^*(S)$ with balanced weight $(\hat{\delta}_T)_{T \in \mathcal{B}^S}$.

Proof First, it is straightforward to see that $\mathcal{B}^S$ is well-defined. Furthermore, suppose for the sake of contradiction that $\sum_{H \in \mathcal{B}: H \cap S = S} \delta_H = 1$. Then, it follows from $\sum_{H \in \mathcal{B}: i \in H} \delta_H = 1$ for every $i \in S$ that there does not exists $H \in \mathcal{B}$ such that $H \cap S \neq \emptyset$ and $H \cap S \subset S$, a contradiction. Hence, we conclude that $\sum_{H \in \mathcal{B}: H \cap S = S} \delta_H < 1$. Second, for each $i \in S$ it holds that:

$$\begin{aligned}
\left(1 - \sum_{\substack{H \in \mathcal{B}: \\ H \cap S = S}} \delta_H\right) \sum_{\substack{T \in \mathcal{B}^S: \\ i \in T}} \hat{\delta}_T &= \sum_{\substack{T \in \mathcal{B}^S: \\ i \in T}} \sum_{\substack{H \in \mathcal{B}: \\ T = H \cap S}} \delta_H \\
&= \sum_{\substack{H \in \mathcal{B}: \\ H \cap S \subset S \\ i \in H}} \delta_H \\
&= \sum_{\substack{H \in \mathcal{B}: \\ H \cap S \subseteq S \\ i \in H}} \delta_H - \sum_{\substack{H \in \mathcal{B}: \\ H \cap S = S \\ i \in H}} \delta_H
\end{aligned}$$

$$= \sum_{\substack{H\in\mathcal{B}:\\ i\in H}} \delta_H - \sum_{\substack{H\in\mathcal{B}:\\ H\cap S=S}} \delta_H$$
$$= 1 - \sum_{\substack{H\in\mathcal{B}:\\ H\cap S=S}} \delta_H,$$

which concludes the proof. □

Given any second best time allocation in a TU-game $(N, v) \in \Gamma_c$, Lemma 4.9 provides a formula to compute the corresponding second best time allocation in any of its reduced games $(S, v^*_{S,x})$ for which the collection $\mathcal{B}^S$ is well-defined. Furthermore, it leads to the following result.

Proposition 4.10 *The contraction core satisfies NMDM-CON for any TU-game* $(N, v) \in \Gamma_c$ *where for every* $x \in CC(N, v)$*, it holds that* $x(N) = \sum_{H\in\mathcal{B}} \delta_H v(H)$ *for some* $\mathcal{B} \in \Lambda^*(N)$ *such that for all* $S \in 2^N\backslash\{\emptyset, N\}$ *where* $s \geq 2$*, the collection* $\mathcal{B}^S$ *given in Lemma 4.9 is well-defined.*

Proof Let $(N, v) \in \Gamma_c$ satisfying all the conditions in Proposition 4.10, $S \in 2^N\backslash\{\emptyset, N\}$ and $x \in CC(N, v)$. We distinguish two cases:

– assume that $s \geq 2$. Take $\mathcal{B} \in \Lambda^*(N)$ with balanced weights $(\delta_H)_{H\in\mathcal{B}}$ such that $x(N) = \sum_{H\in\mathcal{B}} \delta_H v(H)$. Then, by Lemma 4.9 it holds that $\mathcal{B}^S \in \Lambda^*(S)$ with balanced weight $(\hat{\delta}_T)_{T\in\mathcal{B}^S}$.
  Now, we prove that $x(T) \leq v^*_{S,x}(T)$ for each $T \in \mathcal{B}^S$. Given $T \in \mathcal{B}^S$, there exists $H \in \mathcal{B}$ such that $T = H \cap S$. From $x(N) = \sum_{H\in\mathcal{B}} \delta_H v(H)$ and $x(S) \geq v(S)$ for each $S \in 2^N\backslash\{\emptyset, N\}$, it holds that $x(H) = v(H)$, hence $x(T) = v(H) - x(H\backslash T)$. Since $H\backslash T \subseteq N\backslash S$, it holds that $x(T) \leq \max\{v(T \cup Q) - x(Q) : Q \subseteq N\backslash S\} = v^*_{S,x}(T)$.
  Then, we prove that $x(T) \geq v^*_{S,x}(T)$ for each $T \in 2^S\backslash\{\emptyset, S\}$. By contradiction, assume that there exists $T \in 2^S\backslash\{\emptyset, S\}$ such that $x(T) < v^*_{S,x}(T)$. Hence there exists $y^T \in \mathbb{R}^T$ such that $y(T) = v^*_{S,x}(T)$ and $y(T) > x(T)$. Thus, it holds that $y(T) = v(T \cup Q) - x(Q)$ for some $Q \subseteq N\backslash S$. Hence, $y(T) + x(Q) = v(T \cup Q)$ and so, $x(T) + x(Q) < v(T \cup Q)$, a contradiction with $x \in CC(N, v)$ since $T \cup Q \subset N$. We conclude that $x(T) \geq v^*_{S,x}(T)$ for each $T \in 2^S\backslash\{\emptyset, S\}$.
  Thus, $x(T) = v_{S,x}(T)$ for each $T \in \mathcal{B}^S$, and so $x(S) = \sum_{T\in\mathcal{B}^S} \hat{\delta}_T x(T) = \sum_{T\in\mathcal{B}^S} \hat{\delta}_T v^*_{S,x}(T)$. Moreover, $x(T) \geq v^*_{S,x}(T)$ for each $T \in 2^S\backslash\{\emptyset, S\}$ implies that $x^S \in CC(S, v^*_{S,x})$.
– assume that $s = 1$. Take $\mathcal{B} \in \Lambda^*(N)$ with balanced weights $(\delta_H)_{H\in\mathcal{B}}$ such that $x(N) = \sum_{H\in\mathcal{B}} \delta_H v(H)$. Now, we prove that $x^S \leq v^*_{S,x}(S)$. Given $S = \{i\}$, there exists $H \in \mathcal{B}$ such that $i \in H$. From $x(N) = \sum_{H\in\mathcal{B}} \delta_H v(H)$ and $x(S) \geq v(S)$ for each $S \in 2^N\backslash\{\emptyset, N\}$, it holds that $x(H) = v(H)$, hence $x(S) = v(H) - x(H\backslash S)$. Since $H\backslash S \subset N\backslash S$, it holds that $x(S) \leq \max\{v(S \cup Q) - x(Q) : Q \subset N\backslash S\} = v^*_{S,x}(S)$.
  Then, we prove that $x^S \geq v^*_{S,x}(S)$. By contradiction, assume that $x^S < v^*_{S,x}(S)$. Hence, there is $y^S \in \mathbb{R}$ such that $y^S = v^*_{S,x}(S)$ and $y^S > x^S$. Thus, it holds that

$y^S = v^*_{S,x}(S) = v(S \cup Q) - x(Q)$ for some $Q \subset N \setminus S$. Hence, $y(S) + x(Q) = v(S \cup Q)$ and so, $x(S) + x(Q) < v(S \cup Q)$, a contradiction with $x \in CC(N, v)$ since $S \cup Q \subset N$. We conclude that $x^S \in CC(S, v^*_{S,x})$. □

The following example shows that the contraction core does not satisfy *NMDM-CON* on the set Γ_c.

Example 4.11 Consider the TU-game $(N, v) \in \Gamma_c$ such that $N = \{1, 2, 3\}$, $v(\{1\}) = v(\{2\}) = 0$, $v(\{3\}) = 2$, $v(\{1, 2\}) = 6$, $v(\{1, 3\}) = v(\{2, 3\}) = 2$, and $v(\{1, 2, 3\}) = 10$. The second best time allocation is given by $\mathcal{B} = \{\{1, 2\}, \{3\}\}$. Note that for $S = \{1, 2\}$, Lemma 4.9 cannot be applied since $\mathcal{B}^S$ is not well-defined. Then, we consider payoff vector $x = (3, 3, 2) \in CC(N, v)$. The new modified DM-reduced game of (N, v) with respect to S and x is given by $v^*_{\{1,2\},x}(\{1\}) = v^*_{\{1,2\},x}(\{2\}) = 0$ and $v^*_{\{1,2\},x}(\{1, 2\}) = 6$. Hence $x^{\{1,2\}} = (3, 3) \notin CC(\{1, 2\}, v^*_{\{1,2\},x})$ so that the contraction core does not satisfy *NMDM-CON*. Moreover, we can verify that the contraction core satisfies *W-NMDM-CON*.

Proposition 4.12 *The contraction core satisfies NE, IR, W-NMDM-CON and C-SUPA on the set* Γ_c.

Proof It is straightforward to verify that *NE*, *IR* and *C-SUPA* are satisfied. It follows from the second part of the proof of Proposition 4.10 that the contraction core satisfies *W-NMDM-CON*. □

Proposition 4.13 *Let* σ *be a solution concept on* $\Gamma_0 \subseteq \Gamma$ *satisfying IR and W-NMDM-CON. If* $(N, v) \in \Gamma_0$ *and* $x \in \sigma(N, v)$ *then* $x(S) \geq v(S)$ *for any* $S \in 2^N \setminus \{\emptyset, N\}$.

Proof Let σ be a solution concept on $\Gamma_0 \subseteq \Gamma$ satisfying *IR* and *W-NMDM-CON*. Let $x \in \sigma(N, v)$, $S \in 2^N \setminus \{\emptyset, N\}$ and $i \in S$. By *W-NMDM-CON*, $x_i \in \sigma(\{i\}, v^*_{\{i\},x})$. By *IR*, it holds that:

$$\begin{aligned} x_i &\geq v^*_{\{i\},x}(\{i\}) \\ &= \max\{v(\{i\} \cup Q) - x(Q) : Q \subset N \setminus \{i\}\} \\ &\geq v(S) - x(S \setminus \{i\}), \end{aligned}$$

which proves that $x(S) \geq v(S)$ as desired. □

Proposition 4.14 *If* σ *is a solution concept defined on* $\Gamma_0 \subseteq \Gamma_c$ *that satisfies IR and W-NMDM-CON, then for any* $(N, v) \in \Gamma_0$*, any payoff vector* $x \in \sigma(N, v)$ *is efficient, i.e.,* $x(N) = \max_{\mathcal{B} \in \Lambda^*(N)} \sum_{S \in \mathcal{B}} \delta_S v(S)$ *(or* $x \in X^*_{\Lambda^*}(N, v)$*).*

Proof Let σ be a solution concept on $\Gamma_0 \subseteq \Gamma_c$ satisfying *IR* and *NMDM-CON*. Assume that $(N, v) \in \Gamma_0$ and take any $x \in \sigma(N, v)$ and any $y \in X_{\Lambda^*}(N, v)$. Then, there is $\mathcal{B} \in \Lambda^*(N)$ such that $y(N) \leq \sum_{S \in \mathcal{B}} \delta_S v(S)$. It follows from $\mathcal{B} \in \Lambda^*(N)$ and Proposition 4.13 that:

$$\begin{aligned} x(N) &= \sum_{S \in \mathcal{B}} \delta_S x(S) \\ &\geq \sum_{S \in \mathcal{B}} \delta_S v(S) \\ &\geq y(N). \end{aligned}$$

We conclude that $x \in X^*_{\Lambda^*}(N, v)$. □

Proposition 4.15 *If σ is a solution concept defined on Γ_c satisfying IR and W-NMDM-CON, then $\sigma(N, v) \subseteq CC(N, v)$ for any $(N, v) \in \Gamma_c$.*

Proof Take any $x \in \sigma(N, v)$. By Proposition 4.13, it holds that $x(S) \geq v(S)$ for every $S \in 2^N \backslash \{\emptyset, N\}$. Moreover, by Proposition 4.14, $x \in X^*_{\Lambda^*}(N, v)$. So, $x \in CC(N, v)$. □

Proposition 4.16 *If a solution concept σ defined on Γ_c satisfies NE, IR, W-NMDM-CON and C-SUPA, then $CC(N, v) \subseteq \sigma(N, v)$ for any $(N, v) \in \Gamma_c$.*

Proof [10] Let $x \in CC(N, v)$ and define $(N, w) \in \Gamma_c$ as:

$$w(S) = \begin{cases} x(S) & \text{if } |S| \geq 2; \\ v(S) & \text{if } |S| = 1. \end{cases}$$

It holds that $C(N, w) = \{x\}$. By Proposition 4.15, $\sigma(N, w) \subseteq CC(N, w) = C(N, w) = \{x\}$. By *NE*, it holds that $x \in \sigma(N, w)$.
Consider the game $(N, z) \in \Gamma_c$ defined as:

$$\forall S \in 2^N, z(S) = v(S) - w(S).$$

Hence, $z(S) \leq 0$ if $2 \leq |S| < n$, $z(\{i\}) = 0$ for every $i \in N$ and $z(N) = \underline{t}(N, v) \geq 0$ since $(N, v) \in \Gamma_c$ and $x \in CC(N, v)$. Note that $\mathbf{0} \in CC(N, z)$ since $0 = \max_{\mathcal{B} \in \Lambda^*(N)} \sum_{S \in \mathcal{B}} \delta_S z(S) = \sum_{i \in N} z(\{i\})$. By Proposition 4.14, for every $y \in CC(N, z)$ it holds that $y(N) = 0$. Since $z(\{i\}) = 0$ for every $i \in N$, we have $y_i \geq 0$ by *IR* and so, $y = \mathbf{0}$. Thus, $CC(N, z) = \{\mathbf{0}\}$. By Proposition 4.15, it holds that $\sigma(N, z) \subseteq CC(N, z) = \{\mathbf{0}\}$. By *NE*, $\mathbf{0} \in \sigma(N, z)$.
Note that $x(N) + 0 = \max_{\mathcal{B} \in \Lambda^*(N)} \sum_{S \in \mathcal{B}} \delta_S v(S) = \max_{\mathcal{B} \in \Lambda^*(N)} \sum_{S \in \mathcal{B}} \delta_S (w+z)(S)$. So, $x + \mathbf{0} \in X_{\Lambda^*}(N, w + z)$, i.e., $x + \mathbf{0}$ is feasible for $(N, w + z)$. Thus, by *C-SUPA* it follows from $x \in \sigma(N, w)$ and $\mathbf{0} \in \sigma(N, z)$ that $x + \mathbf{0} \in \sigma(N, w + z)$, hence $x \in \sigma(N, v)$. □

Theorem 4.17 *The contraction core is the only solution concept on Γ_c that satisfies NE, IR, W-NMDM-CON and C-SUPA.*

Proof Combine Propositions 4.12, 4.15 and 4.16. □

4.3 Independence of the axioms

The following examples show that the axioms used in the characterization of the contraction core are logically independent on the set Γ_c, i.e., none is implied by the others.

[10] Our proof is inspired from that in Peleg and Sudhölter (2003) in the case where $n \geq 3$. Nevertheless, the main difference is that we do not need to distinguish cases $n = 2$ and $n \geq 3$. Furthermore, while the weaker version of their axiom of consistency is established for $s \in \{1, 2\}$, our weaker version only requires that $s = 1$.

Example 4.18 Consider the solution concept σ_1 on Γ_c such that for any $(N, v) \in \Gamma_c$, $\sigma_1(N, v) = \emptyset$. Obviously, σ_1 violates *NE* but vacuously satisfies *IR*, *W-NMDM-CON* and *C-SUPA*.

Example 4.19 Consider the solution concept σ_2 on Γ_c such that for any $(N, v) \in \Gamma_c$, $\sigma_2(N, v) = CC(N, v)$ if $n \geq 2$ and $\sigma_2(N, v) = X_{\Lambda^*}(N, v)$ if $n = 1$. It is clear that σ_2 satisfies *NE*. It follows from Proposition 4.12 that *W-NMDM-CON* and *C-SUPA* are also satisfied. On one-person games σ_2 violates *IR*.

Example 4.20 Consider the solution concept σ_3 on Γ_c such that for any $(N, v) \in \Gamma_c$, $\sigma_3(N, v) = \{x \in X_{\Lambda^*}(N, v) : x_i \geq v(\{i\})\}$. Clearly, σ_3 satisfies *NE*, *IR* and *C-SUPA*. Proposition 4.15 implies that σ_3 does not satisfy *W-NMDM-CON*.

Example 4.21 For every $(N, v) \in \Gamma_c$, every $S \in 2^N \backslash \{\emptyset\}$ and every $x \in \mathbb{R}^N$, the **excess** of S from x in (N, v) is given by the quantity $e(S, x, v) = v(S) - x(S)$. The excess $e(S, x, v)$ gives the amount of dissatisfaction of coalition S from x in (N, v). We define the vector $\theta(x) = (\theta_1(x), \dots, \theta_{2^n-1}(x))$ whose components are the numbers $(e(S, x, v))_{S \in 2^N \backslash \{\emptyset\}}$ arranged in non-increasing order. For any TU-game $(N, v) \in \Gamma_c$, the **contraction nucleolus**, denoted by $CN(N, v)$, is defined as:

$$CN(N, v) = \left\{x \in X^*_{\Lambda^*}(N, v) : \theta(y) \geq_L \theta(x) \text{ for all } y \in X^*_{\Lambda^*}(N, v)\right\},$$

where $\geq_L$ is the lexicographical ordering. First, since $X^*_{\Lambda^*}(N, v)$ is non-empty, compact and convex, it follows from corollary 5.1.10 in Peleg and Sudhölter (2003) that $CN(N, v)$ consists of a single point. Hence, the contraction nucleolus satisfies *NE*. Second, the contraction nucleolus also satisfies *IR* since it is a subset of the contraction core. Finally, the contraction nucleolus complies with *W-NMDM-CON* since it coincides with the contraction core for every one-person game. Hence, it follows from our axiomatization that the contraction nucleolus does not satisfy *C-SUPA*.

5 Concluding remarks: extended contraction core

We have introduced a new solution concept, the contraction core, that serves as a basis for the investigation of the deterrence of cooperation. This solution concept has permitted to provide a measure of the robustness of cooperation. We have successfully applied the contraction core to oligopolistic markets and we have provided optimal fine imposed by competition authorities for cartel deterrence. We can be convinced that there are many other potential applications of the contraction core.
More generally, we have also provided an axiomatic characterization of the contraction core in order to better understand it. In particular, this has permitted to make comparisons with the core, the aspiration core and the weak least core. We have defined the contraction core on the set of balanced TU-games in order to be consistent with our objective to study the deterrence of cooperation. We argue that it is possible to define an "extended" contraction core by applying the feasibility condition (Definition 2.1) on the set of all TU-games. The "extended" contraction core is then non-empty on the set of all TU-games and coincides with the aspiration core on the set of non-balanced

TU-games. In this case, the axiomatic characterization of the "extended" contraction core can be established in Propositions 4.12, 4.14, 4.15, 4.16 and Theorem 4.17 which hold on the set of all TU-games.

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