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Consistency and the core in games with externalities

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Abstract In the presence of externalities across coalitions, Dutta et al. (J Econ Theory 145:2380–2411, 2010) characterize their value by extending Hart and Mas-Colell reduced game consistency. In the present paper, we provide a characterization result for the core for games with externalities by extending one form of consistency studied by Moulin (J Econ Theory 36:120–148, 1985), which is often referred to as the complement-reduced game property. Moreover, we analyze another consistency formulated by Davis and Maschler (Naval Res Logist Quart 12:223–259, 1965), called the max-reduced game property and a final consistency called the projection-reduced game property. In environments with externalities, we discuss some asymmetric results among these different forms of reduced games.

Keywords Consistency · Core · Games with externalities · Reduced game

1 Introduction

Cooperative game theory is one of the most basic frameworks to analyze coalition formation and to study how we allocate the surplus obtained from the coalition. Many of the traditional models of cooperative game theory consider the worth of a coalition as the surplus obtained by the members of the coalition with no help from the other

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players. This simplification provides a wide variety of sophisticated ideas and insights on allocations such as the Shapley value and the core. Recent works, however, attempt to understand environments in which there is mutual influence among coalitions. In these works, such mutual influence is commonly called *externalities* among coalitions. By using the concept of externalities, we can divide the general field of cooperative games into two classes: games with externalities and games without externalities. Games without externalities, or traditional models, are often referred to as *coalition function form games*, whereas games with externalities are called *partition function form games*.

In the presence of externalities, the allocation of surplus becomes more complicated. Myerson (1977), Bolger (1989), Macho-Stadler et al. (2007) and Albizuri et al. (2005) propose the allocation rules by generalizing the Shapley value to games with externalities. Moreover, Dutta et al. (2010) characterize their value by extending Hart and Mas-Colell consistency to games with externalities.

In contrast to the remarkable progress made in studies on values, there are relatively few works on the core for environments with externalities. One possible reason for this is that a number of types of cores can be defined in the presence of externalities: the definition of the core depends on the "anticipation" of deviating players because of externalities. For example, if some agents who are about to deviate from their original affiliation anticipate the worst reaction from the remaining agents (minimizing the surplus of the deviating agents), the deviating agents may have less incentive to carry out the deviation. The idea of this stability appears in Bloch (1996) and is called the *pessimistic core* in Kóczy (2007), which is closely associated with the concept known as the α-core introduced by Hart and Kurz (1983). Analogous to the pessimistic core, in the presence of externalities, the definition of each core depends on the anticipation for the reaction of the remaining players. Bloch and van den Nouweland (2014) formulate such anticipations for reactions as *expectation functions* and give axiomatic characterizations to them. However, the axiomatic characterizations for the cores have been left open.

In this paper, we provide characterization results for the cores of games with externalities by using some forms of reduced game consistencies. We show that if an expectation function satisfies a certain condition, then we can axiomatize the core based on the expectation function with some axioms. Instead of Hart and Mas-Colell consistency employed by Dutta et al. (2010), we use the other forms of reduced game consistencies: Complement, Max and Projection consistencies. The objective of this paper is to describe what relationships exist between the cores and the consistencies in the presence of externalities. Our result is summarized in Proposition 2 and Table 1.

The remainder of the paper is organized as follows. The next section is devoted to the basic definitions and notations. In Sect. 3, we describe the axioms and offer the axiomatization result. We discuss the differences among some forms of reduced games in Sect. 4. Section 5 concludes this paper with some further remarks.

2 Preliminaries

2.1 Games with externalities

Let $\mathcal{N}$ be a set of all players. We consider a finite player set $N \subsetneq \mathcal{N}$. A *coalition* S is a subset of N. We denote by $|S|$ the number of players in S. For any $S \subseteq N$, a *partition* of S is defined by $\{T_1, \ldots, T_h\}$ where $1 \leq h \leq |S|$, $T_i \cap T_j = \emptyset$ for $i, j = 1, \ldots, h$ $(i \neq j)$, $T_i \neq \emptyset$ for $i = 1, \ldots, h$ and $\bigcup_{i=1}^{h} T_i = S$. We will typically use $\mathcal{P}$ or $\mathcal{Q}$ to denote a partition. Assume that the partition of the empty set $\emptyset$ is $\{\emptyset\}$. For any $S \subseteq N$, let $\Pi(S)$ be the set of all partitions of S. We define an *embedded coalition* of N by $(S, \mathcal{P})$ satisfying $\mathcal{P} \in \Pi(N \backslash S)$. The set of all embedded coalitions of N is given by

$$EC(N) = \{(S, \mathcal{P}) \mid \emptyset \neq S \subseteq N \text{ and } \mathcal{P} \in \Pi(N \backslash S)\}.$$

A *partition function form game* is a pair (N, v), where a *partition function* v is a function that assigns a real number to each embedded coalition, namely, $v : EC(N) \to \mathbb{R}$. Let Γ_A be the set of all partition function form games: $\Gamma_A = \{(N, v) \mid \emptyset \neq N \subseteq \mathcal{N},\ |N| < \infty,\ v : EC(N) \to \mathbb{R}\}$. For any game, we restrict payoff vectors to the following set: $F(N, v) = \left\{x \in \mathbb{R}^N \;\middle|\; \sum_{j \in N} x_j \leq v(N, \{\emptyset\})\right\}$. For a set of games $\Gamma \subseteq \Gamma_A$, a *solution* on Γ is a function σ that associates a subset $\sigma(N, v)$ of $F(N, v)$ with every game $(N, v) \in \Gamma$.

We denote by x_S a *restriction* of $x \in \mathbb{R}^N$ on coalition S, *i.e.*, $x_S = (x_j)_{j \in S} \in \mathbb{R}^S$. To keep our notation simple, for any coalition S and player i, we typically use $S \cup i$ or $S \backslash i$ to denote $S \cup \{i\}$ or $S \backslash \{i\}$.

2.2 The reduced game

In games without externalities, several forms of reduced games are proposed, in many of which remaining players are not influenced by orders of leaving players. However, in the presence of externalities, we may need to consider the possibility that not only an order of leaving players but a partition of leaving players also influences remaining players.

We first focus our attention on the reduced game known as the complement-reduced game. One of the crucial benefits of the complement-reduced game is, as we will elaborate in Lemma 1 and in Sect. 5, that it depends neither on the order of leaving players nor on the partition of leaving players even in the presence of externalities. The complement-reduced game might be thought of as the simplest form of reduced game in the aspect of externalities. Sections 2 and 3 are dedicated to the complement-reduced game. The other types of reduced games are discussed in Sect. 4.

Now, consider $\Gamma \subseteq \Gamma_A$ and $(N, v) \in \Gamma$. Let $S \subseteq N$ $(S \neq \emptyset)$ and $x \in \mathbb{R}^N$.

Definition 1 The *complement-reduced game with respect to S and x* is the game $(S, v^{S,x})$ defined as follows: for any $T \subseteq S$ $(T \neq \emptyset)$ and any $\mathcal{Q} \in \Pi(S \backslash T)$,

$$v^{S,x}(T, \mathcal{Q}) = v(T \cup (N \backslash S), \mathcal{Q}) - \sum_{j \in N \backslash S} x_j.$$

The complement-reduced game describes that a coalition T always obtains the help of all leaving players $N \backslash S$ by paying $(x_j)_{j \in N \backslash S}$ for them. The complement-reduced game was initially introduced by Moulin (1985) for games without externalities. Definition 1 is the simple extension of the original definition to games with externalities.

In the presence of externalities, the order of leaving players is a crucial point of the reduced game. As we mentioned above, the dependence on the order means that different orders result in different worths of each remaining coalition. Below, we show that the complement-reduced game depends neither on the order of leaving players nor on the partition of leaving players. For notational simplicity, let $v^{-i} := v^{N \backslash i, x}$, *i.e.*, v^{-i} means the complement-reduced game after removing i from the original game. Similarly, we use the following notation:

$$(v^{-i_1})^{-i_2} := (v^{N \backslash i_1, x})^{(N \backslash i_1) \backslash i_2, x_{N \backslash i_1}}.$$

Lemma 1 *For any $x \in \mathbb{R}^N$ and any $i_1, i_2 \in N$ $(i_1 \neq i_2)$,*

$$\begin{aligned}(v^{-i_1})^{-i_2} &= (v^{-i_2})^{-i_1} \\ &= v^{N \backslash \{i_1, i_2\}, x}.\end{aligned}$$

Proof For any $T \subseteq N \backslash i_1$ and any $\mathcal{Q} \in \Pi(N \backslash (T \cup i_1))$, we have

$$v^{-i_1}(T, \mathcal{Q}) = v(T \cup i_1, \mathcal{Q}) - x_{i_1}.$$

For any $T' \subseteq N \backslash \{i_1, i_2\}$ and $\mathcal{Q}' \in \Pi(N \backslash (T \cup \{i_1, i_2\}))$,

$$\begin{aligned}(v^{-i_1})^{-i_2}(T', \mathcal{Q}') &= v^{-i_1}(T' \cup i_2, \mathcal{Q}') - x_{i_2} \\ &= v(T' \cup \{i_1, i_2\}, \mathcal{Q}') - x_{i_1} - x_{i_2}.\end{aligned} \tag{1}$$

Similarly, we remove them in the order of i_2, i_1 and obtain the same game as (1).

Next, assume that players i_1 and i_2 simultaneously leave the game. For any $T' \subseteq N \backslash \{i_1, i_2\}$ and any $\mathcal{Q}' \in \Pi(N \backslash (T \cup \{i_1, i_2\}))$, we have

$$v^{N \backslash \{i_1, i_2\}, x}(T', \mathcal{Q}') = v(T' \cup \{i_1, i_2\}, \mathcal{Q}') - x_{i_1} - x_{i_2},$$

which is the same as (1). □

Lemma 1 shows that the following two properties hold even in the presence of externalities: (i) the complement-reduced game is independent of the order of leaving players; (ii) the game obtained by removing players one by one is equivalent to the game obtained by removing players simultaneously. The independence of the order of leaving players has some relation to the two different *path independences* argued by Dutta et al. (2010) and Bloch and van den Nouweland (2014). This will be elaborated in Sect. 5.1.

Definition 1 shows that we can ignore this influence in the complement-reduced game, as all leaving players $N\backslash S$ help the remaining players T and form a coalition $T \cup (N\backslash S)$. This property is unique to the complement-reduced game and not true for the other types of reduced games. This difference will be expanded upon in Sect. 4.

It is straightforward to extend Lemma 1. Consider $T = \{i_1, \ldots, i_t\} \subsetneq N$. For any permutations π, π' of $T = \{i_1, \ldots, i_t\}$, by repeating Lemma 1, we have

$$v^{\pi} = v^{\pi'} = v^{N\backslash T, x},$$

where, for any permutation π'', $v^{\pi''} = (\ldots((v^{-\pi''_1})^{-\pi''_2})\ldots)^{-\pi''_t}$. Player π''_k means the k-th player leaving the game. Hence, we obtain useful notation as follows:

$$v^{-T} := v^{\pi} = v^{\pi'} = v^{N\backslash T, x}.$$

2.3 Expectation functions

To define the core of games with externalities, we introduce the notion of expectation function formulated by Bloch and van den Nouweland (2014). As noted in Sect. 1, there are various definitions of a core in the presence of externalities. This diversity can be represented by different expectation functions.

Definition 2 An *expectation function* is a mapping ψ associating a partition $\mathcal{P}$ such that $\mathcal{P} \in \Pi(N\backslash S)$, with player set N, partition function v and nonempty coalition $S \subseteq N$, formally,

$$\psi(N, v, S) \in \{\mathcal{P}' | \mathcal{P}' \in \Pi(N\backslash S)\}.$$

An expectation function is related to *restriction operators* introduced by Dutta et al. (2010). A restriction operator describes which player moves to which coalition. This will be elaborated in Sect. 5.1.

We introduce four basic expectation functions. An expectation function is:

– the *optimistic* expectation if

$$\psi(N, v, S) \in \underset{\mathcal{P}' \in \Pi(N\backslash S)}{\arg\max}\; v(S, \mathcal{P}').$$

– the *pessimistic* expectation if

$$\psi(N, v, S) \in \underset{\mathcal{P}' \in \Pi(N\backslash S)}{\arg\min}\; v(S, \mathcal{P}').$$

– the *singleton*-expectation if

$$\psi(N, v, S) = \{\{i_{s+1}\}, \ldots, \{i_n\}\}.$$

– the *merge*-expectation if

$$\psi(N, v, S) = \{N\backslash S\}.$$

For the optimistic (pessimistic) expectation function, any expectation function ψ satisfying the condition is optimistic (pessimistic). While each of these four expectation functions simply assigns a partition of $N\backslash S$ to every coalition S, there are alternative behavioral expectation functions. We delegate this topic to Sect. 5.4 and now focus on the following condition of expectation functions.

Bloch and van den Nouweland (2014) introduce *subset consistency* for expectation functions. We change the original definition slightly to suit our framework as follows.

Definition 3 Let Γ be a set of games and $(N, v) \in \Gamma$. An expectation function ψ satisfies *subset consistency* if for any $S \subseteq N$ and any $T \subseteq S$,

$$\psi(N, v, S) = \psi(N, v, T)|_{(N\backslash S)},$$

where $\psi(N, v, T)|_{(N\backslash S)}$ is a partition of $N\backslash S$, the elements of which are the same as $\psi(N, v, T)$.[1]

Subset consistency describes that for a given N, all players within $S \subseteq N$ share the expectation on the behavior of outside players $N\backslash S$. However, as we illustrate in the following example, subset consistency does not satisfy consistency across player sets in the following sense.

Example 1 Let $N = \{1, 2, 3, 4\}$. For any $N' \subseteq N$ and any $S \subseteq N'$

$$\psi(N', v, S) = \begin{cases} \{N'\backslash S\}, & \text{if } N' = N, \\ \{\{i_{s+1}\}, \ldots, \{i_{n'}\}\}, & \text{if } N' \subsetneq N. \end{cases} \tag{2}$$

Namely, if a player (sub)set N' is equal to N, then (2) expects a single coalition of $N'\backslash S$. If not, then it expects a partition of $N'\backslash S$ into singletons. This expectation function satisfies subset consistency because with respect to each $N' \subseteq N$, ψ is the merge- or the singleton-expectations within N'. On the other hand, it is not consistent across player sets (between N and, for example, $N\backslash 4$) as follows:

$$\psi(N\backslash 4, v^{-4}, \{1\}) = \{\{2\}, \{3\}\} \neq \{2, 3\} = \{2, 3, 4\}|_{N\backslash 4} = \psi(N, v, \{1\})|_{N\backslash 4},$$

for any v and x.

Now, our question is: what condition guarantees the consistency across the player sets? To answer this question, we introduce a new property for expectation functions. As we will mention later, this condition is a sufficient condition for the ψ-core to satisfy the complement-reduced game property.

Definition 4 Let Γ be a set of games and $(N, v) \in \Gamma$. An expectation function ψ is *complement-consistent (CC)* if for any $S \subseteq N$ $(|S| \geq 2)$, any $h \in S$, and any $x \in \mathbb{R}^N$,

$$\psi(N, v, S) = \psi(N\backslash h, v^{N\backslash h, x}, S\backslash h).$$

[1] Formally, for any partition $\mathcal{P}$ and coalition $S \subseteq N$, let $\mathcal{P}|_S$ be given by $\mathcal{P}|_S = \{S \cap C \mid C \in \mathcal{P},\ S \cap C \neq \emptyset\} \in \Pi(S)$.

Complement-consistency (CC) requires that coalition S's expectation is equal to coalition $S\backslash h$' s expectation. Note that not only $\psi(N, v, S)$ but also $\psi(N\backslash h, v^{N\backslash h,x}, S\backslash h)$ is a partition of $N\backslash S$.

In Definition 4, we define CC by removing one player. We next consider a slight variant of CC. We call it *strong complement-consistency*, $\widehat{CC}$, and define it as follows: an expectation function ψ is $\widehat{CC}$ if for any $S \subseteq N$ and $T \subsetneq S$ $(T \neq \emptyset)$,

$$\psi(N, v, S) = \psi(N\backslash T, v^{-T}, S\backslash T).$$

Namely, if ψ is $\widehat{CC}$, then we have

$$v(S, \psi(N, v, S)) = v(S, \psi(N\backslash T, v^{-T}, S\backslash T)). \tag{3}$$

The following proposition shows that CC is equivalent to $\widehat{CC}$.

Lemma 2

$$CC \iff \widehat{CC}$$

Proof It is clear that $\widehat{CC} \Rightarrow CC$ holds. We show that $CC \Rightarrow \widehat{CC}$. Let $S \subseteq N$ and $T \subsetneq S$ with $T \neq \emptyset$. Define $T = \{h_1, \ldots, h_t\}$. By CC, we have

$$\begin{aligned}
\psi(N, v, S) &= \psi(N\backslash h_1, v^{-h_1}, S\backslash h_1) \\
&= \psi(N\backslash \{h_1, h_2\}, (v^{-h_1})^{-h_2}, S\backslash \{h_1, h_2\}) \\
&\ldots \\
&= \psi(N\backslash T, (\ldots(v^{-h_1})\ldots)^{-h_t}, S\backslash T) \\
&= \psi(N\backslash T, v^{-T}, S\backslash T),
\end{aligned}$$

where the last equality holds by Lemma 1. □

The four expectation functions listed above (the optimistic, pessimistic, merge- and singleton-expectation functions) are all CC. For the proof, see Proposition 3 and Corollary 2 in the Appendix. The relationship between consistency concepts and expectation functions is summarized in Fig. 1.

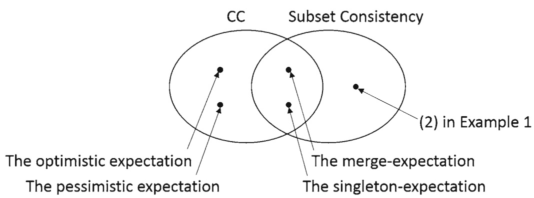

Fig. 1 The relationship between consistency concepts and expectation functions

2.4 The core based on an expectation function

In this subsection, we introduce the core based on an expectation function. Define $X(N, v) = \left\{x \in \mathbb{R}^N \;\middle|\; \sum_{j \in N} x_j = v(N, \{\emptyset\})\right\}$. Then, the core based on an expectation function is given as the following definition.

Definition 5 Let Γ be a set of games and $(N, v) \in \Gamma$. Given an expectation function ψ, the *ψ-core* of game (N, v) is defined as follows:

$$C^{\psi}(N, v) = \left\{ x \in X(N, v) \;\middle|\; \text{for any nonempty } S \subseteq N,\ \sum_{j \in S} x_j \geq v(S, \psi(N, v, S)) \right\}.$$

If ψ is the optimistic, pessimistic, singleton- or merge-expectation function, then the ψ-core means the optimistic core C^{opt}, the pessimistic core C^{pes}, the m-core C^m, or the s-core C^s, respectively.[2] For any expectation function ψ, let $\Gamma_{C^{\psi}}$ denote the class of games in which the nonempty ψ-core exists.[3] Definition 5 is a generalization of the core of usual TU-games without externalities: we implicitly assume that each deviating coalition forms a single coalition without breaking up and restrict our attention to the allocations x feasible in the grand coalition N. We will discuss these assumptions in Sects. 5.2 and 5.3.

3 An axiomatic approach

We introduce the axioms for our characterization results.

Axiom 1 (comp-RGP) *Let Γ be a set of games, $(N, v) \in \Gamma$ and $S \subseteq N$. A solution σ on Γ satisfies the complement-reduced game property (comp-RGP) if for every $x \in \sigma(N, v)$, we have $(S, v^{S,x}) \in \Gamma$ and $x_S \in \sigma(S, v^{S,x})$.*

Axiom 1 is a condition requiring solution σ to be consistent with itself: if $x \in \sigma(N, v)$, namely, x solves the game (N, v), then x_S should solve $(S, v^{S,x})$ for any $S \subseteq N$. Namely, any restriction x_S of x does not contradict the agreement among all players that x is a solution of (N, v). Moreover, Axiom 1 is independent of expectation function ψ. In other words, the choice of an expectation function does not influence the concept of consistency described in Axiom1.

Axiom 2 (NE on Γ) *Let Γ be a set of games. A solution σ on Γ satisfies non-emptiness on Γ (NE on Γ) if for every $(N, v) \in \Gamma$, we have $\sigma(N, v) \neq \emptyset$.*

Axiom 2 states that solution σ should be non-empty for any game in a given class Γ. This axiom may depend on ψ if Γ is specified by ψ.

[2] The terminology of the m-core and the s-core is introduced by Hafalir (2007).

[3] Abe and Funaki (2016) generalize the Bondareva-Shapley condition and define the class $\Gamma_{C^{\psi}}$. The balancedness of each type of core is also studied.

Axiom 3 **(ψ-IR)** *Let ψ be an expectation function. A solution σ on Γ satisfies ψ-individual rationality (ψ-IR) if for every $(N, v) \in \Gamma$, any $x \in \sigma(N, v)$, and every player $i \in N$, we have $x_i \geq v(\{i\}, \psi(N, v, \{i\}))$.*

In the presence of externalities, each player i's solo worth varies depending on the coalition structure of $N \setminus i$. The axiom ψ-IR requires that solution σ should assign to each player at least his individual worth under the expectation ψ and the corresponding coalition structure $\psi(N, v, \{i\})$. Axiom 3 directly depends on ψ.

For any ψ, the ψ-core satisfies ψ-IR because the expectation function ψ is common to both ψ-core and ψ-IR. It is also clear that ψ-core is nonempty on Γ if $\Gamma = \Gamma_{C^\psi}$. For comp-RGP, we have the following result.

Proposition 1 *If an expectation function ψ is CC, the ψ-core satisfies comp-RGP on Γ_{C^ψ}.*

Proof Let $C^\psi(N, v)$ be the ψ-core of (N, v) and $x \in C^\psi(N, v)$. For every nonempty $S \subseteq N$, it suffices to show that $x_S \in C^\psi(S, v^{S,x})$. By Definition 1, for any $T \subseteq S$ $(T \neq \emptyset)$, we have

$$
\begin{aligned}
&\sum_{j \in T} x_j - v^{S,x}(T, \psi(S, v^{S,x}, T)) \\
&\quad = \sum_{j \in T} x_j - \left[v(T \cup (N \setminus S), \psi(S, v^{S,x}, T)) - \sum_{j \in N \setminus S} x_j \right] \\
&\quad = \sum_{j \in T \cup (N \setminus S)} x_j - v(T \cup (N \setminus S), \psi(S, v^{S,x}, T)) \\
&\quad \geq v(T \cup (N \setminus S), \psi(N, v, T \cup (N \setminus S))) - v(T \cup (N \setminus S), \psi(S, v^{S,x}, T)) \quad (4) \\
&\quad = v(T \cup (N \setminus S), \psi(S, v^{S,x}, T)) - v(T \cup (N \setminus S), \psi(S, v^{S,x}, T)) \\
&\quad = 0, \quad (5)
\end{aligned}
$$

where (4) holds because of $x \in C^\psi(N, v)$, and (5) because of (3). □

Now, we offer the axiomatization below. We first show that a well-known result holds even in games with externalities (Lemma 3). This result will be used in the proof of the axiomatization (Proposition 2).

Lemma 3 *Let ψ be an expectation function and σ be a solution on a set of games Γ. If σ satisfies comp-RGP and ψ-IR, then σ satisfies efficiency: for any $x \in \sigma(N, v)$*

$$\sum_{j \in N} x_j = v(N, \{\emptyset\}).$$

Proof This is a simple extension of Peleg (1986) and Tadenuma (1992). Let $(N, v) \in \Gamma$ and $x \in \sigma(N, v)$. Assume that σ is not efficient. Then, there exists $x \in \sigma(N, v)$ such that $\sum_{j \in N} x_j < v(N, \{\emptyset\})$. Let i be a player in N. By comp-RGP, we have

$x_i \in \sigma\left(\{i\}, v^{\{i\},x}\right)$. For any ψ, by ψ-IR, we have $x_i \geq v^{\{i\},x}(\{i\}, \{\emptyset\}) = v(N, \{\emptyset\}) - \sum_{j \in N \setminus i} x_j$. Hence, $\sum_{j \in N} x_j \geq v(N, \{\emptyset\})$, and the desired contradiction has been obtained. □

Proposition 2 *Let ψ be an expectation function. If ψ is CC, then ψ-core C^ψ is the unique function on Γ_{C^ψ} that satisfies comp-RGP, NE on Γ_{C^ψ}, and ψ-IR.*

Proof We prove uniqueness next. Let σ be a solution satisfying the three conditions. The proof consists of two parts: $\sigma \subseteq C^\psi$ and $C^\psi \subseteq \sigma$.

Part 1:

First, we show that $\sigma(N, v) \subseteq C^\psi(N, v)$ for any $(N, v) \in \Gamma_{C^\psi}$. From Lemma 3, it follows that σ satisfies efficiency.

Induction base

For $|N| = 1$, $\sigma(N, v) \subseteq C^\psi(N, v)$ because of efficiency. For $|N| = 2$, let $N = \{i, j\}$. By efficiency, $x_i + x_j = v(N, \{\emptyset\})$ for any $x \in \sigma(N, v)$. By ψ-IR, $x_i \geq v(\{i\}, \{\{j\}\})$ and $x_j \geq v(\{j\}, \{\{i\}\})$. Hence, $\sigma(N, v) \subseteq C^\psi(N, v)$.

Induction proof

We assume that $\sigma(N, v') \subseteq C^\psi(N, v')$ for any $(N, v') \in \Gamma^\psi$ with $|N| \leq k$ $(k \geq 2)$. We show that for any $(M, v) \in \Gamma^\psi$ with $|M| = k+1$, we have $\sigma(M, v) \subseteq C^\psi(M, v)$.

Let $x \in \sigma(M, v)$ and $h \in M$. By comp-RGP, we have $x_{M\setminus h} \in \sigma(M\setminus h, v^{M\setminus h,x})$. By the assumption of induction, $\sigma(M\setminus h, v^{M\setminus h,x}) \subseteq C^\psi(M\setminus h, v^{M\setminus h,x})$. Hence, for any nonempty $S \subseteq M\setminus h$,

$$\begin{aligned}\sum_{j \in S} x_j &\geq v^{M\setminus h,x}(S, \psi(M\setminus h, v^{M\setminus h,\mathcal{P},x}, S)) \\ &= v(S \cup h, \psi(M\setminus h, v^{M\setminus h,\mathcal{P},x}, S)) - x_h \\ &= v(S \cup h, \psi(M, v, S \cup h)) - x_h, \end{aligned} \tag{6}$$

where (6) holds because ψ is CC. Thus, we obtain

$$\sum_{j \in S \cup h} x_j \geq v(S \cup h, \psi(M, v, S \cup h))$$

for any nonempty $S \subseteq M\setminus h$. In addition, by ψ-IR, we have $x_i \geq v(\{i\}, \psi(M, v, \{i\}))$. Hence, $\sigma(M, v) \subseteq C^\psi(M, v)$. By induction, it follows that $\sigma(N, v) \subseteq C^\psi(N, v)$ for all (N, v) in Γ^ψ.

Part 2:

Next, we show that $C^\psi(N, v) \subseteq \sigma(N, v)$ for all $(N, v) \in \Gamma_{C^\psi}$. To prove this, we construct a game (M, u) by using a game $(N, v) \in \Gamma_{C^\psi}$ and a payoff vector $x \in C^\psi(N, v)$. Fix $(N, v) \in \Gamma_{C^\psi}$ and $x \in C^\psi(N, v)$. We define $M := N \cup h$, where $h \in \mathcal{N}$ and $h \notin N$. Define u as follows:

$$\begin{array}{ll} u(\{h\}, \mathcal{P}') = 0 & \text{for all } \mathcal{P}' \in \Pi(M\setminus h), \\ u(S \cup h, \mathcal{P}'') = v(S, \mathcal{P}'') & \text{for all } \mathcal{P}'' \in \Pi(M\setminus(S \cup h)), \\ u(S, \mathcal{P}''') = \sum_{j \in S} x_j, & \text{for all } \mathcal{P}''' \in \Pi(M\setminus S). \end{array} \tag{7}$$

Now, consider $y = (x, 0) \in \mathbb{R}^M$. We will prove the following claims.

Claim 1 $y \in C^{\psi}(M, u)$.

Proof By the definition of y and u, we have

$$\sum_{j \in M} y_j = \sum_{j \in N} x_j = v(N, \{\emptyset\}) = u(M, \{\emptyset\}).$$

First, we show that $v = u^{M \backslash h, y}$. For any $S \subseteq N = M \backslash h$ and any $\mathcal{P}'' \in \Pi(N \backslash S)$, we have

$$\begin{aligned} u^{M \backslash h, y}(S, \mathcal{P}'') &= u(S \cup h, \mathcal{P}'') - y_h \\ &= u(S \cup h, \mathcal{P}'') \\ &= v(S, \mathcal{P}''), \end{aligned}$$

where the last equality holds because of the second line of (7).

Now, for any $S \subseteq N = M \backslash h$, we have

$$\begin{aligned} \sum_{j \in S \cup h} y_j = \sum_{j \in S} x_j &\geq v(S, \psi(N, v, S)) \\ &= u(S \cup h, \psi(M, u, S \cup h)). \end{aligned}$$

The last equality holds because ψ is CC and v is a complement-reduced game of u. In addition, by the third line of (7), for any $S \subseteq N = M \backslash h$ and any $\mathcal{P}''' \in \Pi(M \backslash S)$, we have

$$\sum_{j \in S} y_j = \sum_{j \in S} x_j = u(S, \mathcal{P}''').$$

This completes the proof of Claim 1. □

Claim 2 $\{y\} = C^{\psi}(M, u)$.

Proof If there exists $z \in C^{\psi}(M, u)$ such that $z \neq y$, we must have $\sum_{j \in M} z_j = u(M, \{\emptyset\}) = v(N, \{\emptyset\}) = \sum_{j \in N} x_j = u(N, \{\{h\}\}) \leq \sum_{j \in N} z_j$, and $z_h \geq u(h, \mathcal{P}') = 0$ for any $\mathcal{P}' \in \Pi(M \backslash h)$. Hence, $z_h = 0$.

For any $i \in N$ and any $\mathcal{P}''' \in \Pi(M \backslash i)$, we have $z_i \geq u(i, \mathcal{P}''') = x_i = y_i$ and, also, $\sum_{j \in N} z_j = \sum_{j \in M} z_j = u(M, \{\emptyset\}) = \sum_{j \in M} y_j = \sum_{j \in N} y_j$. Thus, we obtain $z_i = y_i$ for all $i \in N$, *i.e.*, $z = y$. This completes the proof of Claim 2. □

Now, consider $x \in C^{\psi}(N, v)$ and (M, u) again. By the first half of this proof, $\sigma(M, u) \subseteq C^{\psi}(M, u)$. As mentioned above, $C^{\psi}(M, u) = \{y\}$. By connecting them, $\sigma(M, u) \subseteq C^{\psi}(M, u) = \{y\}$. By NE on Γ^{ψ}, we obtain $\sigma(M, u) = C^{\psi}(M, u) = \{y\}$. Furthermore, by comp-RGP and $v = u^{M \backslash h, y}$, we have $x = y_N \in \sigma(N, u^{M \backslash h, y}) = \sigma(N, v)$. Thus, $C^{\psi}(N, v) \subseteq \sigma(N, v)$. □

Proposition 2 states that we can generalize the axiomatization of the core of games without externalities by using expectation function ψ. In Proposition 2, ψ is needed to be CC. More specifically, CC is a sufficient condition for the ψ-core to satisfy comp-RGP. To enhance the intuition behind this requirement, we offer the following example.

Example 2 Let $N = \{1, 2, 3, 4\}$. We consider the expectation function ψ given by (2) in Example 1, which satisfies subset consistency but not CC. Consider the following game: for mutually different players $i, j, k, h \in N$,

$$\begin{aligned}
&v(\{i, j, k, h\}, \emptyset) = 12,\\
&v(\{i, j, k\}, \{\{h\}\}) = 9,\\
&v(\{i, j\}, \{\{k, h\}\}) = 6,\\
&v(\{i, j\}, \{\{k\}, \{h\}\}) = 7,\\
&v(\{i\}, \text{any partition of } N\backslash i) = 0.
\end{aligned}$$

The ψ-core of this game is $C^{\psi} = \{(3, 3, 3, 3)\}$, which is equal to the core based on the merge-expectation. We reduce this game to $\{1, 2, 3\}$ with $x = (3, 3, 3, 3)$, and obtain

$$\begin{aligned}
&v^{-4}(\{i, j, k\}, \emptyset) = 9,\\
&v^{-4}(\{i, j\}, \{\{k\}\}) = 6,\\
&v^{-4}(\{i\}, \{\{j, k\}\}) = 3,\\
&v^{-4}(\{i\}, \{\{j\}, \{k\}\}) = 4,
\end{aligned}$$

for mutually different players $i, j, k \in N\backslash 4$. The ψ-core of this reduced game is empty. This is because the ψ-core is equal to the core based on the singleton-expectation in the player set $N\backslash 4$ ($\subsetneq N$). In contrast, if ψ is CC, then we have a consistent expectation function and the corresponding core: if ψ is the merge-expectation within N, then it should be the merge-expectation within $N\backslash 4$, and, similarly, if it is the singleton-expectation within N, then the singleton-expectation within $N\backslash 4$ as well. This property enables the ψ-core to satisfy comp-RGP.

Note that if we "remove" externalities, this axiomatization coincides with Tadenuma's approach. To see this, consider an expectation function ψ to be a function transforming a game with externalities v into a game without externalities w, by setting

$$w(S) := v(S, \psi(N, v, S)). \tag{8}$$

Proposition 2 shows that if ψ is CC, then this transformation ψ keeps the core's axiomatic characterization unchanged. As we have mentioned, the four expectation functions are all CC. Therefore, we have the following corollary.

Corollary 1 *The four types of cores, i.e., C^{opt}, C^{pes}, C^{s} and C^{m}, can be axiomatized with axioms 1-3 on each class: $\Gamma_{C^{opt}}$, $\Gamma_{C^{pes}}$, $\Gamma_{C^{s}}$ and $\Gamma_{C^{m}}$, respectively.*

An example of expectation function that satisfies neither CC nor subset consistency is

$$\psi(N, v, S) = \begin{cases} \{N\backslash S\} & \text{if } |S| \geq 2, \\ \{\{i_{s+1}\}, \ldots, \{i_n\}\} & \text{if } |S| = 1. \end{cases}$$

This expectation function can be seen as the combination of the merge-expectation and the singleton-expectation. Expectation functions consisting of different expectation rules are typically not CC.

4 The other reduced games

In view of Proposition 2, one might consider that the analogous proof can be adapted for the other types of reduced games. In this section, we show that this conjecture is not necessarily true. To see this, we will extend the max-reduced game and the projection-reduced game, which were formulated by Davis and Maschler (1965) and Funaki and Yamato (2001), respectively. This extension includes two technical difficulties. First, we need a partition as the additional specifier to define the reduced game. We use $v^{S,\mathcal{P},x}$ to denote the reduced game instead of the previous notation $v^{S,x}$. Second, the generalization of the max-reduced game yields two possible extensions: max-I and max-II. The difference between the two is the domain of maximization. For any coalition $S \subseteq N$, the former ignores the partition structure of $N\backslash S$ and chooses $C \subseteq N\backslash S$, whereas the latter chooses C in the partition of $N\backslash S$.

Formally, we consider a set of games $\Gamma \subseteq \Gamma_A$ and a game $(N, v) \in \Gamma$. Let $S \subseteq N$ $(S \neq \emptyset)$, $\mathcal{P} \in \Pi(N\backslash S)$, and $x \in \mathbb{R}^N$.

Definition 6 The *max-reduced game (I) with respect to* $S, \mathcal{P}$ *and* x is the game $(S, v_{m1}^{S,\mathcal{P},x})$ defined as follows: for any $T \subseteq S$ $(T \neq \emptyset)$ and any $\mathcal{Q} \in \Pi(S\backslash T)$,

$$v_{m1}^{S,\mathcal{P},x}(T, \mathcal{Q}) = \begin{cases} \max\limits_{C \subseteq N\backslash S} \left[v(T \cup C, \mathcal{Q} \cup (\mathcal{P}|_{(N\backslash S)\backslash C})) - \sum\limits_{j \in C} x_j \right], & \text{if } T \subsetneq S \\ v(N, \{\emptyset\}) - \sum_{j \in N\backslash S} x_j, & \text{if } T = S \end{cases}.$$

The *max-reduced game (II)*, $(S, v_{m2}^{S,\mathcal{P},x})$, is also defined by replacing the domain of the maximization $C \subseteq N\backslash S$ with $\mathcal{C} \subseteq \mathcal{P}$, formally,

$$v_{m2}^{S,\mathcal{P},x}(T, \mathcal{Q}) = \begin{cases} \max\limits_{\emptyset \subseteq \mathcal{C} \subseteq \mathcal{P}} \left[v(T \cup \bar{C}, \mathcal{Q} \cup (\mathcal{P}\backslash\mathcal{C})) - \sum\limits_{j \in \bar{C}} x_j \right], & \text{if } T \subsetneq S \\ v(N, \{\emptyset\}) - \sum_{j \in N\backslash S} x_j, & \text{if } T = S \end{cases},$$

where $\bar{C} = \bigcup_{C_i \in \mathcal{C}} C_i$.

Definition 7 The *projection-reduced game with respect to* $S, \mathcal{P}$ *and* x is the game $(S, v_p^{S,\mathcal{P},x})$ defined as follows: for any $T \subseteq S$ $(T \neq \emptyset)$ and any $\mathcal{Q} \in \Pi(S\backslash T)$,

Table 1 The relationship between the cores and RGPs

	RGP			
	Max(I)	Max(II)	Projection	Complement
Optimistic core	Yes	Yes	Yes	Yes
Pessimistic core	–	–	–	Yes
s-Core	–	–	–	Yes
m-Core	–	–	–	Yes

Table 2 The relationship between expectation functions and consistency

	Consistency			
	Max(I)	Max(II)	Projection	Complement
Optimistic expectation	–	–	–	Yes
Pessimistic expectation	–	–	–	Yes
Singleton-expectation	Yes	Yes	Yes	Yes
Merge-expectation	Yes	Yes	Yes	Yes

$$v_p^{S,\mathcal{P},x}(T,\mathcal{Q}) = \begin{cases} v(T,\mathcal{Q}\cup\mathcal{P}), & \text{if } T \subsetneq S \\ v(N,\{\emptyset\}) - \sum_{j\in N\setminus S} x_j, & \text{if } T = S \end{cases}.$$

If players leave the game one by one, the max-reduced games (both I and II) and the projection-reduced game are all independent of the order of the leaving players as well as the complement-reduced game. However, if two or more players simultaneously leave the game as a single group, the max-reduced games (I, II) and the projection-reduced game may depend on the partition of the leaving players.[4] This contrasts with the fact that the complement-reduced game is independent of the partition of the leaving players.

The gap between "one-by-one leaving" and "at-once leaving" yields two RGPs. We call them "one-by-one RGP" and "at-once RGP." It is clear that the one-by-one RGP is weaker than the at-once RGP. We restrict our attention to the weaker RGP, *i.e.*, one-by-one RGP, and denote it, simply, RGP hereafter.

Now, we return to the main question of this section: can we adapt the technique of Proposition 2 to the proof of the axiomatizations for the max- and the projection-reduced game? Tables 1 and 2 describe its difficulty. Table 1 describes the relationship between the cores and RGPs, and Table 2 shows the relationship between the four expectation functions and consistencies. There is no ambiguity in defining these consistencies in Table 2. We define them by replacing $v^{N\setminus h,x}$ in Definition 4 with $v_{m1}^{N\setminus h,\{h\},x}$, $v_{m2}^{N\setminus h,\{h\},x}$ or $v_p^{N\setminus h,\{h\},x}$, namely, we use the weaker definition based on one-by-one

[4] Formally, as in Lemma 1, we have $(v_{m1}^{-i_1})_{m1}^{-i_2} = (v_{m1}^{-i_2})_{m1}^{-i_1}$. However, there possibly exist partitions $\mathcal{P}$ and $\mathcal{P}'$ such that $v_{m1}^{N\setminus\{i_1,i_2\},\mathcal{P},x} \neq v_{m1}^{N\setminus\{i_1,i_2\},\mathcal{P}',x}$, where (m1) can be replaced with (m2) or (p).

leaving. We now consider, for example, the max(I)-reduced game. The max(I)-version of Proposition 2 should be as follows:

> If ψ is max(I)-consistent, then ψ-core C^{ψ} is the unique function on $\Gamma_{C^{\psi}}$ that satisfies max(I)-RGP and some axioms.

As Table 2 shows, the singleton-expectation and the merge-expectation satisfy the max(I)-consistency. However, as Table 1 shows, the s-core and the m-core do not satisfy max(I)-RGP. Namely, for each expectation function ψ, either the ψ-core violates the max(I)-RGP or ψ violates the max(I)-consistency, except for the complement-RGP and the complement-consistency (CC). This is the difficulty of the straightforward generalization of the axiomatization using reduced game consistency. In other words, the completion of Proposition 2 is ascribed to the coincidence of the complement reduced game illustrated in Tables 1 and 2. Note that all propositions and examples of Tables 1 and 2 are found in the Appendix.

5 Concluding remarks

5.1 Similarities and differences with path independence

We compare the independence of the reduced game on the order of leaving players, as is described by Lemma 1, with similar notions studied by Dutta et al. (2010) and Bloch and van den Nouweland (2014).

Bloch and van den Nouweland (2014) define path independence for expectation functions. Since their definition is more general than ours, we slightly expand our definition as follows: $\psi(N, v, S, \mathcal{N})$, where $\mathcal{N}$ is a partition of the player set. This extension allows us to consider an expectation depending on the current partition $\mathcal{N}$ from which a coalition S deviates. Now, an expectation function satisfies path independence if for any nonempty disjoint coalitions $S, T \subseteq N$ and any $\mathcal{N} \in \Pi(N)$,

$$\psi(N, v, S \cup T, \{S\} \cup \psi(N, v, S, \mathcal{N})) = \psi(N, v, S \cup T, \{T\} \cup \psi(N, v, T, \mathcal{N})).$$

Namely, if path independence does not hold, the expectation of a coalition depends on the order in which each member's expectation is aggregated. Bloch and van den Nouweland (2014) show that almost all reasonable expectation functions (including the four functions we listed) obey path independence.

Dutta et al. (2010) define *restriction operators*. A restriction operator r specifies a subgame (not a reduced game) $v^{-i,r} : EC(N \backslash i) \to \mathbb{R}$ with respect to a player i. To be more precise, an operator $r^N_{i,S,\mathcal{Q}}$ determines which coalition S in $\mathcal{Q} \cup \emptyset$ to be merged with player i, namely, $v^{-i,r}(S, \mathcal{Q}) = r^N_{i,S,\mathcal{Q}}(\, v(S, \{(S_1 \cup i), S_2, \dots, S_q\}), \dots, v(S, \{S_1, S_2, \dots, (S_q \cup i)\}), v(S, \{S_1, S_2, \dots, S_q, \{i\}\})\,)$. They define path independence as follows: a restriction operator r satisfies path independence if for any (N, v) and any $i, j \in N$,

$$(v^{-i,r})^{-j,r} = (v^{-j,r})^{-i,r}.$$

In words, the subgame obtained by the order i, j should be the same as that by the order j, i. It is important to note that their path independence is (similar, but) not exactly the same as ours: their independence guarantees the same worth of each coalition between two subgames with different paths, whereas ours guarantees that between two reduced games for any $x \in \mathbb{R}^N$. In other words, the path considered by Dutta et al. (2010) focuses on which player moves to which coalition, whereas our path describes which game is specified by player i and agreement x. In this sense, Dutta et al. (2010)'s path independence may be closer to that of expectation functions. In fact, an operator r relates a game with externalities to a game without externalities by defining $w_v^r(S) = v(S, \mathcal{P}_r)$ for any $S \subsetneq N$ and $w_v^r(N) = v(N, \emptyset)$, where $\mathcal{P}_r$ is a partition of $N \backslash S$ specified by r and the $|N| - |S|$ times leavings.

5.2 Deviations with multiple coalitions

In this paper, we assume that deviating coalitions form a single coalition S. As Kóczy (2007) and Bloch and van den Nouweland (2014) noted, this assumption is not necessarily general as forming *multiple coalitions* may be beneficial. In the general framework of multiple coalitions, the assumption of a single coalition can be thought of as an improvement with respect to the *total* payoff of the deviating coalitions: some players deviate by forming some coalitions if the total payoff of the coalitions is greater than their current payoff. However, we note that this approach ignores the case when the total payoff increases, whereas the payoff for *some* coalitions decreases. Moreover, when we consider deviations with multiple coalitions, the non-emptiness of the cores generally varies: an allocation x which prevents players from deviating with a single coalition may allow them to deviate with multiple coalitions. Therefore, in the framework of multiple coalitions, the class of games with the nonempty core shrinks.

5.3 The coalition structure core

The concepts of the cores we used in this paper are extensions of the classical core in games without externalities. In contrast, Greenberg (1994) defines the *coalition structure core* for games in partition function form, which is a set of *outcomes* consisting of a payoff vector x and a partition $\mathcal{P}$ from which no coalition deviates. The core concept described by Kóczy and Lauwers (2004) and the recursive core studied by Kóczy (2007) is based on the coalition structure core. Our extension can be seen as the coalition structure core with restricting partitions to the grand coalition $\{N\}$.

5.4 Alternative expectation functions

Throughout this paper, we used expectation functions associated with partitions, namely, $\psi(N, v, S) \in \Pi(N \backslash S)$. However, the probabilistic approach, as is sometimes seen in the papers studying the generalization of the Shapley value, is also possible, in which $\psi(N, v, S)$ is a probability distribution over $\Pi(N \backslash S)$. Each definition and result in this paper can be straightforwardly adjusted to the probabilistic framework. It is notable that any convex combination of the four expectation functions listed earlier, for instance, 50% for the best partition arg $\max_{\mathcal{P}' \in \Pi(N \setminus S)} v(S, \mathcal{P}')$ and 50% for the worst partition arg $\min_{\mathcal{P}' \in \Pi(N \setminus S)} v(S, \mathcal{P}')$, is still CC.

The expectation functions discussed in this paper are some of the simplest expectation rules. There are expectation functions reflecting other behavioral expectations. For example, Kóczy (2007) proposes the recursive optimism and pessimism and defines the cores based on the expectations. It may be possible to apply our approach to the elaborate expectations. This interesting topic is left for future work.

Appendix

To distinguish each form of reduced game, in this appendix, we use symbols $v_{m1}^{S,\mathcal{P},x}$, $v_{m2}^{S,\mathcal{P},x}$, $v_{p}^{S,\mathcal{P},x}$ and $v_{c}^{S,x}$ to denote max(I), max(II), projection and complement-type of reduced game, respectively. Tables 3 and 4 correspond to Tables 1 and 2, respectively. The number assigned to each cell represents the proposition or example describing the cell, *e.g.*, for the proposition showing that the optimistic core satisfies Max-I RGP, see Proposition 5.

Proposition 3 *If ψ is optimistic or pessimistic, then ψ is CC.*

Table 3 The relationship between expectation cores and RGP (corresponding to Table 1)

	RGP			
	Max(I)	Max(II)	Projection	Complement
Optimistic core	Yes Prop.5	Yes Prop.5	Yes Prop.5	Yes Prop.1
Pessimistic core	− Ex.4	− Ex.4	− Ex.6	Yes Prop.1
s-Core	− Ex.5	− Ex.5	− Ex.6	Yes Prop.1
m-Core	− Ex.4	− Ex.4	− Ex.7	Yes Prop.1

Table 4 The relationship between expectation functions and consistency (corresponding to Table 2)

	Consistency			
	Max(I)	Max(II)	Projection	Complement
Optimistic expectation	− Ex.3	− Ex.3	− Ex.3	Yes Prop.3
Pessimistic expectation	− Ex.3	− Ex.3	− Ex.3	Yes Prop.3
Singleton-expectation	Yes Cor.2	Yes Cor.2	Yes Cor.2	Yes Cor.2
Merge-expectation	Yes Cor.2	Yes Cor.2	Yes Cor.2	Yes Cor.2

Proof We denote by ψ^{opt} the optimistic expectation function. Let (N, v) be a game, and $S \subseteq N$ ($|S| \geq 2$). We define $\mathcal{P}^*$ as follows:

$$\mathcal{P}^* := \psi^{opt}(N, v, S) = \mathop{\arg\max}_{\mathcal{P}' \in \Pi(N \setminus S)} v(S, \mathcal{P}'). \tag{9}$$

For any $h \in S$ and $x \in \mathbb{R}^N$, we have

$$\begin{aligned} v_c^{N\setminus h,x}(S\setminus h, \mathcal{P}^*) &= v(S, \mathcal{P}^*) - x_h \\ &= \max_{\mathcal{P}' \in \Pi(N\setminus S)} [v(S, \mathcal{P}') - x_h] \\ &= \max_{\mathcal{P}' \in \Pi(N\setminus S)} [v_c^{N\setminus h,x}(S\setminus h, \mathcal{P}')], \end{aligned}$$

where the first equality holds by the definition of complement reduced games, the second by (9) and the last by the definition of complement reduced games. Hence, we obtain

$$\mathcal{P}^* = \mathop{\arg\max}_{\mathcal{P}' \in \Pi(N\setminus S)} v_c^{N\setminus h,x}(S\setminus h, \mathcal{P}') = \psi^{opt}(N\setminus h, v_c^{N\setminus h,x}, S\setminus h),$$

and, then, $\psi^{opt}(N, v, S) = \psi^{opt}(N\setminus h, v_c^{N\setminus h,x}, S\setminus h)$, which implies ψ^{opt} is CC.

By replacing max with min, we complete the proof of the pessimistic expectation function ψ^{pes} as well. □

Proposition 4 *If ψ satisfies the following condition: for any games (N, v), (M, w), and nonempty coalitions $S \subseteq N$, $T \subseteq M$,*

$$N\setminus S = M\setminus T \Longrightarrow \psi(N, v, S) = \psi(M, w, T), \tag{10}$$

then ψ satisfies all four types of consistencies: Max-I, Max-II, Projection and Complement.

Proof We prove CC (or, complement consistency). The other types of consistencies are obtained in the same way. Fix a game (N, v). For any $x \in \mathbb{R}^N$ and $h \in N$, we can specify the complement reduced game $(N\setminus h, v_c^{N\setminus h,x})$. For any S such that $h \in S \subseteq N$, we have

$$N\setminus S = (N\setminus h)\setminus(S\setminus h).$$

Using (10), we obtain $\psi(N, v, S) = \psi(N\setminus h, v_c^{N\setminus h,x}, S\setminus h,)$ □

Lemma 4 *If ψ is the singleton-expectation, then ψ satisfies (10).*

Proof We denote by ψ^s the singleton-expectation function. For any nonempty T and S with $T \in S \subseteq N$, and any $w : EC(N\setminus T) \to \mathbb{R}$, we have $\psi^s(N, v, S) = \{\{i\} | i \in N\setminus S\} = \psi^s(N\setminus T, w, S\setminus T)$. □

Lemma 5 *If ψ is the merge-expectation, then ψ satisfies* (10).

Proof This is similar to Lemma 4. Let ψ^m denote the merge-expectation function. We have $\psi^m(N, v, S) = \{N\backslash S\} = \psi^m(N\backslash T, w, S\backslash T)$. □

Corollary 2 *If ψ is the singleton-expectation or the merge-expectation, then ψ satisfies all four types of consistencies.*

Proof See Lemmas 4, 5 and Proposition 4. □

Example 3 Consider the following 4-player game: $N = \{i_1, i_2, i_3, i_4\}$;

$$\begin{aligned}
&v(N, \{\emptyset\}) = 12;\\
&v(\{i, j, k\}, \{\{h\}\}) = 5 \; and \; v(\{h\}, \{\{i, j, k\}\}) = 0, \quad for \; \{i, j, k, h\} = N;\\
&v(\{i, j\}, \{\{k, h\}\}) = 4, \quad for \; \{i, j, k, h\} = N;\\
&v(\{i, j\}, \{\{k\}, \{h\}\}) = 3 \; and \; v(\{k\}, \{\{i, j\}, \{h\}\}) = 1, \quad for \; \{i, j, k, h\} = N;\\
&v(\{i\}, \{\{j\}, \{k\}, \{h\}\}) = 2, \quad for \; \{i, j, k, h\} = N.
\end{aligned}$$

Let $x = (3, 3, 3, 3)$, $S = \{i_1, i_2\}$ and player $h = 1$. For the optimistic expectation function, we have

$$\underset{\mathcal{P}'\in\Pi(N\backslash S)}{\arg\max} \; v(S, \mathcal{P}') = \{\{i_3, i_4\}\},$$

because $\max_{\mathcal{P}'\in\Pi(N\backslash S)} v(S, \mathcal{P}') = \max\{v(S, \{\{i_3, i_4\}\}), v(S, \{\{i_3\}, \{i_4\}\})\} = \max\{4, 3\}$. However, in the Max-I reduced game, we have

$$\underset{\mathcal{P}'\in\Pi(N\backslash S)}{\arg\max} \; v_{m1}^{-h}(S\backslash h, \mathcal{P}') = \{\{i_3\}, \{i_4\}\},$$

because

$$\begin{aligned}
\max_{\mathcal{P}'\in\Pi(N\backslash S)} v_{m1}^{-h}(S\backslash h, \mathcal{P}') &= \max \left\{ \begin{array}{ll} v(S, \{\{i_3, i_4\}\}) - x_h, & v(S\backslash h, \{\{i_1\}, \{i_3, i_4\}\}), \\ v(S, \{\{i_3\}, \{i_4\}\}) - x_h, & v(S\backslash h, \{\{i_1\}, \{i_3\}, \{i_4\}\}) \end{array} \right\}\\
&= \max\{4 - 3, 1, 3 - 3, 2\}\\
&= 2, \qquad (11)
\end{aligned}$$

which is the worth of the bottom-right element in (11). Hence, $\psi^{opt}(N, v, S) = \{\{i_3, i_4\}\} \neq \{\{i_3\}, \{i_4\}\} = \psi^{opt}(N\backslash h, v_{m1}^{-h}, S\backslash h)$. For the optimistic expectation function, this example is still valid for Max-II and Projection consistencies as well. For the pessimistic expectation function, we can generate the example by swapping $v(\{i, j\}, \{\{i, j\}, \{k, h\}\})$ for $v(\{i, j\}, \{\{i, j\}, \{k\}, \{h\}\})$.

Lemma 6 *Let $(N, v) \in \Gamma$. Let $S \subseteq N$, $\mathcal{P} \in \Pi(N\backslash S)$ and $x \in \mathbb{R}^N$. We denote each type of reduced game by $v_{m1}^{S,\mathcal{P},x}$, $v_{m2}^{S,\mathcal{P},x}$, $v_p^{S,\mathcal{P},x}$ and $v_c^{S,x}$, respectively. Then, for any $T \subseteq S$ $(T \neq \emptyset)$ and $\mathcal{Q} \in \Pi(S\backslash T)$, we have*

$$v_{m1}^{S,\mathcal{P},x}(T, \mathcal{Q}) \geq v_{m2}^{S,\mathcal{P},x}(T, \mathcal{Q}),$$

$$v_{m2}^{S,\mathcal{P},x}(T,\mathcal{Q}) \geq v_p^{S,\mathcal{P},x}(T,\mathcal{Q}),$$
$$v_{m2}^{S,\mathcal{P},x}(T,\mathcal{Q}) \geq v_c^{S,x}(T,\mathcal{Q}).$$

Proof The first inequality follows from the domain of maximization: in view of the definitions, for any $\mathcal{P} \in \Pi(N\backslash S)$,

$$\{C|C \in \mathcal{P}\} \text{ (or, Max-II) } \subseteq \{C|C \subseteq N\backslash S\} \text{ (or, Max-I)} .$$

The second (third) inequality holds because we can take $\emptyset$ $(N\backslash S)$ as maximizer C. □

Proposition 5 *The optimistic-core satisfies all types of RGP on $\Gamma_{C^{opt}}$: maxI-RGP, maxII-RGP, projection-RGP and comp-RGP.*

Proof Let $C^{opt}(N,v)$ be the optimistic core of (N,v) and $x \in C^{opt}(N,v)$. We show that the optimistic-core satisfies maxI-RGP. For any $S \subseteq N$, $T \subsetneq S$ $(T \neq \emptyset)$ and $\mathcal{P} \in \Pi(N\backslash S)$, we have

$$\begin{aligned}
&\sum_{j\in T} x_j - \max_{\mathcal{Q}\in\Pi(S\backslash T)} v_{m1}^{S,\mathcal{P},x}(T,\mathcal{Q}) \\
&= \sum_{j\in T} x_j - \max_{\mathcal{Q}\in\Pi(S\backslash T)} \max_{C\subseteq N\backslash S} \left[v(T\cup C, \mathcal{Q}\cup(\mathcal{P}|_{(N\backslash S)\backslash C})) - \sum_{j\in C} x_j \right] \\
&= \sum_{j\in T} x_j - \left[v(T\cup C^*, \mathcal{Q}^*\cup(\mathcal{P}|_{(N\backslash S)\backslash C^*})) - \sum_{j\in C^*} x_j \right] \qquad (12)\\
&= \sum_{j\in T\cup C^*} x_j - v(T\cup C^*, \mathcal{Q}^*\cup(\mathcal{P}|_{(N\backslash S)\backslash C^*})) \\
&\geq \max_{\mathcal{P}'\in\Pi(N\backslash(T\cup C^*))} v(T\cup C^*, \mathcal{P}') - v(T\cup C^*, \mathcal{Q}^*\cup(\mathcal{P}|_{(N\backslash S)\backslash C^*})) \\
&\geq 0, \qquad (13)
\end{aligned}$$

where C^*, $\mathcal{Q}^*$ in (12) are maximizers of the target formula, and (13) holds because $x \in C^{opt}(N,v)$. Similarly, for $T = S$, we have

$$\begin{aligned}
\sum_{j\in S} x_j - v^{S,\mathcal{P},x}(S,\{\emptyset\}) &= \sum_{j\in S} x_j - \left(v(N,\{\emptyset\}) - \sum_{j\in N\backslash S} x_j \right) \\
&= \sum_{j\in N} x_j - v(N,\{\emptyset\}) \\
&= 0.
\end{aligned}$$

By Lemma 6, we can replace $v_{m1}^{S,\mathcal{P},x}$ with $v_{m2}^{S,\mathcal{P},x}$, $v_p^{S,\mathcal{P},x}$ and $v_c^{S,\mathcal{P},x}$, respectively. Then, we obtain the desired proposition. □

Example 4 Consider the following 4-player game: $N = \{1, 2, 3, 4\}$,

$$v(S, \mathcal{P}) = \begin{cases} 12 \text{ if } (S, \mathcal{P}) = (N, \{\emptyset\}), \\ 6 \ \ \text{if } (S, \mathcal{P}) = (\{i, j\}, \{\{k\}, \{h\}\}), \\ 0 \ \ \text{otherwise.} \end{cases}$$

Let $x = (x_1, x_2, x_3, x_4) = (1, 3, 4, 4)$. Then, $x \in C^{pes}(N, v) = C^m(N, v)$. Now, for $S = \{1, 2\}$ and $\mathcal{P} = \{\{3\}, \{4\}\}$, we have the following Max-I reduced game:

$$v_{m1}^{S,P,x}(\{1, 2\}, \{\emptyset\}) = 12 - (4 + 4) = 4,$$
$$v_{m1}^{S,P,x}(\{1\}, \{\{2\}\}) = 6 - 4 = 2,$$
$$v_{m1}^{S,P,x}(\{2\}, \{\{1\}\}) = 6 - 4 = 2.$$

The restriction of x, $x_S = (1, 3)$, is out of the pessimistic core (and the m-core) of the reduced game: $x_S = (1, 3) \notin \{(2, 2)\} = C^{pes}(S, v_{m1}^{S,P,x}) = C^m(S, v_{m1}^{S,P,x})$. We have the Max-II reduced game as well as Max-I.

Example 5 Consider the following 5-player game: $N = \{1, 2, 3, 4, 5\}$,

$$v(S, \mathcal{P}) = \begin{cases} 15 \text{ if } (S, \mathcal{P}) = (N, \{\emptyset\}), \\ 7 \ \ \text{if } (S, \mathcal{P}) = (\{i, j\}, \{\{k\}, \{h, l\}\}), \\ 0 \ \ \text{otherwise.} \end{cases}$$

Let $x = (x_1, x_2, x_3, x_4, x_5) = (2, 2, 4, 4, 3)$. Then, $x \in C^s(N, v)$. For $S = \{3, 4\}$ (who obtain 4 in x) and $\mathcal{P} = \{\{1\}, \{2, 5\}\}$, we have the following Max-I reduced game:

$$v_{m1}^{S,P,x}(\{3, 4\}, \{\emptyset\}) = 15 - (2 + 2 + 3) = 8,$$
$$v_{m1}^{S,P,x}(\{3\}, \{\{4\}\}) = 7 - 2 = 5,$$
$$v_{m1}^{S,P,x}(\{4\}, \{\{3\}\}) = 7 - 2 = 5.$$

Hence, the s-core is empty. We have the same result in Max-II as well as Max-I.

Example 6 Consider the following 4-player game: $N = \{1, 2, 3, 4\}$,

$$v(S, \mathcal{P}) = \begin{cases} 12 \text{ if } (S, \mathcal{P}) = (N, \{\emptyset\}), \\ 4 \ \ \text{if } (S, \mathcal{P}) = (\{i\}, \{\{j, k, h\}\}), \\ 4 \ \ \text{if } (S, \mathcal{P}) = (\{i\}, \{\{j, k\}, \{h\}\}), \\ 3 \ \ \text{if } (S, \mathcal{P}) = (\{i\}, \{\{j\}, \{k\}, \{h\}\}), \\ 0 \ \ \text{otherwise.} \end{cases}$$

Let $x = (x_1, x_2, x_3, x_4) = (3, 3, 3, 3)$. Then, $x \in C^{pes}(N, v) = C^s(N, v)$. For $S = \{1, 2\}$ and $\mathcal{P} = \{\{3, 4\}\}$, we have the following projection reduced game:

$$v_p^{S,P,x}(\{1, 2\}, \{\emptyset\}) = 12 - (3 + 3) = 6,$$

$$v_p^{S,P,x}(\{1\},\{\{2\}\}) = 4,$$
$$v_p^{S,P,x}(\{2\},\{\{1\}\}) = 4.$$

Hence, the pessimistic core and the s-core are empty in the reduced game.

Example 7 Consider the following 4-player game: $N = \{1, 2, 3, 4\}$,

$$v(S,\mathcal{P}) = \begin{cases} 12 \text{ if } (S,\mathcal{P}) = (N,\{\emptyset\}), \\ 3 \quad \text{if } (S,\mathcal{P}) = (\{i\},\{\{j,k,h\}\}), \\ 4 \quad \text{if } (S,\mathcal{P}) = (\{i\},\{\{j,k\},\{h\}\}), \\ 4 \quad \text{if } (S,\mathcal{P}) = (\{i\},\{\{j\},\{k\},\{h\}\}), \\ 0 \quad \text{otherwise.} \end{cases}$$

Let $x = (x_1, x_2, x_3, x_4) = (3, 3, 3, 3)$. Then, $x \in C^m(N, v)$. For $S = \{1, 2\}$ and $\mathcal{P} = \{\{3, 4\}\}$, we have the following projection reduced game:

$$v_p^{S,P,x}(\{1,2\},\{\emptyset\}) = 12 - (3+3) = 6,$$
$$v_p^{S,P,x}(\{1\},\{\{2\}\}) = 4,$$
$$v_p^{S,P,x}(\{2\},\{\{1\}\}) = 4.$$

Hence, the m-core of the reduced game becomes empty.

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