Int J Game Theory (2018) 47:379–415
https://doi.org/10.1007/s00182-017-0580-z

ORIGINAL PAPER

Entrepreneurial Chess

Elwyn Berlekamp[1] · **Richard M. Low**[2]

Accepted: 14 May 2017 / Published online: 22 May 2017

Abstract Although the combinatorial game *Entrepreneurial Chess* (or *Echess*) was invented around 2005, this is our first publication devoted to it. A single Echess position begins with a Black king vs. a White king and a White rook on a quarter-infinite board, spanning the first quadrant of the xy-plane. In addition to the normal chess moves, Black is given the additional option of "cashing out", which removes the board and converts the position into the integer $x+y$, where $[x, y]$ are the coordinates of his king's position when he decides to cash out. Sums of Echess positions, played on different boards, span an unusually wide range of topics in combinatorial game theory. We find many interesting examples.

Keywords Combinatorial games · Chess

Mathematics Subject Classification 91A46

1 Introduction

Following its beginnings in the context of recreational mathematics, combinatorial game theory has matured into an active area of research. Along with its natural appeal, the subject has applications to complexity theory, logic, graph theory and biology

✉ Richard M. Low
richard.low@sjsu.edu

Elwyn Berlekamp
berlek@gmail.com

1 Elwyn and Jennifer Berlekamp Foundation, 5665 College Avenue, Suite #330B, Oakland, CA 94618, USA

2 Department of Mathematics, San Jose State University, San Jose, CA 95192, USA

Table 1 A game where Black uses the DOGMATIC strategy, with initial position in Fig. 1a

White	Black
1. Re2	Kd4
2. Kb2	Kd5
3. Kc3	Kd6
4. Kd4	Kd7
5. Re11	Kd8
6. Ke5	Kd9
7. Kf6	Kd10
8. Ri11	Ke10
9. Kg7	Kf10
10. Kh8	Kg10
11. Ki9	

Using standard chess notation, the rows are indexed by a, b, c, etc. and the columns are indexed by 1, 2, 3, etc

(Fraenkel 1996). For these reasons, combinatorial games have caught the attention of many people and the large body of research literature on the subject continues to increase. The interested reader is directed to Albert and Nowakowski (2009), Albert et al. (2007), Berlekamp et al. (2001), Conway (1976), Guy and Nowakowski (2002), Nowakowski (2015), Nowakowski (1996) and to Fraenkel's excellent bibliography Fraenkel (1996).

In Guy and Nowakowski (2002), the following problem was posed:

> *A King and Rook versus King problem* Played on a quarter-infinite board, with initial position WKa1, WRb2 and BKc3. Can White win? If so, in how few moves? It may be better to ask, "what is the smallest board (if any) that White can win on if Black is given a win if he walks off the North or East edges of the board?" Is the answer 9×11? In an earlier edition of this paper I attributed this problem to Simon Norton, but it was proposed as a kriegsspiel problem, with unspecified position of the WK, and with W to win with probability 1, by Lloyd Shapley around 1960.

With this starting position, White can win if the rook (protected directly or indirectly by the White king) limits and then narrows the moves of the Black king. In Low et al. (2006), Pearson and Berlekamp (2005) it was shown that White has a winning strategy which can be implemented within a 9×11 region (assuming White moves first). For example, if Black utilizes the DOGMATIC strategy described in Sect. 8, we would have the game described in Table 1. White wins after his eleventh move.

Let us consider a board of fixed size where White can move anywhere on or off this board, but Black (and White) cannot move off the bottom or left sides. If Black escapes off the top or right side of the board, he is given a win. Pearson and Berlekamp (2005) created a program to calculate the maximum number of (total) moves it takes Black to escape or White to checkmate Black. We see that from Fig. 2, the smallest board such that Black will be checkmated (no matter who moves first) is of size 11×8. If White moves first, it takes a total of 77 moves to checkmate Black. If Black moves first, it takes a total of 86 moves to checkmate Black. Also, one can see that there is a

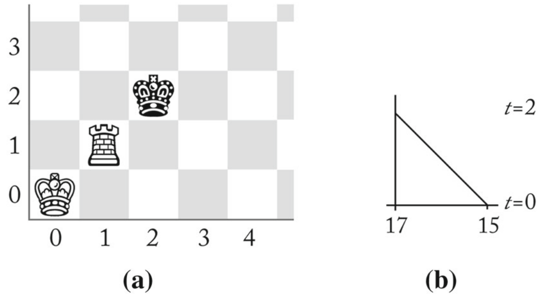


Fig. 1 Initial configuration in the "King and Rook vs. King on a quarter-infinite board" problem and thermograph

NUMBER OF MOVES BEFORE OUTCOME OCCURS

first number: Black moves first; second number: White moves first

bold number: total number of moves before Black escapes

non-bold number: total number of moves before White checkmates Black

Black wins if it passes row													
13				**3,22**	**23,24**	**25,26**	**27**,77	86,77	86,77	86,77	86,77	86,77	86,77
12				**3,20**	**21,22**	**23,24**	**25**,77	86,77	86,77	86,77	86,77	86,77	86,77
11				**3,18**	**19,20**	**21,22**	**23**,77	86,77	86,77	86,77	86,77	86,77	86,77
10				**3,16**	**17,18**	**19,20**	**21**,77	**23**,77	**25**,77	**27**,77	86,77	86,77	86,77
9				**3,14**	**15,16**	**17,18**	**19,20**	**21,22**	**23,24**	**25**,77	86,77	86,77	86,77
8				**3,12**	**13,14**	**15,16**	**17,18**	**19,20**	**21,22**	**23**,77	86,77	86,77	86,77
7				**3,10**	**11,12**	**13,14**	**15,16**	**17,18**	**19,20**	**21**,77	**23**,77	**25**,77	**27**,77
6				**3,8**	**9,10**	**11,12**	**13,14**	**15,16**	**17,18**	**19,20**	**21,22**	**23,24**	**25,27**
5				**3,6**	**7,8**	**9,10**	**11,12**	**13,14**	**15,16**	**17,18**	**19,20**	**21,22**	**23,24**
4				**3,4**	**3,6**	**3,8**	**3,10**	**3,12**	**3,14**	**3,16**	**3,18**	**3,20**	**3,22**
3													
2													
1													
	1	2	3	4	5	6	7	8	9	10	11	12	13
	Black wins if it passes column												

Fig. 2 Number of moves before outcome occurs

segment of board sizes such that it matters who goes first. Finally, observe that Black can eventually escape any board that is too narrow.

Entrepreneurial Chess (or *Echess*), invented by Pearson and Berlekamp (2005), modifies this game so that Black (who would otherwise always lose) can "cash out" by getting a payment of $x + y$ if the Black king is at position $[x, y]$. Cashing out is a move which terminates the game. The present paper gives the first virtually complete analysis of this game. By using results found later in this paper, one can calculate the following: the initial position shown in Fig. 1a has mean 17 with Left sente, and temperature 2.

2 General background

Combinatorial Game Theorists can now find correct analyses and winning strategies for many positions in many games. The theory is most successful in positions which are closely approximated as sums of more localized positions, each of which is a game.

Appropriate analysis of each local game yields some sort of mathematical object or data structure, which is simple enough to be added to other such objects, and yet sophisticated enough to facilitate a correct analysis of their sum. In the (now) classical case, this object is called the game's *canonical form* or equivalently, the *canonical value*. The most important homomorphism from such objects to the real numbers yields the *mean value*. The key measure of dispersion around this mean value is called its *Temperature*, a real number which provides a quantitative measure of the importance of the next move. These terms and a few others we will use in this paper are now standard in Combinatorial Game Theory, as seen in Albert and Nowakowski (2009), Guy and Nowakowski (2002), Nowakowski (2015), Nowakowski (1996). Precise mathematical definitions may be found in introductory texts such as Albert et al. (2007) or in treatises such as Berlekamp et al. (2001), Siegel (2013).

Most games whose sums have been completely analyzed have rather low temperatures, such as 2 or 3. A game whose temperature is -1 is an integer. A game with subzero temperature is a well-understood dyadic number. A game whose temperature is 0 is an infinitesimal. Games with positive temperatures often have canonical forms which are precise but which can be too complicated to be of much use (e.g., Snatzke 2002). When viewed as game-move trees in extensive form, canonical forms can have large breadth as well as large depth.

Yet, complete analysis of many positions in such "hot" games can be facilitated by studies of their means. There is a natural sense in which means can be regarded as scores. Moves which ensure optimal scores are called *orthodox*. Every position in an orthodox form, like the thermograph, requires only one Left follower and one Right follower. Since its breadth is only two, an orthodox form can be vastly simpler than the canonical form. However, victory in many combinatorial games is defined not only by getting the best orthodox score, but also by getting the last move. This typically depends on the analysis of infinitesimal canonical values, which are lost in the orthodox simplification. Although infinitesimals can also have large breadth, there is a homomorphism from them to other games called *atomic weights* which are typically significantly simpler than the infinitesimals from which they were mapped.

Simon Norton's "thermal disassociation" theorem (page 168, WW Berlekamp et al. 2001) essentially proposes a series of increasingly accurate (but increasingly complicated) approximations to an arbitrary canonical form. The first approximation is the mean. The second approximation also includes the appropriate infinitesimal heated by the Temperature. We call this the *primary* infinitesimal. Some moves optimize both the mean and the atomic weight of this infinitesimal. We call such moves *ultra-orthodox*.

3 Summary of results

We show that cooling any Echess Value by one degree (called *chilling*) turns out to be reversible by an operation called *warming*. To distinguish between them, we use "lower-case" for the values and temperatures of the chilled game, but "upper-case" values and temperatures for the warmed games. In general, temperature $=$ Temperature $-$ 1.

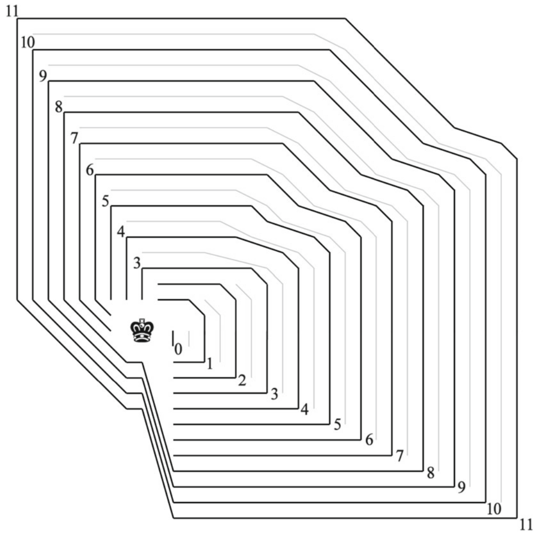


Fig. 3 Contours of the mean value with Black king at [0, 0]; rook at [−2, 1]

After at most a very short opening (lasting at most two or three moves), White's rook will occupy a position whose either x or y coordinate exceeds the Black king's by one. Then begins the middle game, which may last for many moves. We obtain the ultra-orthodox value of each middle game position. If the rook is either behind the Black king, or sufficiently far ahead of him, most of these values near the two kings are only weakly dependent on the precise location of the rook. Their mean values are shown in the contour map of Fig. 3. These values are naturally partitioned into six big regions of the plane, depending on the direction from the White king to the Black king. The values we obtain are ultra-orthodox in every region. In three of the six main regions, they are also canonical. Except in a few very narrow subregions, the original Temperatures are at most 2.

Virtually all Echess games terminate with a canonical Value that is the sum of an integer and a loopy infinitesimal ε, called OVER. The only exception is the unique Value of the position after the rook has been captured. This Value, which is bigger than any finite number, has been discussed by Siegel (pages 31–33, Siegel 2013). We have nothing further to say about it here. Many of the Echess positions discussed in this paper have properties previously encountered only in Berlekamp and Wolfe (1994) analysis of Go.

4 Some typical terminal positions in Echess

Throughout this paper, we assume Left plays BLack and Right plays White. As is common in combinatorial game theory, we assume that whatever position we might be discussing is likely to be played disjunctively as part of a larger overall game (Berlekamp 2002). At any turn, either player may decide to play "elsewhere", in some other summand. Hence, within the particular game under discussion, several

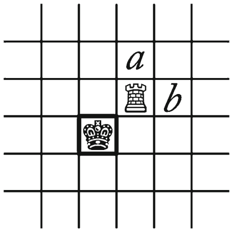


Fig. 4 Two "terminal" positions in Echess

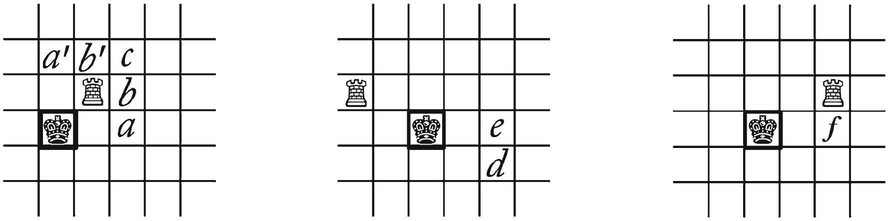


Fig. 5 More positions of value ε. We define e′, d′ and f′ by reflecting both the White king and rook across the diagonal through the Black king

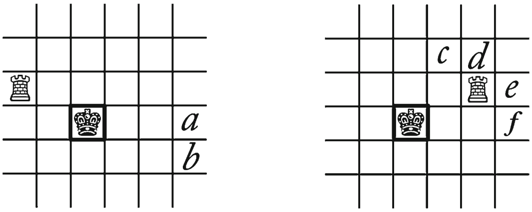


Fig. 6 Some positions of value $\{1\varepsilon|\varepsilon\}$ in Echess

consecutive moves might be made by the same player while his opponent responds "elsewhere".

Figures 4, 5, 6, 7, 8, 9 show positions relatively near the end of the game. In these positions, the square with the thick border is taken as [0, 0]. Each of x and y ranges from some finite negative value to $+\infty$. The locations of the Black king and the White rook are shown explicitly. The White king is presumed to be located at a square with a lowercase letter, and that letter is then also used to denote the position.

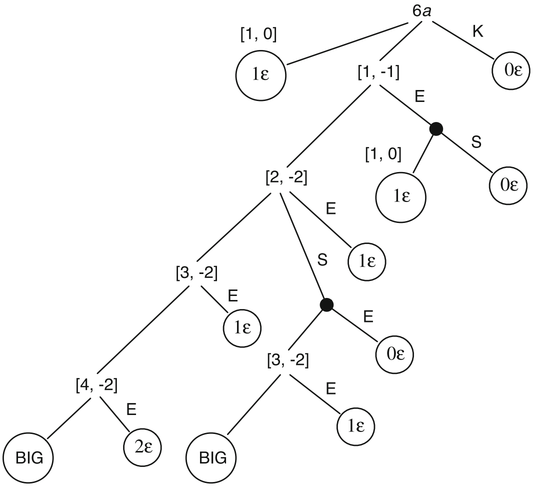
In Fig. 4, the Black king can no longer advance and White can punish any retreating move that Black might consider. So, all of Black's retreating moves are dominated by his option to cash-out to a value of 0. White, on the other hand, can move to and fro between positions a and b. This gives the formal values:

$$a = 0|b$$
$$b = 0|a$$

Since a is a follower of b and b is a follower of a, both a and b are loopy.

To define inequalities among games of this sort, the outcome (with alternating optimal play by both sides) is allowed to be any of three values, ordered from Left's

Fig. 7 Game tree of Fig. 6a



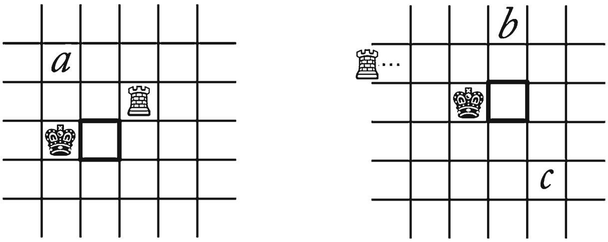
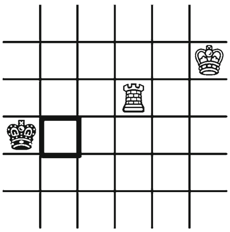


Fig. 8 Some positions which chill to a MINY in Echess

Fig. 9 A "near-terminal" position which chills to $\frac{1}{2}$



perspective as LEFTWIN > DRAW > RIGHTWIN. Let Loutcome(G) be whichever of these three possible values occurs if Left plays first from G, and let Routcome(G) be whichever of these three values occurs if Right plays first from G. Then, we say that

$$G \geq H \iff \forall \text{ games } X, \quad \text{Outcomes } (G + X) \geq \text{ Outcomes } (H + X),$$

for both Loutcome and Routcome. For loop-free games, this reduces to the traditional definitions. However, it also provides many equalities among loopy games. In particular, it is not hard to see that both a and b can be viewed as instances of the same simpler abstract value, called OVER. We denote this value with the symbol ε. OVER is a positive infinitesimal and it satisfies several more equations, including

$$\text{OVER} = \varepsilon = 0|\varepsilon = \varepsilon|\varepsilon = \varepsilon + \varepsilon.$$

The formal negative of OVER is $-\varepsilon = -\varepsilon|0$, and this game is called UNDER. Note that OVER and UNDER do not add up to zero, because their sum is a draw. If α is any conventional loop-free infinitesimal with finite birthday, then it is absorbed by ε in the sense that $\varepsilon + \alpha = \varepsilon$. Similarly, UNDER also absorbs all loop-free infinitesimals with finite birthdays.

Figure 5 shows eight positions, all having value ε. In each case, Black can do no better than to cash-out to value 0. White can do no better than to move to another position in this figure. After a sequence of several such moves, he can reach the rudimentary case of Fig. 4.

Figure 6 shows six positions of value $1\varepsilon|\varepsilon$. As usual in combinatorial game theory, implicit plus signs are omitted, so that 1ε means $1 + \varepsilon$. In each case, Black can move to $[1, 0]$. When translated by 1, this is identical to one of the positions shown in Fig. 5. White can move from each of the positions shown in Fig. 6 to a position of value ε. From Fig. 6a or b, White can move his king to $[2, -1]$. From Fig. 6e or f, White can move his king to $[2, 0]$. From Fig. 6c or d, White can move his rook to $[1, 1]$.

Figure 7 shows the game tree of position Fig. 6a. Edges going downward to the left indicate Black moves. The node reached by such a move is denoted by the resultant position of the Black king. Edges going downward to the right indicate White moves. The moves are denoted by K, E, or S, representing a king move, a rook moving east, or a rook moving south, respectively. In each case, the astute reader will quickly see the appropriate destination of the moving king. At each of the leaves of the tree shown in Fig. 7, we show a circle containing the value of the corresponding position. Most of these are direct translations of positions shown in prior figures. A notable exception occurs after E from $[2, -2]$. Its value is 1ε, because from this position, either player going second can ensure a value at least this good.

The two positions named BIG are not terminal nor are they identical, but they are both clearly very favorable to Black. It will turn out that the details of their complicated values are not too important, because an orthodox White will not allow Black to reach these positions.

5 Freezing and chilling

As explained in Chapter 6 of Berlekamp et al. (2001), *cooling* is a way to map a conventional loop-free game onto another loop-free game which is often more tractable. *Freezing* is cooling by an amount sufficiently large to map the game onto a number, which is called its *mean*. Cooling by 1 is called *chilling*. When ε is chilled, it becomes 0. When $\{1|0\}$ is chilled, it becomes $\{0|1\} = 1/2$. Thus, the positions in Fig. 5 chill to 0, and the positions in Fig. 6 chill to 1/2. If BIG is any sufficiently large game, then when $\{\text{BIG}|0|| - 1\}$ is chilled, it becomes an infinitesimal, $\{\text{BIG}'|0||0\}$. This value is a member of a class known as *Minies*. Minies are negative, but very small. Their *atomic* *weights*, as per Chapter 8 of Berlekamp et al. (2001), are zero. Minies may assume many different values corresponding to different values of the parameter BIG. The larger the BIG, the smaller the magnitude of miny. However, all minies are very small, and their values are very close to each other. For many purposes, all minies behave so much alike that it is convenient to use a single symbol m, to denote any member of this class of games. There are rare occasions when distinctions are needed. However in this paper, we eschew them. Games of this class will be discussed further in Sect. 12.

The S follower of [2, −2] in Fig. 7 chills to $1m$. Figure 8 shows some positions which chill to a MINY.

If White plays first from any of these positions, he can immediately reach a position of value -1ε. If Black plays first, his move creates a threat which, after White's response, leaves a position of value ε. From Fig. 8a–c, Black's first move is to [0, 0]. In Fig. 8a, the size of this threat (to capture the rook) is truly infinite. In Fig. 8b and c, the size of the threat (to continue to [1, 0], escaping local containment) is more modest.

Figure 9 illustrates a position which chills to $1/2$. Note that Black's move to [0, 0] and White's king move to [3, 1] reverse each other.

6 Warming inverts chilling

Infinitesimals vanish when cooled by any positive amount. Thus, all infinitesimals chill to zero, as do many other games such as $\{\frac{1}{2}\ |{-\frac{1}{2}}\}$. So, chilling is a many-to-one mapping which, in general, we could not hope to invert. However, certain classes of games have the property that all of their stopping positions are infinitesimally close to integers. Such values are said to be *integer-ish*. For such games, chilling can be reversed, at least up to the "ish". For some special classes of games, it is even possible to go further and also recover the "ish". The first such game for which this was discovered was *Blockbusting* (Berlekamp 1988), a game closely related to *Domineering*. The second such game (Berlekamp and Wolfe 1994) was *Go*, an Asian board game which for several thousand years has been considered by many to be the most demanding intellectual game ever played.

In both Blockbusting and normal Go, only two infinitesimal values occur, namely 0 and *. Since all stopping positions are integers, chilling is only a two-to-one mapping. In each game, there is a special parity rule (different for Blockbusting than for Go) which facilitates the resolution of this one-bit ambiguity. In Entrepreneurial Chess, things are simpler. There is only one infinitesimal Value, namely ε, which is added to every finite stopping position in which all three pieces are still on the board. So, it is unusually easy and straightforward to define a warming operator which inverts chilling. This operator consists of two steps: (1) overheat the value from 1 to 1, yielding an intermediate result we call the Valu. (2) Add ε, yielding Valu$+\varepsilon =$ Valu$\varepsilon =$ Value. In Entrepreneurial Chess, warming and chilling are homomorphisms. Since chilling gives simpler values without any loss of information, it is our preferred point of view. As we mentioned before, all of the positions in Figs. 4 and 5 chill to 0. Positions in Figs. 6, 7 and 9 chill to $1/2$, and positions in Fig. 8 chill to a MINY.

7 What can we know about a typical position?

When an Echess position is played alone, we would like to know its Left-stop and its Right-stop. These are the cash-out values that result from optimal alternating play if Black starts, or if White starts, respectively. But when many positions (on different boards) in several games are played as a single sum, the overall stop can be quite difficult to compute. However, as in statistics, there is a single numerical parameter, called the *mean*, which usually provides a (relatively easy to compute) good estimate of the outcome. In general, means need not be integers, but they are bounded by the inequalities

$$\text{Left-stop} \geq \text{ Mean } \geq \text{ Right-stop.}$$

Since the mean of a sum is the sum of the means, the means of individual positions are very important.

The minimal data structure of a combinatorial game G, which is sufficient to determine the outcome of $G + X$ for any other game X, is called the canonical form, or equivalently, canonical VALUE of G. We have already encountered several important values, namely 0, $1/2$, and several different minies. If we know the value of a position, the mean can be determined by freezing. However, the Value of a position is often much more difficult to determine than its mean.

8 Black's dogma

For all of the Echess positions that we have examined, a relatively simple strategy for Black proves as good as any. We call this Black strategy, which entails only one-move lookahead, DOGMATIC. Here it is:

1. If possible, capture the rook.
2. If not possible, go northeast.
3. If not possible, go north.
4. If not possible, go east.
5. If not possible, go southeast or northwest.
6. If not possible, cash-out.

Constraining Black to this strategy simplifies the analysis. By optimizing White's play against this strategy, we obtain DOGMATIC results. After obtaining them, it is often straightforward to show that Black could have done no better; that is, the dogmatic results are optimum. Thus, we simplify the discussion by omitting the word "dogmatic". To check our claims, the reader is encouraged to use the dogmatic restrictions first.

In the course of this paper, we will also develop a dogma for White. In general, our dogmatic moves are independent of whatever the summands may be in play, so that each local dogmatic game tree has breadth two, with only one local option for each player. In a sum of several games, a dogmatic player decides which summand to play in based on global parameters that may include recent history, but after that decision is made, he has no choice about what move to make there. Since there may be canonical moves which are inconsistent with whatever dogma is specified, dogmatic strategies may fail to win in some environments. However, they are much simpler. In this paper, we assert (without proof) that our dogmatic strategies are ultra-orthodox.

9 A global view

In general, the temperature of a position is the amount by which the position needs to be cooled in order to freeze it to a number. Following the terminology of the game of Go, a move which raises the temperature is said to have *sente*; a move which lowers the temperature is said to be *gote*. In WW (Berlekamp et al. 2001), pages 159–161, these were replaced by the terms *excitable* and *equitable*. We now advocate the more precise terms *unstable* and *stable*, which are defined in reference to some specified ancestral position, which is often but not always the previous position. A position is then said to be unstable if any of its intermediate ancestors have lower temperature, or stable otherwise. Stable positions can then be further partitioned into *strictly stable* positions and *quasi-stable* positions. The latter must have temperature equal to that of some relevant ancestor; the former do not. This nomenclature evades the difficulties that even the best Go players face when attempting to force marginal cases into a strict dichotomy between sente and gote. Nevertheless, because "sente" and "gote" are in such widespread use, we may use those terms when their meanings are indisputable. It turns out that many entrepreneurial chess positions are either gote or Black sente. For this reason, the mean value is usually much more closely related to the Left-stop than to the Right-stop. Hence, we found that an investigation of dogmatic Left-stops was a fruitful place to begin.

Suppose the Black king is at [0, 0], the White rook at $[k, 1]$, and the White king at $[x, y]$, where $|x| + |y| \gg |k| \gg 0$. Then, when zoomed out to a significant distance, the values and White strategies can be partitioned into six major regions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ as shown in Fig. 10.

On the global map, these regions correspond respectively to the east, the south-southeast, the southwest, the west-northwest, the north, and the northeast. Each of

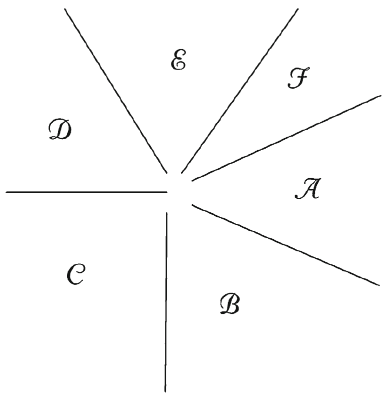


Fig. 10 Very high-level overview of how the plane is partitioned into regions

Table 2 Overview of play when the White king starts from a region's deep interior

Region	r	Targ. end	Param.	Direct.	Start. Val.	End. Val.
$\mathcal{A}$	0 or W	Figure 5d or e	x	-1	x	$(2, -1 \text{ or } 0, 2)$
$\mathcal{B}$	0 or W	Figure 5d	y	$+1$	y	$(2, -1, -1)$
$\mathcal{C}$	EW + S	Figure 5d′	$x + y$	$+2$	$x + y$	$(-1, 2, 1)$
$\mathcal{D}$	EW + S	Figure 5d′	x	$+1$	x	$(-1, 2, -1)$
$\mathcal{E}$	EW + S	Figure 5d′ or e′	y	-1	y	$(-1 \text{ or } 0, 2, 2)$
$\mathcal{F}$	EW+N	Figure 5c or b or b′	$x + y$	-2	$x + y$	$(1, 1, 2)$

these regions is best exemplified by considering one of its "typical" points, located reasonably far from the boundaries of the region, and far away from the Black king and White rook. If $[x, y]$ are the coordinates of such a point, then $|x| + |y|$ is large. If Black plays first and the players alternate turns, then the eventual cash-out value of the position will be its Left-stop $\mathcal{L}$. Dogmatic play by Black will increase at least one of his coordinates at every turn, and correct play by White will ensure that no Black move increases both of her coordinates. So, Black will get exactly $\mathcal{L}$ moves, and White will get either $\mathcal{L}$ or $\mathcal{L} - 1$ moves, depending on who gets the last move. In Go terminology, this depends on whether Black ends in sente (in which case, White gets $\mathcal{L}$ moves) or gote (in which case, White gets only $\mathcal{L} - 1$ moves). In most cases (but not all), Black keeps sente and both White and Black get $\mathcal{L}$ moves. Nearly all of White's moves are king moves, but (in some cases) a few of them, r, must be **r**ook moves.

To determine $\mathcal{L}$ for a "typical point" (located away from the boundaries of a region), we use Table 2. We assume that the Black king starts at $[0, 0]$ and the White roo$k$ starts at $[k, 1]$, where $|k| > 1$. The second column of Table 2 indicates the number and direction(s) of rook moves that White will need to make. In particular, for regions $\mathcal{A}$ and $\mathcal{B}$, White will either have to make one rook move to the west (denoted by W) or not have to make a rook move (denoted by 0). For regions $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$, White will need to make two rook moves (either east or west, and then eventually south or north). The third column of Table 2 refers to Fig. 5, which depicts the final target ending position (up to translation) just before Black cashes-out. The fourth column indicates the critical parameter(s) which the White king changes on each of his moves. Some of his moves may also include a non-critical orthogonal component which does not affect the value. In the fifth column of Table 2, the entries correspond to the change in the key parameter(s) value each time the White king moves. The sixth column indicates the starting value of the key parameter(s). Finally, the entries in the seventh column are triples (x_i, y_i, p_i), where $[x_i, y_i]$ is the final position (modulo translation) of the White king and p_i is the value (modulo translation) of the key parameter(s). We use Table 2 to obtain an equation involving x, y, r, and $\mathcal{L}$. We then solve for $\mathcal{L}$ in each of these cases (see Table 3).

Consider, for example, the optimum play from the following position in Region $\mathcal{C}$ (see Table 4). The starting locations are shown in the second column. The White rook may begin either in the far East or the far West. After Black's second move to $[2, 0]$, the White rook moves to $[6, 1]$. When attacked two moves later, he flees to the far North

Table 3 Left-stop as derived from Table 2, if $\mathscr{L}$ is an integer

Region	Solution of equation for $\mathscr{L}$
$\mathscr{A}$	$(x + r - 2)/2$
$\mathscr{B}$	$-y - 1 + r$
$\mathscr{C}$	$-x - y + 1 + 2r$
$\mathscr{D}$	$-x - 1 + 2r$
$\mathscr{E}$	$(y + r - 2)/2$
$\mathscr{F}$	$(x + y + 2r - 4)/3$

Table 4 A well-played game

	Start	1	2	3	4	5	6	7	8	9
BK	[0, 0]	[1, 0]	[2, 0]	[3, 0]	[4, 0]	[5, 0]	[5, 1]	[5, 2]	[5, 3]	[5, 4]
WK	[−3, −1]	[−2, 0]		[−1, 1]	[0, 2]		[1, 3]	[2, 4]	[3, 5]	[4, 6]
WR	[±10, 1]		[6, 1]			[6, ±10]				

or far South. When the White king eventually moves to [4, 6], he attains the terminal position which is a translation of Fig. 5d′ (by which we denote a reflection of Fig. 5d). Relative to the final position of the Black king, the final position of the White king is [−1, 2]. Black and White each made $\mathscr{L}$ moves, but two of White's moves were rook moves. All White king moves were NE, so, relative to the starting position, White's terminal king position is $[-3, -1] + (\mathscr{L} - 2)[1, 1]$, and Black's terminal king position is $[-3, -1] + (\mathscr{L} - 2)[1, 1] - [-1, 2] = [\mathscr{L} - 4, \mathscr{L} - 5]$. Black took $\mathscr{L}$ moves to get here, so $\mathscr{L} = \mathscr{L} - 4 + \mathscr{L} - 5$ and thus $\mathscr{L} = 9$. More generally, if the White king had started at $[x, y]$ in Region $\mathscr{C}$, he would have ended at $[x + \mathscr{L} - 2, y + \mathscr{L} - 2]$ and the Black king would have ended at $[x + \mathscr{L} - 1, y + \mathscr{L} - 4]$, whence $\mathscr{L} = x + y + 2\mathscr{L} - 5$ and $\mathscr{L} = 5 - x - y$.

Similar calculations obtain other formulas for the values of $\mathscr{L}$ in the remaining regions. The solutions of these equations are shown in Table 3. Of course, these equations cannot give the precise value of $\mathscr{L}$ unless the value is an integer.

When $k < 0$, within the deep interiors of Regions $\mathscr{A}$ and $\mathscr{F}$, when the formula for $\mathscr{L}$ yields an integer, this integer turns out to be the chilled value of the game. The reader may verify this fact by computing Right-stops and observing that they are equal to the Left-stops.

When $k < 0$, in Regions $\mathscr{B}$, $\mathscr{C}$, and $\mathscr{D}$, the mean turns out to be the same integer as the Left-stop. However in the other regions, the mean is often a non-integer. Fortunately, within these regions (excluding certain boundaries), the temperature is no greater than 1. In regions $\mathscr{A}$ and $\mathscr{F}$, the chilled value can be determined by a recursion which we will present in later sections.

10 Overview of the remainder of this paper

We remind the reader that, in the view of Combinatorial Game theorists, any Echess position whose value we seek may be only one component in a gallimaufry, which

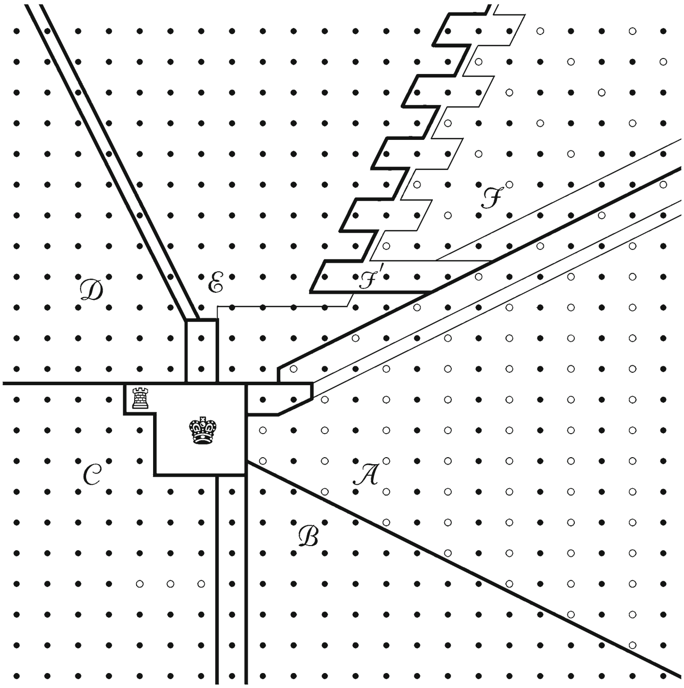


Fig. 11 Regions and their boundaries, Black king at [0, 0]; rook at [−2, 1]

is the (disjunctive) sum of several games, such as in Berlekamp (2002). So from any Echess position, one player might make several consecutive moves while his opponent plays "elsewhere" in some other summand. In the local Echess position, successive moves need not alternate between two players. Thus, stops are insufficient. So we seek ultra-orthodox values, which in some cases will turn out to be canonical as well. For brevity, we sometimes simply call the moves and lines of play which lead to such values "good". The different lines of play to which we refer do not occur because of different canonical options from any position, but rather to the possibility that either player might make the next move from each stable position. From an unstable position, there is no such choice, as an orthodox player will always respond immediately and directly to any destabilizing move.

More generally, since we are interested only in "good" lines of play, that word may sometimes become implicit. A zoomed-in look at Fig. 10 reveals Figs. 11 or 12, which show the partitioning of regions into subregions. We distinguish among these subregions with suffices which may specify the origin, number, and/or direction(s) of the rook's move(s) in good lines of play in which the White king begins within the given region. For example, $\mathcal{A}-$ denotes region $\mathcal{A}$ in Fig. 11 where the rook began with a negative x coordinate; while $\mathcal{A}+$ denotes region $\mathcal{A}$ in Fig. 12 where the rook began with a positive and moderately large x coordinate (e.g., $k = 9$). Since regions in Fig. 11 are typically simplified or degenerate versions of Fig. 12, in many cases we can omit the suffix of + or − because it is irrelevant, or clearly implied. However, especially

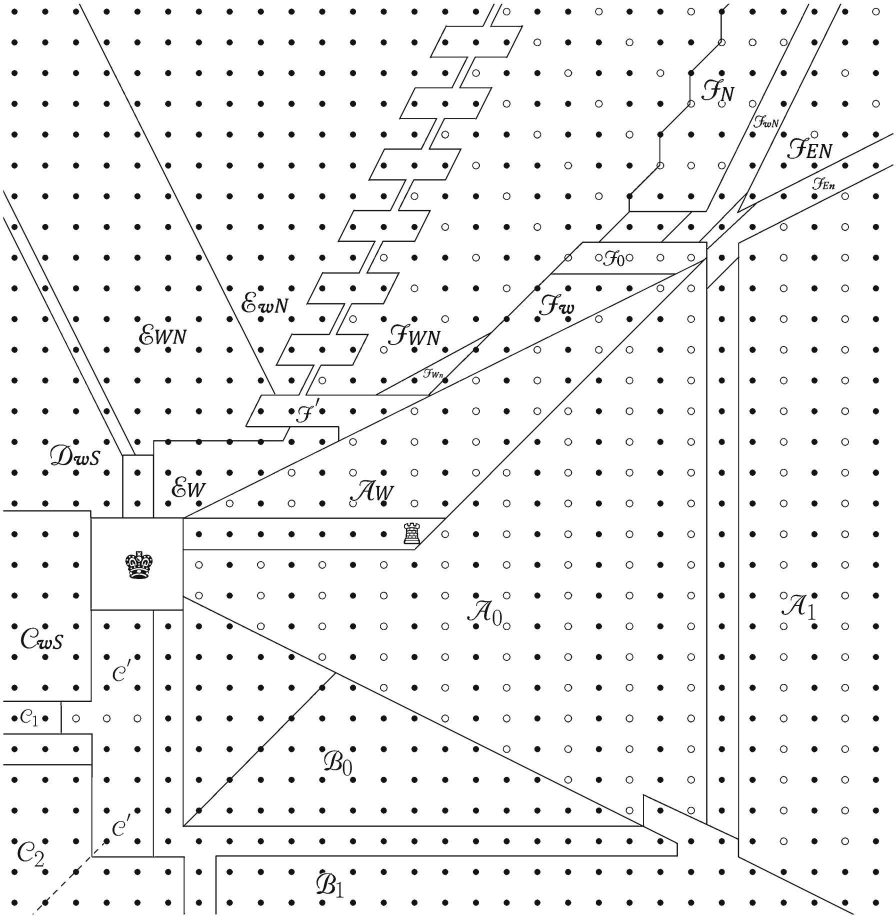


Fig. 12 Regions and their boundaries, Black king at [0, 0]; rook at [9, 1]

in Fig. 12, other suffixes are sometimes needed. $\mathcal{A}_0$ denotes a region wherein, in all (good) lines of play, the rook never moves; $\mathcal{A}_1$ denotes a region wherein in all lines of play, the rook moves exactly once. Capitalized directions, N, S, E, W, indicate a rook move which occurs in all good lines; lower case directions, namely n, s, e, w, indicate that such a rook move will occur in some lines but not others. Thus, in $\mathcal{E}_{WN}$, the rook will always move West and then North; in $\mathcal{E}_{wN}$, there are some (but not all) lines of play in which he can avoid the move West.

As k increases above 9, many of the regions in Fig. 12 grow, but no new regions appear. So, we will first examine the general case of large k before commenting on how some regions in Fig. 12 disappear or merge with each other when k descends from 9 to 2. As k approaches infinity, we will see that of those regions which remain within finite distance of the Black king, only $\mathcal{A}$ maintains any significant difference between Figs. 11 and 12.

In both Figs. 11 and 12, a white dot denotes a position whose value is an integer.

Some narrow regions lie in between major regions and are denoted by descriptions of the relevant contest, as in the row just below $\mathcal{B}_0$ and just above $\mathcal{B}_1$ near the bottom of Fig. 12. We denote this region as $\mathcal{B}_1|\mathcal{B}_0$, meaning that in this contested region, Black's move will translate to a position in $\mathcal{B}_1$; White's move, to a position in $\mathcal{B}_0$. Continuing around the map in the counterclockwise direction, we find other contested border regions: the jagged vertical $\mathcal{C}|\mathcal{B}$, the horizontal $\mathcal{C}_2|\mathcal{C}_1$, the two-point vertical $\mathcal{D}_{wS}|\mathcal{E}_W$, and the north-northwestern region $\mathcal{D}_{wS}|\mathcal{E}_{WN}$. Except for the single-point region $\mathcal{F}_2|\mathcal{F}_0$, other boundaries are uncontested. Some, such as the horizontal 4-point region $\mathcal{E}_{WN}$ and $\mathcal{E}_W$ are actually administered jointly, since although the two sides differ on White's strategy, they both attain the identical optimal result. Other joint borders run diagonally NE to SW within $\mathcal{B}_0$, and $\mathcal{C}_2$.

We will present figures that tabulate values in each irregular region where multiple boundaries come together. In each case, the patterns that arise there persist as one moves away from that irregularity. The value of any point in a narrow border region may be expressed directly as $G = G^L|G^R$. Since G has higher temperature than either G^L or G^R, this expression is already orthodox. When G is frozen, the infinitesimal it gives off is $*$, so the simple expression for G is also ultra-orthodox. In order to save space, we will omit tabulating the values in such border regions.

In the following sections, we evaluate each region in detail. We'll begin with the easiest regions, $\mathcal{A}$ and $\mathcal{B}$. Then, since White can move from $\mathcal{C}$ to $\mathcal{D}$, and from $\mathcal{D}|\mathcal{E}$ to $\mathcal{E}$, we'll investigate $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ in bottom-up order: $\mathcal{E}$, then $\mathcal{D}$, then $\mathcal{C}$. We'll then conclude with $\mathcal{F}$. In each region, if Black king is at [0, 0] and White rook at [k, 1], we'll study $k < 0$, then $k = +\infty$, and finally allow k to decrease to smaller and smaller positive integers.

11 Region $\mathcal{A}$

In Figs. 11 and 12, the simplest row in $\mathcal{A}_0$ is $y = -1$, wherein the White king consistently moves due W while the Black king moves E until they meet in a terminal position seen in Figs. 4 and 5. Since y is fixed, the Values depend only on x. Starting from $V(2) = \varepsilon$, we have the recursion

$$V(x) = \{V(x-1) + 1|V(x-1)\},$$

which chills to

$$v(x) = \{v(x-1)|1 + v(x-1)\} = v(x-1) + \{0|1\} = v(x-1) + \frac{1}{2}.$$

More generally, if the White king starts anywhere within $\mathcal{A}_0$, the White rook need never move, and the value is independent of y because the White king can move NW or SW (which might be needed in order to reach the row $y = -1$ or $y = 0$ in time to halt any further advance of the Black king).

In Fig. 12, if the White king starts on row $y = 2$ in Region $\mathcal{A}_W$, he should not move SW because doing so would block the influence of the White rook and allow the Black king to escape northward. So on this row, the White king moves W until he is close enough to the Black king to bring in the White rook to a position which is either kickable and protected or *confrontational* (i.e., a knight's move ahead of the White king). This region is called $\mathcal{A}_1$ since it requires one rook move westward. As in earlier sections, we denote the number of rook moves by the letter r. In both $\mathcal{A}_W$ and $\mathcal{A}_1$, $r = 1$ and

$$v[x, y] = \frac{x + 2 + r}{2}.$$

In Fig. 12, there is also a column between $\mathcal{A}_0$ and $\mathcal{A}_1$. We call this region $\mathcal{A}_1|\mathcal{A}_0$ because Left's move translates to a position in $\mathcal{A}_1$, but Right's move reaches a position in $\mathcal{A}_0$. Closer analysis reveals each of the chilled values of points in this region to be an *integer* + 3/4, which is consistent with the notion that the number of rook moves could be taken as $r = 1/2$. This simple interpretation is unique to Region $\mathcal{A}$; it does not work along other boundaries. However, it does work in Fig. 12, where the Region $\mathcal{A}_1$ (now located just north of $\mathcal{A}_0$) contains only two points/row. White's best move from this region is king SW. White's single rook move will be made only after the translated position has eventually left Region $\mathcal{A}_1$.

12 Region $\mathcal{B}$

In Fig. 11, the simplest point in Region $\mathcal{B}$ is [2, −2], from which the best White king move is to [2, −1] (whose value is 0). However if Black moves first, he reaches a position which translates to

$$1 + V[1, -2] = 2 + [0, -2]|1 + V[2, -1];$$

so $V[1, -2]$ chills to the hot value BIG |0, where BIG $= 1 + v[0, -2]$. Although the value of this position is complicated, it is easily seen to be bad news for White because the Black king will be able to make several moves before White can stop his advance.

Values such as BIG|0||0 are explored in WW Berlekamp et al. (2001), where they are called *minies*. They occur so frequently in Echess that we'll now give them a special notation:

$$m^0 = \text{BIG}|0||0.$$

This m^0 is a negative game. White can win it if it is played alone. The more interesting question is its effect on the total when added to a sum of other games. The answer is that m^0 is very, very small. It is infinitesimal, in the sense that the sum of any (large) finite number of them remains greater than −1. In WW Berlekamp et al. (2001), there was an attempt to quantify the canonical values of these games precisely in terms of the values of BIG. In Mathematical Go, the quest for this precision was de-prioritized because (in Go, like Echess) the canonical value of the BIG game is often quite complex. What is important is only that BIG exceed small positive numbers such as 2^{-j}; being any larger than that makes at most only a tiny difference. Thus, the interest shifts to focusing on the ancestors of the miny (i.e., earlier games in which the miny occurs as a position). One common sequence of such games (which we also encounter in this paper) satisfies the recursion:

$$m^{n+1} = m^n|0.$$

The placement of n as a superscript (rather than a subscript) is reminiscent of the notation in Mathematical Go, where (for example) BIG|0||0|||0||||0 $= m^0|0||0$ is naturally abbreviated as $m^0|0^2$. The atomic weight of m^n is $-n$. The integer n may be viewed as the number of consecutive Black king moves which White can ignore at a cost of no (chilled) points per move, after which the need for White to respond to Black becomes more urgent.

All of $\mathcal{B}-$ may be regarded as $\mathcal{B}_0$. However, $\mathcal{B}+$ contains a large southern subregion $\mathcal{B}_1$, wherein the White king is too far away to prevent the Black king from kicking the rook. The rook may then flee to the west, after which the position translates to $\mathcal{B}_0$. In $\mathcal{B}_1|\mathcal{B}_0$ (the row between $\mathcal{B}_0$ and $\mathcal{B}_1$), the chilled values are hot because the question of whether $r = 0$ or 1 depends on who plays first. The global maps of Figs. 11 and 12 contain several other subregions (like $\mathcal{B}_1|\mathcal{B}_0$) which are hot but narrow. The fact that they are hot means that an orthodox player will exit any such region immediately after his opponent enters it. Hence, the values within these regions do not propagate. The conventional form, $G = G^L|G^R$, is itself already ultra-orthodox or even canonical. So except for the case of $\mathcal{A}_1|\mathcal{A}_0$ (where the value of the non-integer term is recognizable in its more familiar form 3/4), we will leave the relevant boxes in some forthcoming figures blank. Throughout both $\mathcal{B}_0$ and $\mathcal{B}_1$, the ultra-orthodox value $v[x, y]$ is the sum of an integer $r - y - 1$ and an infinitesimal m^n, for some integer n. Although the infinitesimal term in the canonical value $v[x, y]$ may differ from m^n, this difference necessarily has atomic weight 0.

One way to verify the boundary between $\mathcal{A}$ and $\mathcal{B}$ in Fig. 11 is as follows: First, verify our claimed values along the boundaries $v[x, -1]$ and $v[2, y]$ for the relevant positive x and negative y. Then notice that at all other points, White can do well by moving his king NW. Although other White moves may have higher atomic weight, they can all be viewed as reversible, so king NW is among White's moves which are orthodox. That observation yields the following recursion for all other values with $x > 2$ and $y < -1$:

$$v[x, y] = \{v[x - 1, y]|v[x - 1, y + 1] + 1\}.$$

The solution to this recursion yields the boundary between $\mathcal{A}$ and $\mathcal{B}$, shown in Figs. 11 and 12.

In Region $\mathcal{B}$, $v^R[2, -3]$ includes both of the options $v[2, -2] = 1m^0$ and $v[3, -2] = 1m^1$. In most environments with other summands, the latter option is preferable because it has the better atomic weight. However in some environments, the former option is the only winning move. So, both options must be included in the canonical form. Hence, even though $v[2, -3] = m$, the more precise value m^0 is not canonical. This is true for most of the infinitesimals occurring in Regions $\mathcal{B}$ and $\mathcal{D}$: We can specify their atomic weights simply, but their canonical values might be more complicated.

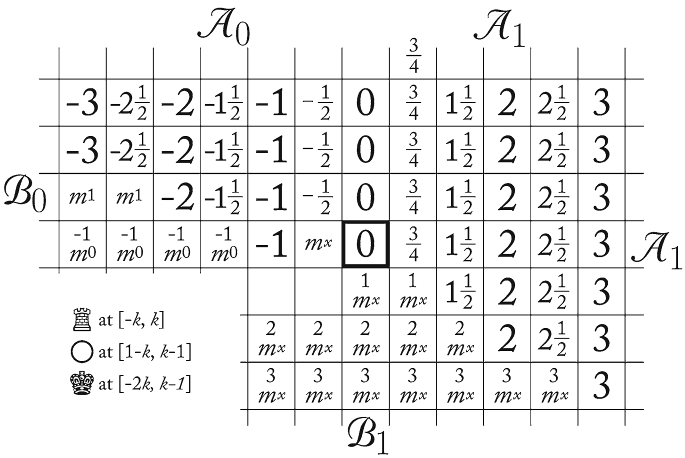


Fig. 13 Values near $\mathcal{A}_1|\mathcal{A}_0$ and $\mathcal{B}_1|\mathcal{B}_0$

In Fig. 12, the eastern end of the horizontal row $\mathcal{B}_1|\mathcal{B}_0$ terminates near the southern end of $\mathcal{A}_1|\mathcal{A}_0$.

The ultra-orthodox values in this vicinity are tabulated in Fig. 13, where the origins of x, y and v have been translated to make the figure independent of k, for all $k \geq 3$. The origin of x and y is the darkened square. The origin of v is denoted by a circle. In Fig. 13, this circle is adjacent to and diagonally SW of the rook. What appears to be the strange irregularity at $[-1, 0]$ is due to the efforts of the White king to prevent Black from kicking the rook. West of the irregularity, White can reach $y = -1$ just in time to block the kick. East of the irregularity, White can arrive at $y = 0$ to protect the kicked rook. However if White starts at the irregular point, he arrives too late to do either. So, this point lies in $\mathcal{B}_1$ rather than $\mathcal{B}_0$ or $\mathcal{B}_1|\mathcal{B}_0$.

The western end of the single row region $\mathcal{B}_1|\mathcal{B}_0$ meets the column $\mathcal{C}|\mathcal{B}$, as shown in Fig. 14. With respect to the origins of this figure, the bottom row of $\mathcal{B}_0$ is now $y = -3$, and each of its values is $-3m^0$.

In this figure, there are two subregions of $\mathcal{B}_0$. The shared boundary between them lies on the diagonal $x - y = 5$. Northwest of this boundary, the eventual threat of Black's king is to catch up with White's, exiting Region $\mathcal{B}_0$ into $\mathcal{C}|\mathcal{B}$. The atomic weight depends only on x. Southeast of this boundary, Black's eventual threat is to kick the rook, exiting Region $\mathcal{B}_0$ to $\mathcal{B}_1|\mathcal{B}_0$. Here, the atomic weight depends only on y.

We defer discussion of the left side of Fig. 14 until Sect. 15, which is about Region $\mathcal{C}$.

13 Region $\mathcal{E}$

In Regions $\mathcal{E}$ and $\mathcal{D}$, unless the two kings are reasonably close (i.e., $0 \leq y \leq 4$ and $0 \leq x \leq 6$), White's orthodox opening move is to bring the rook to confront the Black king, which then kicks the rook, who then flees. This triplet of moves: White, Black,

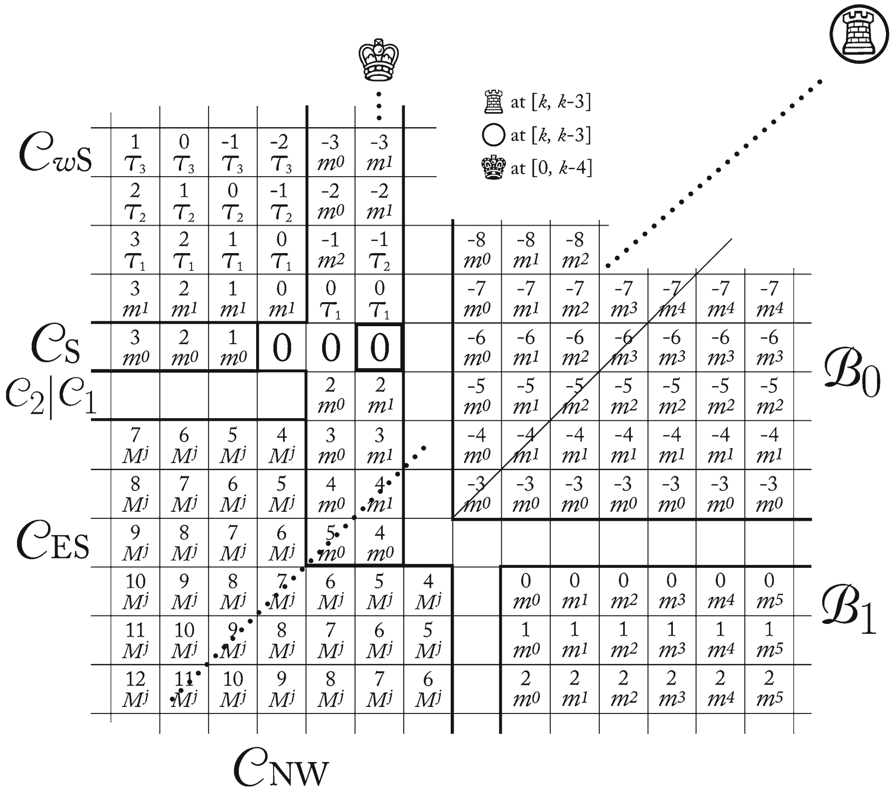


Fig. 14 Values near $\mathcal{B}_1|\mathcal{B}_0$ and $\mathcal{C}_2|\mathcal{C}_1$

White, is most conveniently viewed as a single White move. If instead Black plays first, there will eventually be a confrontation between his king and the rook, soon after which there will be a kick and a flight as before.

Fleeing northward gives an orthodox advantage only if White's king started very near the $\mathcal{E}\mathcal{F}$ boundary. Thus by temporarily excluding such points from the region under consideration, we can assume that the rook flees S. The resulting position is then a diagonal reflection of Region $\mathcal{A}-$ or $\mathcal{B}-$.

Thus, we can view the game as consisting of two separate sequences of play. The *opening* sequence ends with the pair of moves consisting of the kick and the flight. The moves which follow it constitute the *endgame*. It happens that most or all of the opening is at least as hot, and often hotter, than the endgame. So with some minor adjustments when $1 \le k \le 3$, we will view the original value of $v[x, y]$ as the sum of two components.

Region $\mathcal{E}_2$ (ALIAS $\mathcal{E}_{WN}$ in $\mathcal{E}+$ or $\mathcal{E}_{EN}$ in $\mathcal{E}-$)

In the endgame, most of Region $\mathcal{E}-$ reflects diagonally into Region $\mathcal{A}$. The row y in $\mathcal{E}$ reflects into the corresponding column of $\mathcal{A}$, yielding an endgame component of value $(y-2)/2$. To this must be added the opening component. In the opening sequence, White (if he wishes) may ignore some initial moves while Black gains one point per move. Eventually, Black's next move becomes a threat to translate the position from Region $\mathcal{E}$ to the hotter Region $\mathcal{D}$. So an orthodox White player must answer this threat. Evidently, the chilled value of this opening is of the form $2m$, by which we mean $2m^n = 2 + m^n$ for some integer n, whose value depends on the horizontal distance from the $\mathcal{DE}$ boundary.

The presumption that the kicked rook fled southward allowed us to reflect into Region $\mathcal{A}-$, avoiding the potential complications of Region $\mathcal{A}+$. However, now that we have found endgame values on each row in $\mathcal{E}_2$, we can consider the possibility that the kicked rook might flee northward. This move is essential if the White king starts adjacent to the $\mathcal{EF}$ boundary. If y is even, the rook's desired destination is directly in the path of his SW-bound king, where he can be protected just in time. If y is odd, the rook's desired destination is one square S of the king's projected SW path. This latter location also works for White king locations whose latitude is at least as eastward as the rook. For locations westward of the rook, any location sufficiently far north is good enough. For example, he might as well move to the same row as the location of his king. If White follows these guidelines, we can assume that from every position in $\mathcal{E}_2$, the rook always flees north. Then for all positive k, $\mathcal{E}+$ can be renamed $\mathcal{E}_{WN}$.

Region $\mathcal{E}_1$

Region $\mathcal{E}_1$ appears when y is small enough that White can avoid the vertical rook move. Instead, he brings the king down to $y = 2$ and then terminates the position with his single horizontal rook move.

The southern boundary of $\mathcal{E}_2$ lies on the row $y = 4$; $x = 1, 2, 3$, and 4. These points also lie in $\mathcal{E}_1$, a region within which White can move the king S (or SE or SW) until it reaches $y = 2$ (one or two squares ahead of the Black king), and which White can play his single rook move to reach the integral stop. Although $\mathcal{E}_2$ does not include the points $y = 4$; $x = 5, 6$, $\mathcal{E}_1$ does. In $\mathcal{E}_1$, the mean value of row y is $y - 1$, for $y = 2, 3$, and 4.

When k is negative, among the values shown in Fig. 15 are Region $\mathcal{E}_1$ and its vicinity. When $14 \le k \le \infty$, all values in this figure in Regions $\mathcal{B}$, $\mathcal{E}$, and $\mathcal{F}$ remain unchanged.

In Fig. 12, the points with $y = 1$ and $0 < x < k$ are hot. However, in Fig. 11, for $x \ge 4$, the points $[x, 1]$ are in $\mathcal{A}$, but $[2, 1]$ and $[3, 1]$ form their own small region $\mathcal{E}|\mathcal{A}$. From these points, Black may begin by moving his king SE. However (from either of these two positions), White can reverse Black's SE move by playing his rook one square S, yielding the value at $[1, 2]$ or $[2, 2]$ respectively. Although there are other plausible variations of play, the canonical values are as just claimed.

Region $\mathcal{E}_{wN}$

Let's look again at Fig. 12, when $k > 3$, in Region $\mathcal{E}$ when $x \approx k/2$ and y is positive and large. In this region, there are canonical lines of play in which the westward rook move is avoided. If the horizontal distance from the White king to the $\mathcal{D}|\mathcal{E}$ boundary is sufficiently large (e.g., more than k), then the value of the initial component is not $2+m$. Instead, it is 2 plus a negative number of small magnitude, $\lambda = -2^{-(k-2)}$. Since such numbers are very common in Echess, to compress them, we define $j = k - 2$ and $\lambda_j = -2^{-j}$.

Figure 16 illustrates some of the values in the opening sequences in $\mathcal{E}_{wN}$. In this figure, the locations of both White pieces are fixed, but the Black king can occupy any

10	$8M^{ON}$	$7M^{ON}$	$6m^0$	$6m^1$	$6m^2$	$6m^3$	$6m^4$	$6m^5$	$6m^6$	$6m^7$	$6m^8$	$6m^9$	$6m^{10}$	$6.$	$6\frac{1}{2}$	$6\frac{3}{4}$
9	$8M^{ON}$	$7M^{ON}$	$6(1\frac{1}{2})$	$5\frac{1}{2}m^0$	$5\frac{1}{2}m^1$	$5\frac{1}{2}m^2$	$5\frac{1}{2}m^3$	$5\frac{1}{2}m^4$	$5\frac{1}{2}m^5$	$5\frac{1}{2}m^6$	$5\frac{1}{2}m^7$	$5\frac{1}{2}$	$5\frac{3}{4}$	$5\frac{7}{8}$	$6.$	$6\frac{1}{2}$
8	$8M^{ON}$	$7M^{ON}$	$6M^{ON}$	$5m^0$	$5m^1$	$5m^2$	$5m^3$	$5m^4$	$5m^5$	$5m^6$	$5m^7$	$5m^8$	$5.$	$5\frac{1}{2}$	$5\frac{3}{4}$	$6.$
7	$8M^{ON}$	$7M^{ON}$	$6M^{ON}$	$5(1\frac{1}{2})$	$4\frac{1}{2}m^0$	$4\frac{1}{2}m^1$	$4\frac{1}{2}m^2$	$4\frac{1}{2}m^3$	$4\frac{1}{2}m^4$	$4\frac{1}{2}m^5$	$4\frac{1}{2}$	$4\frac{3}{4}$	$4\frac{7}{8}$	$5.$	$5\frac{1}{2}$	$5\frac{3}{4}$
6	$8M^{ON}$	$7M^{ON}$	$6M^{ON}$	$5M^{ON}$	$4m^0$	$4m^1$	$4m^2$	$4m^3$	$4m^4$	$4m^5$	$4m^6$	$4.$	$4\frac{1}{2}$	$4\frac{3}{4}$	$4\frac{7}{8}$	$4\frac{15}{16}$
5	$8M^{ON}$	$7M^{ON}$	$6M^{ON}$	$5M^{ON}$	$4(1\frac{1}{2})$	$3\frac{1}{2}m^0$	$3\frac{1}{2}m^1$	$3\frac{1}{2}m^2$	$3\frac{1}{2}m^3$	$3\frac{1}{2}$	$3\frac{3}{4}$	$3\frac{7}{8}$	$3\frac{15}{16}$	$3\frac{31}{32}$	$4.$	$4\frac{1}{2}$
4	$8M^{ON}$	$7M^{ON}$	$6M^{ON}$	$5M^{ON}$	$4M^{ON}$	$3m^0$	$3m^1$	$3m^2$	$3m^3$	$3m^4$	$3m^5$	$3m^6$	$3.$	$3\frac{1}{2}$	$3\frac{3}{4}$	$4.$
3	$8M^{ON}$	$7M^{ON}$	$6M^{ON}$	$5M^{ON}$	$4M^{ON}$	$3M^{ON}$	$2m^0$	$2m^1$	$2m^2$	$2m^3$	$2.$	$2\frac{1}{2}$	$2\frac{3}{4}$	$3.$	$3\frac{1}{2}$	4
2	$8M^{ON}$	$7M^{ON}$	$6M^{ON}$	$5M^{ON}$	$4M^{ON}$	$3M^{ON}$	$1m^0$	$1m^1$	$1.$	$1\frac{1}{2}$	$1\frac{3}{4}$	$2.$	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4
1	$9M^{ON}$							$1m^0$	$1m^1$	$1.$	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4
0	$10M^{ON}$	$9M^{ON}$	$8M^{ON}$	$7M^{ON}$				0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4
−1	$11M^{ON}$	$10M^{ON}$	$9M^{ON}$	$8M^{ON}$				0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4
−2	$12M^{ON}$	$11M^{ON}$	$10M^{ON}$	$9M^{ON}$	$8m^0$	$8m^1$		$1m^0$	$1m^1$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4
−3	$13M^{ON}$	$12M^{ON}$	$11M^{ON}$	$10M^{ON}$	$9m^0$	$9m^1$		$2m^0$	$2m^1$	$2m^2$	$2m^3$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4
−4	$14M^{ON}$	$13M^{ON}$	$12M^{ON}$	$11M^{ON}$	$10m^0$	$10m^1$		$3m^0$	$3m^1$	$3m^2$	$3m^3$	$3m^4$	$3m^5$	3	$3\frac{1}{2}$	4
−5	$15M^{ON}$	$14M^{ON}$	$13M^{ON}$	$12M^{ON}$	$11m^0$	$11m^1$		$4m^0$	$4m^1$	$4m^2$	$4m^3$	$4m^4$	$4m^5$	$4m^6$	$4m^7$	4
	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	10

Fig. 15 Values with Black king at [0, 0] and rook at [−4, 1]

of the annotated squares. All such squares have the same endgame value, $y/2+1$, but the values of the openings differ, as shown in the figure. We might use an accounting system which decomposes the mean value of Black's current position into its cash term, which is the number of moves he has advanced from the origin, and an accrual term of value $2+\lambda$. If the origin is translated to the Black king's current position, from which the rook is then at $[k', 1]$, the accrual term is the translated value of the opening. Its right follower is 1, the value attained if the rook moves into confrontation. The value λ satisfies the recursion:

$$\lambda_{n+1} = \lambda_n | 0.$$

With appropriate initial conditions we shall soon discuss, the solution is $\lambda_n = -2^{-n}$. In Fig. 16, when the translated value of the opening is $2+\lambda$, the temperature is λ; the Temperature is $1+\lambda$. Each Black move decreases the temperature, yielding a new stable position. When $k' = 3$, the temperature is $-1/2$, identical to the temperature

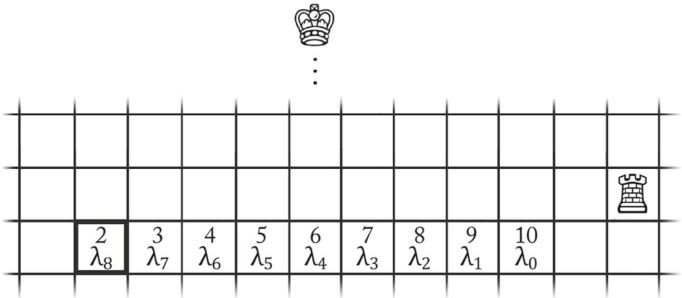


Fig. 16 Some values of the opening sequence in $\mathcal{E}_{wN}$

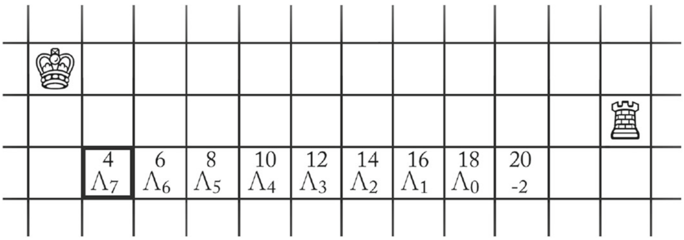


Fig. 17 Some values in $\mathcal{D}$+

of the endgame. So if White decides to move when $k' = 3$ or 2, White can move his king instead of his rook. From $k' = 3$, Black's king move and White's king move have incentive equal to $-1/2$. They can be viewed as reversals of each other, in either order. From $k' = 2$, Black's kick and White's flight are again an orthodox reversal, whether or not they were preceded or followed by another White king move at $t = -1/2$.

14 Region $\mathcal{D}$

Figure 17 is similar to Fig. 16 except that now we are in Region $\mathcal{D}$, and the figure shows the full orthodox Valu, not only its opening component. We now start with the Black king at [0, 0], the White king at $[-1, 2]$, and the rook at $[k, 1]$. If instead K starts at $[x, y]$ in Region $\mathcal{D}$, then the values in Fig. 16 should be incremented by $1 - x$. As before, if $k > 3$, White's best first move is to confront Black's king, get kicked and flee away vertically, leaving a position whose value is of the form $1 - x + m$. But because each eastward Black move now increments $-x$ as well as gaining the usual point by translation, the mean value of the initial position is now $3 - x + \Lambda$, where the mean value of Λ (like λ) is -2^{-i}, where $i = k - 3$. However, whereas the game λ is a number of temperature -2^{-i}, the game Λ is hotter, of temperature $1 - 2^{-i}$. Thermographs of translations of Λ_2, Λ_1, and Λ_0 are among those shown in Fig. 18. In accordance with the formula, the mean of Λ_0 is -1, and its temperature is 0. Its chilled value is a negative infinitesimal of atomic weight -1, like m^1. Formally, when $k = 2$ and $i = -1$, we have $\Lambda_{-1} = -2$. This happens to give the correct mean value, but the wrong temperature. Λ_{-1} is actually $-2 + m^0$.

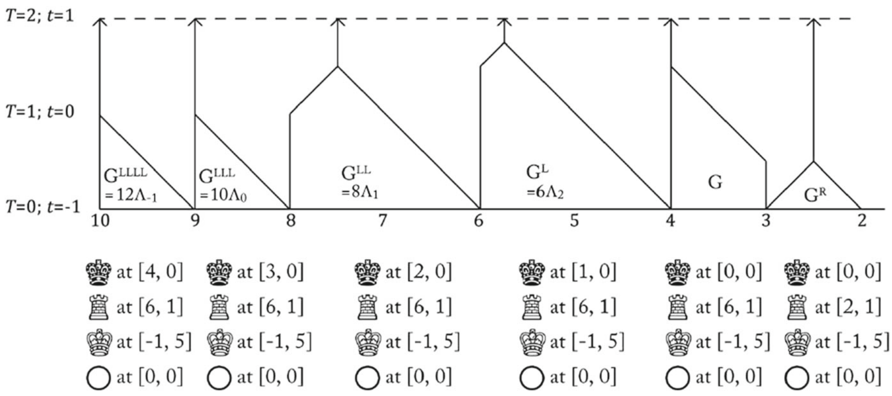


Fig. 18 Some thermographs

When, as in Fig. 18, the y-coordinate of the Black king exceeds 2, the values are increased by an infinitesimal whose atomic weight is the same as m^{y-2}.

Region $\mathcal{D}|\mathcal{E}_2$

Unlike the boundary between $\mathcal{A}$ and $\mathcal{B}$, or between $\mathcal{E}_{WN}$ and $\mathcal{E}_{wN}$, there are transitional points within the contested region $\mathcal{D}|\mathcal{E}_2$. All such points have odd values of y. The thermograph of one such point is depicted as "G" in Fig. 18. Although its Right follower is presented in the figure as Black king at [0, 0] and rook at [2, 1], we know that the kick is reversed by the flight, so that this G^R is equivalent to Black king at [1, 0] and rook at [2, −2], a position whose value is more easily seen to be a half-integer.

As one progresses upwards along any column of $\mathcal{D}$, one encounters one such transitional point, below which the temperature is $1 + \lambda_j$. At the transition point, $t = \frac{1}{2}$. Above the transition point, the values are integers plus infinitesimals having temperature 0. Below the transitional point, the mean value is $2-x+\Lambda_j$. Immediately above the transition point, the mean value is the integer $2 - x = 1 + y/2$, which increments by 1/2 thereafter until it crosses the boundary from $\mathcal{E}_{WN}$ into $\mathcal{E}_{wN}$.

Quenching

To reduce an original (warm) position depicted as Λ_j to the number -2^{-j}, it is necessary to cool it by 2. Just as cooling by 1 is called chilling, cooling by 2 is called *quenching*. In Echess, we have discovered one special case under which quenching can be reversed. When k goes to infinity, Λ obviously approaches 0 and the chilled temperature approaches 1. As the surreal numbers which appear elsewhere in combinatorial game theory distinguish among different values of infinity, some readers may find it interesting to specify the infinite value of k in greater detail. The most interesting value is k = ON, defined as ON = {ON|}. ON is so big that it satisfies ON − 1 = ON. When $k = j$ = ON (in Region $\mathcal{D}$), the value $V[x, y]$ quenches to the game

$$\text{UNDER} = \text{UNDER}|0,$$

where UNDER = −OVER and OVER is denoted by ε (which we have used in an earlier section of this paper). Whenever White decides to stop paying two points per

Black move, he moves his rook into confrontation. In this particular case, restoration by heating works; we can say that when $k = \text{ON}$ or $k < -1$ in Region $\mathcal{D}$, we have $v[x, y] = -x + 2 + m^{y-2} + \int_1^1 \varepsilon$.

15 Region $\mathcal{C}$

The partition of $\mathcal{C}$ into several subregions is shown in Fig. 12. Figure 14 shows some values in region $\mathcal{C}$ when k is a large positive number. The Black king is located at $[0, k-4]$; the White rook and the baseline of values are both at $[k, k-3]$.

Region $\mathcal{C}_1$

Since the value of a terminal position is the sum of its x and y coordinates, all of the values shown would have been unchanged if we had instead placed the circle a single king's move NW of the rook, which will be the terminal position if the White king starts at $[-2, 0]$. If Black goes first, his king will get $k+1$ moves ($k-1$ East and then 2 North), ending at $[k-1, k-2]$; White will also get $k+1$ moves (1 with his fleeing rook and k moves as his king treks from $[-2, 0]$ to $[k-2, k]$). On the other hand, if White goes first, he can do no better. Unless he makes an early rook move, his king will be blocked by the Black king, and White's king can then only move in a direction that lacks any eastward component. Nor does an early rook move prove any better. As White's rook is already optimally positioned, it has nowhere better to go. So, the value at $[-2, 0]$ is precisely an integer, namely 0. Similar arguments also reveal zero values at $[-1, 0]$ and $[0, 0]$. From all three positions, if Black starts, White's first move of K to $[-1, 1]$ is as good as any. If the White king starts at $[0, 0]$ and White plays first, he might try the confrontational R to $[2, k-3]$, then kicked northward. However, this terminates with Black cashing out for the same value as the circled position on which the rook started, namely 0. Although this Black strategy avoids the blocked king, it costs him a third rook move.

Since $\mathcal{C}$ is generally the hottest of the six primary regions, we regard the appearance of these three integer values at $[0, 0]$, $[-1, 0]$ and $[-2, 0]$ as a surprise. White's moves (east and west among these positions) are all canonical, although he could also play another move which Black could reverse.

If the White king is initially positioned anywhere along the row with $y = 0$, his rook is already optimally positioned. Outside of $\mathcal{C}'$, the next lower row, $\mathcal{C}_2|\mathcal{C}_1$ with $y = -1$, is hot. If Black moves first, he can kick the White rook before White's king is able to reach Region $\mathcal{D}$. Then, White's rook does best to flee eastward, and will need to make a vertical second move later. However, if White moves first, to the row $y = 0$, he will then be able to reach Region $\mathcal{D}$ just in time before his rook is kicked, to which it responds vertically with its single move. If $y < -1$, then White's rook will need to make at least two moves in all orthodox lines of play.

Orthodox play in $\mathcal{C}_{wS}$

If $x \leq -2$, in $\mathcal{C}_{wS}$, the mean is

$$\mu = \lambda_{y-1} - x - y,$$

where $\lambda_n = -2^{-n}$. We apply this formula even when $y = 0$ and $\lambda = -2$, thus obtaining the correct mean of 0 at $[-2, 0]$. We find it useful to view the orthodox

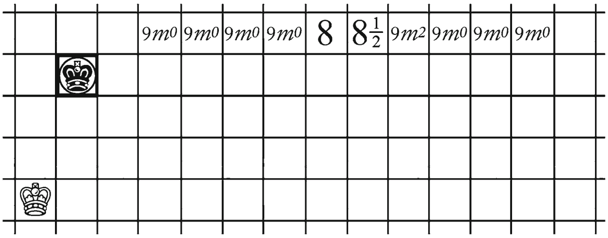


Fig. 19 Some values in column $x = -1$ of $\mathcal{C}'$

Value as the sum of two terms. The integer $-(x + y)$ is the primary term and the secondary term is the game Λ. Its mean is the same as the number $\lambda_{y-1} = -2^{-(y-1)}$, but its temperature is one degree higher than λ's. The formula also holds on the the bottom row of Region $\mathcal{D}$, where its mean value coincides with the mean values found there. From the perspective of orthodox Values, that row is shared between $\mathcal{C}$ and $\mathcal{D}$. However, $\mathcal{D}$'s claim to that row prevails because White's best move is not king NE nor rook eastward, but rook to confrontation.

In $\mathcal{C}_{wS}$, if White moves his king NE, he changes from $[x, y]$ to $[x - 1, y - 1]$, increasing the temperature. If instead, Black advances his king, he moves the origin one unit NE. Since rook and the circle around it are unchanged, this has the same effect as changing $[x, y]$ to $[x - 1, y - 1]$. So, each White king move that stays within the region can be reversed by Black. On the other hand, White might instead choose to move his rook. This leaves the horizontal component of the origin and of the king's $[x, y]$ position unchanged. So the rook move effects only the column in Fig. 14. Since the circle representing the origin of values stays with the rook, the primary term of the Value is unchanged. So, the rook's best move is to $\mathcal{C}_1$, where $y = 0$, and the secondary part of the value changes from Λ_{y-2} to -2. White's rook move improves the mean by $2 + \Lambda_{y-1}$. If Black had moved first, changing $[x, y]$ to $[x - 1, y - 1]$, he would have changed the formal fractional part of the mean from Λ_{y-1} to Λ_{y-2}, a difference of Λ_{y-1}. So evidently, from $[x, y]$ in $\mathcal{C}_{wS}$, either player can improve the mean by $2\Lambda_{y-1}$. This is equal to the Temperature and to both orthodox incentives. The position is stable, because although White might destabilize it, Black's orthodox reply will reverse its mean back to its prior value. Black's moves from $\mathcal{C}_{wS}$ are all stabilizing. In Go terminology, these Black moves are gote. However, once we exit $\mathcal{C}_{wS}$, either to $\mathcal{D}$ or to $\mathcal{C}_1$, the situation changes.

In $\mathcal{C}_1$, $y = 0$, the secondary term is -2, and the infinitesimal term is m^0. In $\mathcal{C}_{wS}$, if $y = 1$, the secondary term is -1, and the infinitesimal term is m^1.

Region $\mathcal{C}'$

Region $\mathcal{C}'$ is the subregion of $\mathcal{C}$ with $x \geq -1$ and $y \geq -4$. It requires special treatment because in some situations, the White king's northeastern trek runs into Black's. As seen in Fig. 14, all values in this region have nonpositive temperatures. The assiduous reader is invited to verify that these results are consistent with Figs. 19 and 20, which exhibit two sequences of values as a function of the location of the rook.

Region $\mathcal{C}_2$

South of $\mathcal{C}_1 = \mathcal{C}_S$ lies the Region $\mathcal{C}_2$, wherein White cannot prevent Black from eventually kicking the rook twice. As shown in Figs. 12 and 14, this region is partitioned

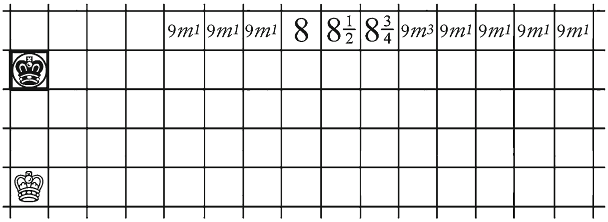


Fig. 20 Some values in column $x = 0$ of $\mathcal{C}'$

into two parts by the dotted diagonal line running through the rook. In $\mathcal{C}_{ES}$, the first time the rook is kicked, he can flee East; in $\mathcal{C}_{NW}$, he can flee North. The option to flee north is important on files such as $x = 0$ or -1, because it avoids the White king running into the Black king.

Once kicked, the rook will flee East to a position in $\mathcal{C}_1$, or North to a diagonal reflection of such a position.

The means in $\mathcal{C}_2$ are all equal to the Left-stops. They are all integers. The Temperature is 2. Black gains two points from each move preceding the kick. White could typically also gain two points by playing his king NE. The question is how many such two-point moves either player can let the other take. This is related to the atomic weight of the quenched Value of the position, where quenching means cooling by 2.

From positions in $\mathcal{C}_2$ for which $|x| > 1$, if White wants to exit the region, he can do so immediately by playing his rook the appropriate number of squares horizontally to reach $\mathcal{C}_1$.

But if Black seeks to exit the Region, he can do so only by playing $j = k - 2$ preliminary moves before he can kick the rook. So the atomic weight of the quenched game is $-j$. In Fig. 14, we denote the secondary terms of the values by M^j, which is m^j heated by one. If $k = \text{ON}$, $j = \text{ON}$. Regions $\mathcal{C}_{wS}$ and $\mathcal{C}_1$ vanish and all of $\mathcal{C}$, except $\mathcal{C}'$, becomes $\mathcal{C}_2$, wherein the secondary term of every value is M^{ON}. This same result also holds if $k < 0$.

16 Region $\mathcal{F}$

We find it instructive to examine the play from two of the westernmost points in $\mathcal{F}$ in detail, with the Black king starting at [0, 0]. For the convenience of chess players, we annex another row at the bottom of the chessboard, and denote its squares by a0, b0, c0, ... so that the digits in chess notation correspond to the value of y, while $\{a, b, c, \ldots\} = \{1, 2, 3, \ldots\}$, correspondingly.

In each of these four skirmishes (Fig. 21), the White rook may be viewed as starting at h1 = [7, 1].

If K starts at g6, then no matter who goes first, the second player can ensure that the stopping position is 4 (or better for him). Hence, that value is 4. Similarly, if the king starts at e5, then either player going second can ensure that the value is 3 1/2 (or better for him). After three full moves by each player, the warmed temperature is 1/2 and the next player (who went first originally) may spend his next move taking his half

White	Black
1. K(g6)f5	b0
2. Rd1	c0
3. Rd3	c1
4. Ke4	c2

Black	White
1. b0	Rd1
2. c0	Rd3
3. c1	K(g6)f5
4. c2	Ke4

White	Black
1. K(e5)d4	b0
2. Rd1	c0
3. Rd3	c1

Black	White
1. b0	Rd1
2. c0	Rd3
3. c1	K(e5)d4

Fig. 21 Four short games

point, to which the second player need not respond. Or in some cases, the first player may have moved to his desired integer earlier, and the second player might cease to respond then, rather than continuing to reverse the temperature back to 1/2.

Similar results occur further NNE-ward. For even y, the westernmost point in $\mathcal{F}$ is at $x = (y+6)/2$; for odd y, it is at $x = (y+3)/2$. In both cases, the value immediately to the W in $\mathcal{E}$ is only slightly less. In Fig. 11, where $k < 0$, it is only infinitesimally less. In Fig. 12, which has the same $\mathcal{E}\mathcal{F}$ boundary as in Fig. 11, the "slightly less" becomes $-2^{-(k-2)} = \lambda_j$.

A recursion for values in $\mathcal{F}$—

The defining property of $\mathcal{F}$ is that White's best king move is SW. This property can also be imposed on the subset of Region $\mathcal{A}$ for which $y > 2$. We are thereby able to find a simple recursion for all values in $\mathcal{F}$, because

$$v[x, y] = \{v^L[x, y] - 1|v^R[x, y] + 1\}$$

becomes $v[x, y] = \{v[x-1, y]|v[x-1, y-1]+1\}$. The initial conditions are given by known values at points just over the boundary in $\mathcal{A}$ or $\mathcal{E}$. Crossing over the southwestern boundary enters $\mathcal{E}_1$ at $y = 4$, where $v = 3m$.

The solution of this recursion is simplified by an important theorem (in WW), which states that if there is any number in the interval between G^L and G^R, then the value of $G = G^L|G^R$ is the simplest such number. Among integers, the simplest is the one of least magnitude. Among non-integers, the simplest number is the one whose denominator is the smallest power of two. Thus, for example, $\{3\frac{1}{2}|3\frac{7}{8}\} = \{3\frac{1}{2}|3\frac{15}{16}\} = \{3\frac{1}{2}|4m\} = \{3\frac{1}{2}|4\} = 3\frac{3}{4}$. Similarly, $\{3\frac{1}{2}|4\frac{1}{2}\} = \{4m|4\frac{1}{2}\} = 4$. But since there is no number between the infinitesimals m^1 and m^2, the simplest number theorem does not apply, and we find that $\{m^2|m^1\} = \{m^2|0\} = m^3$. We need not invoke that here, because there are no infinitesimals in Region $\mathcal{F}$. For $x \leq 10$ and $y \leq 10$, the solution to this recursion is shown in Fig. 15. Within Region $\mathcal{F}$ of Fig. 11, we often have $v[x, y] = 1 + v[x-2, y-1]$, from which we can state that $v[x, y] = n + v[x-2n, y-n]$, for all n sufficiently small (where the latter's coordinates still lie within $\mathcal{F}$).

Subregions of $\mathcal{F}$—

In Fig. 11, the values in the deep interior of $\mathcal{F}$ are given by $(x+y)/3$, rounded up to the nearest quarter integer. In our earlier discussion of stops, we justified this value when it is an integer $\mathcal{L}$. We call this region $\mathcal{F}_2$ because at every point within it, all canonical lines of play entail two rook moves. To its west, for even y, $\mathcal{F}$ abuts $\mathcal{E}$, but for odd y, it is separated from $\mathcal{E}$ by a subregion of $\mathcal{F}$, which we will call $\mathcal{F}'$. This region contains values which need to be measured in eighths, and even a unique spot at [8, 5] which requires 32nds. By excluding $\mathcal{F}'$ from $\mathcal{F}_2$, we can say that the value of the westernmost point of every row of $\mathcal{F}_2$ is an integer, as depicted by a white dot in Fig. 11. To its south, $\mathcal{F}_2$ is separated from $\mathcal{A}$ by another subregion of $\mathcal{F}$. We call this subregion $\mathcal{F}_{En}$, because all of its points have canonical positions which eventually translate into $\mathcal{E}_1$, thus eliminating the need for a vertical king move in that canonical line of play. Each row of $\mathcal{F}_{En}$ contains two points, whose values are integers plus λ_3 and λ_4, respectively.

Region $\mathcal{F}+$

As before, we assume Black king at [0, 0] and White rook at $[k, 1]$, but we now consider positive k (rather than negative k). We will start with $k = \infty$ and proceed downward until $k = 2$. Before studying this region, we will enlarge it!

Renaming top of $\mathcal{A}_1+$ as $\mathcal{F}_E$

Many types of infinity appear in ONAG and WW, the largest of which is so big that subtracting one from it leaves it unchanged. It is called ON = {ON|}. If $k = \text{ON}$, then the values are everywhere identical to those with $k = -2$, except for the large subset of Region $\mathcal{A}+$ with $y > 0$. The Region $\mathcal{A}_1-$, formerly located only along the top of $\mathcal{A}-$, where $v = (y + 2)/2$, now extends all the way down to just above the hot row $y = 1$. Within this larger $\mathcal{A}_1+$, if $x \geq 6$, from all rows with $y \geq 4$, king SW is an orthodox move. For $y \geq 6$, we now propose to to rename the region consisting of the top three rows of $\mathcal{A}_1$ as $\mathcal{F}_E$. To this end, we assume that the players alternate moves, with White playing to integers and Black reversing back to half-integers. Then from each of the top three rows in any fixed column of $\mathcal{A}$, the southwestward-trekking White king will be at $y = 2$, 3, or 4 when Black's move reduces the horizontal distance between them from 7 to 6. White then brings in the rook to one move shy of a confrontation. Black's next two moves confront and kick, but White's king continues his SW trek to arrive just in time to protect the rook, ending with $y = 2$, 1, or 0 accordingly as he began on the top, next-to-top, or third row from the top of $\mathcal{A}_1$.

Notice that $\mathcal{F}$'s top 3-row land-grab cannot be extended to the fourth row, because that would require a king move in a direction E rather than SE. Observe also that White could have played his rook move earlier, although its destination needs to be in exactly the same place.

Surveying the northeast, far and near, when k is large but finite

As shown in Fig. 12, when k is large and finite, the region that was formerly $\mathcal{F}_2$ is now split into two parts: a western subregion $\mathcal{F}_{WN}$ where the rook's horizontal move must be westward, and an eastern region $\mathcal{F}_{EN}$, where the rook's horizontal move must be eastward. For values of y significantly bigger than k, somewhere between $\mathcal{F}_{WN}$ and $\mathcal{F}_{EN}$ lies a new region, $\mathcal{F}_N$, where no horizontal rook move is needed. We will now examine this more precisely in Figs. 22 and 23. Appropriate translations relocate the origin in both of these figures to make its values independent of k. With respect to this origin, the rook is located at $[-k, -k]$; the Black king at $[-2k, -k-1]$, and the v-origin at $[-k-1, -k-1]$. If the Black king treks eastward while the White king treks SW from [0, 0], White will arrive just in time to protect his unmoved rook.

In Fig. 22, we show the dotted diagonal line running northeast from the rook through [0, 0]. The portion of Region $\mathcal{F}$ on or south of this diagonal in Fig. 22 is identical to what it was when $k = \text{ON}$. Its southernmost portion is the Region $\mathcal{F}_E$, which was stolen from $\mathcal{A}_1$. Above $\mathcal{F}_E$ is the subregion $\mathcal{F}_{En}$, wherein every game has some

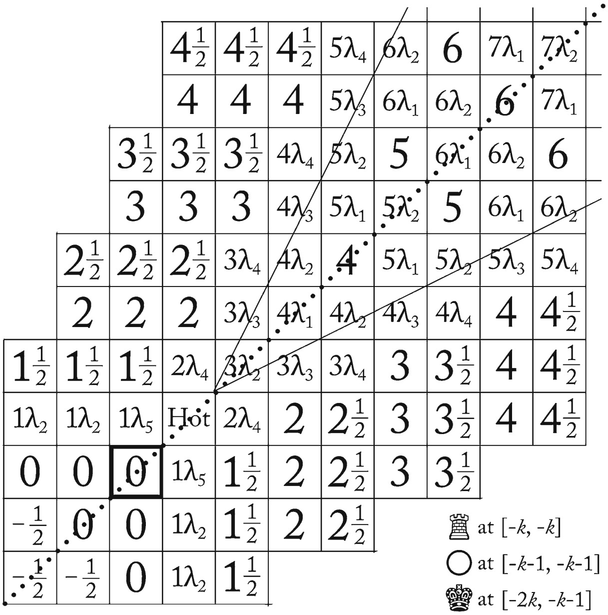


Fig. 22 Values in $\mathcal{F}_1$ and the far Northeast

canonical lines of play which require a vertical rook move and others which don't. Above that is the subregion $\mathcal{F}_2$, which we now write as $\mathcal{F}_{EN}$.

The purpose of our creation of $\mathcal{F}_E$ from the top three rows of $\mathcal{A}_1$ will now be revealed. When the rook is kicked, the values on the entire map are symmetric with respect to reflection through the diagonal containing the rook and the Black king. So, we can reflect $\mathcal{F}_{wN}$ into $\mathcal{F}_{En}$, and $\mathcal{F}_E$ into $\mathcal{F}_N$, thereby explaining all of Fig. 22 except for [0, 1], [1, 1], and [1, 0].

A relationship along the diagonals

White's best move in Fig. 22 is . always king SW. Black's best move is to advance his king. This reduces k, yielding this relationship along each diagonal:

$$v[x, y] = \{-1 + v'[x+1, y+1] | 1 + v[x-1, y-1]\},$$

where v' has a decremented value of k. For large enough k ($k > 10$ is amply sufficient), v is independent of k, so $v' = v$. (Although we are logically dependent on how this diagonal relationship evolves when k is sufficiently decremented that v' differs from v, we defer examination of that until later in this section.) When $k =$ ON, for large x and y and solution to the diagonal relationship, working southwest from any integer i is observed to be $i = (i+1)\lambda_0, i\lambda_1, (i-1)\lambda_2, (i-2)\lambda_3, \ldots$ until the diagonal encounters another integer. This requires that the value be consistent with the next encountered integer, which happens when the magnitude of the difference between any pair of adjacent values along the diagonal is less than 1.

This observation generates solutions to the missing values at [0, 1] and [1, 0]. In particular, since we know from our discussion of stops that $v[6, 5] = 5 = 6\lambda_0$, we find

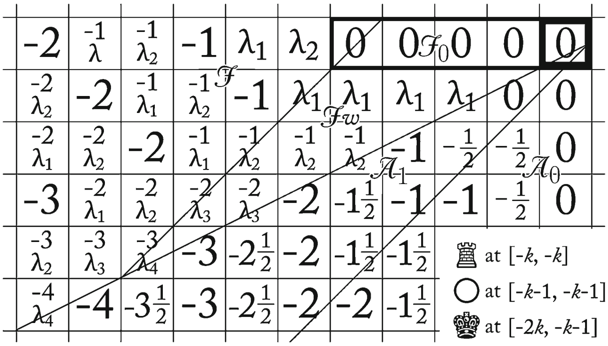


Fig. 23 Values in $\mathcal{F}_0$ and $\mathcal{F}_w$

$v[5, 4] = 5\lambda_1$; $v[4, 3] = 4\lambda_2$; $v[3, 2] = 3\lambda_3$; $v[2, 1] = 2\lambda_4$; $v[1, 0] = 1\lambda_5 = 31/32$. Similarly, from $v[3, 1] = 2 = 3\lambda_0$, we find $v[2, 0] = 2\lambda_1 = 1\ 1/2$, and $v[1, -1] = 1\lambda_2 = 3/4$. These values all reflect across the diagonal to the row $y = 1$.

The southwestern-most integer point on the shared diagonal with $x = y$ is $v[4, 4] = 4 = 5\lambda_0$, from which we verify that $v[3, 3] = 4\lambda_1$, and $v[2, 2] = 3\lambda_2$. If our prior line of argument were continued, it would lead to the assertion that $v[1, 1] = 1\ 7/8$?? But unlike all the prior results that we obtained by working down the diagonal, this fails to check out because $v[0, 0]$ is so extraordinarily favorable to White. In fact, $v[1, 1]$ is the unique point in the region $\mathcal{F}_2|\mathcal{F}_0$. Its value is

$$v[1, 1] = \{-1 + v[2, 2]|1 + v[0, 0]\} = 13/4|1.$$

Its temperature is $+3/8$. Unlike all other values in $\mathcal{F}$, it is hot.

We conclude our discussion of Fig. 22 by observing that the western boundary of $\mathcal{F}_N$ is shared with the western portion of $\mathcal{F}_{EN}$. From the points on that boundary, White has two different strategies, which both yield the same value.

Region $\mathcal{F}_w$ in the near southwest

Figure 23 shows the details of the region $\mathcal{F}_w$, where x ranges from -1 down to -4. The five points just above this region lie in $\mathcal{F}_0$ because from any of these points, no rook move is required. White's king descends SW until he reaches $y = 3$ when $k = 2$ (in the region which is then $\mathcal{E}_0$), from which he can defend the kicked rook by a single move whose direction has a southern component.

As in Fig. 22, values propagate southwest from $\mathcal{F}_0$, until the diagonals collide with Region $\mathcal{A}_1$, which dominates because its strategy gives White better values there. The points on the western boundary, namely $[-4, 0]$, $[-5, -1]$, $[-6, -2]$, $[-7, -3]$, and $[-8, -4]$, are shared with the adjacent region $\mathcal{F}_{WN}$.

Just as $\mathcal{F}_N$ splits apart $\mathcal{F}_{WN}$ and $\mathcal{F}_{EN}$, so $\mathcal{F}_w$ and $\mathcal{F}_0$ split apart $\mathcal{F}_{Wn}$ and $\mathcal{F}_{En}$.

17 The descent of *k*

Let us return to Fig. 12, with the origin at the Black king, and view the northern half-plane as seen locally from there. We consider a sequence of many different starting

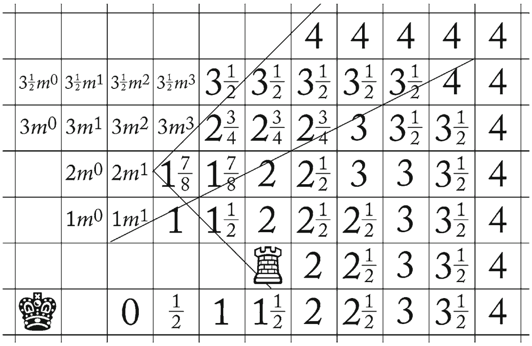


Fig. 24 Nearby values with Black king at [0, 0], rook at [5, 1]

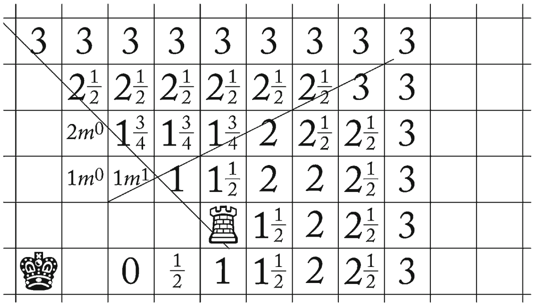
positions, with the rook coming in closer and closer from the far East. If k is odd, the southernmost point of $\mathcal{E}_{wN}$ is $[k-5, k-3]$. If k is even, they are $[k-5, k-2]$ and $[k-4, k-2]$. These are the points at which $\mathcal{E}_{wN}$ encroaches into $\mathcal{E}_{WN}$ from the north northeast. From the east northeast, encroachment into $\mathcal{F}_{wN}$ and $\mathcal{F}_{WN}$ comes from $\mathcal{F}_w$, whose lowest unique point is at $[2k-7, k-2]$. So we may distinguish between "low" rows, in which in $\mathcal{E}$ and $\mathcal{F}$ are unchanged by such encroachment(s), and "high" rows, which are. In particular, the top row of Fig. 15, with $y = 10$, is "low" if $k \geq 14$, but "high" if $k \leq 13$. So if $k \geq 14$ and $x \geq -3$, the values shown on this row remain valid, even if the tabulation were extended rightward until just before $x = 21$. That point lies in Region $\mathcal{A}$, where it becomes effected by the difference between $\mathcal{A}_0$ for negative k and $\mathcal{A}_1$ for positive k.

However, when $k = 13$, $\mathcal{E}_{wN}$ encroaches into $\mathcal{E}_2$ at [6, 10], changing $6m^{10}$ to $6\lambda_{11}$.

We view the descents of $\mathcal{E}_{wN}$ and $\mathcal{F}_w$ as the beachheads of bigger invasions, which continue regularly as k decreases down through 9. At $k = 8$, the southwestern-most point of $\mathcal{F}_w$ comes into view, invading $\mathcal{F}'$ and changing $v[8, 5]$ from $4\lambda_5$ to $4\lambda_4$. This spearhead point of $\mathcal{F}_w$, at $[2k-8, k-3]$, continues to lead the invasion through $\mathcal{F}'$ into $\mathcal{E}_W$ at $k = 7$ and $k = 6$. But as seen in Fig. 24, at $k = 5$, it encounters a newly important diagonal line running northwestward from the rook. If the White king starts south of that line, he cannot reach the rook by trekking southwestward, southward, or southeastward. However, from points on or above that line, he can. So south of that line, $\mathcal{E}_W$ prevails. On and above that line, as seen in Fig. 25, when $k \leq 4$, parts of $\mathcal{E}_W$ and $\mathcal{E}_{WN}$ are converted into new regions, $\mathcal{E}_0$ and $\mathcal{E}_w$, which are uncompetitive for $k \geq 5$. However, $\mathcal{F}_w$ does remain competitive on and above the diagonal running northwestward from the rook. In particular, when $k = 3$, it correctly yields $v[2, 2] = 3/4$.

Meanwhile, in Region $\mathcal{E}$, decreasing k yields transitions from m to λ. We have noted in Fig. 15 that the non-integer part of $v[6, 10]$ dropped from m^{10} to λ_{11} as k decremented from 14 to 13, corresponding to the change from $\mathcal{E}_2$ to $\mathcal{E}_{wN}$. More generally, in Fig. 12, for $x \leq 0$, every column of $\mathcal{E}_{WN} = \mathcal{E}_2$ contains $2k-7$ points, and for $y \geq k-3$, every row of $\mathcal{E}_2$ contains $k-3$ points if y is even or $k-4$ points if y is odd. The non-integer terms in the values on every high row of $\mathcal{E}_2$ range from m^0 to at most m^{k-2}. This maximum value of k in $\mathcal{E}_2$ can also be viewed from our prior

Fig. 25 Nearby values with Black king at [0, 0], rook at [4, 1]



discussion of Fig. 16. A term of value m^{k-1} would reduce to m^0 after $k-1$ moves, which cannot occur if the rook is then already being kicked.

So if the White king's starting position is fixed at $[x, y]$ in Region $\mathcal{E}$, there is a sense in which a closer rook is advantageous to White. That is because if he is able to wait long enough, he may be able to avoid the horizontal rook move.

However, we shall now see that if the White king's starting position is fixed at $[x, y]$ in the more distant Regions $\mathcal{F}_{EN}$ or $\mathcal{F}_{eN}$, the closer rook may be more advantageous to Black. This is because the kicked rook's preferred destination depends on the location of his king. So in some positions, White can do better if he can defer the kick until later.

More specifically, a term of canonical value λ_n will become an integer after n successive Left moves. This is feasible if $k \geq n+2$. But if $k \leq n+1$, then the kick will occur too soon, so the term's value will be converted from λ_j to m^j, where $j = k-2$.

Other regions are also effected by the descent of k. As exemplified in Fig. 12, if $y \geq k+3$, the width of $\mathcal{F}_{WN}$ depends only on the parity of y. If the point shared with $\mathcal{F}_N$ is excluded, the width of a high row with even y is $\lfloor 3(k-4)/2 \rfloor$, and the width of a high row with an odd y is $\lfloor 3(k-5)/2 \rfloor$. So as k descends through 5 to 4, $\mathcal{F}_{WN}$ gets squeezed out of existence as $\mathcal{F}_N$ begins its acquisition of $\mathcal{F}'$.

When $k < 4$, traditional regional borders become blurred.

When $k = 3$, the border Region $\mathcal{D}|\mathcal{E}_2$ merges into $\mathcal{E}_2$. In this particular case, $\mathcal{C}_{wS}$ vanishes and $\mathcal{C}_1$ directly abuts $\mathcal{D}$. Finally, as seen in Fig. 26, when $k = 2$, the row $\mathcal{C}_1$ vanishes and diagonal symmetry between y and $x-1$ prevails at all points more than two king moves away from the origin.

When $k = 2$, the even rows of $\mathcal{F}_N$ merge with $\mathcal{E}$ (!). Yet the influence of the odd rows of the former $\mathcal{F}'$ are still evident in Fig. 26, and their diagonal reflections help explain the otherwise puzzling incursions of infinitesimals into the top of Region $\mathcal{A}$, whose interior numerical stronghold is strong enough to prevail in its reflection in $\mathcal{E}$.

Evidently, if we had been clairvoyant, at the top of Region $\mathcal{A}_1$ in Fig. 22, instead of stating the values as $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$ and $4\frac{1}{2}$, we might instead have given them as $2\lambda_1$, $3\lambda_1$, $4\lambda_1$ and $5\lambda_1$. That naming would foretell the conversion of λ_1 to m^0, when $k = 2$.

The transitions from λ to m in Region $\mathcal{F}$ as k declines from 7 to 2 merits further discussion. In Fig. 26, Black's next move will kick the rook.

	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	10
10	$6m^8$	5	5	5	5	5	5	5	5	5	5	5	5	$6m^0$	$7m^0$	$7m^0$
9	$6m^7$	$5m^7$	$4\frac{1}{2}$	$4\frac{1}{2}$	$4\frac{1}{2}$	$4\frac{1}{2}$	$4\frac{1}{2}$	$4\frac{1}{2}$	$4\frac{1}{2}$	$4\frac{1}{2}$	$4\frac{1}{2}$	$5m^0$	$5m^0$	$6m^0$	6	$7m^0$
8	$6m^6$	$5m^6$	4	4	4	4	4	4	4	4	4	4	$5m^0$	$6m^0$	$6m^0$	6
7	$6m^5$	$5m^5$	$4m^5$	$3\frac{1}{2}$	$3\frac{1}{2}$	$3\frac{1}{2}$	$3\frac{1}{2}$	$3\frac{1}{2}$	$3\frac{1}{2}$	$3\frac{1}{2}$	$4m^0$	$4m^0$	$5m^0$	5	$6m^0$	$6m^0$
6	$6m^4$	$5m^4$	$4m^4$	3	3	3	3	3	3	3	3	$4m^0$	$5m^0$	$5m^0$	$5m^0$	$5m^0$
5	$6m^3$	$5m^3$	$4m^3$	$3m^3$	$2\frac{1}{2}$	$2\frac{1}{2}$	$2\frac{1}{2}$	$2\frac{1}{2}$	$2\frac{1}{2}$	$3m^0$	$3m^0$	$4m^0$	$4m^0$	$4m^0$	4.	$5m^0$
4	$6m^2$	$5m^2$	$4m^2$	$3m^2$	$2m^0$	$2m^0$	$2m^0$	$2m^0$	$2m^0$	$2m^0$	$3M^0$	$3m^0$	3.	$4m^0$	4	$4\frac{1}{2}$
3	$6m^1$	$5m^1$	$4m^1$	$3m^1$	$2m^1$	1.	1.	1.	1.	1.	$2m^0$	$3m^0$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$
2	$6m^0$	$5m^0$	$4m^0$	$3m^0$	$2m^0$	$1m^0$	$\frac{1}{2}$	$\frac{1}{2}$	1.	1.	$2m^0$	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$
1	9(4)	8(4)	7(4)	6(4)					$\frac{1}{2}$	1.	$2m^0$	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$
0	$10M^0$	$9M^0$	$8M^0$	$7M^0$				0	$\frac{1}{2}$	1.	$2m^0$	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$
−1	$11M^0$	$10M^0$	$9M^0$	$8M^0$				0	$1m^0$	1.	$2m^0$	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$
−2	$12M^0$	$11M^0$	$10M^0$	$9M^0$	$8m^0$	$7m^0$	6(6)	5(5)	$2M^0$	$2m^1$	$2m^0$	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$
−3	$13M^0$	$12M^0$	$11M^0$	$10M^0$	$9M^0$	$8M^0$	$7M^0$	6(4)	$3m^0$	$3m^1$	$3m^2$	$3m^3$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$
−4	$14M^0$	$13M^0$	$12M^0$	$11M^0$	$10M^0$	$9M^0$	$8M^0$	7(4)	$4m^0$	$4m^1$	$4m^2$	$4m^3$	$4m^4$	$4m^5$	4	$4\frac{1}{2}$
−5	$15M^0$	$14M^0$	$13M^0$	$12M^0$	$11M^0$	$10M^0$	$9M^0$	8(4)	$5m^0$	$5m^1$	$5m^2$	$5m^3$	$5m^4$	$5m^5$	$5m^6$	$5m^7$

Fig. 26 Global Values with $k = 2$. Values with Black king at [0, 0] and rook at [2, 1]. If the temperature of the position is less than or equal to 1, its chilled value is shown. If the temperature is greater than 1, then its mean, followed by its temperature in *parentheses*, is shown

Before the kick, both half-integers and minies can have v^R = integer. The distinction between them is that half-integers require that after the kick and flight, v = half-integer. If $y \gg x \in \mathcal{F}$, the rook does best to flee eastward to create a new position in Eastern $\mathcal{F}_N$, where half-integers have $3k = 2x + 3 - y$, and integers have $3k = 2x + 2 - y$, or $3k = 2x + 4 - y$. The half-integer formula can be restated as $3(k + y - 1)/2 = x + y$. So to be able to reach a good half-integer value post-kick, the pre-kick value of $x + y$ must be congruent to 1, modulo 3. This is satisfied by the locations of the half-integer values in $\mathcal{F}_2$ for large k. But, mod 3, the locations of $\mathcal{F}_2$ where $x + y$ is congruent to 2 are only quarter-integers for large k. But in $k = 2$, on the next Black move they get kicked, and then White can do no better than return them to integers. So evidently, locations in $\mathcal{F}_2$ whose chilled values had formal fractional terms of λ_2 when $k \geq 3$, now all increase to the miny m^0 when $k = 2$.

This same phenomena occurs earlier. When $k \leq n$, the fraction λ_n increases to a negative infinitesimal of a form we call m. The question of how long White can play elsewhere until the value becomes an integer now reverts to the question of how long White can play elsewhere until his indefensible rook gets kicked. Assuming $k > 1$, the answer is evidently $j = k - 2$. Thus, for example, when k changes from 6 to 5, λ_5 becomes m^3. This is consistent with the diagonal relationship described earlier.

If we had computed values by induction, incrementing k upwards from $k = 1$, we would find apparent chaos for small k before the asymptotic pattern emerges when k reaches the large single digits. By describing the results working backwards from infinity, we believe we have been able to provide a much better understanding of what would otherwise have appeared unduly complex.

18 Greater accuracy

We say a game is *barely* frozen when it is cooled by its Temperature. The game's primary infinitesimal is the difference between its barely frozen result and its mean. The shape at the top of the thermograph reveals only the sign of this primary infinitesimal, which can be either positive, negative, or fuzzy. The most common fuzzy infinitesimal, by far, is $*$, called STAR. The most common positive infinitesimals are tinies; the most common negative infinitesimals are minies. All of these very common primary infinitesimals have zero atomic weight. But in this paper, we encounter many primary infinitesimals which are negative and have negative atomic weights. These include m^n and M^n. The former can be viewed as the latter, cooled by one.

Nearly all of the values we've presented are ultra-orthodox approximations to the canonical. They all have the correct means, and their primary infinitesimals have the correct atomic weights. Since every Λ and λ with positive subscript has STAR as its primary infinitesimal, for them the distinction between orthodox and ultra-orthodox is trivially satisfied. When $k \geq 7$, in Region $\mathcal{A}$ and in all of $\mathcal{F}$ excepting the lone point at $\mathcal{F}_2|\mathcal{F}_0$, the temperatures are negative. Such games are dyadic rationals. Their orthodox forms are unique, and identical to the canonical forms. This also holds for Region $\mathcal{E}_{wN}$. In the remainder of Region $\mathcal{E}$, the temperatures are zero, but the values we have found there for the primary infinitesimals also turn out to be canonical. In Region $\mathcal{B}$, the values we have found are ultra-orthodox, but not necessarily canonical. Here, there may be canonical options of less desirable atomic weights, which are excluded from the simpler and more tractable orthodox forms, m^n.

In Region $\mathcal{D}$, the temperature is $1 + \lambda_i$, where $i = k - 3$. If White confronts, after the rook kick and flight, the temperature drops to zero, and the atomic weight is $2 - y$. Hence, in Region $\mathcal{D}$, we can obtain a better-than-ultra-orthodox approximation to the canonical value by adding m^{2-y} as another term.

When $k > 3$, temperatures in $\mathcal{D}$ are positive, and $\mathcal{D}$ abuts $\mathcal{C}_{wS}$. But when $k = 3$, Region $\mathcal{C}_{sW}$ vanishes. Region $\mathcal{D}$ then abuts Region $\mathcal{C}_1$. In both $\mathcal{C}_1$ and $\mathcal{D}$, the temperature is 0. The ultra-orthodox value is the sum of an integer, $2 - x$ and an infinitesimal, m^{y-1}. From White's perspective, this has atomic weight one better than m^{y-2}, which occurs more commonly in descendants of positions in Region $\mathcal{D}$. The reason is that, unlike Region $\mathcal{D}$ when k is larger, Black's move which decreases the translated k from 3 to 2 now costs only one point, while leaving the temperature

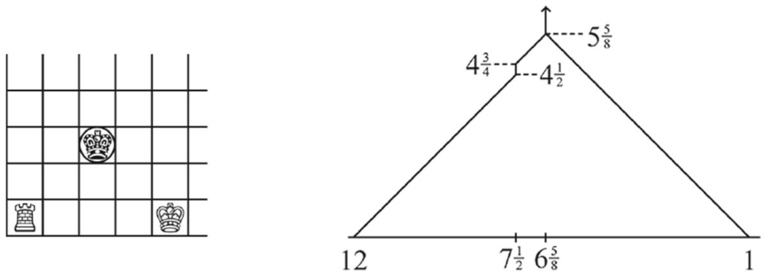


Fig. 27 An Echess position with temperature $5\frac{5}{8}$

unchanged. So there is a sense in which the temperature of Λ_i could be viewed as $t = 1 - 2^{-i}(1 + m^1)$, which is infinitesimally cooler than its real-valued temperature.

19 The hottest finite temperature

We conjecture that the hottest finite temperature in Echess is $5\frac{5}{8}$. Figure 27 shows an example of such a position and its thermograph. We challenge the interested reader to compose a hotter example.

Acknowledgements In 2005, Mark Pearson, then a student in Berlekamp's graduate course on Combinatorial Game Theory, determined the means and temperatures shown in Figs. 15 and 26. All of our subsequent calculations have been done by hand. The authors wish to thank Eugenia Lee for creating the beautiful figures found throughout this paper. We are also indebted to David Berlekamp, both for his technical prowess in dealing with several mutually incompatible document preparation technologies, and for his help in debugging the mathematics in some of the sections of this paper. We are grateful for the valuable comments made by the referee and Editor. Finally, the authors wish to thank Bernhard von Stengel for his help in resolving some unforeseen typesetting problems as this paper went to press.

References

Albert MH, Nowakowski RJ (2009) Games of no chance 3 (Berkeley, CA). Math. Sci. Res. Inst. Publ., vol 56, Cambridge Univ. Press, Cambridge

Albert MH, Nowakowski RJ, Wolfe D (2007) Lessons in play: an introduction to combinatorial game theory. A. K. Peters, USA

Berlekamp ER (2002) The 4G4G4G4G4 problems and solutions. In: More games of no chance (Berkeley, CA, 2000), Math. Sci. Res. Inst. Publ., vol 42, Cambridge Univ. Press, Cambridge, pp 231–241

Berlekamp ER (1988) Blockbusting and domineering. J Comb Theory Ser A 49:67–116

Berlekamp ER, Conway JH, Guy RK (2001) Winning ways for your mathematical plays, Volume 1 (of 4), AK. Peters, USA

Berlekamp ER, Wolfe D (1994) Mathematical go: chilling gets the last point. AK Peters, USA

Conway JH (1976) On numbers and games. Academic, London

Fraenkel AS (1996) Combinatorial games: selected biography with a succinct gourmet introduction. In: Games of no chance (Berkeley, CA, 1994), Math. Sci. Res. Inst. Publ., vol 29. Cambridge Univ. Press, Cambridge

Guy RK, Nowakowski RJ (2002) Unsolved problems in combinatorial games. In: More games of no chance (Berkeley, CA, 2000), Math. Sci. Res. Inst. Publ., vol 42. Cambridge Univ. Press, Cambridge, pp 457–473

Guy RK, Nowakowski RJ (2002) More games of no chance (Berkeley, CA 2000), Math. Sci. Res. Inst. Publ., vol 42. Cambridge Univ. Press, Cambridge

Low RM, Stamp M (2006) King and rook vs. king on a quarter-infinite board. Integers. 6, #G3:8 **(electronic)**

Nowakowski RJ (2015) Games of no chance 4 (Berkeley, CA), Math. Sci. Res. Inst. Publ., vol 63. Cambridge Univ. Press, Cambridge

Nowakowski RJ (1996) Games of no chance (Berkeley, CA, 1994), Math. Sci. Res. Inst. Publ., vol 29. Cambridge Univ. Press, Cambridge

Pearson M, Berlekamp ER (2005) Entrepreneurial chess: some king and rook versus king combinatorial game theory problems (**unpublished**)

Snatzke RG (2002) Exhaustive search in the game Amazons. In: More games of no chance (Berkeley, CA, 2000), Math. Sci. Res. Inst. Publ., vol 42. Cambridge Univ. Press, Cambridge, pp 261–278

Siegel AN (2013) Combinatorial game theory, graduate studies in mathematics, vol 146. American Mathematical Society, Providence, RI