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Pure-strategy Nash equilibria in large games: characterization and existence

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Abstract In this paper, we first characterize pure-strategy Nash equilibria in large games restricted with countable actions or countable payoffs. Then, we provide a counterexample to show that there is no such characterization when the agent space is an arbitrary atomless probability space (in particular, Lebesgue unit interval) and both actions and payoffs are uncountable. Nevertheless, if the agent space is a saturated probability space, the characterization result is still valid. Next, we show that the characterizing distributions for the equilibria exist in a quite general framework. This leads to the existence of pure-strategy Nash equilibria in three different settings of large games. Finally, we notice that our characterization result can also be used to characterize saturated probability spaces.

Keywords Large games · Pure-strategy Nash equilibrium · Characterization · Atomless probability space · Saturated probability space

1 Introduction

A *large game* models its agent space with an atomless probability space which captures the predominant characteristic in a large conflicting economy whereby a single player is negligible but a group of players are influential. Over the past few decades, research on large games has been fruitful. Various results on the existence or non-existence

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of pure or mixed strategy Nash equilibria are determined in various settings of large games.[1]

However, most studies on large games focus on showing the existence or non-existence of Nash equilibria but few pay attention to characterizing the equilibria, which, from our point of view, might be a loss in the literature. From this paper, we can see that a good characterization result helps explain equilibria from another perspective which enhances our understanding of them. Moreover, it can also provide an alternative and even easier way to show the existence of equilibria.[2]

We start by considering a generalized large game where the agent space is divided into countable (finite or countably infinite) different subgroups and each player's payoff depends on her own action and the action distribution in each of the subgroups.[3] In such a large game, a pure-strategy action profile that assigns an action to each player is called a *(pure-strategy) Nash equilibrium* if no player has the incentive to deviate from her assigned action. A *(pure-strategy) equilibrium distribution* is a distribution on the action space that is induced by a pure-strategy Nash equilibrium.

If such a large game is further restricted by having (i) *a countable action space* or (ii) *a countable payoff space* or (iii) *a saturated probability space of agents*, then a given distribution on the action space is shown to be an equilibrium distribution if and only if for every Borel, closed or open subset of actions, the players in each subgroup playing actions in it are no more than the players having a best response in the set. We also show through a counterexample that if both actions and payoffs are uncountable and the agent space is a general probability space, say the Lebesgue unit interval, then a similar characterization result is *not* valid.

Following these characterization results, we proceed to show the existence of the characterizing distribution for the equilibrium. Our result (Theorem 5) reveals that the characterizing distributions do exist and they exist in a much more general framework than the equilibria. In particular, there is no need to impose any further restrictions on the agent, action or payoff space other than the regular conditions that define a large game. This result, together with the previous characterization results, leads to the existence of pure-strategy Nash equilibria in three settings of large games under countability or saturation assumption. These existence results generalize or parallel the corresponding results in Khan and Sun (1995, 1999) and also include a new scenario showing the existence of pure-strategy Nash equilibria in large games endowed with at most countably many different payoffs while dropping any countability or saturation restrictions on the agent or action space.

Throughout the paper, we present quite a number of results on the characterization or existence of pure-strategy equilibria in large games. However, our paper is not

[1] Interested readers can refer to (Khan and Sun 1999, 2002), Kalai (2004), Khan et al. (2013).

[2] Blonski (1999) provides a characterization result for the case of two actions and the result of Blonski (2005) is confined to a finite action space. Our paper works on both countable and uncountable action spaces.

[3] The large game discussed here is a generalization to the large non-anonymous games discussed in (Khan and Sun 1999, 2002).

tedious and our proofs are mostly elementary. This can be seen as another advantage in considering the existence of pure-strategy equilibria via characterization.[4]

The paper is organized as follows. Section 2 introduces the large game model. Section 3 presents all the characterization results. Section 4 shows the existence of characterizing equilibria and hence, also pure-strategy equilibria. Section 5 contains some concluding remarks.

2 Large game model

Let $(T, \mathscr{T}, \lambda)$ be an atomless probability space of *players* and I a countable (finite or countably infinite) index set. Let $(T_i)_{i\in I}$ be a measurable partition of T with positive λ-measures $(\alpha_i)_{i\in I}$, i.e., $\alpha_i = \lambda(T_i) > 0$ for all $i \in I$. For each $i \in I$, let λ_i be the *re-scaled probability measure* obtained from the restriction $\lambda|_{T_i}$ of λ on T_i, namely $\lambda_i(S) = \lambda(S)/\lambda(T_i)$ for any measurable set S in T_i. By introducing this partition, we imply that the players are divided into I groups.

Let the *action space*, denoted by A, of the game be a Polish space with $\mathscr{B}(A)$ its Borel σ-algebra and $\mathscr{M}(A)$ the set of all Borel probability measures on A. Suppose that all the players in each group $i \in I$ choose their *actions* from a common compact subset A_i of A. Without loss of generality, we assume that $(A_i)_{i\in I}$ are disjoint of each other.[5] For ease of notation, we define an action correspondence $K : T \twoheadrightarrow A$ for all players such that $K(t) = A_i$ for all $t \in T_i$. Let $\mathscr{M}(A_i)$ be the set of all Borel probability measures on A_i endowed with the topology of weak convergence of probability measures and $\prod_{i\in I} \mathscr{M}(A_i)$ the usual product space endowed with the product topology. For ease of notation, we let $\Theta := A \times \prod_{i\in I} \mathscr{M}(A_i)$ and $\Theta_i := A_i \times \prod_{i\in I} \mathscr{M}(A_i)$, $i \in I$.[6]

The *payoff function* (or simply, *payoff*) of each player depends on her own action as well as on the distribution of actions played by the players in each of the groups. Mathematically, we let the space of *payoffs* be the space of all continuous real-valued functions on Θ which is denoted by $\mathcal{C}(\Theta)$ and endowed with the topology of compact convergence.

Definition 1 Given player space T and action space A, a *large game* is a measurable mapping U from T to $\mathcal{C}(\Theta)$.[7] A measurable function $f : T \to A$ is called a *pure-strategy profile* if $f(t) \in K(t)$ for all $t \in T$. A pure-strategy profile f is called a *pure-strategy (Nash) equilibrium* if for λ-almost all $t \in T$,

[4] The proof of our first result uses Bollobas and Varopoulos (1975)'s extension of the famous marriage theorem (or the Hall's theorem) and the proof of the third result relies on Keisler and Sun (2009)'s result on the distributional properties of correspondence on saturated probability spaces.

[5] If initially, $(A_i)_{i\in I}$ are not disjoint, we can always introduce a disjoint set of action sets $(A'_i)_{i\in I}$ by adding an index dimension to the original action sets while keeping the same topological structure. For example, if $A_1 = A_2 = \{a, b\}$, we can let $A'_1 = \{(1, a), (1, b)\}$ and $A'_2 = \{(2, a), (2, b)\}$.

[6] Unless otherwise specified, any topological space discussed in this paper is tacitly understood to be equipped with its Borel σ-algebra (the σ-algebra generated by the family of open sets) and measurability is defined in terms of it.

[7] Such a large game is often called a large non-anonymous game in the literature. See e.g., Khan and Sun (2002).

$$U(t)[f(t), (\lambda_i f_i^{-1})_{i\in I}] \geq U(t)[a, (\lambda_i f_i^{-1})_{i\in I}] \text{ for all } a \in K(t),$$

where f_i is the restriction of f to T_i. A distribution $\mu \in \mathscr{M}(A)$ is called an *equilibrium distribution* if there exists a pure-strategy equilibrium f such that $\mu = \lambda f^{-1}$.

Given a pure-strategy profile $f : T \to A$ and its induced distribution $\mu := \lambda f^{-1}$, let f_i be the restriction of f to T_i and μ_i the *re-scaled probability measure* of μ on A_i. Since $(A_i)_{i\in I}$ are disjoint sets, $f_i^{-1}(B) = f^{-1}(B)$ for all $B \in A_i$ and hence, for any $B \in \mathscr{B}(A_i)$, $\mu_i(B) = \frac{\mu(B)}{\mu(A_i)} = \frac{\lambda f^{-1}(B)}{\lambda f^{-1}(A_i)} = \frac{\lambda f_i^{-1}(B)}{\lambda f_i^{-1}(A_i)} = \frac{\lambda f_i^{-1}(B)}{\lambda(T_i)} = \lambda_i f_i^{-1}(B)$. Thus, we have $\mu_i = \lambda_i f_i^{-1}$ for all $i \in I$.

Recall that a *correspondence* F from T to A, denoted by $F : T \twoheadrightarrow A$, is called *measurable* if for each closed subset C of A, the set

$$F^{-1}(C) := \{t \in T : F(t) \cap C \neq \emptyset\}$$

is measurable in $\mathscr{T}$. A function f from T to A is said to be a *measurable selection* of F if f is measurable and $f(t) \in F(t)$ for all $t \in T$. When F is measurable and closed valued, the classical Kuratowski-Ryll-Nardzewski Theorem [see e.g., Aliprantis and Border (1999, p 567)] shows that F has a measurable selection.

Given an arbitrary probability measure $\mu \in \mathscr{M}(A)$, the *best responses* of player t facing the collective behavior μ is given by

$$B^{\mu}(t) := \arg\max_{a\in K(t)} U(t)(a, (\mu_i)_{i\in I})$$

where μ_i is the re-scaled probability measure of μ on A_i. By the Measurable Maximum Theorem [Aliprantis and Border (1999, p 570)], B^{μ} is a *measurable correspondence* from T to A, has nonempty compact values and admits a measurable selection. Let $B_i^{\mu} : T_i \twoheadrightarrow A_i$ be the restriction of B^{μ} to T_i.

3 Characterizing large games

Unless otherwise specified, throughout this section, we follow all the notations defined in the last section.

3.1 Large games with countable actions

Our first result is on large games with countable actions.

Theorem 1 *Let $\mu \in \mathscr{M}(A)$ and μ_i the re-scaled probability measure of μ on A_i. If the action space A in the large game U is countable, then the following statements are equivalent:*

(i) *μ is an equilibrium distribution;*
(ii) *for each $i \in I$, $\mu_i(C) \leq \lambda_i[(B_i^{\mu})^{-1}(C)]$ for every subset C in A_i;*

(iii) *for each* $i \in I$, $\mu_i(D) \leq \lambda_i[(B_i^{\mu})^{-1}(D)]$ *for every finite subset* D *in* A_i.

To prove this theorem, we need the following lemma from Khan and Sun (1995), which is a special case of the famous marriage theorem offered by Bollobas and Varopoulos (1975).[8]

Lemma 1 (Khan and Sun 1995, Theorem 4)[9] *Let* $(T, \mathcal{T}, \lambda)$ *be an atomless probability space,* I *a countable index set,* $(T_i)_{i \in I}$ *a family of sets in* $\mathcal{T}$ *and* $(\alpha_i)_{i \in I}$ *a family of non-negative numbers. Then, the following two statements are equivalent*

- $\lambda(\bigcup_{i \in D} T_i) \geq \sum_{i \in D} \alpha_i$ *for all finite subsets* D *of* I*;*
- *there is a family of sets,* $(S_i)_{i \in I}$*, in* $\mathcal{T}$ *such that for all* $i, j \in I, i \neq j$*, one has* $S_i \subseteq T_i$*,* $\lambda(S_i) = \alpha_i$ *and* $S_i \cap S_j = \emptyset$.

Proof of Theorem 1 (i)$\Rightarrow$(ii): Let μ be an equilibrium distribution. Then by definition, there exists a Nash equilibrium $f : T \to A$ such that $\mu = \lambda f^{-1}$. Notice that for each $i \in I$, $f_i(t) \in B_i^{\mu}(t)$ for all $t \in T_i$. Thus, for any $i \in I$ and for every $C \subseteq A_i$,

$$\mu_i(C) = \lambda_i(f_i^{-1}(C)) = \lambda_i\Big(\{t \in T_i : f_i(t) \in C\}\Big)$$
$$\leq \lambda_i(\{t \in T_i : B_i^{\mu}(t) \cap C \neq \emptyset\}) = \lambda_i\Big[(B_i^{\mu})^{-1}(C)\Big].$$

(ii) $\Rightarrow$ (iii): It is obvious.

(iii)$\Rightarrow$(i): Suppose (iii) holds. Fix an arbitrary $i \in I$. Since A is a countable set, it can be written as $A := \{a_1, a_2, \ldots\} = \{a_j\}_{j \in \mathbb{N}}$. For each $j \in \mathbb{N}$, let $\beta_j := \mu_i(\{a_j\})$ and $T_i^j := (B_i^{\mu})^{-1}(\{a_j\}) = \{t \in T_i : a_j \in B_i^{\mu}(t)\}$. Let D be an arbitrary finite subset of $\mathbb{N}$. Observe that $(B_i^{\mu})^{-1}(\bigcup_{j \in D}\{a_j\}) = \bigcup_{j \in D} T_i^j$. Statement (iii) tells that $\sum_{j \in D} \beta_j = \mu_i(\bigcup_{j \in D}\{a_j\}) \leq \lambda_i(\bigcup_{j \in D} T_i^j)$. Thus, by Lemma 1, there exists a family of sets, $(S_j)_{j \in \mathbb{N}}$, such that for all $j, k \in \mathbb{N}, k \neq j$, one has $S_j \subseteq T_i^j$, $\lambda_i(S_j) = \beta_j$ and $S_j \cap S_k = \emptyset$.

Now, define a measurable function $h_i : T_i \to A$ such that for all $j \in \mathbb{N}$ and for all $t \in S_j$, $h_i(t) = a_j$. Since for any $j \in \mathbb{N}$, $t \in S_j$ implies that $a_j \in (B_i^{\mu})(t)$, we have $h_i(t) \in B_i^{\mu}(t)$ for all $t \in T$. Furthermore, $\lambda_i(h_i^{-1}(\{a_j\})) = \lambda_i(S_j) = \beta_j = \mu_i(\{a_j\})$ for all $j \in \mathbb{N}$, which implies $\lambda_i h_i^{-1} = \mu_i$. Repeat the above arguments for all $i \in I$ and define a measurable function $h : T \to A$ by letting $h(t) = h_i(t)$ if $t \in T_i$. Thus, it is clear that h is a pure strategy Nash equilibrium and $\mu = (\mu_i)_{i \in \mathbb{N}} = \lambda h^{-1}$ is the equilibrium distribution induced by h. □

Remark 1 Note that μ is an equilibrium distribution if and only if there exists a measurable selection f of B^{μ} such that $\mu = \lambda f^{-1}$. Hence, if μ is an equilibrium distribution, then $\mu_i(C) = \lambda_i(f_i^{-1}(C)) = \lambda_i\{t \in T_i : f_i(t) \in C\}$, is simply the proportion of players playing their actions in C. Therefore, the above theorem literally says that a

[8] This lemma was also used by Yu and Zhang (2007) to show the existence of pure-strategy equilibria in games with countable actions.

[9] Throughout the paper, we refer to results previously available in the literature as "Lemma".

distribution on the product action space is an equilibrium distribution if and only if for any subset or any finite subset of the actions, there are fewer players in each group playing their actions in the subset than having a best response in it. It should be noted that the case that $|I| = 1$ and A is finite in our theorem is the main result in Blonski (2005).

3.2 Large games with countable payoffs

In the last section, we characterize large games with a countable set of actions. One may wonder if we can allow an action space without the countability restriction. The answer is “yes” provided that there are only countably many payoff functions in the game or equivalently, all the players in each group play a common payoff function.

Definition 2 The players in a large game U is said to be *homogeneous* if for each group $i \in I$, $U(t)$ is same for all $t \in T_i$.

Since the total number of elements in a countable collection of countable sets is still countable, this definition of homogeneity is equivalent to assuming that in each group, there are at most countably many payoff functions for its players.

Theorem 2 *Let $\mu \in \mathscr{M}(A)$ and μ_i the re-scaled probability measure of μ on A_i. If the players in the large game U are homogeneous, then the following statements are equivalent:*

(i) *μ is an equilibrium distribution;*
(ii) *for each $i \in I$, $\mu_i(C) \leq \lambda_i[(B_i^{\mu})^{-1}(C)]$ for every Borel subset C in A_i;*
(iii) *for each $i \in I$, $\mu_i(D) \leq \lambda_i[(B_i^{\mu})^{-1}(D)]$ for every closed subset D in A_i;*
(iv) *for each $i \in I$, $\mu_i(O) \leq \lambda_i[(B_i^{\mu})^{-1}(O)]$ for every open subset O in A_i.*

To prove this theorem, we first introduce the following well known lemma which can be obtained by appropriately adjusting the proof of Theorem 3.11 in Skorokhod (1956).

Lemma 2 (Skorokhod 1956, Theorem 3.11) *Let $(T, \mathcal{T}, \lambda)$ be an atomless probability space and A a Polish space. Then, for any $\nu \in \mathscr{M}(A)$, there exists a measurable function $f : T \to A$ such that $\lambda f^{-1} = \nu$.*

Proof of Theorem 2 Firstly, we want to make sure that for each $i \in I$ and every $C \in \mathscr{B}(A_i)$, $(B_i^{\mu})^{-1}(C)$ is measurable. To see this, fix any $i \in I$. The homogeneous condition, i.e., $U(t)$ is fixed for all $t \in T_i$, implies that $B_i^{\mu}(t)$ is the same for all $t \in T_i$. Thus, we can let $C_i := B_i^{\mu}(t)$ for all $t \in T_i$. Then, for any $C \in \mathscr{B}(A_i)$, we have

$$\left(B_i^{\mu}\right)^{-1}(C) = \left\{t \in T_i : B_i^{\mu}(t) \cap C \neq \emptyset\right\} = \begin{cases} T_i & \text{if} \quad C_i \cap C \neq \emptyset; \\ \emptyset & \text{otherwise,} \end{cases}$$

which is measurable.

(i)⇒(ii): Suppose μ is now an equilibrium distribution. By assumption, there exists a Nash equilibrium $f : T \to A$ such that $\mu = (\lambda_i f_i^{-1})_{i \in I}$ and $f(t) \in B^{\mu}(t)$ for all $t \in T$. Therefore, for any $C \in \mathscr{B}(A_i)$,

$$\begin{aligned}\mu_i(C) &= \left(\lambda_i f_i^{-1}\right)(C) = \lambda_i\left(\{t \in T_i : f_i(t) \in C\}\right) \\ &\leq \lambda_i\left(\{t \in T_i : B_i^{\mu}(t) \cap C \neq \emptyset\}\right) \\ &= \lambda_i\left[\left(B_i^{\mu}\right)^{-1}(C)\right].\end{aligned}$$

(ii) ⇒ (iii): It is obvious.

(iii) ⇒ (iv): Let O be an open set in A_i. Then there is an increasing sequence $\{F_n\}_{n=1}^{\infty}$ of closed sets in A_i such that $O = \bigcup_{n=1}^{\infty} F_n$. For each n, we have $(B_i^{\mu})^{-1}(F_n) \subseteq (B_i^{\mu})^{-1}(O)$, which implies that $\mu_i(F_n) \leq \lambda_i[(B_i^{\mu})^{-1}(F_n)] \leq \lambda_i[(B_i^{\mu})^{-1}(O)]$. Thus, $\mu_i(O) \leq \lambda_i[(B_i^{\mu})^{-1}(O)]$.

It remains to show (iv) ⇒ (i). Recall that for all $i \in I$, the set $C_i := B_i^{\mu}(t)$ for any $t \in T_i$ is compact and hence, also complete and separable. Fix any $i \in \mathbb{N}$. By the fact that the set $(A_i - C_i)$ is open, we have

$$1 - \mu_i(C_i) = \mu_i(A_i - C_i) \leq \lambda_i[(B_i^{\mu})^{-1}(A_i - C_i)] = 0, \tag{1}$$

which gives $\mu_i(C_i) = 1$ for all i. Therefore, by Lemma 2, there exists a measurable function $f_i : T_i \to C_i$ such that $\mu_i = \lambda_i f_i^{-1}$. By definition, $f_i \in B_i^{\mu}$.

Define $f : T \to A$ by letting $f(t) = f_i(t)$ for all $t \in T_i$ and all $i \in I$. Thus, f is a measurable selection of B^{μ} and $\mu = (\mu_i)_{i \in I} = (\lambda_i f_i^{-1})_{i \in I}$ is an equilibrium distribution. □

3.3 Large games without countability restrictions

Now one may ask, does such a characterization result exist for a large game without the countability restriction on action or payoff space? Our next result gives a negative answer to this question.

Theorem 3 *Let $\mu \in \mathscr{M}(A)$ and μ_i the re-scaled probability measure of μ on A_i. There exists a large game U such that the following statements are* not *equivalent:*

(i) *μ is an equilibrium distribution of U;*
(ii) *for each $i \in I$, $\mu_i(C) \leq \lambda_i[(B_i^{\mu})^{-1}(C)]$ for every Borel subset C in A_i.*

To show this result, we need only to give one counterexample.

Example 1 Consider a large game U given as follows. Let the space of players be the Lebesgue unit interval $T = [0, 1]$ endowed with its Boral σ-algrbra and the Lebesgue measure λ. Let the action space A be the interval $[-1, 1]$ and let the payoffs be given by $U(t)(a, \mu) = -|t - |a||$ where $t \in T$, $a \in A$ and $\mu \in \mathscr{M}(A)$.[10]

Let η be the uniform distribution on $[-1, 1]$. Thus, given η, the best response set for player t is:

$$B^{\eta}(t) = \arg \max_{a \in [-1,1]} U(t)(a, \eta) = \{t, -t\}.$$

[10] This payoff function is similar to a payoff function used in Khan et al. (1997).

Let C be an arbitrary Borel subset in A and let $C_1 = C \cap (0, 1]$ and $C_2 = C \cap [-1, 0]$. Let $\tilde{C}_2 = \{t \in [0, 1] : -t \in C_2\}$ be the positive reflection of C_2 on interval [0, 1]. Then,

$$\begin{aligned}\lambda[(B^\eta)^{-1}(C)] &= \lambda\Big(\{t \in T : B^\eta(t) \cap C \neq \emptyset\}\Big)\\ &= \lambda\{t \in T : t \in C_1 \text{ or } -t \in C_2\}\\ &\geq \max\{\lambda(C_1), \lambda(\tilde{C}_2)\}\\ &\geq \frac{\lambda(C_1) + \lambda(\tilde{C}_2)}{2}.\end{aligned}$$

Since η is the uniform distribution on $[-1, 1]$, $\eta(C) = \eta(C_1 \bigcup C_2) = \eta(C_1) + \eta(C_2) = \frac{\lambda(C_1)+\lambda(\tilde{C}_2)}{2}$. Therefore, we have

$$\lambda[(B^\eta)^{-1}(C)] \geq \eta(C).$$

Now we shall prove by contradiction that η cannot be an equilibrium distribution.

Suppose η is an equilibrium distribution. Then, by definition, there exists a measurable selection f of B^η such that $\lambda f^{-1} = \eta$ and $f(t) \in B^\eta(t)$ for all $t \in T$. Let $D = f^{-1}((0, 1])$. Then,

$$f(t) = \begin{cases} t, & t \in D \\ -t, & t \notin D. \end{cases}$$

Note that $f^{-1}(D) = \{t : f(t) \in D\} = \{t : t \in D\} = D$. Hence, $\lambda(D) = \lambda(f^{-1}(D) = \eta(D) = \frac{\lambda(D)}{2}$. So $\lambda(D) = 0$.

Now Let $E = f^{-1}([-1, 0])$. Then, E is the complement event of D on T, i.e. $E = T \setminus D$. Hence $\lambda(E) = \lambda(T \setminus D) = \lambda(T) - \lambda(D) = 1$. On the other hand because $\lambda f^{-1} = \eta$, $\lambda(E) = \lambda(f^{-1}([-1, 0])) = \eta([-1, 0]) = 1/2$. This is a contradiction. Therefore, η cannot be an equilibrium distribution.

3.4 Large games with agent space being a saturated probability space

Although a general characterization result for equilibria in large games does not hold as we have seen from last section, we notice that if we assume the agent space to be a saturated probability space, then we can still have a similar characterization result. This result follows easily from the work of Sun (1996) and Keisler and Sun (2009).

We first introduce the concept of a saturated probability space.[11]

Definition 3 **(i)** A probability space $(T, \mathscr{T}, \lambda)$ is called *essentially countably generated* or *countably generated (modulo null sets)* if there is a countable set $\{X_n \in \mathscr{T} : n \in \mathbb{N}\}$ such that for any $Y \in \mathscr{T}$, there is a set Y' in the σ-algebra generated by $\{X_n : n \in \mathbb{N}\}$ with $\lambda(Y \triangle Y') = 0$, where $\triangle$ denotes the symmetric difference in $\mathscr{T}$.

[11] The concept of saturated was firstly introduced by Hoover and Keisler (1984). For a more detailed explanation on the properties of the space, see, e.g., Keisler and Sun (2009) and Sun and Zhang (2015).

(ii) A probability space $(T, \mathscr{T}, \lambda)$ is called *saturated* if for any subset $C \in T$ with $\lambda(C) > 0$, the re-scaled probability space $(C, \mathscr{T}_C, \lambda_C)$ is not countably-generated, where $\mathscr{T}_C := \{C \cap C' : C' \in \mathscr{T}\}$ and λ_C is the probability measure derived from the restriction of λ to $\mathscr{T}_C$.

Note that in our Example 1, the Lebesgue unit interval [0, 1] endowed with its σ-algebra of Lebesgue measurable sets and the Lebesgue measure, is a countably-generated probability space and hence, *not* saturated.

Theorem 4 *If the agent space $(T, \mathscr{T}, \lambda)$ of a large game U is a saturated probability space, then the four statements (i)-(iv) in Theorem 2 are still equivalent when the homogeneous condition is removed.*

To prove the above theorem, we shall refer to the following lemma which is analogous to Proposition 3.5 of Sun (1996)

Lemma 3 *Let F be a closed valued measurable correspondence from a saturated probability space $(\Omega, \mathscr{F}, P)$ to a Polish space X. Let ν be a Borel probability measure on X. Then the following statements are equivalent:*

(i) *there is a measurable selection f of F such that $Pf^{-1} = \nu$;*
(ii) *for every Borel set C in X, $\nu(C) \leq P(F^{-1}(C))$;*
(iii) *for every closed set D in X, $\nu(D) \leq P(F^{-1}(D))$;*
(iv) *for every open set O in X, $\nu(O) \leq P(F^{-1}(O))$.*

Proof This lemma is analogous to Proposition 3.5 of Sun (1996). It follows easily by combining Theorem 3.6 (P3) of Keisler and Sun (2009) and Proposition 3.5 of Keisler and Sun (2009). □

Proof of Theorem 4 For any $i \in I$, notice that B_i^{μ} is a compact valued (and hence closed valued) measurable correspondence from an atomless Loeb probability space $(T_i, \mathscr{T}_i, \lambda_i)$ to the Polish space A. Thus, by applying Lemma 3 to B_i^{μ}, we see that $\mu_i = \lambda_i f_i^{-1}$ for some f_i being a measurable selection of B_i^{μ} if and only if for every Borel (closed, or open) set H in A_i, $\mu_i(H) \leq \lambda_i[(B_i^{\mu})^{-1}(H)]$.

Since the above result holds for all $i \in I$, thus $\mu = (\mu_i)_{i \in I}$ is an equilibrium distribution if and only if for each $i \in I$ and every Borel (closed, or open) set H in A_i, $\mu_i(H) \leq \lambda_i[(B_i^{\mu})^{-1}(H)]$. □

4 Existence of equilibria in large games

The above characterization results enable us to understand equilibria in large games from another perspective. Moreover, these characterization results also enable us to prove the existence of pure-strategy equilibria by showing the existence of characterizing equilibria distributions.

Theorem 5 *There exists in every large game U a distribution $\mu \in \mathscr{M}(A)$ such that for each $i \in I$,*

$$\mu_i(E) \leq \lambda_i[(B_i^{\mu})^{-1}(E)] \text{ for every Borel set } E \text{ in } A_i.$$

where μ_i is the re-scaled probability measure of μ on A_i.

Proof of Theorem 5 Let $\mu_i \in M(A_i)$ for all $i \in I$. For easy notation, we use $\bar{\mu}$ to denote the distribution vector $(\mu_1, \mu_2, \cdots)$, i.e., $\bar{\mu} := (\mu_i)_{i \in I}$. Also, define $B^{\bar{\mu}}(t) := \arg\max_{a \in K(t)} U(t)(a, (\mu_i)_{i \in I})$ which is the best response correspondence.

Now for each group $i \in I$, let $B_i^{\bar{\mu}} : T_i \twoheadrightarrow A_i$ be the restriction of $B^{\bar{\mu}}$ to T_i and $U_i : T_i \rightarrow \mathcal{C}(\Theta)$ the restriction of U to T_i. Define $V_i : T_i \rightarrow \mathcal{C}(\Theta_i)$ by letting $V_i(t) = U_i(t)|_{\Theta_i}$, where $U_i(t)|_{\Theta_i}$ is the restriction of $U_i(t)$ to Θ_i and $\mathcal{C}(\Theta_i)$ is also endowed with the topology of compact convergence. Thus, we also have $B_i^{\bar{\mu}}(t) = \arg\max_{a \in A_i} V_i(t)(a, (\mu_i)_{i \in I})$. As mentioned earlier in the paper, each topological space is endowed with its Borel σ-algebra on which we define the measurability.

We next claim that V_i is also measurable. To verify this, we first define $W_i : \mathcal{C}(\Theta) \rightarrow \mathcal{C}(\Theta_i)$ by letting $W_i(u) = u|_{\Theta_i}$ for all $u \in \mathcal{C}(\Theta)$. Thus, $V_i = W_i \circ U_i$ and hence, we only need to show that W_i is measurable. Let d be the usual metric on $\mathbb{R}$. Given an element f of $\mathcal{C}(\Theta_i)$, a compact subset D of Θ_i and a number $\epsilon > 0$, let $B_{\Theta_i}(f, D, \epsilon) = \{g \in \mathcal{C}(\Theta_i) : \sup\{d(f(x), g(x)) | x \in D\} < \epsilon\}$. Thus, the sets $B_{\Theta_i}(f, D, \epsilon)$ form a basis for the topology of compact convergence on $\mathcal{C}(\Theta_i)$.(See, eg, p 283 in Munkres (2000)) Hence, we only need to show that $W_i^{-1}(B_{\Theta_i}(f, D, \epsilon))$ is measurable. To see this, let $\Delta = \{u \in \mathcal{C}(\Theta) : u|_D = f\}$ and note that

$$\begin{aligned}
W_i^{-1}(B_{\Theta_i}(f, D, \epsilon)) &= \{h \in \mathcal{C}(\Theta) : h|_{\Theta_i} \in B_{\Theta_i}(f, D, \epsilon)\} \\
&= \{h \in \mathcal{C}(\Theta) : \sup\{d(f(x), h(x)) | x \in D\} < \epsilon\} \\
&= \bigcup_{u \in \Delta} \{h \in \mathcal{C}(\Theta) : \sup\{d(u(x), h(x)) | x \in D\} < \epsilon\} \\
&= \bigcup_{u \in \Delta} B_{\Theta}(u, D, \epsilon).
\end{aligned}$$

Since $B_{\Theta}(u, D, \epsilon)$ is open by the definition of the topology on $\mathcal{C}(\Theta)$, $W_i^{-1}(B_{\Theta_i}(f, D, \epsilon))$ is also open and hence, measurable. Thus, our claim is verified.

For all $i \in I$, define $\Gamma_i^{\bar{\mu}} : \mathcal{C}(\Theta_i) \twoheadrightarrow A_i$ by letting $\Gamma_i^{\bar{\mu}}(u) = \arg\max_{a \in A_i} u(a, (\mu_i)_{i \in I})$ for all $u \in \mathcal{C}(\Theta_i)$. Thus, we have $B_i^{\bar{\mu}}(t) = \Gamma_i^{\bar{\mu}}(V_i(t))$ for all $t \in T_i$. By the Berge's Maximum Theorem, $\Gamma_i^{\bar{\mu}}$ is upper semicontinuous.[12] Thus, $(\Gamma_i^{\bar{\mu}})^{-1}(F)$ is measurable for all closed set $F \in A$.[13] It is also straightforward to verify that $V_i^{-1}[(\Gamma_i^{\bar{\mu}})^{-1}(F)] = (B_i^{\bar{\mu}})^{-1}(F)$ for any closed set $F \in A$. Since V_i is measurable, $\lambda_i V_i^{-1}$ is a Borel probability measure on $\mathcal{C}(\Theta_i)$.

Let $\bar{\eta} := (\eta_i)_{i \in I} \in \prod_{i \in I} \mathscr{M}(A_i)$. Define $\Phi : \prod_{i \in I} \mathscr{M}(A_i) \twoheadrightarrow \prod_{i \in I} \mathscr{M}(A_i)$ as

$$\Phi(\bar{\mu}) = \{\bar{\eta} : \eta_i(E) \leq \lambda_i[(B_i^{\bar{\mu}})^{-1}(E)] \text{ for each } i \in I \text{ and any } E \in \mathscr{B}(A_i)\}.$$

[12] Note that the map $f_{\bar{\mu}} : A \times \mathscr{U} \rightarrow R$ defined by $f_{\bar{\mu}}(a, u) = u(a, (\bar{\mu}_i)_{i \in I})$ is continuous (see Theorem 46.10 in Munkres (2000)).

[13] See e.g., Lemma 16.4 in Aliprantis and Border (1999).

It is easy to see that Φ is nonempty,[14] closed-valued and convex-valued.

Now we want to show that Φ is upper semicontinuous or, equivalently, has a closed graph. To this end, we choose a sequence $\{(\bar{\mu}^m, \bar{\eta}^m)\}_{m\in\mathbb{N}}$ from $\left(\prod_{i\in I}\mathscr{M}(A_i)\times\prod_{i\in I}\mathscr{M}(A_i)\right)$ with $\bar{\eta}^m\in\Phi(\bar{\mu}^m)$ for each m and converging to $(\bar{\mu}^0, \bar{\eta}^0)$. We need to show that $\bar{\eta}^0\in\Phi(\bar{\mu}^0)$.

Fix any $i\in I$. Let F be a closed subset of A_i and let $\Lambda_m := (\Gamma_i^{\bar{\mu}^m})^{-1}(F)$ and $\Lambda_0 := (\Gamma_i^{\bar{\mu}^0})^{-1}(F)$. Since $\Gamma_i^{\bar{\mu}^0}$ is upper semicontinuous and F is closed, Λ_0 is also closed. Since Θ_i is compact, $\mathcal{C}(\Theta_i)$ is metrizable and we let $\hat{d}$ be one of the compatible metrics on $\mathcal{C}(\Theta_i)$. For all $k=1,2,\ldots$, let $G_k=\{u\in\mathcal{C}(\Theta_i):\hat{d}(u,\Lambda_0)\}<\frac{1}{k}\}$.

Fix any k. We claim that $\Lambda_m\subset G_k$ for large enough m. To see this, let $u_m\in\Lambda_m$, which, by the definition of Λ_m, implies that there is an $a_m\in F$ such that $u_m(a_m,\bar{\mu}^m)=\max_{a\in A_i}u_m(a,\bar{\mu}^m)$. Since $\bar{\mu}^m\to\bar{\mu}^0$ and u_m is uniformly continuous on $A_i\times\prod_{i\in I}\mathscr{M}(A_i)$[15], when m is large enough, we have $|u_m(a_m,\bar{\mu}^0)-\max_{a\in A_i}u_m(a,\bar{\mu}^0)|<\frac{1}{k}$. Thus, it is straightforward to find a continuous real function $u'_m\in\mathcal{C}(\Theta_i)$ such that $u'_m(a_m,\bar{\mu}^0)=\max_{a\in A_i}u'_m(a,\bar{\mu}^0)=\max_{a\in A_i}u_m(a,\bar{\mu}^0)$ and $\hat{d}(u_m,u'_m)<\frac{1}{k}$.[16] Thus, $u'_m\in\Lambda_0$ and $u_m\in G_k$.

Hence, the above result and our hypothesis imply that $\bar{\eta}_i^m(F)\le\lambda_i V_i^{-1}(\Lambda_m)\le\lambda_i V_i^{-1}(G_k)$ for large enough m. Since $\bar{\eta}_i^m(F)\to\bar{\eta}_i^0(F)$, we have $\bar{\eta}_i^0(F)\le\lambda_i V_i^{-1}(G_k)$. Since $G_k\downarrow\Lambda_0$, we have $\bar{\eta}_i^0(F)\le\lambda_i V_i^{-1}(\Lambda_0)=\lambda_i V_i^{-1}[(\Gamma_i^{\bar{\mu}^0})^{-1}(F)]=\lambda_i[(B_i^{\bar{\mu}^0})^{-1}(F)]$.

Now, we want to show the above result holds for all Borel set $E\in A$. To verify this, recall that every probability measure on a Polish space is regular.[17] Therefore, we have

$$\begin{aligned}\bar{\eta}_i^0(E)=\bar{\eta}_i^0(E\cap A_i)&=\sup\{\bar{\eta}_i^0(F): F\text{ is closed and }F\subseteq E\cap A_i\}\\&\le\sup\{\lambda_i[(B_i^{\bar{\mu}^0})^{-1}(F)]: F\text{ is closed and }F\subseteq E\cap A_i\}\\&\le\lambda_i[(B_i^{\bar{\mu}^0})^{-1}(E\cap A_i)]=\lambda_i[(B_i^{\bar{\mu}^0})^{-1}(E)].\end{aligned}$$

Since the above arguments hold for all $i\in I$, we conclude that $\bar{\eta}^0\in\Phi(\bar{\mu}^0)$. Therefore Φ also has a closed graph. Hence, by the Ky Fan fixed point theorem in Fan (1952), there is a fixed point $\bar{\mu}^*\in\Phi(\bar{\mu}^*)$.

Define a probability measure μ such that $\mu|_{A_i}=\mu_i^*$ for all $i\in I$ and 0 otherwise. Then, μ is the probability measure that we seek. □

Remark 2 Theorem 5 does not impose any restrictions on the agent, payoff and/or action spaces and hence, is a quite general result.

14 By the Measurable Maximum Theorem, $B_i^{\bar{\mu}}$ admits a measurable selection g_i and hence, $\bar{\eta}=(\lambda_i g_i^{-1})_{i\in I}$ is a trivial element of $\Phi(\bar{\mu})$.

15 Continuous real function on compact metric space is also uniformly continuous.

16 Just let u'_m be a little bit bigger than u_m around the area of a_m.

17 See Theorem 10.7 in Aliprantis and Border (1999).

Combining Theorems 1, 2, 3, and 5 leads us to the existence of pure-strategy equilibria in large games.

Theorem 6 *If a large game U satisfies one of the following three conditions:*

(a). *the action space A is a countable set;*
(b). *all the players in each group share a common payoff;*
(c). *the agent space $(T, \mathcal{T}, \lambda)$ is a saturated probability space,*

then there exists a pure-strategy equilibrium for the game.

Remark 3 By allowing $|I|$ to be countably infinite and the action space A to be Polish, our case (a) is a generalization to Theorem 10 in Khan and Sun (1995) and Theorem 3.2 in Yu and Zhang (2007) and our case (c) strengthens Theorem 1 in Khan and Sun (1999). Moreover, our case (b) is new.

Remark 4 The existence results in Theorem 6 are obtained easily. However, this is not the case if we want to prove these results directly. Actually, the direct proofs on the existence of equilibria for the three settings of large games need to be constructed individually and each of them may well involve a lot of effort (see, eg, Khan and Sun (1995, 1999)).

Finally, we notice that similar to Theorem 4.6 in Keisler and Sun (2009), we can have a global characterization of saturated probability spaces using our characterization result. For this purpose, it suffices to consider the case where I is a singleton, i.e., all players share a common action space A.[18]

Proposition 1 *Let $(T, \mathcal{T}, \lambda)$ be an atomless probability space and A an uncountable compact metric space. Then, $(T, \mathcal{T}, \lambda)$ is saturated if and only if for every large game U with player space $(T, \mathcal{T}, \lambda)$ and action space A, any Borel probability distribution μ on A which satisfies $\mu(D) \leq \lambda[(B^{\mu})^{-1}(D)]$ for every closed subset D in A must be an equilibrium distribution.*

Proof The necessity part ("only if" part) has already been shown by Theorem 4. We only need to show the sufficiency part here. Suppose the condition in the theorem holds. Now, by Theorem 5 we know that for every large game U there exists a μ on A which satisfies $\mu(D) \leq \lambda[(B^{\mu})^{-1}(D)]$ for every closed subset D in A. Thus it is an equilibrium distribution by the condition of the theorem. Hence there is a Nash equilibrium for the game. Therefore, by Theorem 4.6 in Keisler and Sun (2009), $(T, \mathcal{T}, \lambda)$ must be saturated. □

Remark 5 Because of Theorem 2, Proposition 1 is still valid if the condition "for every closed subset D" in the last row is replaced by "for every open subset O" or "for every Borel subset C".

[18] It is also straightforward to generalize this result to the case where I is any finite or countable set.

5 Concluding remarks

In this paper, we provide a unified framework for characterizing equilibrium distributions in large games. Our framework also leads to an easy way of proving the existence of Nash equilibria in large games. Our division of the agent space into countably many groups and the corresponding existence results are new and can be practically useful. It is noticed that our characterization framework can also be used to characterize saturated probability spaces. We hope our method used here can be applied to other situations, for example, large games with traits as discussed in Khan et al. (2013) and games with public and private information as discussed in Fu et al. (2007). We may address this issue in subsequent work.

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