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Axiomatizing core extensions

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Abstract We give an axiomatization of the aspiration core on the domain of all TU-games using a relaxed feasibility condition, non-emptiness, individual rationality, and generalized versions of the reduced game property (consistency) and superadditivity. Our axioms also characterize the C-core (Guesnerie and Oddou, Econ Lett 3(4):301–306, 1979; Sun et al. J Math Econ 44(7–8):853–860, 2008) and the core on appropriate subdomains. The main result of the paper generalizes Peleg's (J Math Econ 14(2):203–214, 1985) core axiomatization to the entire family of TU-games.

Keywords Core extensions · Axiomatization · Aspiration core · C-core · Consistency

JEL Classification C71

1 Introduction

Cooperative game theory is ideally equipped to deal with issues regarding coalition formation. Nevertheless, its two main solution concepts, the core (Gillies 1959) and the Shapley value (Shapley 1953), assume that all players will work together in a single group. Perhaps not surprisingly, the axiomatization literature typically restricts attention to solution concepts that select a way to distribute the worth of the *grand coalition* among its members. Any payoff vector exceeding such amount is simply discarded

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as unfeasible.[1] With such feasibility restriction, coalition formation becomes a mute point. Moreover, the intuitive and appealing properties used to characterize the core then lead to contradictions when applied to the domain of non-balanced games.

In this article we investigate the role of the feasibility condition in the axiomatization of the core. We consider a larger class of solution concepts, those satisfying a relaxed version of feasibility which allows for non-trivial coalition formation. Then, we show that the aspiration core, a non-empty core extension (Bennett 1983; Cross 1967; Albers 1979), is the only solution in this class that satisfies non-emptiness, individual rationality, and some appropriately-modified versions of superadditivity and consistency on the domain of all transferable utility games.

The standard superadditivity and consistency properties (see, for example, Peleg 1986) implicitly depend on grand coalition feasibility. We replace them with similar axioms that are compatible with non-trivial coalition formation. First, traditional reduced games (Davis and Maschler 1965) make an exception in their definition to ensure that payoff vectors "add up" to the worth of the grand coalition. We use a more general version of consistency (e.g. see Moldovanu and Winter 1994), one that treats all coalitions in the same way. Second, following the lines of Aumann (1985) and Hart (1985), we impose a feasibility requirement on superadditivity. Both axioms coincide with their classical versions when applied to the family of balanced games.

On appropriate subdomains, our axioms uniquely characterize the C-stable solution (Guesnerie and Oddou 1979) (also known as C-core in the work of Sun et al. 2008) and the core. In particular, on the subdomain of balanced games our results replicate, and thus generalize, Peleg's (1986) core axiomatization. This also posits the aspiration core as a very natural core extension, as it shares several intuitive properties with the core. As opposed to core axiomatizations that hold on the entire domain of TU-games (e.g. Hwang and Sudhölter 2001), our axioms are not incompatible on the domain of non-balanced games. We characterize a solution concept, the aspiration core, which is non-empty for every TU-game and coincides with the core on the domain of balanced games.

This article is organized as follows. Notation and basic definitions are introduced in Sect. 2 and axioms are listed in Sect. 3. The main results are given in Sect. 4, Sect. 5 discusses axiom independence, and Sect. 6 concludes by relating our work with previous literature.

2 Definitions and notation

2.1 TU-games

Given a finite set of agents $\mathcal{U}$, a *cooperative TU-game* is an ordered pair (N, v) where N is a non-empty subset of $\mathcal{U}$ and $v : 2^N \longrightarrow \mathbb{R}$ is a function such that $v(\emptyset) = 0$. Γ denotes the space of all cooperative TU-games. Let $\mathcal{N} = \{S \subseteq N \mid S \neq \emptyset\}$ be the set of *coalitions* of (N, v). For every $S \in \mathcal{N}$, we call $v(S)$ the *worth* of coalition S.

[1] Examples of this literature include Peleg (1985); Peleg (1986); Peleg (1989), Keiding (1986), Tadenuma (1992), Serrano and Volij (1998), Voorneveld and Van Den Nouweland (1998), and Hwang and Sudhölter (2001).

Possible outcomes of a game (N, v) are described by vectors $x \in \mathbb{R}^N$ that assign a *payoff* x_i to every $i \in N$. For every $S \in \mathcal{N}$ and $x \in \mathbb{R}^N$, define $x(S) = \sum_{i \in S} x_i$ and let $x^S \in \mathbb{R}^S$ be such that $x_i^S = x_i$ for every $i \in S$. The *generating collection* of $x \in \mathbb{R}^N$ is defined as $\mathcal{GC}(x) = \{S \in \mathcal{N} \mid x(S) = v(S)\}$. A payoff vector x is an *aspiration* of the game (N, v) if $x(S) \geq v(S)$ for every $S \in \mathcal{N}$ and $\bigcup_{S \in \mathcal{GC}(x)} S = N$. We denote the set of aspirations of (N, v) by $Asp(N, v)$.

2.2 Feasibility

We define feasibility by taking into account all possible arrangements of agents devoting fractions of their time to different coalitions, not just the grand coalition. Let (N, v) be an arbitrary TU-game. Define a *production plan* for N as a vector $\lambda \in [0, 1]^{\mathcal{N}}$ such that $\sum_{S \ni i} \lambda_S = 1$ for every $i \in N$. We interpret λ_T as the fraction of time during which coalition T is active. The requirement that $\sum_{S \ni i} \lambda_S = 1$ is a time-feasibility condition, under the assumption that every agent is endowed with one unit of time. Let $\Lambda(N)$ denote the set of all production plans for N.[2] Define the worth of any production plan $\lambda \in \Lambda(N)$ as

$$v(\lambda) = \sum_{S \in \mathcal{N}} \lambda_S v(S).$$

Definition 2.1 The set of *feasible* payoff vectors of (N, v) is

$$X^*_\Lambda(N, v) = \{x \in \mathbb{R}^N \mid x(N) \leq v(\lambda) \text{ for some } \lambda \in \Lambda(N)\}.$$

Classical axiomatization literature works with the set

$$X^*(N, v) = \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\},$$

which only contains payoff vectors that are feasible when the grand coalition forms. Clearly, $X^*(N, v) \subseteq X^*_\Lambda(N, v)$.

The following subset of $X^*_\Lambda(N, v)$ contains payoff vectors that are feasible when agents cannot divide their time among various coalitions, and thus only disjoint coalitions can form. A family of coalitions $\pi \subseteq \mathcal{N}$ is a *partition of* N if $\bigcup_{P \in \pi} P = N$ and for every $P, Q \in \pi$ such that $P \neq Q$, $P \cap Q = \emptyset$. Let $\Pi(N)$ denote the family of all partitions of N. For every partition $\pi \in \Pi(N)$ define its worth as

$$v(\pi) = \sum_{P \in \pi} v(P),$$

and for every TU-game (N, v) let

$$X^*_\Pi(N, v) = \{x \in \mathbb{R}^N \mid x(N) \leq v(\pi) \text{ for some } \pi \in \Pi(N)\}.$$

[2] For every $\lambda \in \Lambda(N)$, the components of λ are known in the literature as *balancing weights* and the set $\{S \in \mathcal{N} \mid \lambda_S > 0\}$ as a (*strictly*) *balanced family of coalitions*.

Remark 2.2 Notice that every partition $\pi \in \Pi(N)$ (in particular $\{N\} \in \Pi(N)$) can be naturally identified with the production plan $\lambda^{\pi} \in \Lambda(N)$ defined as $\lambda_S^{\pi} = 1$ if $S \in \pi$ and $\lambda_S^{\pi} = 0$ otherwise. Thus, for every $(N, v) \in \Gamma$,

$$X^*(N, v) \subseteq X_{\Pi}^*(N, v) \subseteq X_{\Lambda}^*(N, v).$$

2.3 Efficiency

The set of *efficient* payoff vectors for every $(N, v) \in \Gamma$ is defined as

$$X_{\Lambda}(N, v) = \arg\max\{x(N) \mid x \in X_{\Lambda}^*(N, v)\}.$$

A production plan $\hat{\lambda} \in \Lambda(N)$ is *efficient* if $v(\hat{\lambda}) = \max\{v(\lambda) \mid \lambda \in \Lambda(N)\}$.

This definition of efficiency differs from the one typically used in the literature, which implicitly assumes that forming the grand coalition is Pareto-optimal. Peleg (1986), for example, defines the set of efficient payoff vectors of a TU-game (N, v) as

$$X(N, v) = \{x \in X^*(N, v) \mid x(N) = v(N)\} = \arg\max\{x(N) \mid x \in X^*(N, v)\}.$$

2.4 Solution concepts

Fix a family of games $\Gamma_0 \subseteq \Gamma$. A *solution concept* on Γ_0 is a mapping σ that assigns to every game $(N, v) \in \Gamma_0$ a (possibly empty) set $\sigma(N, v) \subseteq X_{\Lambda}^*(N, v)$. The following are the definitions of the solution concepts that are our main object of study.

The *core* (Gillies 1959) is defined as

$$C(N, v) = \{x \in X^*(N, v) \mid \forall_{S \in \mathcal{N}}\ x(S) \geq v(S)\}.$$

The subdomain of *balanced* TU-games is denoted by

$$\Gamma_c = \{(N, v) \in \Gamma \mid C(N, v) \neq \emptyset\}.$$

Bondareva (1963) and Shapley (1967) showed that $(N, v) \in \Gamma_c$ if and only if forming the grand coalition is an efficient production plan. Therefore, outside of Γ_c, it is natural to consider production plans different from $\lambda^{\{N\}}$. For example, changing the definition of the core by using the sets $X_{\Pi}^*(N, v)$ and $X_{\Lambda}^*(N, v)$ instead of $X^*(N, v)$ generates two different solution concepts.

The *C-core* (Sun et al. 2008) or *C-stable set* (Guesnerie and Oddou 1979) is defined as

$$cC(N, v) = \{x \in X_{\Pi}^*(N, v) \mid \forall_{S \in \mathcal{N}}\ x(S) \geq v(S)\}.$$

This definition leads to a new family of games, those with a non-empty C-core. The subdomain of *C-balanced* TU games is denoted by

$$\Gamma_{cc} = \{(N, v) \in \Gamma \mid cC(N, v) \neq \emptyset\}.$$

The *aspiration core* or *balanced aspiration set* (Bennett 1983) (see also Cross 1967; Albers 1979) is defined as[3]

$$AC(N, v) = \{x \in X^*_\Lambda(N, v) \mid \forall_{S \in \mathcal{N}}\ x(S) \geq v(S)\}.$$

Remark 2.3 Given the new feasibility condition, it would be natural to define the set $AC'(N, v) = \{x \in X^*_\Lambda(N, v) \mid \forall_{S \in \mathcal{N}}\ \forall_{\lambda \in \Lambda(S)}\ x(S) \geq v(\lambda)\}$ as the aspiration core. Nevertheless, the two definitions are equivalent. Clearly $AC'(N, v) \subseteq AC(N, v)$. Conversely, let us assume that $x \in AC(N, v)$ and fix an arbitrary $S \in \mathcal{N}$. For every $T \subseteq S$ we have $x(T) \geq v(T)$. Then, for every $\lambda \in \Lambda(S)$, multiplying the inequality by λ_T and adding over all the subsets of S yields $x(S) \geq v(\lambda)$. Thus, $x \in AC'(N, v)$ as desired.

Remark 2.4 Coalitions formed must integrate in a production plan that makes a given $x \in AC(N, v)$ feasible, i.e., a production plan $\lambda \in \Lambda(N)$ such that $x(N) = v(\lambda)$. Such coalitions necessarily belong to the generating collection $\mathcal{GC}(x)$.

Remark 2.5 Bennett (1983) shows that $AC(N, v) \neq \emptyset$ for every $(N, v) \in \Gamma$.

Remark 2.6 Notice that Remark 2.2 and the previous definitions imply that, for every $(N, v) \in \Gamma$,

$$C(N, v) \subseteq cC(N, v) \subseteq AC(N, v).$$

Proposition 2.7 *If* $(N, v) \in \Gamma_c$, *then* $X^*(N, v) = X^*_\Pi(N, v) = X^*_\Lambda(N, v)$. *Also, if* $(N, v) \in \Gamma_{cc}$, *then* $X^*_\Pi(N, v) = X^*_\Lambda(N, v)$.

The proof of this proposition uses standard techniques and is left to the reader.

Remark 2.8 Applying Proposition 2.7 to the definition of the solution concepts implies that whenever the C-core is not empty, it coincides with the aspiration core. Similarly, whenever the core is not empty, it coincides with the aspiration core. Thus, Remark 2.5 implies that the aspiration core is a non-empty core extension.

3 The axioms

Let Γ_0 be an arbitrary subset of Γ. The following are the axioms relevant to our results:

Non-emptiness (NE) A solution σ on Γ_0 satisfies *NE* if for every $(N, v) \in \Gamma_0, \sigma(N, v) \neq \emptyset$.

[3] Bennett (1983) originally defines the aspiration core (which she calls the *set of balanced aspirations*) as the set of minimal sum aspirations and goes on to show the equivalence with the definition above.

Individual rationality (IR) A solution σ on Γ_0 satisfies *IR* if for every $(N, v) \in \Gamma_0$, every $x \in \sigma(N, v)$, and every $i \in N$, $x_i \geq v(\{i\})$.

We now present two versions of reduced games and their corresponding consistency axioms. Fix $(N, v) \in \Gamma$, $S \in \mathcal{N}$, and $x \in \mathbb{R}^N$. Define the *DM-reduced game* (Davis and Maschler 1965) of (N, v) with respect to S and x as $(S, v^x) \in \Gamma$ such that

$$v^x(T) = \begin{cases} 0 & \text{if } T = \emptyset \\ v(N) - x(N \setminus S) & \text{if } T = S \\ \max\{v(T \cup Q) - x(Q) \mid Q \subseteq N \setminus S\} & \text{otherwise} \end{cases}$$

DM-consistency (DM-CON) A solution σ on Γ_0 satisfies *DM-CON* if for every $(N, v) \in \Gamma_0$, every $S \in \mathcal{N}$, and every $x \in \sigma(N, v)$, $(S, v^x) \in \Gamma_0$ and $x^S \in \sigma(S, v^x)$.

Since we do not assume that the coalition of all players forms, we use a version of reduced game that does not give special treatment to the grand coalition. The *modified reduced game* of (N, v) with respect to S and x (used, among others, by Moldovanu and Winter 1994; Hokari and Kibris 2003) is the game $(S, v_*^x) \in \Gamma$ such that

$$v_*^x(T) = \begin{cases} 0 & \text{if } T = \emptyset \\ \max\{v(T \cup Q) - x(Q) \mid Q \subseteq N \setminus S\} & \text{otherwise} \end{cases}$$

MDM-consistency (MDM-CON) A solution σ on Γ_0 satisfies *MDM-CON* if for every $(N, v) \in \Gamma_0$, every $S \in \mathcal{N}$, and every $x \in \sigma(N, v)$, it is true that $(S, v_*^x) \in \Gamma_0$ and $x^S \in \sigma(S, v_*^x)$.

Remark 3.1 Note that if $v \in \Gamma_c$ and $x \in C(N, v)$ then the two versions of reduced game coincide. Indeed, for every $S \in \mathcal{N}$, the games (S, v^x) and (S, v_*^x) differ at most on the worth assigned to S. To show that $v^x(S) = v_*^x(S)$, notice that $v^x(S) = v(S \cup (N \setminus S)) - x(N \setminus S) \leq \max\{v(S \cup Q) - x(Q) \mid Q \subseteq N \setminus S\} = v_*^x(S)$. Conversely, as $x \in C(N, v)$, for every $Q \subseteq N \setminus S$ we have $v^x(S) = x(S) \geq v(S \cup Q) - x(Q)$, so $v^x(S) \geq v_*^x(S)$. We conclude that the core satisfies *MDM-CON* on Γ_c because, as Peleg (1986) shows, the core satisfies *DM-CON* on Γ_c.

The last axiom is an extension of the usual additivity for single-valued solution concepts. The standard version follows.

Superadditivity (SUPA) A solution σ on Γ_0 satisfies *SUPA* if every pair of games $(N, v_A), (N, v_B) \in \Gamma_0$, every $x_A \in \sigma(N, v_A)$ and every $x_B \in \sigma(N, v_B)$ satisfy $x_A + x_B \in \sigma(N, v_A + v_B)$ whenever $(N, v_A + v_B) \in \Gamma_0$.

In a similar fashion to Aumann's (1985) axiomatization of the NTU value, we add a feasibility requirement. When working on the domain Γ_c, such condition is redundant as feasibility is trivially satisfied.

Conditional Superadditivity (C-SUPA) A solution σ on Γ_0 satisfies *C-SUPA* if for every $(N, v_A), (N, v_B) \in \Gamma_0$, every $x_A \in \sigma(N, v_A)$ and every $x_B \in \sigma(N, v_B)$, then

$x_A + x_B \in \sigma(N, v_A + v_B)$ whenever $(N, v_A + v_B) \in \Gamma_0$ and $x_A + x_B$ is feasible for $(N, v_A + v_B)$.

Example 3.2 The aspiration core does not satisfy *SUPA*. For example, let $N = \{1, 2\}$ and define $(N, v_A), (N, v_B) \in \Gamma$ as follows. For every $S \subsetneq N$ let $v_A(S) = v_B(S) = 0$. Also, let $v_A(N) = -v_B(N) = 2$. Then, $(1, 1) \in AC(N, v_A)$ and $(0, 0) \in AC(N, v_B)$, but $(1, 1) + (0, 0) \notin AC(N, v_A + v_B)$. In this sense, *SUPA* is a stronger axiom than *C-SUPA*.

Remark 3.3 Notice that consistency axioms require the corresponding reduced game to lie in the domain of games where the solution is defined. There is no such requirement for superadditivity axioms. Therefore, if a solution σ on $\Gamma_1 \subseteq \Gamma$ satisfies *C-SUPA* (or *SUPA*), the axiom is immediately inherited by σ when defined on any subdomain $\Gamma_0 \subseteq \Gamma_1$.

Remark 3.4 Peleg (1986) shows that the core satisfies *SUPA* on Γ_c. Therefore, as *C-SUPA* coincides with *SUPA* on Γ_c by Proposition 2.7, the core satisfies *C-SUPA* on Γ_c.

4 Axiomatizations

Proposition 4.1 *The aspiration core satisfies NE, IR, MDM-CON, and C-SUPA on* Γ.

Proof *NE* is satisfied by Remark 2.5, *IR* is satisfied by definition, and Hokari and Kibris (2003) proved that the aspiration core satisfies *MDM-CON* on Γ. It is straightforward to verify that *C-SUPA* is also satisfied. □

Proposition 4.2 *Let* σ *be a solution concept defined on* $\Gamma_0 \subseteq \Gamma$ *satisfying IR and MDM-CON. If* $(N, v) \in \Gamma_0$ *and* $x \in \sigma(N, v)$*, then* $x(S) \geq v(S)$ *for every* $S \in \mathcal{N}$.

Proof Let σ be a solution concept on Γ_0 satisfying *IR* and *MDM-CON*. Let $x \in \sigma(N, v)$, $S \in \mathcal{N}$ and choose any $i \in S$. By *MDM-CON*, $x_i \in \sigma(\{i\}, v_*^x)$, so *IR* implies

$$x_i \geq v_*^x(\{i\}) = \max\{v(Q \cup \{i\}) - x(Q) \mid Q \subseteq N \setminus \{i\}\} \geq v(S) - x(S \setminus \{i\}).$$

This means that $x(S) \geq v(S)$, as desired. □

The following proposition generalizes Lemma 5.5 in Peleg (1986) to the whole family of TU games Γ.

Proposition 4.3 *If* σ *is a solution concept defined on* $\Gamma_0 \subseteq \Gamma$ *that satisfies IR and MDM-CON then, for every* $(N, v) \in \Gamma_0$*, every payoff vector in* $\sigma(N, v)$ *must be efficient.*

Proof Assume $(N, v) \in \Gamma_0$ satisfies *IR* and *MDM-CON*, $x \in \sigma(N, v)$ and $y \in X_\Lambda^*(N, v)$. Then, there is a $\lambda^y \in \Lambda(N)$ such that $y(N) \leq v(\lambda^y)$. Then, Proposition 4.2 implies that

$$x(N) = \sum_{R \in \mathcal{N}} \lambda_R^y x(R) \geq \sum_{R \in \mathcal{N}} \lambda_R^y v(R) = v(\lambda^y) \geq y(N),$$

so x is efficient. □

Proposition 4.4 *If the solution concept σ defined on $\Gamma_0 \subseteq \Gamma$ satisfies IR and MDM-CON, then $\sigma(N, v) \subseteq AC(N, v)$ for every $(N, v) \in \Gamma_0$.*

Proof This is an immediate consequence of combining Proposition 4.2 and feasibility. □

Proposition 4.5 *Let $\mathcal{U}$ have at least three elements. If a solution concept σ defined on Γ satisfies NE, IR, MDM-CON and C-SUPA, then $AC(N, v) \subseteq \sigma(N, v)$ for every $(N, v) \in \Gamma$.*

Proof Let $x \in AC(N, v)$.

Case $|N| \geq 3$: Define $(N, w) \in \Gamma_c$ as

$$w(S) = \begin{cases} x(S) \text{ if } |S| \geq 2 \\ v(S) \text{ if } |S| = 1 \end{cases}$$

Note that $C(N, w) = \{x\}$. Then, by Proposition 4.4 and Remark 2.8, $\sigma(N, w) \subseteq AC(N, w) = C(N, w) = \{x\}$. *NE* then implies $x \in \sigma(N, w)$.

Consider now the game $(N, z) \in \Gamma$ defined as

$$z(S) = v(S) - w(S) \text{ for every } S \in \mathcal{N} \tag{1}$$

The vector $\mathbf{0} \in \mathbb{R}^N$ is in $AC(N, z)$ because, by definition of (N, z), every $S \in \mathcal{N}$ satisfies $0 \geq z(S)$, and the production plan associated with partition $\{\{i\} \mid i \in N\}$ makes $\mathbf{0}$ feasible in (N, z). Furthermore, given $\mathbf{0} \in AC(N, z)$, Proposition 4.3 implies $y(N) = 0$ for every $y \in AC(N, z)$. Then, as the aspiration core is individually rational and $z(\{i\}) = 0$ for every $i \in N$, $AC(N, z) = \{\mathbf{0}\}$. Again, Proposition 4.4 implies $\sigma(N, z) \subseteq AC(N, z) = \{\mathbf{0}\}$, so *NE* implies $\mathbf{0} \in \sigma(N, z)$.

Note that $x + \mathbf{0} \in X^*_\Lambda(N, w+z)$ as $x \in AC(N, v)$, so *C-SUPA* implies $x \in \sigma(N, v)$ as we wanted.

Case $|N| = 2$ **and** $|AC(N, v)| > 1$: In this case $\sum_{|S|=1} v(S) < v(N)$. Let $x = (x_1, x_2) \in AC(N, v)$ and define $\tilde{x} = (x, 0) \in \mathbb{R}^3$. Let $d \in \mathcal{U} \setminus N$, a non-empty set because $|\mathcal{U}| \geq 3$. Consider the game $(N \cup \{d\}, \tilde{v}) \in \Gamma_c$ defined by

$$\tilde{v}(S) = \begin{cases} v(S \setminus \{d\}) & \text{if } |S| \leq 2 \text{ and } S \neq N \\ \sum_{i \in N} v(\{i\}) & \text{if } S = N \\ v(N) & \text{if } S = N \cup \{d\} \end{cases}$$

Using the case $|N| \geq 3$ and Remark 2.8, conclude that $\tilde{x} \in C(N \cup \{d\}, \tilde{v}) = AC(N \cup \{d\}, \tilde{v}) = \sigma(N \cup \{d\}, \tilde{v})$. It is simple to verify that $(N, \tilde{v}^{\tilde{x}}_*) = (N, v)$. Then, use *MDM-CON* to conclude that $x = \tilde{x}_N \in \sigma(N, \tilde{v}^{\tilde{x}}_*) = \sigma(N, v)$ as we wanted.

Case $|N| \leq 2$ **and** $|AC(N, v)| = 1$: By Proposition 4.4, $\sigma(N, v) \subseteq AC(N, v) = \{x\}$, so *NE* implies $x \in \sigma(N, v)$. □

We are now ready to state our main results.

Theorem 4.6 *Let $\mathcal{U}$ have at least three elements. The aspiration core is the only solution concept on Γ that satisfies NE, IR, MDM-CON, and C-SUPA.*

Proof Combine Propositions 4.1, 4.4, and 4.5. □

Remark 4.7 The proof of our Theorem 4.6 is very similar to the one used by Peleg (1986) to obtain his core axiomatization. The similarity of the arguments highlights the importance of the definition of feasibility, the key to determining which solution concept arises. It should also be emphasized that Peleg's proof uses the *converse consistency* of the core, while we do not. A solution σ on $\Gamma_0 \subseteq \Gamma$ satisfies modified Davis–Maschler converse consistency (*MDM-CC*) if for every $(N, v) \in \Gamma_0$ and every $x \in X^*_\Lambda(N, v)$, if $T \in \{S \subseteq N \mid |S| = 2\}$ implies $(T, v^x_*) \in \Gamma_0$ and $x^T \in \sigma(T, v^x_*)$, then $x \in \sigma(N, v)$. The aspiration core does not satisfy *MDM-CC*.

Relaxing the definition of feasibility implies that other solution concepts may satisfy the axioms mentioned in Theorem 4.6. For example, define

$$\widetilde{X}^*(N, v) = \{x \in \mathbb{R}^N \mid \forall_{i \in N}\ \exists_{S \in \mathcal{N}} \text{ s.t. } S \ni i \text{ and } x(S) \leq v(S)\}$$

and substitute the set $X^*(N, v)$ by $\widetilde{X}^*(N, v)$ when defining the terms "feasibility" and "solution concept." We argue then that the set of aspirations, which now can be seen as a solution concept, complies with the axioms at hand. Indeed, $Asp(N, v)$ satisfies *IR*, *NE* (Bennett 1983), and *MDM-CON* (Hokari and Kibris 2003). To see that it satisfies *C-SUPA* note that, if $x_A \in Asp(N, v_A)$ and $x_B \in Asp(N, v_B)$ are such that $x_A + x_B \in X^*_\Lambda(N, v_A + v_B)$, then $x_A + x_B \in AC(N, v_A + v_B) \subseteq Asp(N, v_A + v_B)$.[4]

Consider now the following adaptation of *C-SUPA* to the new notion of feasibility.

Conditional* Superadditivity (C*-SUPA) A solution σ on $\Gamma_0 \subseteq \Gamma$ satisfies *C*-SUPA* if for every $(N, v_A), (N, v_B) \in \Gamma_0$, every $x_A \in \sigma(N, v_A)$, and every $x_B \in \sigma(N, v_B)$, then $x_A + x_B \in \sigma(N, v_A + v_B)$ whenever $(N, v_A + v_B) \in \Gamma_0$ and $x_A + x_B \in \widetilde{X}^*(N, v_A + v_B)$.

The following result characterizes the set of aspirations in the relaxed feasibility setting.

Theorem 4.8 *Assume that feasibility is defined using the set $\widetilde{X}^*(N, v)$ and let $\mathcal{U}$ have at least three elements. Then, the aspiration set is the only solution concept on Γ that satisfies NE, IR, MDM-CON, C*-SUPA and MDM-CC.*

Proof It is immediate to see that the aspiration set satisfies NE, IR and C*-SUPA on Γ. In addition, the aspiration set also satisfies MDM-CON and MDM-CC (Hokari and Kibris 2003). The uniqueness portion can be obtained by replicating the steps of the proof of Theorem 4.6, with the following modifications. Using a reasoning similar to that of Proposition 4.4, one can show that if a solution concept σ (defined as above)

[4] If any vector in $\mathbb{R}^N$ is considered feasible, similar arguments show that the set of unblocked vectors, $UB(N, v) = \{x \in \mathbb{R}^N \mid \forall_{S \in \mathcal{N}}\ x(S) \geq v(S)\}$, satisfies the four axioms.

satisfies IR and MDM-CON then $\sigma(N, v) \subseteq Asp(N, v)$. Conversely, in the case $|N| \geq 3$ of Proposition 4.5, the definition of the game w should be changed to

$$\widetilde{w}(S) = \begin{cases} v(S) \text{ if } |S| \geq 2 \\ x(S) \text{ if } |S| = 1. \end{cases}$$

This guarantees that $Asp(N, \widetilde{w}) = \{x\}$ and thus $x \in \sigma(N, \widetilde{w})$. *MDM-CC* is then used in the next step to show that $\mathbf{0} \in \sigma(N, \widetilde{z}) = \sigma(N, v - \widetilde{w})$.[5] Note that, for every $S \in \mathcal{N}$ with $|S| = 2$, $Asp(S, \widetilde{z}^{\mathbf{0}}_{*}) = AC(S, \widetilde{z}^{\mathbf{0}}_{*}) = \{\mathbf{0}_S\}$ and thus, by NE, $\mathbf{0}_S \in \sigma(S, \widetilde{z}^{\mathbf{0}}_{*})$ which, by MDM-CC implies that $\mathbf{0} \in \sigma(N, \widetilde{z})$. By *C*-SUPA*, $\mathbf{0} \in \sigma(N, v)$. Finally, as $Asp(N, v) = AC(N, v)$ whenever $|N| \leq 2$, the proof concludes as in Proposition 4.5. □

Remark 4.9 *MDM-CC* only plays a marginal role in the previous proof. This axiom is only used at the end and can be replaced, for example, by the simple requirement that $\sigma(N, v) \cap C(N, v) \neq \emptyset$ whenever $C(N, v) \neq \emptyset$. Thus, one might suspect that relaxing feasibility (using $\widetilde{X}^*(N, v)$) implies that only the first four axioms of Theorem 4.8 are needed to uniquely characterize the aspiration set.[6] This issue remains an open question.

In the rest of this section we study characterizations of solution concepts over smaller families of games such as Γ_c and Γ_{cc}. Given that the aspiration core coincides with the core on the class of balanced games, the following theorem shows that the axioms that uniquely characterize the aspiration core on the domain of all games, uniquely characterize the core on the domain of balanced games.

Theorem 4.10 *Let $\mathcal{U}$ have at least three elements. The core is the unique solution concept defined on Γ_c that satisfies NE, IR, MDM-CON, and C-SUPA.*

Proof By definition the core satisfies *NE* and *IR*. By Remark 3.1 the core satisfies *MDM-CON*. By Proposition 4.1 the aspiration core satisfies *C-SUPA* on Γ, so Remarks 2.8 and 3.3 imply the core satisfies *C-SUPA* on Γ_c. Now, let a solution σ on Γ_c satisfy the axioms and fix a game $(N, v) \in \Gamma_c$. Then Proposition 4.4 and Remark 2.8 imply $\sigma(N, v) \subseteq AC(N, v) = C(N, v)$. On the other hand, in the proof of Proposition 4.5, $(N, v) \in \Gamma_c$ implies the game z defined in (1) is in Γ_c. Hence, the proof remains valid on the domain of balanced games and $C(N, v) = AC(N, v) \subseteq \sigma(N, v)$. Thus, $\sigma(N, v) = C(N, v)$. □

We now show that, even if we use a weaker version of feasibility, the previous result still holds.

Theorem 4.11 *Assume that feasibility is defined using the set $\widetilde{X}^*(N, v)$ and let $\mathcal{U}$ have at least three elements. Then, the core is the unique solution concept defined on Γ_c that satisfies NE, IR, MDM-CON, and C*-SUPA.*

[5] The previous proof does not work here because $\mathbf{0}$ is not the only element of $Asp(N, \widetilde{z})$.

[6] We thank an anonymous referee for suggesting this conjecture.

Proof Let σ be a solution on Γ_c satisfying *NE, IR, MDM-CON* and *C*-SUPA*. Given the weaker feasibility, similar arguments to Propositions 4.2 and 4.4 show that, for any $(N, v) \in \Gamma_c, \sigma(N, v) \subseteq Asp(N, v)$. Let $x \in \sigma(N, v)$. Then, for every coalition $S \in \mathcal{N}$, *MDM-CON* implies that $(S, v_*^x) \in \Gamma_c$. In particular, if $|S| = 2$, $x^S \in \sigma(S, v_*^x) \subseteq Asp(S, v_*^x) = C(S, v_*^x)$. As the core satisfies *MDM-CC*, we conclude that $x \in C(N, v)$. Once $\sigma(N, v) \subseteq C(N, v)$, the proof concludes imitating Theorem 4.10. □

Theorem 4.6 can also be used to obtain a characterization of the C-core on the domain Γ_{cc} as follows. To the best of our knowledge, this is the first axiomatization of the C-core in the literature.

Theorem 4.12 *Let $\mathcal{U}$ have at least three elements. The C-core is the unique solution concept defined on Γ_{cc} that satisfies NE, IR, MDM-CON, and C-SUPA.*

Proof By definition the C-core satisfies *NE* and *IR*. Reasoning as in Theorem 4.10, Proposition 4.1 and Remarks 2.8 and 3.3 imply the C-core satisfies *C-SUPA* on Γ_{cc}. We now show that the C-core satisfies *MDM-CON* on Γ_{cc}. Let $(N, v) \in \Gamma_{cc}$, $x \in cC(N, v)$ and $S \in \mathcal{N}$. By definition, there must exist $\pi \in \Pi(N)$ such that $x(N) \leq v(\pi)$. However, as $x \in cC(N, v)$, $x(N) = \sum_{P\in\pi} x(P) \geq \sum_{P\in\pi} v(P) = v(\pi)$. Hence, $x(N) = v(\pi)$ and $x(P) = v(P)$ for every $P \in \pi$. Let $\bar{\pi} \in \Pi(S)$ be defined by

$$\bar{\pi} = \{\bar{P} \subseteq S \mid \bar{P} = P \cap S \text{ for some } P \in \pi\}.$$

Then, for every $\bar{P} = P \cap S \in \bar{\pi}$ we have

$$x(\bar{P}) = v(\bar{P} \cup (P \setminus S)) - x(P \setminus S) \leq v_*^x(\bar{P}),$$

and

$$x(S) = \sum_{\bar{P}\in\bar{\pi}} x(\bar{P}) \leq \sum_{\bar{P}\in\bar{\pi}} v_*^x(\bar{P}) = v_*^x(\bar{\pi}).$$

Hence, $x^S \in X_\Pi(S, v_*^x)$. By Proposition 4.1 the aspiration core satisfies *MDM-CON* on Γ and thus $x(T) \geq v_*^x(T)$ for every $T \subseteq S$. It follows that $x^S \in cC(S, v_*^x)$.

Similar to the proof of Theorem 4.10, Propositions 4.4 and 4.5 are adaptable to work on Γ_{cc}, so every solution satisfying the axioms on this subdomain must coincide with the C-core. □

5 Independence of the axioms

The following examples show that no axiom in our aspiration core characterization, Theorem 4.6, is implied by the others. They can be easily adapted to work on the subdomains Γ_c and Γ_{cc}, so the axioms in Theorems 4.10 and 4.12 are also independent from each other.

Example 5.1 Consider the solution concept σ_1 on Γ such that $\sigma_1(N, v) = \emptyset$ for every $(N, v) \in \Gamma$.σ_1 violates *NE* but vacuously satisfies *IR*, *MDM-CON*, and *C-SUPA*. Therefore *NE* is independent of the other axioms.

Example 5.2 Consider the solution concept σ_2 on Γ such that $\sigma_2(N, v) = X^*_\Lambda(N, v)$ for every $(N, v) \in \Gamma$. It satisfies *NE* because $AC(N, v) \subseteq X^*_\Lambda(N, v)$ is non-empty by Proposition 4.1. It satisfies *C-SUPA* by definition. We now show that it satisfies *MDM-CON*. For every $(N, v) \in \Gamma$, every $S \in \mathcal{N}$ and every $x \in X^*_\Lambda(N, v)$, there exists $\lambda \in \Lambda(N)$ such that $x(N) \leq v(\lambda)$. Consider the vector $\bar{\lambda}$ defined for every $\emptyset \neq T \subseteq S$ as

$$\bar{\lambda}_T = \sum_{\substack{R \subseteq N \\ R \cap S = T}} \lambda_R.$$

Then $\bar{\lambda} \in \Lambda(S)$ as

$$\sum_{\substack{T \subseteq S \\ T \ni i}} \bar{\lambda}_T = \sum_{\substack{T \subseteq S \\ T \ni i}} \sum_{\substack{R \subseteq N \\ R \cap S = T}} \lambda_R = \sum_{\substack{R \subseteq N \\ R \ni i}} \lambda_R = 1.$$

Additionally, $x_S \in X^*_\Lambda(S, v^x_*)$ because

$$\begin{aligned}
x(S) &= \sum_{T \subseteq S} \bar{\lambda}_T \, x(T) = \sum_{T \subseteq S} \sum_{\substack{R \in \mathcal{N} \\ R \cap S = T}} \lambda_R \, x(T) \\
&= \sum_{R \in \mathcal{N}} \lambda_R \, x(R \cap S) + \sum_{R \in \mathcal{N}} \lambda_R \, x(R \setminus S) - \sum_{R \in \mathcal{N}} \lambda_R \, x(R \setminus S) \\
&= \sum_{R \in \mathcal{N}} \lambda_R \, x(R) - \sum_{R \in \mathcal{N}} \lambda_R \, x(R \setminus S) = x(N) - \sum_{R \in \mathcal{N}} \lambda_R \, x(R \setminus S) \\
&\leq v(\lambda) - \sum_{R \in \mathcal{N}} \lambda_R \, x(R \setminus S) = \sum_{R \in \mathcal{N}} \lambda_R \, [v(R) - x(R \setminus S)] \\
&\leq \sum_{R \in \mathcal{N}} \lambda_R \, v^x_*(R \cap S) = \sum_{T \subseteq S} \sum_{\substack{R \in \mathcal{N} \\ R \cap S = T}} \lambda_R \, v^x_*(T) \\
&= \sum_{T \subseteq S} \bar{\lambda}_T \, v^x_*(T) = v^x_*(\bar{\lambda}).
\end{aligned}$$

It is also clear that σ_2 is not individually rational, so *IR* is independent of the other axioms.

Example 5.3 Consider the solution concept σ_3 on Γ such that $\sigma_3(N, v) = \{x \in X^*_\Lambda(N, v) \mid x_i \geq v(\{i\}) \; \forall i \in N\}$ for every $(N, v) \in \Gamma$. σ_3 clearly satisfies *NE*, *IR*, and *C-SUPA*. Therefore our results imply that σ_3 does not comply with *MDM-CON*.

Example 5.4 Following Schmeidler's (1969) procedure on the set of aspirations we now recall the definition of the aspiration nucleolus (Bennett 1981). For every

$(N, v) \in \Gamma$ and every $x \in \mathbb{R}^N$, let $e(v, x) \in \mathbb{R}^{\mathcal{N}}$ be defined by $e_S(v, x) = v(S) - x(S)$ for every $S \in \mathcal{N}$. Define also $\theta(e(v, x)) \in \mathbb{R}^{\mathcal{N}}$ as the non-increasing rearrangement of the components of $e(v, x)$. The aspiration nucleolus of (N, v) is then defined as

$$Asp\ \nu(N, v) = \{x \in Asp(N, v) \mid \theta(e(v, x)) \preccurlyeq_L \theta(e(v, y))\ \forall y \in Asp(N, v)\}$$

where $\preccurlyeq_L$ denotes the lexicographic order. Bennett (1981) shows that the concept satisfies *NE*, while Hokari and Kibris (2003) show that it complies with *MDM-CON*. The aspiration nucleolus also satisfies *IR* as Sharkey (1993) shows it is a subsolution of the aspiration core. Hence, our axiomatization implies that the aspiration nucleolus is not conditionally superadditive.

6 Final comments and related literature

Keiding (2006) gives another axiomatization of the aspiration core. We share with his work the use of *MDM-CON*. However, he adds a class of auxiliary non-transferable utility games to the domain of TU-games, while our results hold within the family Γ of TU-games.

Among the first core axiomatizations are Peleg (1986); Peleg (1989), Tadenuma (1992), and Voorneveld and Van Den Nouweland (1998), (for TU games) and Peleg (1985) (for NTU games). While important contributions to the literature, these papers worked with the family of balanced games Γ_c, so there is some circularity when they use the core to define their domain of games.[7] This is why it is of particular importance that our aspiration core axiomatization holds on the entire domain of TU-games, Γ. Hwang and Sudhölter (2001) solved an important difficulty by providing an axiomatic characterization of the core on the entire domain of TU-games, but their axioms characterize the empty solution outside the domain of balanced games. Closer to our work is Orshan and Sudhölter (2010) axiomatization of the *positive core*, a non-empty core extension. However, they still assume that the grand coalition forms. Unlike the concepts we study, if a game is not balanced every vector in the positive core can be improved upon by some coalition. Modifying the feasibility constraint allows us to characterize a natural extension of the core to non-balanced games while also suggesting a family of coalitions that are likely to form.

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[7] Our C-core axiomatization is subject to the same type of criticism, but we also provide an axiomatization of the aspiration core, a solution concept that extends the C-core outside its natural domain, Γ_{cc}.

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