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Gerrymander-proof representative democracies

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Abstract This article is devoted to the analysis of electoral systems involving two step procedures. It appears that designers are able to manipulate the result of these type of elections by gerrymandering, except in a very few cases. When imposing an unanimity condition on every jurisdiction's voting rule, it is shown that, for any finite number of candidates, a two step voting rule that is gerrymander-proof necessarily gives every voter the power of overruling the unanimity. A characterization of the set of gerrymander proof rules is provided in the case of two candidates.

Keywords Gerrymandering · Manipulation · Federal voting rules

JEL Classification D71 · D72

1 Introduction

This article examines the problem of *manipulation by gerrymandering* in constitutions equipped with two step elections, such as Great-Britain, Canada or the US. In these constitutions, the country is divided into jurisdictions in which elections are held at the first step in order to choose the jurisdictional winners. In the second step, the government is appointed by an aggregation procedure over all the jurisdictional

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winners. It is a well-know fact that these constitutions are subject to manipulation via gerrymandering by the election designers.

Gerrymandering is a term that describes the deliberate rearrangement of the boundaries of electoral districts to influence the outcome of elections. The original gerrymander was created in 1812 by Massachusetts governor Elbridge *Gerry*, who crafted a district for political purposes that looked like a sala-*mander*. The purpose of gerrymandering is to concentrate opposition votes into a few districts or gain more seats for the majority in surrounding districts (called packing), or to diffuse minority strength across many districts (called dilution). In this article, we assume that gerrymandering allows changing the number of voters in each jurisdiction while rearranging boundaries. Gerrymandering with no constraints on the number of voters can also be referred to as *malapportionment*. It is true that malapportionment is no longer permitted in the US since 1964 following the case of *Wesberry vs. Sanders*,[1] but it is still allowed in other countries, such as France, where electoral laws allow for the jurisdictions' population to differ more or less by 20%.

A gerrymander-proof constitution (i.e. non manipulable by gerrymandering) is such that the outcome of the election is the same wherever the location of the voters is when they cast their ballot. Manipulability by gerrymandering, such as the one considered in this article, is different from the standard notion of manipulation à la Gibbard–Satterthwaite (G-S) (see Gibbard 1973; Satterthwaite 1975). Indeed, the standard G-S manipulation refers to manipulation by misrepresenting preferences, whereas herein, we assume that the voters' appointed jurisdictions can be modified, but when doing so, the votes of the voters remain the same. We look for electoral rules which are gerrymander-proof.

Notice that from a logical standpoint, the problem addressed in this article concerns not only gerrymandering but other methods of manipulation by the movement of the voters. First, voters themselves could manipulate the election whenever they have the right to choose the jurisdiction they vote in. For instance, in the US presidential elections, American university students can choose to vote either in the state they come from or the one they study in. In the US again, some states allow voters to vote if they have a secondary residence in the state. In the same way, a French deputy, Jacques Dominati, has been found guilty of allowing voters to vote in his district instead of their home districts in the Paris municipal election of 1989 and 1995. Finally, giving nomadic populations in Europe the right to vote wherever they are at the time of the European elections, as advocated by some European deputees,[2] constitutes possible sources of manipulation by movement (nomadic populations are estimated to be about 10 million, the size of a median european country). Building gerrymander-proof electoral rules would then prevent such a manipulation.

Second, vote swapping constitutes another possible way of manipulation. Vote swapping was proposed by Nader's supporters during the 2000 US presidential election and it consists of exchanging votes between two voters in two different states.

[1] We thank an anonymous referee for raising this point.

[2] Alima Boumedienne-Thiery and Olivier Duhamel (2000), two former members of the European Parliament, recommend: "in a context of extreme mobility, it is fundamental that the individual be a citizen wherever he is, as more and more persons will spend a considerable amount of their time abroad."

Vote swapping aims at beating a third candidate in states in which the polls show a very tight result. Although quite recent, vote swapping may become an important issue when organizing elections as this method seems to encounter great success across organized groups of electors. In particular, when judging the case against the site voteexchange2000.com,[3] the 9th US Court of Appeal concluded on August 6, 2007, that "vote-swapping mechanisms as well as the communication and vote swaps that websites enabled are constitutionally protected."[4] The practice is now spreading across Canada (see, for instance, voteswapcanada.ca) whose electoral authorities have legalized vote swapping but warned the voters about the dangers of such practices. In Great-Britain, as well this practice has become popular since the 2005 elections of the Members of Parliament (see, for instance, votedorset.net, helpbeathoward.org.uk, tacticalvoter.net, …).[5] Note that vote swapping requires the number of voters in each jurisdiction to remain the same before and after the manipulation, while the version of gerrymandering we study here does not impose such a constraint. Nevertheless, finding gerrymander-proof constitutions would also prevent manipulations by vote swapping.

Surprisingly, little attention has been devoted to the study of two step procedures from an axiomatic point of view. Major exceptions are the works by Murakami (1966, 1968), Fishburn (1971, 1973) and Fine (1972). All these authors started from the characterization of majority rule between two alternatives by May (1952), and explored the consequences of dropping or relaxing some axioms. In particular, Murakami (1966) showed that a *neutral*, *monotonic*, and *non dictatorial* decision rule can always be described as a finite hierarchy of councils using majority rule; each council reports its decision to a higher council, whose decision is in turn reported to a higher council, and so on, until the supreme council is reached.

Departing from the axiomatic approach, a few contributions have also proposed normative conditions to judge the quality of a federal system (as two step procedures), the earliest being due to Penrose (1946) and Banzhaf (1968) who suggested that every voter should be given the same probability to influence the outcome. They show that the optimal number of mandates for a state should be proportional to the square root of its population. Maaser and Napel (2007) have generalized these results. More recently, Felsenthal and Machover (1999), Beisbart et al. (2005), as well as Barberà and Jackson (2006) have come with the suggestion that a federal state should maximize, on average, the total utility of the citizens. Feix et al. (2004) proposed that the way mandates are attributed to a state should minimize the probability of electing a candidate who does not obtain the majority. There is also a long tradition of contributions on the rounding off problem (see Balinski and Young 1982). This tradition assumes from the start that the voting rule is the proportional rule.

In this article, we study two step voting rules in a general framework that does not restrict the particular voting method used at each step of the procedure. First, voters submit one vote for one candidate in their jurisdiction, the votes being aggregated by

[3] All websites offering vote swapping have now been grouped under votepair.org.

[4] For more details, see the case 06 55517 *Porter vs. Jones*, on http://www.ca9.uscourts.gov.

[5] For more on vote swapping, the reader can check the paper by Hartvigsen (2006).

the jurisdictional voting rules. Then, the locally elected candidates are aggregated at the second step, with a federal voting rule. We restrict the federal rule from using information about the jurisdiction members, and especially about the size of the jurisdictions. This allows for candidates to be elected by *rotten boroughs*, i.e. jurisdictions in which the number of voters is very small.[6] This is a clear limit of the model, because such practices are not possible in modern politics. But our results provide a first step toward the analysis of federal constitutions and we discuss in the conclusion alternatives one may explore to weaken our version of gerrymander proofness.

In the case of an arbitrary finite number of candidates, we show the existence of gerrymander-proof rules, although the set of rules is very small. When imposing a jurisdictional unanimity condition, the rules that remain are rather undemocratic. The first set of gerrymander-proof rules is the Constant Constitution, stating that the winner is always the same, regardless of the votes. The second set of rules exhibits an undesirable feature, namely every voter in society is given the power to overrule the unanimous vote.

In the more specific case of two candidates, we show that the set of gerrymander-proof rules consists exactly of the Constant Constitution and the so called Unanimity Rules, which require unanimity for a candidate to be elected, while any other outcome elects his rival. Next, the unanimity condition is weakened into a condition called Local Faithfulness, following Young (1974)'s terminology, stating that when a jurisdiction is formed by a single voter, the winner in that jurisdiction should be the one chosen by that voter. We find a new class of gerrymander-proof rules, the rather strange Parity Rules, that has not yet, to the best of our knowledge, been studied. A Parity Rule designates in every jurisdiction as well as at the aggregate level, a candidate according to the parity of the votes the candidate receives.

The rest of the article is organized as follows. In the next section, we describe the formal framework and present several normative properties that may be imposed on them. Section 3 provides the main theorem of the article for the general case of any finite number of candidates, while Sect. 4 provides a characterization result in the case of two candidates. In Sect. 5 we compare our contribution to those of Laffond and Lainé (1999, 2000), Chambers (2008) and Perote Peña (2005), which consider similar issues and we conclude.

2 The general framework

2.1 Notations and definitions

Let $A = \{a_1, \ldots, a_p\}$ be a finite and fixed set of candidates, $N = \{1, \ldots, n\}$ the fixed set of voters, with $n \geq 3$, and $J = \{J_1, \ldots, J_m\}$ the set of jurisdictions, with $m \geq 2$. We assume that $n > m$.

Let σ be a partition function from N to $\{1, \ldots, m\}$. Formally, for all $i \in N,\ \sigma(i) = j \Leftrightarrow i \in J_j$, with $\bigcup_{j=1,m} J_j = N$ and $J_j \cap J_k = \emptyset$ when $j \neq k$. In what follows,

[6] A classical example is the borough of Old Sarum, a populated city which ended with 3 houses and 7 voters in the nineteenth century.

we consider partition functions in Σ, defined as the set of all partitions such that $\sigma^{-1}(j) \neq \emptyset$ for all $j \in \{1, \ldots, m\}$. There is no empty jurisdiction and there is at least one jurisdiction with strictly more than one voter.

Voters vote for one unique candidate in the jurisdiction they reside in and votes are taken as given. We assume that when voters move, they do not change the candidate they vote for. Thus what we are looking at is manipulation by gerrymandering, it is not manipulation à la Gibbard–Satterthwaite which refers to manipulation of the outcome by misrepresenting preferences.

$\pi \in A^n$ denotes a *vote profile*. Typically, a vote profile is identified with a vector of a's, b's, …where the ith coordinate indicates voter i's vote. $\pi|_i$ denotes voter i's vote and for any subset S of N, we denote by $\pi|_S$ the restriction of π to S. Once the vote profile is given, the winner in jurisdiction J_j is chosen via the social choice function f_j:

$$\begin{aligned} f_j : \Sigma \times A^n &\to A \\ (\sigma, \pi) &\to z \in A \end{aligned}$$

We impose the following very mild condition on the functions $\{f_j\}_{j=1,m}$:

Jurisdiction Sovereignty

If $[\sigma(i) = j \Leftrightarrow \sigma'(i) = j]$ and $[\pi|_{J_j} = \pi'|_{J_j}]$ then $f_j(\sigma, \pi) = f_j(\sigma', \pi')$ for all j

In words, the result of an election in jurisdiction J_j is independent of what happens in other jurisdictions. The set of all social choice functions satisfying Jurisdiction Sovereignty is denoted by $\mathcal{F}$.

The m jurisdictional winners constitute a jurisdictional vote profile $\Pi \in A^m$, called a *federal profile*. The federation then appoints a federal winner using the federal social choice function g defined as follows:

$$\begin{aligned} g : \qquad\qquad A^m &\to A \\ \Pi = (z_1, \ldots, z_m) &\to z \in A \end{aligned}$$

Function g's domain is restricted to the set of all jurisdictional elected candidates. It does not include any other type of information such as the number of voters who elected each candidate, the margin of victory, etc.

A *federal constitution* is given by a $(m+1)$-tuple $C = (g, f_1, \ldots, f_m)$, with $f_j \in \mathcal{F}$ for all j. The federal winner of the election will be denoted by:

$$g(f_1(\sigma, \pi), \ldots, f_m(\sigma, \pi)) = g(f(\sigma, \pi))$$

Although functions f_j and g are social choice functions, the combination of these in a two steps procedure does not generate a social choice function. Strictly speaking, a social choice function only has the preferences of the voters as an argument, whereas, in order to define the voting procedure of the federal constitution, it is necessary to take the partition of the voters as an argument in addition to the vote profile.

The following example illustrates the previous remark, by showing how a vote profile π can yield different results. Assume there are 2 candidates, $A = \{a, b\}$, and there are 5 voters in the society, partitioned into 3 jurisdictions J_1, J_2, J_3. Assume furthermore that the jurisdictional rules as well as the federal voting rule are the majority rule. In case of a tie in a jurisdiction (resp. at the federal level), a will arbitrarily be designated as the winner in that jurisdiction (resp. in the federation). The local voting rules thus satisfy Jurisdiction Sovereignty. Consider now the voting profile $\pi = (a, a, a, b, b)$ where voters 1 to 3 vote for a while voters 4 and 5 vote for b. Partition σ is such that $\sigma(1) = \sigma(2) = \sigma(3) = 1,\ \sigma(4) = 2$ and $\sigma(5) = 3$. Thus the winner in J_1 is a, while the winner in J_2 and J_3 is b, so that $\Pi = (a, b, b)$. Therefore, $g(f(\sigma, \pi)) = g(\Pi) = b$. Consider now another partition σ' such that $\sigma'(1) = 1,\ \sigma'(2) = \sigma'(3) = \sigma'(4) = 2$ and $\sigma'(5) = 3$. In that case, the winner in J_1 and in J_2 is a, while the winner in J_3 is b. Therefore, $\Pi' = (a, a, b)$, so that $g(f(\sigma', \pi)) = a$.

2.2 Properties

We define properties we will impose on the different social choice functions. The first two axioms concern the jurisdictional functions f_j:

Local Faithfulness

$$\text{If } J_j = \{i\} \text{ then } f_j(\sigma, \pi) = \pi|_i$$

When a jurisdiction consists of a single voter, it should choose the alternative he voted for.

Local Unanimity

$$\text{If } \pi|_i = \{z\} \text{ for all } i \in J_j \text{ then } f_j(\sigma, \pi) = z$$

A candidate should be chosen in his jurisdiction whenever he receives the votes of all voters. Local Unanimity implies Local Faithfulness. Both Local Unanimity and Local Faithfulness are silent about what happens at the federal level.

Individual gerrymander-proofness

$$\text{For all } \pi \in A^n, \text{ all } i \in N, g(f(\sigma, \pi)) = g(f(\sigma', \pi)),$$
$$\text{for all } \sigma, \sigma' \in \Sigma \text{ such that } \sigma(h) = \sigma'(h),\ h \neq i \text{ and } \sigma(i) \neq \sigma'(i)$$

No single voter can affect alone the outcome by changing jurisdiction while voting for the same candidate.

Gerrymander-proofness (G-P)

$$\text{For all } \pi \in A^n,\ g(f(\sigma, \pi)) = g(f(\sigma', \pi)) \text{ for all } \sigma,\ \sigma' \in \Sigma$$

The location of the voters among the different jurisdictions should not influence the choice of the winner. As a consequence, no coalition of voters can manipulate the result of a federal election by changing jurisdictions. As mentioned in the introduction, this definition applies to partitions in which the number of voters in each jurisdiction can be different. One can think of alternative definitions in which some constraints would restrict the set of possible partitions. We discuss this issue in our conclusion. In any case, a G-P rule is non manipulable even with more restrictive partitions, the reverse is not true.

Proposition 1 *For any set of candidates A, a constitution C is gerrymander-proof if and only if it is individually gerrymander-proof.*

The proof of Proposition 1 is standard and left to the reader. The purpose of Proposition 1 is to weaken the implications of Gerrymander-proofness. Indeed, one could consider G-P to be a strong condition, as it forbids coalitions to change the outcome of the election. However, Individual G-P seems relatively weak as it just states that no single voter can change the outcome on his own. Proposition 1 states that these two facts are logically identical.

2.3 Pivotal voters

It seems reasonable to require that constitutions do not concentrate the decision in the hands of a unique voter. We express part of this idea with the following definitions, stating that one voter alone cannot overrule unanimity.

Let the *Unanimous vote profile* π_z be defined as the vote profile such that $\pi|_i = z$ for all i in N, z in A.

Pivotal voter in $\boldsymbol{\pi_z}$: Voter i is called pivotal in π_z with $z \in A$ if for all σ there exists $y \in A$, $z \neq y$ and π such that $\pi|_i = y$ and $\pi|_h = z$, $h \neq i$ and $g(f(\sigma, \pi_z)) \neq g(f(\sigma, \pi))$.

Pivotal voter: Voter i is called pivotal if there exists an alternative $z \in A$ such that i is pivotal in π_z.

A pivotal voter is thus pivotal for at least one unanimous vote profile. Note that a pivotal voter i does not have the power to impose his choice on society, he can only change the winner on his own by voting for y at some unanimous profile π_z (but his choice could be a third candidate). Furthermore, being pivotal is different from having a veto power, as pivotal voters have power only against the unanimity.

3 A result on gerrymander-proof constitutions

In this section the main theorem of the article is presented. It says that any constitution satisfying G-P and Local Unanimity necessarily gives every voter the power of being pivotal, unless it is the Constant Constitution, the definition of which is provided below.

Constant Constitution: A constitution $C = (g, f_1, \ldots, f_m)$ is called constant if there exists $z \in A$ such that for all $\sigma \in \Sigma$ and $\pi \in A^n$, $g(f(\sigma, \pi)) = z$.

It is a voting rule where the winner is the same whatever the votes of the voters. A Constant Constitution is a large class of rules, as it is defined irrespective of the properties satisfied by the jurisdictional rules f_j or by the federal rule g. Among others we define the subclass of *LU constant constitutions* as all constant constitutions for which every jurisdiction rule f_j satisfies Local Unanimity.

We now turn to some lemmata which will be useful to state and prove the main theorem.

Lemma 1 *Let A contain any finite number p of alternatives. Assume a federal constitution $C = (g, f_1, \ldots, f_m)$ satisfies Local Unanimity and G-P. Then if one voter is pivotal in π_a, every voter is.*

Proof Assume voter i is pivotal in π_a by voting for candidate b. Consider the vote profile π such that every voter votes for a except voter i who votes for b, and the partition σ such that $J_1 = \{i\}$. Then $g(f(\sigma, \pi_a)) \neq g(f(\sigma, \pi))$ where, by Local Unanimity, $g(f(\sigma, \pi)) = g(b, a, \ldots, a)$. Consider then the vote profile π' where every voter votes for a except voter $h (h \neq i)$ who votes for b, and the partition σ' such that $J_1 = \{h\}$. Then $g(f(\sigma', \pi')) = g(b, a, \ldots, a) = g(f(\sigma, \pi))$ and hence $g(f(\sigma', \pi')) \neq g(f(\sigma, \pi_a))$. But G-P implies that $g(f(\sigma', \pi')) = g(f(\sigma, \pi'))$. Thus voter h is also pivotal in π_a. □

Lemma 2 *Let A contain any finite number p of alternatives. Assume a federal constitution $C = (g, f_1, \ldots, f_m)$ satisfies Local Unanimity and G-P. For any two federal profiles Π and $\Pi' \in A^m$ such that Π and Π' both contain the same s different alternatives, with $s < m$, we have $g(\Pi) = g(\Pi')$.*

Proof The case $m = 2$ is trivial as necessarily $s = 1$. Consider $m > 2$ and assume, without loss of generality, that the federal profile Π contains the s alternatives a_1 to a_s. Because $s < m$, the federal profile Π has at least one alternative that is present more than once. Consider any federal profile Π' and a jurisdiction k such that $\Pi'|_j = \Pi|_j$ for all $j \neq k$ and $\Pi'|_k \neq \Pi|_k$, and such that Π' contains also the same s different alternatives a_1 to a_s. For instance, assume $\Pi_{ex} = a_1, a_2, \ldots, a_{s-1}, a_s, a_s, \ldots a_s$. Then $\Pi'_{ex} = a_1, a_2, \ldots, a_{s-1}, a_s, a_s, \ldots a_s, a_1$ is such a federal profile, while $\Pi'_{ex2} = a_s, a_2, \ldots, a_{s-1}, a_s, a_s, \ldots a_s, a_s$ is not.

We show that $g(\Pi) = g(\Pi')$. Assume, without loss of generality, that $\Pi = \Pi_{ex}$ and $\Pi' = \Pi'_{ex}$, thus in J_m, a_s has been replaced by a_1. Let $\pi = \{a_1, a_2, \ldots, a_{s-1}, \underbrace{a_s, \ldots, a_s}_{n-s}, a_1\}$. Putting the two a_1 voters together in J_1 will generate the federal profile Π, while separating them will generate Π'. Hence, by G-P we have

$$g(a_1, a_2, \ldots, a_{s-1}, a_s, a_s, \ldots a_s) = g(a_1, a_2, \ldots, a_{s-1}, a_s, a_s, \ldots a_1)$$

By extension of this argument, any federal profiles Π and Π' such that $\Pi'|_j = \Pi|_j$ for all $j \neq k$ and $\Pi'|_k \neq \Pi|_k$, and such that Π' has the same s different alternatives, elect the same winner.

In order to conclude, notice that if Π contains s different alternatives, then any federal profile Π' containing the same s alternatives can be obtained from Π by a sequence of replacements of one alternative by another, each of these replacements leaving the new federal profile with exactly the s same different alternatives. $\square$

A last lemma will serve. It says that when the constitution C satisfies Local Unanimity and G-P, then the federal social choice function g is anonymous. Let us first define anonymity.

Federal Anonymity: A federal social choice function $g : A^m \longrightarrow A$ satisfies anonymity if and only if $g(\Pi) = g(\mu(\Pi))$ where Π is a federal vote profile in A^m and μ is a permutation on $\{1, \ldots, m\}$.

Federal Anonymity implies that no jurisdictions has more power than another.

Lemma 3 *Let A contain any finite number p of alternatives. Consider a constitution $C = (g, f_1, \ldots, f_m)$ which satisfies G-P and Local Unanimity. Then g satisfies Federal Anonymity.*

Proof Given Local Unanimity, it is always possible to find $\pi \in A^n$ and $\sigma \in \Sigma$ in order to generate any federal profile Π by forming homogenous partitions, for which any voter in J_j votes for $\Pi|_j$.

Any permutation μ on the name of the jurisdictions generates a federal profile $\mu(\Pi)$ that can be obtained as the result of the corresponding permutation on the set of voters, i.e. the partition σ' such that $\sigma(i) = j$ and $\mu(j) = j' \Longrightarrow \sigma'(i) = j'$. Since C satisfies G-P, $g(f_j(\sigma, \pi)) = g(f_{\mu(j)}(\sigma', \pi))$. Consequently, $g(\Pi) = g(\mu(\Pi))$, so g is anonymous. $\square$

The main theorem can now be stated.

Theorem 1 *Let A contain any finite number p of alternatives. If a constitution $C = (g, f_1, \ldots, f_m)$ satisfies G-P and Local Unanimity then C is either a LU Constant Constitution or in C every voter is pivotal.*

Proof of Theorem 1 The sketch goes as follows. Assume C satisfies Local Unanimity and G-P and no voter is pivotal. Then we show that C is LU Constant Constitution. Hence, if C satisfies G-P, Local Unanimity and is not a LU Constant Constitution, then in C there is at least one pivotal voter. By Lemma 1, the theorem will then be proved. The proof is by induction over natural numbers.

Step 1: All federal profiles containing one or two alternatives give the same winner

Assume that C satisfies Local Unanimity, G-P, and no voter is pivotal. Let π_{a_j} be the unanimous vote profile for a_j. By Local Unanimity, $g(f(\sigma, \pi_{a_j})) = g(a_j, \ldots, a_j)$. Call this winner z. Let π be such that $\pi|_i = a_j$ for any $i \neq h$ and $\pi|_h = a_k$, and σ' such that $J_m = \{h\}$. By Local Unanimity, $g(f(\sigma, \pi)) = g(a_j, \ldots, a_j, a_k)$. As no voter is pivotal, $g(a_j, \ldots, a_j, a_k) = g(a_j, \ldots, a_j) = z$.

Given Lemma 2, we have

$$g(a_j, \ldots, a_j) = \cdots = g(a_j, \ldots, a_j, a_k, \ldots, a_k) = \cdots = g(a_j, a_k, \ldots, a_k)$$

Furthermore, $g(a_j, a_k, \ldots, a_k)$ can be generated by the vote profile π' such that $\pi'|_1 = a_j$ and $\pi'|_i = a_k$ for $i \neq 1$ using σ' such that $J_1 = \{1\}$. Again, no voter is pivotal, so when we switch from π' to π_{a_k}, the unanimous profile for a_k, the result of the election is still z, so $g(a_k, \ldots, a_k) = z$. Hence if $g(f(\sigma, \pi_{a_j})) = z$, then $g(f(\sigma, \pi_{a_k})) = z$ for any $a_k \in A$ and $g(f(\sigma, \pi)) = z$ if π contains exactly two different alternatives.

Step 2: If all federal profiles containing s or less alternatives give the same winner then all federal profiles containing $s + 1$ alternatives give the same winner

Assume $g(f(\sigma, \pi)) = z$ for any π containing s or less different alternatives in the federal profile, with $s < m$. Consider the vote profile such that one voter votes for a_1, one votes for a_2, ..., one votes for a_{s-2}, $n - s$ voters vote for a_{s-1} and the two last voters vote for any alternative $a_i \in A$. Forming homogeneous jurisdictions and using Local Unanimity, we get $g(a_1, a_2, \ldots, a_{s-2}, \underbrace{a_{s-1}, \ldots, a_{s-1}}_{m-s+1}, a_i)$. This federal profile contains $s - 1$ different alternatives if $a_i \in \{a_1, \ldots a_{s-1}\}$ and s alternatives if $a_i \in \{a_s, \ldots a_p\}$, so the winner is z by hypothesis. Hence

$$g(a_1, a_2, \ldots, a_{s-2}, \underbrace{a_{s-1}, \ldots, a_{s-1}}_{m-s+1}, f_m) = z \text{ whatever the value of } f_m$$

Consider now the same partition of the voters, and assume the last two voters vote respectively for a_s and a_{s+1}. This vote profile will generate a federal profile of the form $g(a_1, a_2, \ldots, a_{s-2}, a_{s-1}, \ldots, a_{s-1}, f_m)$ so z is the winner. Using the same vote profile and choosing σ such that $\pi|_{J_{m-1}} = a_s$ and $\pi|_{J_m} = a_{s+1}$, by G-P we get

$$g(a_1, a_2, \ldots, a_{s-2}, \underbrace{a_{s-1}, \ldots, a_{s-1}}_{m-s}, a_s, a_{s+1}) = z$$

If $s + 1 < m$ Lemma 2 helps to conclude. If $s + 1 = m$ or $s \geq m$, the federal profile cannot contain more than m alternatives. Lemma 3 thus concludes. □

Theorem 1 gives a negative answer to the question raised in this article, that of the existence of "democratic" constitutions that are not manipulable by gerrymandering or by movement of voters. All rules share the weakness of letting every voter be pivotal if the constitution is not constant.

We next examine what this implies in the case of two candidates.

4 The case of two candidates

The G-S result about manipulation by misrepresentation of preferences holds with three or more alternatives. It is no longer true in the case of two alternatives, the majority rule being a non manipulable voting rule (see Moulin 1988). In contrast, two stage voting rules do not escape from possibilities of gerrymandering when restricted to the case of two candidates. Consider therefore $A = \{a, b\}$.

4.1 Local Unanimity

In this section we characterize the set of rules which satisfy both G-P and Local Unanimity. To do this, we first define what we call the class of Unanimity Rules. For notational convenience, when $z \in A$ is either one of the two candidates, we denote by $\bar{z}$ his opponent. For instance, if $z = b$, then $\bar{z} = a$.

Unanimity Rule: Let $(x, z) \in A^2$. A constitution is a Unanimity Rule (x, z) if

– For all J_j, $[\pi|_i = x \ \forall i \in J_j \implies f_j(\sigma, \pi) = x]$ and $[\exists h \in J_j,\ \pi|_h = \bar{x} \implies f_j(\sigma, \pi) = \bar{x}]$

– $[f_j(\sigma, \pi) = x \ \forall J_j \in J \implies g(f(\sigma, \pi)) = z]$ and $[\exists J_{j'} \in J,\ f_{j'}(\sigma, \pi) = \bar{x} \implies g(f(\sigma, \pi)) = \bar{z}]$.

In words, in every jurisdiction, x is elected if he is unanimously chosen, otherwise $\bar{x}$ is. At the federal level, z is elected if x is unanimously chosen, $\bar{z}$ is chosen otherwise. As $A = \{a, b\}$, Unanimity Rule is a class of four rules only: Unanimity Rule (a, a); Unanimity Rule (a, b); Unanimity Rule (b, a) and Unanimity Rule (b, b). For instance, Unanimity Rule (a, a) is such that a is elected at the jurisdictional level only when a receives unanimity, otherwise b is. At the federal level, if a is unanimous then a is the winner, otherwise b is. Therefore, when every voter votes for a, the federal winner is a. In any other case, the federal winner is b.

In turn, Unanimity Rule (b, a) is such that a is elected as the final winner if and only if everyone votes for b. In any other case, b is elected.

Theorem 2 *When $A = \{a, b\}$, a federal constitution C satisfies Local Unanimity and G-P if and only if C is a LU Constant Constitution or is a Unanimity Rule.*

Proof of Theorem 2 In what follows, we denote by π_0 the unanimous profile for a, $\pi_k^{l_1 \ldots l_k}$ denotes the vote profile such that the k voters $l_1, \ldots, l_k$ have switched their vote to b and $\pi_n^{l_1 \ldots l_n} = \pi_n$ is the unanimous profile for b.

LU Constant Constitutions and Unanimity Rules satisfies Local Unanimity and G-P. We show the other implication holds.

Assume C satisfies Local Unanimity and G-P. By Theorem 1, C is either the LU Constant Constitution or in C, every voter is pivotal. By Lemma 2, $g(f(\sigma, \pi_1^{l_1})) = \ldots = g(f(\sigma, \pi_{n-1}^{l_1 \ldots l_{n-1}}))$. Call this winner z. It remains examining what happens with profiles π_0 and π_n. Recalling that $\bar{z}$ denotes the opponent of z we have four cases:

(1) $g(f(\sigma, \pi_0)) = z$ and $g(f(\sigma, \pi_n)) = z$;
(2) $g(f(\sigma, \pi_0)) = z$ and $g(f(\sigma, \pi_n)) = \bar{z}$;
(3) $g(f(\sigma, \pi_0)) = \bar{z}$ and $g(f(\sigma, \pi_n)) = z$;
(4) $g(f(\sigma, \pi_0)) = \bar{z}$ and $g(f(\sigma, \pi_n)) = \bar{z}$.

We show here that case (4) cannot hold. Assume there exists a sequence $l_1 \ldots l_n$ such that $g(f(\sigma, \pi_0)) = \bar{z}$, $g(f(\sigma, \pi_1^{l_1})) = z, \ldots, g(f(\sigma, \pi_{n-1}^{l_1 \ldots l_{n-1}})) = z$ and $g(f(\sigma, \pi_n^{l_1 \ldots l_n})) = \bar{z}$. Consider σ such that $J_m = \{l_1, l_n\}$. By Local Unanimity,

$g(f(\sigma, \pi_0)) = g(a, \ldots, a, a) = \bar{z}$. In $\pi_1^{l_1}$ we have $g(f(\sigma, \pi_1^{l_1})) = g(a, \ldots, a, f_m) = z$. Hence, $f_m = b$. Now let all the other voters switch their votes. We will reach $g(f(\sigma, \pi_{n-1}^{l_1 \ldots l_{n-1}})) = g(b, \ldots, b, f_m) = g(b, \ldots, b) = z$ and this is a contradiction. Hence case (4) is excluded.

Case (1) corresponds to the LU Constant Constitution. In cases (2) and (3) all voters are pivotal. It remains proving that these cases generate the four possible Unanimity Rules.

Consider case (3). Given Local Unanimity, we get $g(f(\sigma, \pi_0)) = g(a, \ldots, a) = \bar{z}$ for all σ. Consider σ such that $J_i = \{l_i\}$ for $i \leq m-1$. Then $\pi_1^{l_1}$ gives $g(f(\sigma, \pi_1^{l_1})) = g(b, a, \ldots, a) = z$. According to Federal Anonymity (see Lemma 3), any federal profile containing one b only will select z as the winner.

We also get $g(f(\sigma, \pi_2^{l_1 l_2})) = g(b, b, a, \ldots, a) = z$. Again, Federal Anonymity says that whenever the federal profile is formed of two b's and $m-2$ a's, the winner is z. Going on until $\pi_{m-1}^{l_1, \ldots, l_{m-1}}$ we reach $g(b, \ldots, b, a) = z$. By Federal Anonymity, whenever the profile is formed of $m-1$ b's and one a, the winner is z.

Finally, in π_n we have $g(b, \ldots, b, b) = z$. Hence $\bar{z}$ can win the election only for profile π_0. This corresponds to Unanimity Rule (a, a) or Unanimity Rule (a, b), according to the value of z.

In the same way, case (2) will lead to Unanimity Rule (b, a) and Unanimity Rule (b, b). □

Consistently with the statement of Theorem 1, Unanimity Rules offer a pivotal power to every voter.

Remark 1 Although not a specific requirement, the rules characterized in Theorem 2, if not in the set of the LU Constant Constitution, all have in common that the rules at the jurisdiction level are identical, i.e. $f_j = f_{j'}$ for all j, j'.

4.2 Local Faithfulness

In this subsection, we explore the consequences of weakening Local Unanimity into Local Faithfulness. This exercise emphasizes the role of the G-P axiom, exonerating the unanimity condition for the negative results of the previous sections.

We say that C is a **LF Constant Constitution** if it is a Constant Constitution for which every jurisdiction rule f_j satisfies Local Faithfulness.

Next we define what we call the class of Parity Rules. In words, a parity rule is such that at both the jurisdictional and the federal levels, the winner is designated uniquely by the parity of the votes he or his opponent receives.

Parity Rule: Let $(x, y, z) \in A^3$. A constitution C is a Parity Rule (x, y, z) if

- in every jurisdiction, x is the jurisdictional winner if the number of voters for x is *odd*. $\bar{x}$ wins if that number is *even*.
- at the federal level, if the number of jurisdictions that have chosen y is *odd* (y is either x or $\bar{x}$), then z is the winner. $\bar{z}$ is the winner if that number is *even*.

This class of rules may look rather peculiar and has, to the best of our knowledge, never been studied. It bases the election on the parity of the votes. As $A = \{a, b\}$, Parity Rule is a class of eight rules: Parity Rule (a,a,a); Parity Rule (a,a,b); …; Parity Rule (b,b,a) and Parity Rule (b,b,b).

For instance, Parity Rule (a,b,a) is the following: in every jurisdiction, if the number of votes for a is odd then a is the winner, otherwise b is. At the federal level, if the number of jurisdictions that voted for b is odd, then a is the winner, otherwise b is. As a matter of illustration of this rule, assume $N = \{1, 2, 3, 4\}$, $m = 3$ and the partition of society is given by σ such that $J_1 = \{1, 2\}$, $J_2 = \{3\}$, $J_3 = \{4\}$. Then, according to the Parity Rule (a,b,a) we have

Vote profile	Federal profile	Federal winner
$\pi_a = (a, a, a, a)$	b, a, a	a
$\pi_1 = (b, a, a, a)$	a, a, a	b
$\pi_2 = (b, b, a, a)$	b, a, a	a
$\pi_3 = (b, b, b, a)$	b, b, a	b
$\pi_b = (b, b, b, b)$	b, b, b	a

One interesting feature of this class of rules is that every time a voter changes his vote, he changes the outcome of the election. Thus if two voters change their vote at the same time they both neutralize themselves.

Theorem 3 *When $A = \{a, b\}$, a constitution C satisfies Local Faithfulness and G-P if and only if C is a LF Constant Constitution, a Unanimity Rule or a Parity Rule.*

The (long and tedious) proof is available on request from the corresponding author.

Theorem 3 confirms the insights of Theorem 1. Its contribution, apart from yielding another set of strange rules, is to show that the non-democratic feature of Theorem 1 is due to the G-P condition rather than to Local Unanimity. Of course, we do not advocate that a society should use such a decision rule, we just argue that whenever constitutions with two-step elections want to escape manipulation by gerrymandering, by movement of voters or by vote swapping, they are left with very few ugly options.

Remark 2 As for Theorem 2 we notice that, if C is not in the class of the Constant Constitutions, all the rules characterized imply that all the jurisdiction rules are identical, $f_j = f_{j'}$.

5 Comments and conclusion

5.1 A comprehensive comparison of recent results

For a long time, the works of Murakami (1966, 1968), Fishburn (1971, 1973) and Fine (1972) were the only contributions to the axiomatic analysis of multi level voting rules. Clearly, the 2000 US presidential elections, as well as the debates about the European constitutions prompted by the Nice treaty have revived the analysis of two step voting rules in social choice theory. In this section we place our results into perspective by mentioning recent contributions on the same issue.

Laffond and Lainé (1999, 2000) started to analyze this issue in a series of papers since 1999, focusing on its relationship with the Ostrogorski's Paradox (see Nurmi 1999) and the use of tournament solutions. Their model assumes that voters have strict preferences over a set X of at least 3 alternatives. The number of voters in the society is odd, as well as the number of jurisdictions. Moreover, each jurisdiction has the same number of voters. Tournament solutions can then be used in a two step procedure: the preferences of the voters in a jurisdiction can be aggregated into a tournament via the majority rule, and the m tournaments obtained in this way are aggregated into a federal tournament, the solution of which will select a federal winner. The question is whether a tournament solution exists such that its direct implementation to the preference profile of n voters coincides with the two step procedure described above. Lainé and Laffond clearly show that direct and representative democracy may lead to mutually inconsistent decisions when the society uses the majority rule. The strength of their results comes from their ability to obtain an impossibility result with equal size constituencies, while we need single voter jurisdictions to build our proofs.

Chambers (2008) reaches some general impossibility results by imposing slightly different conditions. Three main assumptions, departing from our study, are made in that contribution. First, he assumes that every elected candidate at the jurisdictional level has a weight equal to the size of his jurisdiction at the federal level. Second, the jurisdictional voting rules are all identical. This assumption therefore restricts the set of rules one can look for. Unanimity and Anonymity are imposed on both f and g from the start. Third, Chambers assumes that the rule f should be defined for any finite number of voters. Thus the voting rule used is defined for a variable population size, hence for any number of jurisdictions. As Chambers notices in his conclusion, his results might not hold in the case of a fixed number n of voters or of a fixed number m of jurisdictions. In this framework, the non manipulability condition imposed is called "Representative Consistency", stating that the indirect voting procedure should yield the same result as the direct voting rule. Chambers provides characterization results for any finite number of alternatives, with a general theorem stating that Anonymity, Representative Consistency and Unanimity together define the class of Priority Rules, which coincides with our Unanimity Rules (a,a) and (b,b) in the case $A = \{a, b\}$.

The work by Perote Peña (2005) uses concepts close to ours and to Chambers' ones: the f_j's are identical, the number of jurisdiction may vary (though the total number of voters is fixed), the Anonymity condition is needed together with a gerrymander-proof assumption (here called Independence of Institution Formation, IIF) and the Unanimity condition. The main contribution is that Perote-Peña assumes that the voters can express their whole preferences (represented by a weak ordering or a linear ordering) during the voting process. He proves that IFF, Unanimity and Neutrality[7] are incompatible with Anonymity. However, when dropping Anonymity, he shows the existence of some rules, the characterization of which remain open.

[7] For a definition of this classical property, see May (1952) or Young (1974).

5.2 Concluding remarks

This article was devoted to the study of manipulation by gerrymandering, by the movement of voters or by vote swapping in federal constitutions. It appears that the only voting procedure that is non manipulable when imposing a local unanimity condition and requiring, in addition, that voters should not have the power of overruling unanimous profiles, is the class of Constant Constitutions. Some further characterizations are provided in the case of two candidates. The main drawback of our result comes from the flexibility offered to the planner in his manipulation possibilities as jurisdictions are free to be of any size. A natural extension would be to consider a condition that imposes equal size jurisdictions, to fit with the US congress case:[8]

Equiproportional gerrymander-proofness (EG-P)

$$\text{For all } \pi \in A^n,\ g(f(\sigma,\pi)) = g(f(\sigma',\pi)) \ \text{ for all } \sigma,\ \sigma' \in \Sigma \ s.t.$$
$$for\ all\ j = 1,\ldots m\ \#\{i \in N \mid \sigma(i) \in J_j\} = \#\{i \in N \mid \sigma'(i) \in J_j\} = \frac{n}{m}$$

Following this definition, the reader can check that Theorem 2 still holds in the particular case of m jurisdictions of exactly two voters. However, the possibility of extending the current proofs to equal size jurisdictions with three or more voters seems to require additional assumptions or different techniques.

This definition allows only for vote swaps, which could be apprehended in a more general way without restrictions on the number of voters in each jurisdiction:

Constant size gerrymander-proofness (CSG-P)

$$\text{For all } \pi \in A^n,\ g(f(\sigma,\pi)) = g(f(\sigma',\pi)) \ \text{ for all } \sigma,\ \sigma' \in \Sigma \ s.t.$$
$$for\ all\ j = 1,\ldots m\ \#\{i \in N \mid \sigma(i) \in J_j\} = \#\{i \in N \mid \sigma'(i) \in J_j\}$$

Another route would be to take the US presidential election as a model: the federal social choice function g could vary with σ, potentially giving a higher weight to bigger jurisdictions.

In any case, all the results in this literature seem to show that the possibility of gerrymandering in two step electoral systems, is a robust fact. Whatever the framework, all conclusions go in the same direction. We conjecture that the modifications sketched above would not change the picture. As a consequence, it seems that the noble principle defended by the two European deputies, according to which every European citizen should have the right to vote in any country he is at the time of the European election, would generate undesirable manipulations of a new kind, in addition to the possible manipulations already known of in direct electoral systems.

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[8] The following definitions were suggested by two anonymous referees.

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References

Balinski M, Young P (1982) Fair representation. Yale University Press, New Haven

Banzhaf J (1968) One man, 3.312 votes: a mathematical analysis of electoral college. Villanova Law Rev 13:304–332

Barberà S, Jackson MO (2006) On the weights of nations: assigning voting weights in a heterogeneous union. J Polit Econ 114:317–339

Beisbart C, Bovens L, Hartmann S (2005) An utilitarian assessment of alternative decision rules in the council of ministers. Eur Union Polit 6:395–418

Boumedienne-Thiery A, Duhamel O (2000) Le droit de vote des étrangers, une question européenne. Le Monde, 27 April 2000

Chambers C (2008) Consistent representative democracy. Games Econ Behav 62:348–363

Feix MR, Lepelley D, Merlin VR, Rouet JL (2004) The probability of conflicts in a U.S. presidential type election. Econ Theory 23:227–257

Felsenthal DS, Machover M (1999) Minimizing the mean majority deficit: the second square-root rule. Math Soc Sci 37:25–37

Fine K (1972) Some necessary and sufficient conditions for representative decision on two alternatives. Econometrica 40:1083–1090

Fishburn PC (1971) The theory of representative majority decision. Econometrica 39:279–284

Fishburn PC (1973) The theory of social choice. Princeton University Press, Princeton

Gibbard A (1973) Manipulation of voting schemes: a general result. Econometrica 41:587–601

Hartvigsen D (2006) Vote trading in public elections. Math Soc Sci 52:31–48

Laffond G, Lainé J (1999) A general impossibility theorem on representative democracy. In: de Swart H (ed) Logic, game theory and social choice, proceedings of the international conference LGS99. Tilburg University Press, Tilburg, pp 504–521

Laffond G, Lainé J (2000) Representation in majority tournaments. Math Soc Sci 39:35–53

Maaser N, Napel S (2007) Equal representation in two-tier voting systems. Soc Choice Welf 28:401–402

May K (1952) A set of independent, necessary and sufficient conditions for simple majority decision. Econometrica 20:680–684

Moulin H (1988) Axioms of cooperative decision making, monograph of the econometric society. Cambridge University Press, Cambridge

Murakami Y (1966) Formal structure of majority decision. Econometrica 34:709–718

Murakami Y (1968) Logic and social choice. Dover Publications Inc, New York

Nurmi H (1999) Voting oaradoxes, and how to deal with them?. Springer, Berlin

Penrose LS (1946) The elementary statistics of majority voting. J Roy Stat Soc 109:53–57

Perote Peña J (2005) Gerrymendering proof social welfare functions. Universidad de Zaragoza, Mimeo

Satterthwaite M (1975) Strategy proofness and Arrow's conditions. J Econ Theory 10:187–217

Young HP (1974) An axiomatization of Borda's rule. J Econ Theory 9:43–52