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The covering values for acyclic digraph games

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Abstract We introduce a novel covering method to compute values for acyclic digraph games, and we call the values obtained by this method the covering values. These values may be considered as natural extensions of the component efficient solutions for line-graph games studied by van den Brink et al. (Econ Theory 33:349–364, 2007), and the tree values studied by Khmelnitskaya (Theory Decis 69(4):657–669, 2010a). With the new method, we reinterpret the tree values proposed by Khmelnitskaya (2010a). Besides, we propose the covering values in the digraph game with general acyclic digraph structures presenting flow situations when some links may merge while others split into several separate ones. We give axiomatizations of these values, and interpret these values in terms of dividend distributions.

Keywords TU game · Covering value · Efficiency · Harsanyi dividend · Acyclic digraph game

1 Introduction

A group of players form cooperation and obtain payoffs. If the utility can be transferred costlessly between them, then we can describe this situation with a *cooperative game with transferable utility*, or a *TU game*, which is a pair $\langle N, v\rangle$, where $N = \{1, \dots, n\}$ is a nonempty, finite set, called the *player set*, and $v : 2^N \mapsto \mathbb{R}$ is a *characteristic function*, defined on the power set 2^N of N, satisfying $v(\emptyset) = 0$. An element of N

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and a subset S of N are called a *player* and a *coalition*, respectively. The associated real number $v(S)$ is called the *worth* of a coalition S. We denote by $\mathcal{G}_N$ the set of all these TU-games with player set N. A *payoff vector* of an n-person TU-game is an n-dimensional vector, giving a payoff to any player $i \in N$. A *value* is a function ξ that assigns to any game $\langle N, v\rangle \in \mathcal{G}_N$ a payoff vector $\xi(v) \in \mathbb{R}^n$.

In standard cooperative game theory it is usually assumed that any coalition of players may form. However, in many practical situations the collection of feasible coalitions is restricted by some social, economical, hierarchical, communicational, or technical structure. Examples are, *games with communication structure* (Myerson 1977), *games under precedence constraints* (Faigle and Kern 1992), and *games with permission structure* (Gilles et al. 1992; van den Brink 1997). In this paper we restrict our consideration to classes of acyclic digraph games in which all players are partially ordered and a possible communication via bilateral agreements between participants is presented by an acyclic digraph. Many economic and social situations can be modeled by means of a digraph. Transportation networks that are used to ship commodities from their production centers to their markets can be most effectively analyzed when viewed as digraphs that possess additional structure. Following Myerson (1977), we assume that for a given game with cooperation structure, cooperation is possible only among connected players.

van den Brink et al. (2007) restricted themselves to a special type of games with limited communication structure, called *line-graph games*, in which the communication structure is given by a linear ordering on the set of players. We introduce a novel covering method to compute values for digraph games. These values may be considered as natural extensions of the component efficient (CE) solutions for line-graph games studied in van den Brink et al. (2007) and the tree values studied in Khmelnitskaya (2010a, b). With the new method, we regain the rooted tree and sink tree values studied in Khmelnitskaya (2010a, b). Furthermore, we extend the values in acyclic digraphs, and study the distribution of Harsanyi dividends.

The structure of the paper is as follows. Basic definitions and notation are introduced in Sect. 2. Section 3 introduces some CE solutions that we will use later. Section 4 introduces flows and weights in acyclic digraphs. Section 5 shows that an acyclic digraph can be covered by line-graphs, rooted trees or sink trees. Section 6 finds rooted and sink covering values through two different approaches. In Sect. 7 we discuss the two solutions in terms of distribution of the Harsanyi dividends.

2 Preliminaries

We refer a lot to Bondy and Murty (2008) for the definitions and notation from graph theory. As introduced by Myerson (1977), a *cooperation structure* on a player set N is specified by a an undirected graph D without loops. Later this notion is extended to a directed graph, such as Khmelnitskaya (2010a, b) and van den Brink et al. (2007). A *directed graph*, or *digraph* D, is an ordered pair $(V(D), A(D))$ consisting of a set $V := V(D) = N$ of *vertices* and a set $A := A(D)$, disjoint from $V(D)$, of *arcs*, together with an *incidence function* ψ_D that associates with each arc of D an ordered pair of vertices of D. If a is an arc and $\psi_D(a) = (i, j)$, then we say that i

dominates j. Vertex i is the *tail* of a and vertex j its *head*, or i is a *parent* of j, and j is a *child* of i. The vertices in D which dominate vertex i are its *inneighbors*, those that are dominated by the vertex its *outneighbors*. These sets are denoted by $N_D^-(i)$ and $N_D^+(i)$, respectively. For a set $S \subseteq N$, we denote $N_D^-(S) = \cup_{i \in S} N_D^-(i) \backslash S$ and $N_D^+(S) = \cup_{i \in S} N_D^+(i) \backslash S$ as the inneighbors and outneighbors of S, respectively. The *indegree* $d_D^-(i)$ of vertex i in D is the number of arcs with head i, and the *outdegree* $d_D^+(i)$ of i is the number of arcs with tail i. A vertex with indegree zero is called a *source*, one with outdegree zero a *sink*. An *undirected graph* or a *graph* G is a digraph ignoring the orders of vertices in arcs. In other words, we will not distinguish between (i, j) and (j, i), we denote them as $\{i, j\}$, and call it an *edge* of G. For each digraph D, it is associated with an undirected graph G, whose vertex set is $V(D)$ and edge set $E(G) = \{\{i, j\} | (i, j) \in A(D)\}$. We call G the *underlying graph* of D.

A digraph E is called a *subgraph* of a digraph D if $V(E) \subseteq V(D)$, $A(E) \subseteq A(D)$, and ψ_E is the restriction of ψ_D to $A(E)$, that is, for every arc $a \in A(E)$, we have $\psi_E(a) = \psi_D(a)$. A *directed path* P in a digraph D is a subgraph of D whose vertices can be arranged in a linear sequence $(i_1, i_2, \ldots, i_k)$, where $i_j \neq i_{j'}$, for $j, j' = 1, 2, \ldots, k$, $j \neq j'$, and $A(P) = \{(i_j, i_{j+1}) | j = 1, 2, \ldots, k-1\}$. In a digraph D we say that i is a *predecessor* of j and j is a *successor* of i if there is a directed path from i to j. For any vertex $i \in N$ we denote by $P_D(i)$ the set of all predecessors of i in D, and by $S_D(i)$ the set of all successors of i in D. Moreover, $\overline{P}_D(i) := P_D(i) \cup i$ and $\overline{S}_D(i) := S_D(i) \cup i$. A *directed cycle* C in a digraph D is a subgraph of D whose vertices can be arranged in a cyclic sequence $(i_1, i_2, \ldots, i_{k+1})$, $k \geq 2$, where $i_1 = i_{k+1}$ and $i_j \neq i_{j'}$ for $j, j' = 1, 2, \ldots, k$, $j \neq j'$, and $A(C) = \{(i_j, i_{j+1}) | j = 1, 2, \ldots, k\}$. A directed graph is *acyclic*, if it does not contain any directed cycle.

An undirected graph G is *connected* if, for every partition of its vertex set into two nonempty sets X and Y, there is an edge with one end in X and the other end in Y; otherwise G is *disconnected*. A digraph is connected if its underlying graph is connected. An acyclic connected graph is called a *tree*. A *rooted tree* (see Fig. 2b) $T(r)$ is a tree T with a specified vertex r, called the *root* of T, together with an orientation in which every vertex but the root has indegree one. The root of a rooted tree is a source. A digraph T is a *sink tree*, if the digraph T', formed by the same set of links with T but with opposite orientation, appears to be a rooted tree; in this case the root of T' turns to a sink in T. A *line-graph* (see Fig. 2a) is a digraph that contains links only between subsequent vertices.

The combination of a TU-game $\langle N, v \rangle \in \mathcal{G}_N$ and a communication graph D is a so-called *graph game* or *digraph game*, depending on the graph D being directed or not. The set of all games endowed with a cooperation structure D on a fixed player set N is denoted as $\mathcal{G}_N^{\mathcal{D}}$. A value of a game with a graph structure is called a G-value.

For any digraph D on N and any coalition $S \subseteq N$, the *subgraph* of D on S is the digraph $D|_S$ with vertex set S and arc set $\{(i, j) \in A(D) | i, j \in S\}$, respectively. Given a digraph D, a coalition $S \subseteq N$ is said to be *connected* if the subgraph $D|_S$ is connected. A coalition $S \subseteq N$ is called a *component* of N if $D|_S$ is maximally connected in D. A subcoalition $S' \subseteq S \subseteq N$ is called a *component* of S if $D|_{S'}$ is maximally connected in $D|_S$. By S/D we denote the set of components of S and let $(S/D)_i$ be the component of S containing player $i \in S$.

Following Myerson (1977), we assume that for a given game with cooperation structure $\langle v, D\rangle$, cooperation is possible only among connected players and consider a *restricted game* $v^D \in \mathcal{G}_N$ defined as

$$v^D(S) = \sum_{C \in S/D} v(C), \quad \text{for all } S \subseteq N.$$

3 Basic component efficient values

In this section we list some CE values in line-graph and tree games, which we will apply later. The reason we call them "basic" values is that we can gain more complex values by combinations of them in acyclic digraph games. We want to show that acyclic digraph games can be covered by line-graph, rooted or sink tree games.

A G-value ξ is *CE* if, for any graph game $\langle v, D\rangle$, for all $C \in N/D$,

$$\sum_{i \in C} \xi_i(v, D) = v(C).$$

For a permutation $\pi : N \mapsto N$, assigning rank number $\pi(i) \in N$ to a player $i \in N$, let $\pi^i = \{j \in N | \pi(j) \leq \pi(i)\}$ be the set of all players with rank number smaller or equal to the rank number of i, including i itself. Then the *marginal value vector* $m^\pi(v) \in \mathbb{R}^n$ of game v and permutation π is given by $m_i^\pi(v) = v(\pi^i) - v(\pi^i \backslash \{i\}),\ i \in N$. By p we denote the permutation on N relevant to the natural ordering from 1 to n, i.e., $p(i) = i,\ i \in N$, and by q the permutation relevant to the reverse ordering $n, n-1, \ldots, 1$, i.e., $q(i) = n + 1 - i,\ i \in N$.

Different values for games with cooperation structure presented by line-graphs are studied in van den Brink et al. (2007). Along with the Myerson value for graph game $\langle v, D\rangle$, the authors consider three other solution concepts, namely, the *upper equivalent solution* given by

$$\xi_i^{UE}(v, D) = m_i^p(v), \quad \text{for all } i \in N,$$

the *lower equivalent solution* given by

$$\xi_i^{LE}(v, D) = m_i^q(v), \quad \text{for all } i \in N,$$

and the *equal loss solution* given by

$$\xi_i^{EL}(v, D) = \frac{m_i^p(v) + m_i^q(v)}{2}, \quad \text{for all } i \in N.$$

The previous three solutions for line-graph games are characterized via component efficiency and one of the three link deletion axioms. Please refer to the original paper to get the details.

A G-value ξ is *successor equivalent* (SE) if, for any rooted tree digraph game $\langle v, D\rangle$, for every link $(i, j) \in A(D)$, for all $k \in \overline{S}_D(j)$,

$$\xi_k(v, D\backslash(i, j)) = \xi_k(v, D).$$

A G-value ξ is *predecessor equivalent* (PE) if, for any sink tree digraph game $\langle v, D\rangle$, for every link $(i, j) \in A(D)$, for all $k \in \overline{P}_D(i)$,

$$\xi_k(v, D\backslash(i, j)) = \xi_k(v, D).$$

The *tree value* was first introduced in Demange (2004) and later axiomatized in Khmelnitskaya (2010a, b). It is the unique G-value that satisfies CE and SE. For any rooted tree digraph game $\langle v, D\rangle$, it is given by

$$t_i(v, D) = v(\overline{S}_D(i)) - \sum_{\{j|(i,j)\in A(D)\}} v(\overline{S}_D(j)), \quad i \in N. \tag{1}$$

The *sink value* was introduced in Khmelnitskaya (2010a, b). It is the unique G-value that satisfies CE and PE. For any sink tree digraph game $\langle v, D\rangle$, it is given by

$$s_i(v, D) = v(\overline{P}_D(i)) - \sum_{\{j|(j,i)\in A(D)\}} v(\overline{P}_D(j)), \quad i \in N.$$

4 Flows and weights in acyclic digraphs

We will analyze the structure of acyclic digraph games first. The analysis is similar to that in a network in graph theory. For more about network flow please refer to Bondy and Murty (2008). In an acyclic digraph game $\langle v, D\rangle$, we define two weight functions u and w, which are called *inweight* and *outweight* function of i, respectively:

$$\sum_{j\in N_D^-(i)} u(ji) = 1, \qquad \sum_{j\in N_D^+(i)} w(ij) = 1. \tag{2}$$

In fact, the sums in Eq. 2 are not necessarily 1. It is sufficient if the sum of the weights $u(ji)$, $j \in N_D^-(i)$ (or $w(ij)$, $j \in N_D^+(i)$) is a fixed value. We let them be 1 for simplicity. For every arc $(i, j) \in A(D)$, the values $u(ij)$, $w(ij)$ represent the inweight and outweight on the arc (i, j), respectively.

Given the total inflow f_{total} and the weight functions u, w, we can define a non-negative real-valued function f on the arc set $A(D)$. For every arc $(i, j) \in A(D)$, the value $f(ij)$ represents the flow on arc (i, j). We can simply assume that the total inflow f_{total} of D is 1, that is, let $R(D)$ be the source set of D, then

$$f_{\text{total}} = \sum_{i\in R(D)} \sum_{j\in N_D^+(i)} f(ij) = 1. \tag{3}$$

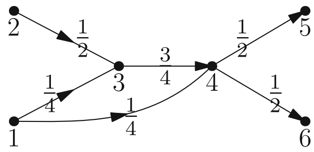

Fig. 1 A flow in an acyclic digraph

In fact it does not matter how much the total inflow is. We just assume it is 1 for simplicity. The thing that matters is that the flow is transferable, and the allocation from one vertex i to its outneighbors is determinate. The total inflow is allocated first among the vertices in the source set, then goes to the sinks from the sources, passing through all the arcs in $A(D)$ and all the vertices in $V(D)$. Except the sources and sinks, for every vertex i in $V(D)$, the inflow of i equals the outflow of i, that is,

$$\sum_{j \in N_D^+(i)} f(ij) = \sum_{j \in N_D^-(i)} f(ji). \tag{4}$$

For every vertex $i \in V(D)$ which is not a source or sink, we use $f_i = \sum_{j \in N_D^+(i)} f(ji) = \sum_{j \in N_D^-(i)} f(ij)$ to represent the inflow of i. And the inflows from $j \in N_D^-(i)$ to i are proportional to the weight $u(ji)$, the outflows from i to $j \in N_D^+(i)$ are proportional to the weight $w(ij)$, that is

$$\begin{aligned} \frac{f(ji)}{f_i} &= u(ji), \quad \text{for every vertex } j \in N_D^-(i); \\ \frac{f(ij)}{f_i} &= w(ij), \quad \text{for every vertex } j \in N_D^+(i). \end{aligned} \tag{5}$$

Example 4.1 A flow in an acyclic digraph

In Fig. 1, a flow is indicated. The source set is $\{1, 2\}$, the total flow

$$f_{\text{total}} = f(13) + f(14) + f(23) = \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1.$$

Take vertex 3 as an example. The flow in 3 is $f_3 = \frac{3}{4}$. The weight functions of vertex 3 are $u(23) = \frac{2}{3}$, $u(13) = \frac{1}{3}$ and $w(34) = 1$. □

Then, by Eq. 5 we know that in a digraph D, once the total inflow and the weight functions u, w for every vertex $i \in V(D)$ are given, we can calculate the flow on every arc in $A(D)$. In the following, all of the operations run with the condition that the total flow f_{total} and the weight functions u, w are given. Initially, all the coalitions work in full power. When some edges or vertices are deleted, the flow will change, and the worth of the coalitions will change correspondingly. For a coalition $S \subseteq N$, when the flow that comes from the source set of D to the source set of the subgraph $D|_S$ changes, we assume that the worth of S changes proportionally.

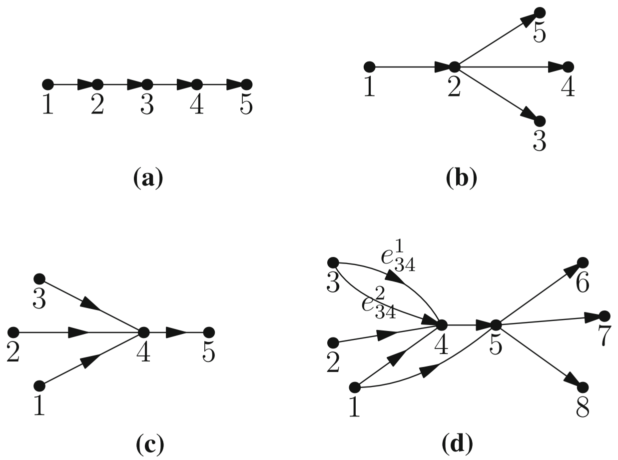

Fig. 2 A line-graph, a rooted tree, a sink tree and an acyclic digraph. **a** A line-graph, **b** a rooted tree, **c** a sink tree and **d** an acyclic digraph

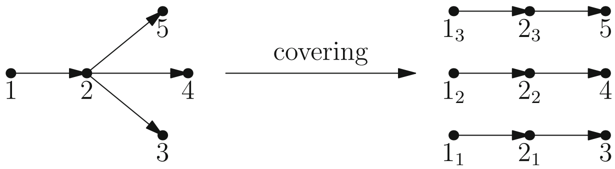

Fig. 3 The covering of a rooted tree with line-graphs

5 Coverings of trees and acyclic digraphs

In order to use the results in tree games in Khmelnitskaya (2010a, b) and line-graph games in van den Brink et al. (2007), we need to study the relationship between them and acyclic digraph games. In graph theory, a line-graph is actually a union of directed paths, in which all the components contain consecutive linearly ordered players. A line-graph is a special case of a rooted tree or sink tree, and a rooted or sink tree is a special case of an acyclic digraph. We will define a *covering* to represent this kind of relationship. A *covering* of a digraph D is a family $\mathcal{F}$ of subgraphs of D, not necessarily arc-disjoint, such that

$$\cup_{F\in\mathcal{F}} A(F) = A(D).$$

Under our definition, we can cover acyclic digraphs with rooted or sink trees, and cover rooted or sink trees with line-graphs. First of all, let's see the figures below. They are a line-graph, a rooted tree, a sink tree and an acyclic digraph, respectively.

In a rooted tree T, there is a unique path from the root to each sink. For example, in Fig. 2b, there is a unique path from the root 1 to each of the sinks in $\{3, 4, 5\}$. In Fig. 3, the rooted tree in Fig. 2b is covered with three line-graphs. In this sense, we say that a rooted tree can be *covered* by line-graphs. We can see that some vertices have been used in the covering more than once. Immediately, we get the following lemma.

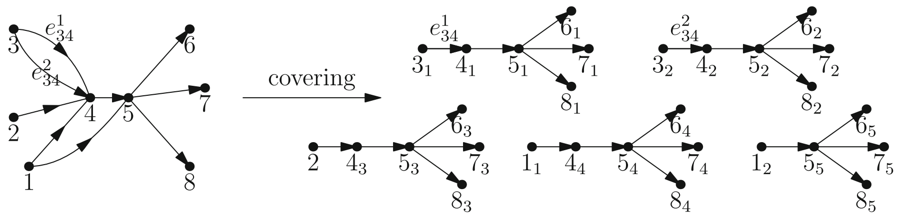

Fig. 4 A covering of an acyclic digraph with rooted trees

Lemma 5.1 *Let T be a rooted tree whose root is r, and let T have k sinks $\{s_1, s_2, \ldots, s_k\}$. Then T can be uniquely covered by k line-graphs, which are paths from r to s_i $(i \in \{1, 2, \ldots, k\})$, respectively.* □

For sink trees, we have a similar lemma. We will omit it here.

Remark 5.2 Using the depth-first search (DFS, see Bondy and Murty 2008) method, we can get the covering of a rooted or sink tree in $O(|V(T)|)$ time, which is proportional to the number of vertices in the tree. Two or more links with the same pair of ends are said to be *parallel edges*. In the definition of Khmelnitskaya (2010a, b), there is a unique path from the root to any vertex in a rooted tree. In the sequel, we can extend them to trees with parallel edges. Besides, there may be multiple paths from one vertex to another, too. □

Next we consider acyclic digraphs, see Fig. 2d. Arcs e^1_{34} and e^2_{34} are parallel edges from vertex 3 to vertex 4. Besides, there are two paths from vertex 1 to vertex 5, which are 145 and 15, respectively. For simplicity, we assume that the acyclic digraphs are connected. Since we can cover trees with line-graphs, as long as we can cover acyclic digraphs with rooted trees or sink trees, we are able to cover them with line-graphs. See the covering of an acyclic digraph in Fig. 2d with rooted trees.

We can easily know that the covering is not unique. All the rooted trees in Fig. 4 can be covered with smaller rooted trees. Although we do not have efficient algorithms to find this covering, it does not matter. In fact, we do not care about what exactly the covering is. We can define values as long as we make sure that such a covering exists. Then, we have the following lemma.

Lemma 5.3 *Let D be an acyclic digraph whose source set is $R := \{r_1, r_2, \ldots, r_m\}$ and sink set is $S := \{s_1, s_2, \ldots, s_n\}$. Then the digraph D can be covered by rooted trees $\mathcal{T} := \{T_{r_i j}\}(i \in \{1, 2, \ldots, m\}, j \in \{1, 2, \ldots, k_i\})$, where $k_i (i \in \{1, 2, \ldots, m\})$ is the number of different rooted trees with the root r_i in the digraph D.*

Proof We prove this lemma by induction on the number of vertices of the acyclic digraph. If $n = 1$ or 2, the result holds obviously. Suppose that the covering exists for acyclic digraphs with no more than $n - 1$ vertices. We now show that it holds for acyclic digraphs with n vertices. In fact, choose a vertex $r \in R$, and let $D' := D \backslash \{r\}$. Then D' has $n - 1$ vertices. By the induction hypothesis, we know that D' can be covered by rooted trees. Let the rooted tree covering of D' be $\mathcal{T}'$, and let $\mathcal{T}^r := \{T \in \mathcal{T}' | V(T) \subset \overline{S}_D(r)\}$. For every $j \in N^+_D(r)$, $\overline{S}_D(j)$ can be covered by a set of rooted trees $\mathcal{T}^j$. For every $T \in \mathcal{T}^j$, let T^{rj} be the rooted tree obtained by adding one vertex r to $V(T)$ and one arc (r, j) to $A(T)$. Let $\mathcal{T}^{rj} := \{T^{rj} | T \in \mathcal{T}^j\}$. Then $\cup_{j \in N_D^+(r)} \mathcal{T}^{rj}$ is a rooted tree covering of $\overline{S}_D(r)$. Let $\mathcal{T} := (\mathcal{T}' \backslash \mathcal{T}^r) \cup (\cup_{j \in N_D^+(r)} \mathcal{T}^{rj})$. Then $\mathcal{T}$ is a rooted tree covering of D. The lemma is thus proved. □

For convenience, we assume that any rooted tree T is not a subgraph of another tree T' in the covering $\mathcal{T}$, or we can delete T from $\mathcal{T}$, and $\mathcal{T}' := \mathcal{T} \backslash T$ is still a covering. We have a similar conclusion for the coverings of acyclic digraphs with sink trees. We omit the details here. By Lemmas 5.1 and 5.3, we can easily see that every acyclic digraph can be covered by line-graphs, too.

Now we have clarified the relationships among line-graphs, rooted trees, sink trees and acyclic digraphs in structure. We are able to investigate the game values of acyclic digraphs based on the study of line-graphs and trees.

6 Covering values of acyclic digraph games

Theorem 6.1 (Khmelnitskaya 2010a, b) *On the class of rooted-tree digraph games there is a unique G-value that satisfies CE and SE, and for any rooted tree digraph game $\langle v, D \rangle$ it is given by Eq. 1.*

From Theorem 6.1, we know that the two axioms of CE and SE uniquely define a G-value for a rooted tree digraph game. In order to find values in an acyclic digraph game, we have two kinds of ways. One way is to make use of Theorem 6.1 and Lemma 5.3, the other way is to modify the axioms for acyclic digraph games. We can uniquely define a G-value for an acyclic digraph game in both ways. Besides, we can prove that the two values are the same value in fact, and both are extensions of the tree values in Khmelnitskaya (2010a, b). In the following we will determine the acyclic digraph values in two ways, respectively.

6.1 The covering method

6.1.1 Explanations of rooted and sink tree values

By Lemma 5.1 we know that a rooted tree T can be covered by line-graphs. Let the covering be $\mathcal{L}$. For every $L \in \mathcal{L}, i \in V(L)$, we can compute the lower equivalent solution

$$\xi_i^{LE}(v, L) = m_i^q(v), \quad i \in N.$$

We define a solution of the game $\langle v, T \rangle$,

$$t_i(v, T) := \sum_{\{L \in \mathcal{L} | i \in V(L)\}} \xi_i^{LE}(v, L) + \Delta_i,$$

where

$$\Delta_i := \left(v(\overline{S}_T(i)) - \sum_{\{L\in\mathcal{L}|i\in V(L)\}} v(\overline{S}_L(i)) \right) - \sum_{\{j|(i,j)\in A(T)\}} \left(v(\overline{S}_T(j)) - \sum_{\{L\in\mathcal{L}|j\in V(L)\}} v(\overline{S}_L(j)) \right).$$

Let

$$a_i := v(\overline{S}_T(i)) - \sum_{\{L\in\mathcal{L}|i\in V(L)\}} v(\overline{S}_L(i))$$

$$b_i := \sum_{\{j|(i,j)\in A(T)\}} \left(v(\overline{S}_T(j)) - \sum_{\{L\in\mathcal{L}|j\in V(L)\}} v(\overline{S}_L(j)) \right),$$

we can rewrite $\Delta_i = a_i - b_i$. Then

$$\begin{aligned} t_i(v, T) &= \sum_{\{L\in\mathcal{L}|i\in V(L)\}} m_i^q(v) + \Delta_i \\ &= \sum_{\{L\in\mathcal{L}|i\in V(L)\}} (v(q^i) - v(q^i\backslash\{i\})) + \Delta_i \\ &= \sum_{\{L\in\mathcal{L}|i\in V(L)\}} (v(\overline{S}_L(i)) - v(S_L(i))) + \Delta_i \\ &= v(\overline{S}_T(i)) - \sum_{\{j|(i,j)\in A(T)\}} v(\overline{S}_T(j)), \end{aligned}$$

which coincides with the tree value defined in Khmelnitskaya (2010a, b).

We may consider Δ_i as the total benefit (possibly negative) of joining the different line-graphs at player i.

When $\Delta_i > 0$, we can apply Burt's structural hole theory (Burt 1992), which argues that social capital is created by a network in which people can broker connections among otherwise disconnected segments. The players in $\overline{S}_T(i)$ scattered themselves in $\overline{S}_L(i)$, $L \in \mathcal{L}$ before. We may consider they were not themselves well connected or organized. The lack of connections among them is the structural hole. The structural hole provides player i with opportunities, the involvement of player i strengthens the connections among them. Player i brokers connections among $\overline{S}_L(i)$, $L \in \mathcal{L}$, the structure hole is "filled" by player i, and Δ_i is thus created. The term a_i is the difference between the worth of $\overline{S}_T(i)$ before and after the involvement of player i. And the term b_i may be considered as some compensation to the children of player i.

When $\Delta_i \leq 0$, the involvement of player i is not that successful, a loss is produced. There may be many reasons for a loss to come into being. For example, maybe there is not a structural hole among the players in $\overline{S}_T(i)$; maybe player i is not good at brokering; maybe this loss is temporary; maybe there exist conflicts among the players in $\overline{S}_T(i)$ that can not be resolved.

Example 6.2 The explanation of the rooted tree value

We will explain the rooted tree value in the graph game with communication structure shown in Fig. 3. As shown in the covering, player 1 has three choices. He may form three line-graph coalitions, $L_1 = \{1, 2, 3\}$, $L_2 = \{1, 2, 4\}$, or $L_3 = \{1, 2, 5\}$. In each line-graph coalition, he gets the marginal value $\xi_1^{LE}(v, L_i)$, $i = 1, 2, 3$. If the players in different line-graph coalitions can not cooperate, he can only earn the payoff in one of the three line-graph coalitions. He is not satisfied, a greater coalition must be better for him. Then he tries to form the grand coalition, with all the players in it. In the grand coalition $\{1, 2, 3, 4, 5\}$, he gets the sum of the payoffs in the line-graphs, i.e., $\sum_{i=1}^{3} \xi_1^{LE}(v, L_i)$.

If $\Delta_1 > 0$, we may consider Δ_1 as the gain. By Burt's structural hole theory (Burt 1992), player 1 brokers connections among L_i, $i = 1, 2, 3$, social capital is created. In Δ_1, a_1 is the social capital created by "filling" the structural hole, and b_1 is paid to player 2 by player 1, since player 2 also contributes to the connections.

If $\Delta_1 \leq 0$, we may consider Δ_1 as the cost, where a_1 is the cost of player 1, and b_1 is the compensation from player 2, since player 2 also earns more by forming the grand coalition. We can see that

$$\sum_{i=1}^{5} \Delta_i = v(\overline{S}_T(1)) - \sum_{i=1}^{3} v(\overline{S}_{L^i}(1)),$$

which equals a_1 in Δ_1. This means that the cost a_1 in Δ_1 is undertaken by all the players. Players in $\{3, 4, 5\}$ do not bear any cost since their payoffs do not increase. □

Similarly, for a sink tree game $\langle v, T\rangle$, and a line-graph covering $\mathcal{L}$ of T, we can define another solution

$$s_i(v, T) = \sum_{\{L\in\mathcal{L}|i\in V(L)\}} \xi_i^{UE}(v, L) + \Delta_i,$$

which coincides with the sink value defined in Khmelnitskaya (2010a, b), where

$$\Delta_i = \left(v(\overline{P}_T(i)) - \sum_{\{L\in\mathcal{L}|i\in V(L)\}} v(\overline{P}_L(i))\right) - \sum_{\{j|(j,i)\in A(T)\}} \left(v(\overline{P}_T(j)) - \sum_{\{L\in\mathcal{L}|j\in V(L)\}} v(\overline{P}_L(j))\right).$$

6.1.2 Covering values of acyclic digraph games

If the flow f in an acyclic digraph D is given, then by Lemma 5.3, correspondingly we can define all the flows of the rooted trees in the covering of D. Denote the set of rooted trees in the covering found in Lemma 5.3 of D by $\mathcal{T}$. Then by Theorem 6.1, in every rooted tree game $\langle v_T, T\rangle, T \in \mathcal{T}$, there is a unique G-value that satisfies CE and SE, and it is given by

$$t_i(v, T) = v_T(\overline{S}_T(i)) - \sum_{j\in N_T^+(i)} v_T(\overline{S}_T(j)), \quad \text{for all } i \in V(T), \tag{6}$$

where $\langle v_T, T\rangle$ is the corresponding game of $\langle v, D\rangle$ restricted on the rooted tree T.

Equation 6 gives the unique value for every vertex in the rooted tree T. Let the set of rooted trees containing vertex i in $\mathcal{T}$ be $\mathcal{T}^i$. By summing up Eq. 6 over $\mathcal{T}^i$, we get

$$\sum_{T\in\mathcal{T}^i} t_i(v, T) = \sum_{T\in\mathcal{T}^i} v_T(\overline{S}_T(i)) - \sum_{T\in\mathcal{T}^i}\sum_{j\in N_T^+(i)} v_T(\overline{S}_T(j)), \quad \text{for all } i \in V(D).$$

We define a new value,

$$d_i(v, D) := \sum_{T\in\mathcal{T}^i} t_i(v, T) + \Delta_i, \tag{7}$$

where

$$\Delta_i := \left(v(\overline{S}_D(i)) - \sum_{T\in\mathcal{T}^i} v_T(\overline{S}_T(i))\right) - \sum_{j\in N_D^+(i)} \left(\frac{f(ij)}{f_j} v(\overline{S}_D(j)) - \sum_{\{T\in\mathcal{T}^i | j\in V(T)\}} v_T(\overline{S}_T(j))\right). \tag{8}$$

In Eq. 8, $f(ij)$ is the flow amount directly from vertex i to j, or the flow amount on arc (i, j). If multiple arcs occur from vertex i to j here, then $f(ij)$ is their sum of flows. And f_j is the total flow amount that goes through vertex j from all of its parents.

It is obvious that

$$N_D^+(i) = \cup_{T\in\mathcal{T}^i} N_T^+(i).$$

So we have

$$\sum_{j\in N_D^+(i)} \sum_{\{T\in\mathcal{T}^i | j\in V(T)\}} v_T(\overline{S}_T(j)) = \sum_{T\in\mathcal{T}^i} \sum_{j\in N_T^+(i)} v_T(\overline{S}_T(j))$$

Then Eq. 7 becomes

$$d_i(v, D) = v(\overline{S}_D(i)) - \sum_{j\in N_D^+(i)} \frac{f(ij)}{f_j} v(\overline{S}_D(j)), \quad \text{for all } i \in V(D). \tag{9}$$

We call the class of values d_i defined in (9) the class of *rooted covering value*. The outcome of the rooted covering value does not necessarily satisfy component efficiency, we have the following proposition.

Proposition 6.3 *Let D be an acyclic digraph. Let $C \in N/D$ be a component of N, and let R_C be its source set. Then for the digraph game $\langle v, D\rangle$, the outcome of the rooted covering value satisfies component efficiency if and only if*

$$v(C) = \sum_{i \in R_C} v(\overline{S}_D(i)), \quad \textit{for any } C \in N/D.$$

Proof Obviously, $D|_C$ is an acyclic digraph, and $C = \cup_{i \in R_C} \overline{S}_D(i) = \overline{S}_D(R_C)$. Let f_j^i be the total flow amount that goes from i to j. Note that f_j^i is different from $f(ij)$. Since there may be multiple paths from i to j, and f_j^i is the sum of the flow amount over all of such paths, $f(ij)$ is just the flow amount on arc (i, j). Since $\sum_{i \in R_C} \frac{f_j^i}{f_j} = 1$, we have

$$\sum_{j \in \overline{S}_D(R_C)} d_j(v, D) = \sum_{i \in R_C} \sum_{j \in \overline{S}_D(i)} \frac{f_j^i}{f_j} d_j(v, D) \tag{10}$$

Simply multiplying $\frac{f_j^i}{f_j}$ and rooted covering value d_j of j, we have the following equality:

$$\frac{f_j^i}{f_j} d_j(v, D) = \frac{f_j^i}{f_j}\left(v(\overline{S}_D(j)) - \sum_{k \in N_D^+(j)} \frac{f(jk)}{f_k} v(\overline{S}_D(k))\right). \tag{11}$$

Summing up Eq. 11 over $\overline{S}_D(i)$, we get

$$\begin{aligned}\sum_{j \in \overline{S}_D(i)} \frac{f_j^i}{f_j} d_j(v, D) &= \frac{f_i^i}{f_i} d_i(v, D) \\ &\quad + \sum_{j \in S_D(i)} \frac{f_j^i}{f_j}\left(v(\overline{S}_D(j)) - \sum_{k \in N_D^+(j)} \frac{f(jk)}{f_k} v(\overline{S}_D(k))\right) \\ &= d_i(v, D) + \sum_{j \in N_D^+(i)} \frac{f(ij)}{f_j} v(\overline{S}_D(j)).\end{aligned} \tag{12}$$

Summing up Eq. 12 over $i \in R_C$, we get

$$\sum_{i \in R_C} \sum_{j \in \overline{S}_D(i)} \frac{f_j^i}{f_j} d_j(v, D) = \sum_{i \in R_C}\left(d_i(v, D) + \sum_{j \in N_D^+(i)} \frac{f(ij)}{f_j} v(\overline{S}_D(j))\right) \tag{13}$$

The definition of rooted covering value in Eq. 9 shows that

$$\sum_{i\in R_C}\left(d_i(v,D)+\sum_{j\in N_D^+(i)}\frac{f(ij)}{f_j}v(\overline{S}_D(j))\right)=\sum_{i\in R_C}v(\overline{S}_D(i)) \tag{14}$$

By Eqs. 10, 13 and 14, we know that

$$\sum_{j\in\overline{S}_D(R_C)}d_j(v,D)=\sum_{i\in R_C}v(\overline{S}_D(i)) \tag{15}$$

The outcome of the rooted covering value for the digraph game $\langle v, D\rangle$ satisfies component efficiency if and only if for every component C in N,

$$v(C)=\sum_{j\in\overline{S}_D(R_C)}d_j(v,D). \tag{16}$$

And by Eq. 15 we know that Eq. 16 holds if and only if $v(C)=\sum_{i\in R_C}v(\overline{S}_D(i))$, thus the proof is complete. □

Specially, if $|R_C|=1$, the outcome of the rooted covering value satisfies component efficiency.

We consider the expression $v_e=v(C)-\sum_{i\in R_C}v(\overline{S}_D(i))$, which is the difference between the worth of the coalition C and the sum of the payoffs of the players (equal to $\sum_{i\in R_C}v(\overline{S}_D(i))$ by Eq. 15) in the rooted covering value. If the difference is zero, we can easily see that the outcome of the rooted covering value satisfies component efficiency. However, this is not always the fact. We call the expression given above the *extra value* of the component C. The extra value may be positive, negative or zero. What does the extra value mean? We want to explain this in the following example.

Example 6.4 The explanation of the extra value
We consider a digraph game with Fig. 2d as the communication graph. This digraph contains only one component. Let $U=\{4,5,6,7,8\}$, $U_1=\overline{S}_D(1)$, $U_2=\overline{S}_D(2)$, $U_3=\overline{S}_D(3)$. U takes part in three different coalitions, U_1, U_2 and U_3, and the payoff of U is the sum of his payoffs in them. In this digraph game, $v_e=v(N)-\sum_{i=1,2,3}v(U_i)$. The coalitions U_1, U_2, U_3 are a covering of the grand coalition N, they have U as the common players. The players in $\{1,2,3\}$ are the sources of the digraph, they have special positions. In a sense, we can say they are the creators of the coalitions. The cooperation in N is not possible without their involvements. They all have their own successors, despite some of them have intersections. However, they are not successors of each other. If we consider this predecessor–successor relationship as some superior–subordinate relationship, then they are all superiors, they have their own subordinates (there may be intersections). But they are in the same level, they do not have to be subordinated to anyone. This situation creates some difficulty for the formation of the grand coalition. Perhaps we may consider v_e as the difference between the collective interest and individual interest. The rooted covering value

describes a situation where the individual interest prevails. We can not tell which is better for the group N, since v_e may be positive, zero or negative. If $v_e = 0$, this means that although U_1, U_2 and U_3 have U as the common members, they are kind of independent. Their cooperation does not create extra value, the value of the grand coalition is just the sum of their values. If $v_e > 0$, this means that the grand coalition gains extra value. If $v_e < 0$, this means that the grand coalition loses extra value. We may just consider U_1, U_2, U_3 as three different departments in one company D. The group U consists of the common employees in these departments, and U gets paid from all of the three departments. The payoff of U is the sum of his payoffs in the three departments. Having paid the wages of his employees, the company gets extra money v_e. □

In some cases, if the grand coalition does form, the total worth is $v(N)$, what can we do now? A small modification can be made to the covering value to get a CE value.

Example 6.5 The allocation of the extra value

If the grand coalition forms, who deserves to earn ($v_e \geq 0$) or has to afford ($v_e \leq 0$) the extra value? We will show this in the game in Example 6.4. We say the players in the set $B = \{1, 2, 3\}$ are responsible for the extra value. The set B may be considered as the boss set, they are the bosses of the company. When the company makes profit, they earn extra money; when the company operates badly, they lose extra money. The extra value v_e is allocated among the set B according to their payoffs. Let the payoffs of the bosses be d_1, d_2, d_3 respectively. Then the boss i, $i = 1, 2, 3$ get extra value

$$v_e^i = \frac{v_e d_i}{d_1 + d_2 + d_3}.$$

□

In general, let $C \in N/D$ be a component of N, and let R_C be its source set. Then we can define a CE covering value, such that

$$d_i' = \begin{cases} d_i + \frac{d_i v_e^C}{\sum_{j \in R_C} d_j} & \text{if } i \in R_C; \\ d_i & \text{if } i \in C \backslash R_C, \end{cases}$$

where v_e^C is the extra value of the component C. It is easy to check that the newly defined value d' is CE.

The rooted covering value does not necessarily satisfy the axiom of SE, either. That is because if we delete an arc a from D, the new digraph $D' = D - a$ may be still connected.

Similarly, we can cover an acyclic digraph with sink trees or line-graphs. We can define different covering values which are the combinations of the sink values or line-graph values. We will not discuss them in detail here.

6.2 Axiomatic characterizations

In this section, our approach to the value is close to that of Myerson (1977) based on ideas of efficiency and a certain link deletion property.

In Sect. 6.1.2, we define a unique G-value for an acyclic digraph game $\langle v, D\rangle$, under the condition that all the subgames on the rooted trees in D's coverings satisfy the two axioms: CE and SE. We wish to find new axioms to substitute SE and CE, such that with the new axioms, we can uniquely define a value for an acyclic digraph game. In this section, we find such axioms, and define a value which coincides with the rooted covering value.

Denote $E = \{(j, k) \in A(D) | j \in N_D^-(\overline{S}_D(i)), k \in \overline{S}_D(i)\}$, let $D' = D\backslash E$. A G-value ξ is *modified successor equivalent* (MSE) if, for any acyclic digraph game $\langle v, D\rangle$, for every vertex $i \in V(D)$, we have

$$\begin{aligned}\xi_j(v, D') &= \frac{\sum_{k\in N_{D'}^-(j)} u(kj)}{\sum_{k\in N_D^-(j)} u(kj)}\xi_j(v, D)\\ &= \xi_j(v, D) \sum_{k\in N_{D'}^-(j)} u(kj), \quad \text{for all } j \in S_D(i),\end{aligned}$$

and $\xi_i(v, D') = \xi_i(v, D)$.

MSE is a natural extension of SE defined in Khmelnitskaya (2010a, b). It shows that the payoffs of the players in the group $\overline{S}_D(i)$ change accordingly in a situation when the group is isolated. In the new digraph D', player i finds his way to create as much flow amount as he had in D, he then takes on the responsibility to supply flows for the group. But for other players in the group who can't create flow, their flow amount may decrease in D' than that they had in D. Since they could have received flows through E which are now cut off. In the new digraph game $\langle v, D'\rangle$, the payoff of player i keeps the same, and the payoffs of other players decrease according to how much the flow amount going through them has decreased by.

A G-value ξ is *modified covering efficient* (MCE) if, for any acyclic digraph game $\langle v, D\rangle$, for any component $C \in N/D$, and its corresponding source set R_C,

$$\sum_{i\in V(C)} \xi_i(v, D) = \sum_{i\in R_C} v(\overline{S}_D(i)).$$

MCE is a variant of component efficiency. In the acyclic digraph game $\langle v, D\rangle$, the players in the source set R_C are in special positions. They do not have any predecessors, they all have some successors (some in common). We may consider this predecessor–successor relationship as some kind of superior–subordinate relationship. Since all players in R_C do not have any superiors, this situation makes it difficult for them to work without frictions. They have the motive to seek their independence from the coalition C. We think that there does be cooperation across C, since $\overline{S}_D(i), i \in R_C$ have players in common. Nevertheless, the coalition C is not necessarily thus formed.

Coalitions $\overline{S}_D(i), i \in R_C$ form instead. The sum of the payoffs of all the players in C is only a simple addition over the worth of $\overline{S}_D(i),\ i \in R_C$.

Note that MCE does not depend on the weight functions u, w and flow function f, but MSE does depend on them. It turns out that the two axioms of MCE and MSE uniquely define a G-value on the class of acyclic digraph games.

Theorem 6.6 *On the class of acyclic digraph games, there is a unique G-value that satisfies MCE and MSE, and for any acyclic digraph game $\langle v, D\rangle$ with weight functions u, w and flow function f defined in Sect. 4, it is given by*

$$d_i(v, D) = v(\overline{S}_D(i)) - \sum_{j\in N_D^+(i)} u(ij)v(\overline{S}_D(j)), \quad \textit{for all } i \in N. \tag{17}$$

From now on we refer to the G-value d as the *digraph value*.

Proof D' is as defined in MSE. At first, we show that MCE and MSE on acyclic digraph games define uniquely a G-value that satisfies these axioms. Consider a G-value ξ satisfying MCE and MSE, and let $\langle v, D\rangle \in \mathcal{G}_N^{\mathcal{D}}$.

For a vertex $i \in V(D)$, let C^i be the component in N/D' containing player i. By the acyclic digraph structure of D, $C^i = \overline{S}_D(i)$, and $R_{C^i} = \{i\}$. Because of MCE, it holds that

$$\sum_{j\in C^i} \xi_j(v, D') = v(C^i). \tag{18}$$

Then MSE implies that

$$\begin{aligned}\sum_{j\in C^i} \xi_j(v, D') &= \sum_{j\in \overline{S}_D(i)} \xi_j(v, D')\\ &\overset{MSE}{=} \xi_i(v, D) + \sum_{j\in S_D(i)} \frac{\sum_{k\in N_{D'}^-(j)} u(kj)}{\sum_{k\in N_D^-(j)} u(kj)}\xi_j(v, D)\\ &\overset{(18)}{=} v(C^i).\end{aligned} \tag{19}$$

We get $|V(D)|$ equations of type (19), with $\xi_j(v, D),\ j \in N$ as variables. We know that $|V(D)| = |N|$. Due to the acyclic digraph structure of D, we can arrange the order of the $|V(D)|$ equations such that the coefficient matrix is an upper triangular matrix with ones on the diagonal. So the $|N|$ equations of type (19) are linearly independent and, therefore, uniquely determine $\xi(v, D)$.

Secondly, we verify now that the digraph value d, defined by (17), satisfies MCE and MSE. The value d coincides with the covering value defined in Sect. 6.1.2. And by the proof of Proposition 6.3 (Eq. 15), we know that MCE is satisfied. Next, observe that, due to the acyclic graph structure of D, for any player $i \in V(D)$, the sets $\overline{S}_{D'}(i)$ and $\overline{S}_D(i)$ coincide. Therefore, by definition (17), it follows immediately that d meets MSE. In fact, by (17), we have

$$
\begin{aligned}
d_i(v, D') &= v(\overline{S}_{D'}(i)) - \sum_{j\in N^+_{D'}(i)} u(ij)v(\overline{S}_{D'}(j)) \\
&= v(\overline{S}_D(i)) - \sum_{j\in N^+_D(i)} u(ij)v(\overline{S}_D(j)) \\
&= d_i(v, D),
\end{aligned}
$$

$$
d_j(v, D') = v_{D'}(\overline{S}_{D'}(j)) - \sum_{k\in N^+_{D'}(i)} v_{D'}(\overline{S}_{D'}(k)) \tag{20}
$$

$$
\begin{aligned}
&= \sum_{k\in N^-_{D'}(j)} u(kj)\left(v(\overline{S}_D(j)) - \sum_{k\in N^+_D(j)} u(jk)v(\overline{S}_D(k))\right) \\
&= d_j(v, D) \sum_{k\in N^-_{D'}(j)} u(kj) \quad \text{for all } j \in S_D(i).
\end{aligned} \tag{21}
$$

Equations 20 and 20 show that the value d satisfies the axiom of MSE. □

Given weight functions u, w and flow function f, since $\frac{f(ij)}{f_j} = u(ij)$, we know that the value axiomatized above coincides with the value defined in (9).

Example 6.7 Calculation of the digraph value

In Fig. 5, it is a sink tree game provided with a flow. We find the digraph values in this game in two methods.

Firstly, we consider the subgames restricted on the line-graphs which are in the line-graph covering of the sink tree. We give their tree values (same with the lower equivalent values),

$$
\begin{aligned}
&t(5_1) = v(5_1), \quad t(4_1) = v(\overline{S}(4_1)) - v(5_1), \quad t(1) = v(\overline{S}(1)) - v(\overline{S}(4_1)); \\
&t(5_2) = v(5_2), \quad t(4_2) = v(\overline{S}(4_2)) - v(5_2), \quad t(2) = v(\overline{S}(2)) - v(\overline{S}(4_2)); \\
&t(5_3) = v(5_3), \quad t(4_3) = v(\overline{S}(4_3)) - v(5_3), \quad t(3) = v(\overline{S}(3)) - v(\overline{S}(4_3)).
\end{aligned}
$$

Besides, because the worth of a coalition is proportional to the flow amount that goes to its source set, we know that

$$
\begin{aligned}
v(5_1) = v(5_2) = v(5_3) &= \frac{1}{3}v(5); \\
v(\overline{S}(4_1)) = v(\overline{S}(4_2)) = v(\overline{S}(4_3)) &= \frac{1}{3}v(\overline{S}(4)).
\end{aligned}
$$

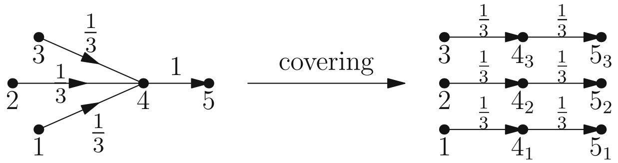

Fig. 5 Digraph values in sink tree

The covering value is defined by summing up the tree values over all the trees in the covering plus an extra value. We have

$$
\begin{aligned}
d(5) &= \sum_{i=1,2,3} t(5_i) + \Delta_5 = v(5) + 0 = v(5);\\
d(4) &= \sum_{i=1,2,3} t(4_i) + \Delta_4\\
&= \sum_{i=1,2,3} (v(\overline{S}(4_i)) - v(5_i)) + \left(v(\overline{S}(4)) - \sum_{i=1,2,3} v(\overline{S}(4_i))\right)\\
&\quad -\left(v(5) - \sum_{i=1,2,3} v(5_i)\right)\\
&= v(\overline{S}(4)) - v(5); \qquad (22)\\
d(1) &= v(\overline{S}(1)) - \frac{1}{3} v(\overline{S}(4));\\
d(2) &= v(\overline{S}(2)) - \frac{1}{3} v(\overline{S}(4));\\
d(3) &= v(\overline{S}(3)) - \frac{1}{3} v(\overline{S}(4)).
\end{aligned}
$$

Secondly, by Theorem 6.6, we can directly calculate the digraph values of all the players in the example, which coincide with Eq. 22. □

Consider another two axioms:

Denote $E = \{(j,k) \in A(D) | j \in \overline{P}_D(i), k \in N_D^+(\overline{P}_D(i))\}$, let $D' = D \backslash E$. A G-value ξ is *modified predecessor equivalent* (MPE) if, for any acyclic digraph game $\langle v, D\rangle$, for every vertex $i \in V(D)$, we have

$$
\begin{aligned}
\xi_j(v, D') &= \frac{\sum_{k \in N_{D'}^+(j)} w(jk)}{\sum_{k \in N_D^+(j)} w(jk)} \xi_j(v, D)\\
&= \xi_j(v, D) \sum_{k \in N_{D'}^+(j)} w(jk), \quad \text{for all } j \in P_D(i),\\
\text{and} \quad \xi_i(v, D') &= \xi_i(v, D).
\end{aligned}
$$

A G-value ξ is *modified covering efficient* (MCE') if, for any acyclic digraph game $\langle v, D\rangle$, for any component $C \in N/D$, and its corresponding sink set S_C,

$$
\sum_{i \in C} \xi_i(v, D) = \sum_{i \in S_C} v(\overline{P}_D(i)).
$$

It turns out that the two axioms of MCE' and MPE uniquely define a G-value on the class of acyclic digraph game.

Theorem 6.8 *On the class of acyclic digraph games there is a unique G-value that satisfies MCE' and MPE. For any acyclic digraph game $\langle v, D\rangle$ with weight functions u, w and flow function f defined in Sect. 4, it is given by*

$$s_i(v, D) = v(\overline{P}_D(i)) - \sum_{j \in N_D^-(i)} w(ji)v(\overline{P}_D(j)), \quad \textit{for all } i \in N, \tag{23}$$

From now on we refer to the G-value s as the sink digraph value.

The proof of Theorem 6.8 is similar to that of Theorem 6.6, and we skip it here.

7 Distribution of Harsanyi dividends

In the sequel, for the cardinality of a given set N we use lowercase letters like $n = |N|$.

Shapley (1953) introduced a well-known basis for the game space $\mathcal{G}^N$ called unanimity games. With every coalition $S \subseteq N$, there is associated its *unanimity game* $\langle N, u_S\rangle$ defined by

$$u_S(T) = \begin{cases} 1, & \text{if } S \subset T; \\ 0, & \text{otherwise.} \end{cases}$$

It is well-known that any game $\langle N, v\rangle$ can be represented as a linear combination of the unanimity games $\langle N, u_S\rangle$, $S \subseteq N$, such that

$$v = \sum_{S \subseteq N} \Delta_S^v u_S,$$

where Δ_S^v is the so-called dividend with respect to the coalition S, $S \subseteq N$. The *dividends* Δ_S^v, $S \subseteq N$, of a game $\langle N, v\rangle$, as defined by Harsanyi (1958), are of the form

$$\Delta_S^v = \sum_{T \subseteq S} (-1)^{s-t} v(T), \quad \text{for all } S \subseteq N.$$

Moreover,

$$v(S) = \sum_{T \subseteq S} \Delta_T^v, \quad \text{for all } S \subseteq N. \tag{24}$$

By (24), the worth of any coalition is equal to the sum of Harsanyi dividends of the coalition itself and all its proper subcoalitions. The Harsanyi dividend of a coalition has a natural interpretation as the extra revenue from cooperation among the players of the coalition that they did not already realize by cooperating in smaller coalitions. How the value distributes the dividend of a coalition provides important information concerning the interest of different players to create the coalition. van den Brink et al. (2007) discussed the distribution of Harsanyi dividends in line-graphs, and Herings et al. (2008) discussed it in trees. We consider now the digraph and sink digraph values with respect to the distribution of Harsanyi dividends.

The digraph value d of any acyclic digraph game $\langle v, D\rangle$ can be equivalently represented in terms of restricted games as

$$d_i(v, D) = v^D(\overline{S}_D(i)) - \sum_{j \in N_D^+(i)} u(ij)v^D(\overline{S}_D(j)). \tag{25}$$

Let

$$\begin{aligned}
\mathcal{S} &= \{S \subseteq \overline{S}_D(i)\},\\
\mathcal{S}_1 &= \{S \subseteq \overline{S}_D(i) | i \in S\},\\
\mathcal{S}_2 &= \{S \subseteq S_D(i) | \forall j \in N_D^+(i),\ S \nsubseteq S_D(j)\},\\
\mathcal{S}_3 &= \{S \subseteq S_D(i) | \exists j \in N_D^+(i)\ s.t.\ S \subseteq S_D(j)\},\\
I_S &= \{j \in N_D^+(i) | S \subseteq S_D(j)\}.
\end{aligned}$$

We can easily see that

$$\begin{aligned}
\mathcal{S} &= \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3,\\
\mathcal{S}_i \cap \mathcal{S}_j &= \emptyset, \quad \text{for } i \neq j,\ i, j = 1, 2, 3.
\end{aligned}$$

From Eq. 25 and the representation of the worth of a coalition via Harsanyi dividends in Eq. 24, it follows that for any acyclic digraph game $\langle v, D\rangle \in \mathcal{G}_N^{\mathcal{D}}$, the digraph value in terms of the distribution of Harsanyi dividends is given by

$$\begin{aligned}
d_i(v, D) &= \sum_{S \subseteq \overline{S}_D(i)} \Delta_S^{v^D} - \sum_{j \in N_D^+(i)} u(ij) \sum_{S \subseteq S_D(j)} \Delta_S^{v^D}\\
&= \sum_{S \in \mathcal{S}_1} \Delta_S^{v^D} + \sum_{S \in \mathcal{S}_2} \Delta_S^{v^D} + \sum_{S \in \mathcal{S}_3} \Delta_S^{v^D} - \sum_{j \in N_D^+(i)} u(ij) \sum_{S \subseteq S_D(j)} \Delta_S^{v^D}\\
&= \sum_{S \in \mathcal{S}_1} \Delta_S^{v^D} + \sum_{S \in \mathcal{S}_2} \Delta_S^{v^D} + \sum_{S \in \mathcal{S}_3} \left(1 - \sum_{j \in I_S} u(ij)\right) \Delta_S^{v^D}, \quad \text{for all } i \in N.
\end{aligned}$$

The digraph value of player i in terms of dividends consists of three parts. In the following we will explain them, respectively.

- The first part is the sum of dividends of coalitions in $\mathcal{S}_1$. For any coalition $S \in \mathcal{S}_1$, it has player i as the source. The players in S have player i as the only supplier of flows, and they turn in the dividend of S to player i as a return.
- The second part is the sum of dividends of coalitions in $\mathcal{S}_2$. For any coalition $S \in \mathcal{S}_2$, its source set has more than one members. If the players in its source set have conflicts, it is difficult for the coalition S to form. The coalition S might have not been formed if player i had not been involved. So S turns in its dividend to

player i. If we apply Burt's Structural hole theory (Burt 1992), this part of dividends may be also considered as the social capital created by player i who brokers connections among players in coalitions in $\mathcal{S}_2$.

- The third part is a linear combination of dividends of coalitions in $\mathcal{S}_3$. For any coalition $S \in \mathcal{S}_3$, the coefficient of $\Delta_S^{v^D}$ is related to the flow amount going from player i to his children in I_S. Although player i is a predecessor of S, I_S also contains predecessors of S, and the distance from the players in I_S to S are nearer than that from player i to S. Player i may trust I_S to pass flows to S, although he is in charge of the distribution of the dividend of S, he has to pay to every player $j \in I_S$ part of the dividend $u(ij)\Delta_S^{v^D}$ for their work to pass flows. The coefficient $1 - \sum_{j \in I_S} u(ij)$ is less than 1, it may be positive, negative or zero.

Note that in the special case of rooted tree games, the second and third part of the digraph value are zero, which means that the digraph value of a rooted tree graph assigns dividend of any connected coalition to its root. As discussed above, in the general case of acyclic digraph games, the digraph value assigns more dividends to the source player. In a coalition $S = \overline{S}_D(i)$, player i is given a strong motive to create S.

For the sink digraph value, we have a similar analysis, and we omit the details here.

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