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A characterization of the Kalai–Smorodinsky bargaining solution by disagreement point monotonicity

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Abstract We provide a new axiomatization of the Kalai–Smorodinsky bargaining solution, which replaces the axiom of individual monotonicity by disagreement point monotonicity and a restricted version of Nash's IIA.

Keywords Bargaining · Kalai–Smorodinsky solution · Disagreement point monotonicity

JEL Classification C78 · D74

1 Introduction

The classic *bargaining problem*, originated in Nash (1950), is defined as a pair (S, d), where $S \subset \mathbb{R}^2$ is the *feasible set*, representing all possible (v-N.M) utility agreements between the two players, and $d \in S$, the *disagreement point*, is a point that specifies the players' utilities in case they do not reach a unanimous agreement on some point of S. The following assumptions are made on (S, d):

- S is compact and convex;
- $d < x$ for some $x \in S$;[1]
- For all $x \in S$ and $y \in \mathbb{R}^2 : d \leq y \leq x \Rightarrow y \in S$.

Denoting by $\mathcal{B}$ the collection of all such pairs (S, d), a *solution* is a function $\mu \colon \mathcal{B} \to \mathbb{R}^2$ that satisfies $\mu(S, d) \in S$ for all $(S, d) \in \mathcal{B}$. Given a feasible set S, the *weak Pareto frontier* of S is $WP(S) \equiv \{x \in S | y > x \Rightarrow y \notin S\}$ and the *strict Pareto frontier* of

[1] Vector inequalities: xRy iff $x_i R y_i$ for both $i \in \{1, 2\}$, $R \in \{>, \geq\}$; $x \gneq y$ iff $x \geq y \& x \neq y$.

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S is $P(S) \equiv \{x \in S | y \gneq x \Rightarrow y \notin S\}$. A function $f : \mathbb{R} \to \mathbb{R}$ is a *positive affine transformation* if $f(x) = \alpha x + \beta$, for some $\alpha > 0$ and $\beta \in \mathbb{R}$. Let F_A denote the set of these functions. Also, let $\pi(a, b) \equiv (b, a)$.

Letting (S, d) denote a generic element of $\mathcal{B}$, the *Nash solution* to the problem (S, d) is the unique maximizer of $(x_1 - d_1)(x_2 - d_2)$ over $S_d \equiv \{x \in S | x \geq d\}$. Nash showed that this is the unique solution that satisfies the following axioms, in the statements of which (S, d) and (T, d') are arbitrary elements of $\mathcal{B}$.

Weak Pareto optimality (WPO): $\mu(S, d) \in WP(S)$.[2]

Individual rationality (IR): $\mu_i(S, d) \geq d_i$ for all $i \in \{1, 2\}$.

Independence of equivalent utility representations (IEUR): $(f_1, f_2) \in F_A \times F_A \Rightarrow (f_1, f_2) \circ \mu(S, d) = \mu((f_1, f_2) \circ S, (f_1, f_2) \circ d)$.[3]

Symmetry (SY): $[\pi S = S]\&[\pi d = d] \Rightarrow \mu_1(S, d) = \mu_2(S, d)$.[4]

Independence of irrelevant alternatives (IIA): $[S \subset T]\&[d = d']\&[\mu(T, d') \in S] \Rightarrow \mu(S, d) = \mu(T, d')$.

Whereas the first four axioms are widely accepted, IIA has raised some criticism. The idea behind a typical such criticism is that the bargaining solution could, or even should, depend on the shape of the feasible set. In particular, Kalai and Smorodinsky (1975) noted that when the feasible set expands in such a way that for every feasible payoff for player 1 the maximal feasible payoff for player 2 increases, it may be the case that player 2 loses from this expansion under the Nash solution.

We now introduce additional notation in order to state Kalai's and Smorodinsky's idea formally. Let (S, d) denote a generic element of $\mathcal{B}$. For each $x \in S_d$, let $g_i^S(x_j)$ be the maximal possible payoff for i in S given that j's payoff is x_j, where $\{i, j\} = \{1, 2\}$. The point $a(S, d)$, called the *ideal point* of (S, d), is defined by $a_i = \max\{x_i | x \in S_d\}$. The monotonicity axiom considered by Kalai and Smorodinsky is stated as follows, where (S, d) and (T, d') are arbitrary elements of $\mathcal{B}$.

Individual monotonicity (IM):

$$[d = d']\&[a_j(S, d) = a_j(T, d')]\&[g_i^S(x_j) \leq g_i^T(x_j) \; \forall x \in S_d \cap T_{d'}] \\ \Rightarrow \mu_i(S, d) \leq \mu_i(T, d')$$

The axiom IM states that if for every utility level that player j may demand the maximum feasible utility level that player i can simultaneously reach is increased, then

[2] A natural strengthening of this axiom is Pareto optimality (PO), which requires $\mu(S, d) \in P(S)$ for all $(S, d) \in \mathcal{B}$.

[3] If $f_i : \mathbb{R} \to \mathbb{R}$ for each $i = 1, 2$, $x \in \mathbb{R}^2$, and $A \subset \mathbb{R}^2$, then: $(f_1, f_2) \circ x \equiv (f_1(x_1), f_2(x_2))$ and $(f_1, f_2) \circ A \equiv \{(f_1, f_2) \circ a | a \in A\}$.

[4] A feasible set S that satisfies $\pi S = S$ is called *symmetric*.

player i should (weakly) benefit. Beyond showing that the Nash solution violates IM, Kalai and Smorodinsky proposed another solution and proved that it is the unique solution that satisfies WPO, SY, IEUR, and IM. This solution is given by choosing for every $(S, d) \in \mathcal{B}$ the highest point in S (according to the standard partial order on $\mathbb{R}^2$) that lies on the line segment connecting d and $a(S, d)$. Let K denote this solution.

We propose an alternative axiomatization of K that does not involve IM. In addition to WPO, IEUR, and SY, we impose the following two axioms, in the statements of which (S, d), (S', d'), and (T, d') are arbitrary elements of $\mathcal{B}$.

Restricted independence of irrelevant alternatives (R.IIA): $[S \subset T]\&[d = d']\&[a(S, d) = a(T, d')]\&[\mu(T, d') \in S] \Rightarrow \mu(S, d) = \mu(T, d')$.

This axiom was introduced by Roth (1977). Its requirement from the solution is the same as the one of IIA, but it applies only to problems that in addition to the conditions stipulated by IIA share the same ideal point.

Disagreement point monotonicity (DIM): $[a(S, d) \notin S]\&[S' = S]\&[d'_i > d_i]\&[d'_j = d_j] \Rightarrow \mu_i(S', d') > \mu_i(S, d)$.

This is a fairness axiom. Interpreting d_i as player i's "outside option", DIM requires i's solution-payoff to be an increasing function of it. Note that the requirement that this axiom applies only to problems that do not contain their ideal point is hardly ever binding: clearly no bargaining theory is needed for solving such "rectangular situations", in which each player can be assigned his first-best utility. The requirement from the solution to strictly increase in the disagreement point d_i was introduced by Livne (1989). The less demanding weak-inequality-version of this axiom was introduced by Thomson (1987).

The main contribution of our characterization is that it takes the "monotonicity" from Kalai's and Smorodinsky's IM, and applies it to the disagreement point instead of to the feasible set. This has two advantages. First, it highlights the importance of the disagreement point in bargaining. Second, it reinforces a well-known pattern: K behaves very well with respect to various monotonicity conditions.[5]

2 The main result

We start with two auxiliary lemmas, the statements of which will involve the following axiom. In the axiom's statements (S, d) is an arbitrary element of $\mathcal{B}$.

Strong individual rationality (S.IR): $\mu(S, d) > d_i$ for all $i \in \{1, 2\}$.

Lemma 1 $WPO \,\&\, R.IIA \,\&\, DIM \Rightarrow IR$.

[5] For example, as was shown by Thomson (1983), when one considers an environment with a variable number of agents, K can be characterized in terms of *population monotonicity*. Informally, this axiom requires that "if there is one more mouth to feed, then everybody should contribute" (see Thomson 1983 for details).

Proof Let $(S, d) \in \mathcal{B}$ and let μ be a solution that satisfies the three axioms. Assume by contradiction, and without loss of generality (wlog), that $\mu_1(S, d) < d_1$. Let l be the line segment with endpoints d and $\mu(S, d)$. Let S' be the set of the points $x \in S$ such that (1) x lies to the right of l, and (2) $x_2 \geq d_2$. Note that $(S', d) \in \mathcal{B}$, $S' \subset S$, and $a(S', d) = a(S, d)$. Hence, by R.IIA, $\mu(S', d) = \mu(S, d)$. Next, let:

$$S'' \equiv \{x \in \mathbb{R}^2 | x \geq (\mu_1(S, d), d_2) \quad \text{and} \quad x \leq y \quad \text{for some y} \in \text{S}\}$$

Note that $(S'', d) \in \mathcal{B}, a(S', d) = a(S'', d)$, and $S' \subset S''$. Moreover, $WP(S'') = WP(S')$, so by WPO $\mu(S'', d) \in S'$. Hence, by R.IIA, $\mu(S', d) = \mu(S'', d)$. Therefore, $\mu(S'', d) = \mu(S, d)$. Finally, consider a shift-to-the-left of d in S'': by DIM this change should strictly decrease player 1's payoff, but this is not possible in S'', a contradiction. □

Lemma 2 *WPO & SY & IEUR & R.IIA & DIM* $\Rightarrow$ *S.IR.*

Proof Let μ be a solution satisfying the axioms. Assume by contradiction that it violates S.IR. Then, there exists an element $(S, d) \in \mathcal{B}$ and an $i \in \{1, 2\}$ such that $\mu_i(S, d) \leq d_i$; by Lemma 1, it must be that $\mu_i(S, d) = d_i$. Assuming wlog that $i = 1$, WPO implies $\mu(S, d) = (d_1, a_2(S, d))$. Let $T \equiv \text{conv}\{d, (d_1, a_2(S, d)), (a_1(S, d), d_2)\}$. By R.IIA $\mu(T, d) = \mu(S, d) = (d_1, a_2(S, d))$. However, by IEUR, WPO, and SY it follows that $\mu(T, d)$ is the midpoint of T's hypotenuse, a contradiction. □

We are now ready to state and prove the main result.

Theorem 1 *The Kalai–Smorodinsky solution is the unique solution that satisfies WPO, SY, IEUR, R.IIA, and DIM.*

Proof It is easy to check that K satisfies all the axioms. We now prove uniqueness. Let μ be a solution satisfying the axioms and let (S, d) be an arbitrary element of $\mathcal{B}$. Assume by contradiction that $\mu(S, d) \neq K(S, d)$. Note that $\mu(S, d) \neq K(S, d)$ implies that $a(S, d) \notin S$ (otherwise, we would have $\mu(S, d) = K(S, d) = a(S, d)$).[6] Therefore, DIM applies to all the bargaining problems to be constructed in the remainder of the proof.

Since K satisfies PO, it follows that $\mu_i(S, d) < K_i(S, d)$ for some $i \in \{1, 2\}$. Since μ satisfies WPO, it follows that there exists exactly one such i. Suppose then, wlog, that $\mu_1(S, d) < K_1(S, d)$ and $\mu_2(S, d) \geq K_2(S, d)$. In fact, it must be that $\mu_2(S, d) > K_2(S, d)$.[7]

Let $V \equiv \text{conv}\{\mu(S, d), d, (d_1, a_2(S, d)), (a_1(S, d), d_2)\}$. By R.IIA, $\mu(V, d) = \mu(S, d)$. Lemma 2 guarantees that V is a convex hull of *four* points.[8] Therefore,

[6] To see this, suppose that $a(S, d) \in S$ and let $\nu \in \{\mu, K\}$. By Lemma 1 $\nu(S, d) \in R \equiv \text{conv}\{d, a(S, d), (a_1(S, d), d_2), (d_1, a_2(S, d))\}$ and hence by R.IIA $\nu(S, d) = \nu(R, d)$. By WPO, SY, and IEUR, $\nu(R, d) = a(S, d)$.

[7] If $\mu(S, d) \in WP(S)$, $\mu_1(S, d) < K_1(S, d)$, and $\mu_2(S, d) = K_2(S, d)$, then $\mu_2(S, d) = a_2(S, d)$ and therefore $K(S, d) = a(S, d)$, in contradiction to $a(S, d) \notin S$.

[8] Without S.IR we could not exclude the possibility $\mu(S, d) \in \{(d_1, a_2(S, d)), (a_1(S, d), d_2)\}$.

there exists a $d'_2 > d_2$ such that $K(V, d') = \mu(V, d)$, where $d' = (d_1, d'_2)$.[9] By DIM, $\mu(V, d) \neq \mu(V, d')$. Let $W \equiv \text{conv}\{\mu(V, d), d', (d_1, a_2(S, d')), (a_1(S, d'), d'_2)\}$. We obtain:

$$\mu(W, d') = \mu(V, d') \neq \mu(V, d) = K(V, d') = K(W, d')$$

The first equality is by R.IIA, the second equality is by definition of d', and the third equality follows from the definition of K. However, by WPO, SY, and IEUR, $\mu(W, d') = K(W, d')$, a contradiction. □

If we replaced DIM by its weak-inequality version (see Thomson 1987), then the conclusion of Theorem 1 would not hold. In fact, the Nash solution satisfies the weak-inequality version of DIM, as well as the rest of the axioms of Theorem 1.[10]

2.1 Independence of the axioms

As was just noted, the Nash solution satisfies all the axioms of Theorem 1 besides DIM. The *disagreement solution* $\mu(S, d) \equiv d$ satisfies all the axioms besides WPO. Letting $m(S, d) \equiv \frac{1}{2}[a(S, d) + d]$, the solution $\mu(S, d) \equiv m(S, d) + (\epsilon, \epsilon)$, where ϵ is the maximal number such that the expression on the right-hand-side is in S, satisfies all the axioms besides IEUR.[11] The *generalized individually monotonic solution* (due to Peters and Tijs 1985) satisfies all the axioms besides SY. To describe this solution, some preliminary definitions are needed. For $(S, d) \in \mathcal{B}$ there exists a unique pair $(f_1, f_2) \in F_A \times F_A$ such $(f_1, f_2) \circ (S, d)$ is a *normalized problem*, in the sense that its disagreement point is $(0, 0)$ and its ideal point is $(1, 1)$. Call this pair (f_1, f_2) the *normalizing transformation for* (S, d). Let Ψ be the set of strictly increasing and continuous functions $\psi\colon [0, 1] \to [0, 1]$ that satisfy $\psi(0) = 0$ and $\psi(1) = 1$. Given $\psi \in \Psi$, let $G(\psi)$ denote the graph of ψ. Given $\psi \in \Psi$, the associated (generalized individually monotonic) solution works as follows: Given (S, d), normalize it by applying its normalizing transformation (f_1, f_2), take the intersection point $WP((f_1, f_2) \circ S) \cap G(\psi)$ and "pull it back" by applying to it (f_1^{-1}, f_2^{-1}). The resulting point is the solution to (S, d). Denote this solution μ_ψ. Note that $K = \mu_\psi$ for the identity function $\psi(t) = t$.

I now turn to describe a solution that satisfies all the axioms besides R.IIA. Let $\alpha \in (0, 1)$ and $S^* \equiv \{(x, y) \in \mathbb{R}^2_+ | x^2 + y^2 \leq 1, x \leq \alpha\}$. Let $\mathcal{S}^* \equiv \{(f_1, f_2) \circ S^* | (f_1, f_2) \in F_A \times F_A\}$. Take a $\psi \in \Psi$ such that $\psi(\frac{1}{2}) \neq \frac{1}{2}$. Let μ^* be the solution such that $\mu^*(S, d) = \mu_\psi(S, d)$ if $S \in \mathcal{S}^*$ and $\mu^*(S, d) = K(S, d)$ otherwise. It is easy to check that μ^* satisfies all the axioms besides R.IIA. Though formally making the point it is supposed to make, the solution μ^* is just an artificial construct. One may further ask whether there exist well-known, or well-behaved solutions that satisfy all the axioms of Theorem 1 besides R.IIA. I am not aware of such a solution on the entire

[9] If V were a convex hull of three points such a d' would not have existed, because K satisfies S.IR.

[10] The fact that the Nash solution satisfies the weak inequality version of DIM was proved by Thomson (1987), the rest is due to Nash (1950).

[11] This solution appears in de Clippel (2007).

domain of bargaining problems considered here; however, there is no shortage of such solutions on the sub-domain of *strictly comprehensive problems*—the problems with feasible sets S that satisfy $WP(S) = P(S)$. For example, the *equal-area solution* is one such solution (see Anbarci and Bigelow 1994 for details).[12]

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[12] Another relevant example is the Perles–Maschler solution. This solution is defined on the domain of strictly comprehensive problems with feasible sets that contain only individually rational points (i.e. every (S, d) in this domain satisfies $S = S_d$). On this domain, this solution satisfies all the axioms besides R.IIA (see Perles and Maschler 1981 for details).