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Network formation under negative degree-based externalities

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Abstract While a relationship in a social or business network should be mutually beneficial, it is ambiguous whether the relationship benefits or harms the rest of the network. This paper focuses on the situation where any new relationship imposes a negative externality on the rest of the network. We model this by assuming an agent's payoff from a relationship is a decreasing function of the number of relationships the other agent maintains. We solve for the socially efficient and stable networks. While in general the two diverge, we demonstrate that they coincide when agents are able to make transfers to their partners.

Keywords Networks · Externalities · Degree-based utility · Distance-based utility

1 Introduction

When two agents form a relationship in a social or business network, we presume that it must benefit both parties. However, whether or not this new relationship benefits the rest of the network is ambiguous. This paper focuses on the situation where any new relationship imposes a negative externality on the other agents in the network. This is particularly applicable to competitive environments, such as buyers and sellers, or when the players are constrained by a limited resource, such as time.

A particularly important example of this is employment networks. As a stylized fact, roughly half of those currently employed were told about their job by a friend or relative.[1] A professional develops a network of business contacts both to be informed

[1] See Ioannides and Loury (2004) for a survey on empirical research on social networks in labor markets. In the networks literature, Calvo-Armengol and Jackson (2004) is an important paper that models the impact of networks on unemployment.

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of relevant employment opportunities and to increase the probability she is selected should she apply. However, the value of this contact depends critically on its exclusivity. It is a great advantage to have a present employee forward your resume to her boss; however, it is much less helpful if the contact forwards additional resumes.

Another example where relationships impose a negative externality is a network of buyers and sellers.[2] When a seller forms a relationship with a new buyer, that seller's other buyers face increased competition. When a buyer forms a new relationship with a seller, she improves her bargaining position with the other sellers she has a relationship with. Similarly, automobile manufacturers maintain long-term relationships with suppliers. When a manufacturer develops a new relationship, its other suppliers face increased competition.

A classic example discussed in Jackson and Wolinsky (1996) is academic coauthors. An academic benefits from having a coauthor as it increases her research output. However, if her coauthor works on a new project, she has less time to devote to their project and the value of the collaboration is diminished.[3]

This paper is concerned with what networks will form and whether these networks are socially efficient when each link imposes a negative externality on the other members of the network. Towards this aim, we solve for the stable and efficient networks for a class of utility functions called degree based. A utility function is *degree based* if the payoff from a relationship is a function of the number of other relationships the agent is maintaining. As we are concerned with network formation under negative externalities, we assume this function is decreasing. In this environment, externalities can only be negative. An agent can only be harmed by an indirect relationships as it weakly increases the number of relationships maintained by her neighbors.

This paper completely characterizes the socially optimal and equilibrium set of networks for additively-separable, degree-based utility functions. First, we show that the regular network is socially optimal for a degree based utility function. A network is regular if all agents have the same number of connections. Next, we consider a refinement of pairwise stability, called strong pairwise stability, where an agent is able to unilaterally drop a link, bilaterally add a link, or do both simultaneously. This is critical when externalities are negative. An agent's relationships make her less valuable to potential partners. If she wishes to form a new relationship, she may need to sever one or more of her current relationships to make herself more valuable. Strong pairwise stability provides a new type of profitable deviation: exchanging a less valuable relationship for a more valuable relationship. Moreover, this deviation is reasonable since if an agent has the power to unilaterally sever a relationship and bilaterally form a new relationship, then she should have the power to do both simultaneously.[4] We

[2] See, for example, Kranton and Minehart (2001).

[3] It is worth noting that in all of these examples externalities will not be exclusively negative. For example, if A and B are co-authors, then A is harmed if B starts a project with C. However, if C starts a new project, then she has less time to devote to her project with B. Since B now has more time to devote to A, A benefits from this indirect relationship. Similarly, in an employment network, an agent may learn of a job she is not interested in from one partner and pass along the information to a second partner.

[4] Note that strong pairwise stability is not necessary when externalities are positive. When externalities are positive, her current relationships make her more valuable to prospective partners, and she should never need to drop a profitable relationship in order to form a more profitable one.

demonstrate that the unique, strongly-pairwise-stable network is also a type of regular network; however, it is socially inefficient as each agent maintains too many links.

The paper concludes with a positive result. We extend strong pairwise stability to allow agents to make transfers to their neighbors. This concept emphasizes the cooperative nature of a social network and is motivated by Bloch and Jackson (2007). We demonstrate that this is enough to regain efficiency. Specifically, with only a mild assumption on marginal utilities, we show that a network is strongly pairwise stable with transfers if and only if it is the socially optimal regular graph.

This paper is a contribution to the literature on network formation pioneered by Jackson and Wolinsky (1996) and Bala and Goyal (2000).[5] There are a number of papers that consider network formation when externalities can only be positive but comparatively few that consider models with negative externalities. In particular, distance based utility functions, where the payoff two agents receive from being connected is a function of the distance they are apart in the network, is a common utility function in the literature.[6] In such a model externalities can only be positive as any new link weakly decreases the distance between all agents. Bloch and Jackson (2007) demonstrate that the unique non-trivial efficient network for a distance-based utility model is the star network. Two papers that model the impact of both positive and negative externalities on network formation are Currarini (2007) and Goyal and Joshi (2006).

The remainder of this article is organized as follows. Section 2 introduces the network game. Section 3 introduces degree based utility functions and completely characterizes the socially optimal and equilibrium networks for any degree based utility function. Section 4 concludes, and the Appendix provides several of the more technical proofs.

2 Modeling networks

Given a set of players $\mathcal{N} = \{1, \ldots, N\}$, we model the network they form as a graph where each node represents a player and each edge represents a pairwise relationship.[7] We fix the set of players, so the graph is completely defined by the set of edges. Specifically, a network g is a set of unordered pairs of players $\{i, j\}$. $\{i, j\} \in g$ indicates that there is a *link* between players i and j in the network g. The interpretation is that players i and j maintain a direct relationship. Edges are not directed as relationships in this model are reciprocal. Let g^N denote the set of all subsets of $\mathcal{N}$ of size 2. The network g^N is called the *complete* network as all combinations of players maintain a link. Define $G = \left\{g : g \subseteq g^N\right\}$ to be the set of all possible networks. The *utility* player i receives from a network is a function $u_i : G \to \mathbb{R}_+$.

[5] See Jackson (2004) and Jackson (2008) for excellent surveys. See also Goyal and Vega-Redondo (2004), Hojman and Szeidl (2008), and Bramoulle and Kranton (2007) for some more recent papers on network formation.

[6] Jackson and Wolinsky (1996) connections model is the most famous example. Several other examples that use a distance-based payoff function are Bala and Goyal (2000), Watts (2001), Dutta and Jackson (2000), Matsubayashi and Yamakawa (2006), and Hojman and Szeidl (2008).

[7] Jackson (2004) provides background and a more detailed discussion of this model of networks.

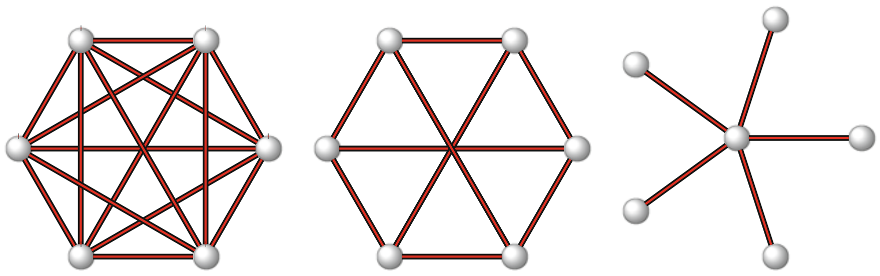

Fig. 1 A complete, a 3-regular, and a star network

A *path* between players i and j in a network $g \in G$ is a sequence of players $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k \in \{1, \ldots, K-1\}$, with $i_1 = i$ and $i_K = j$. Two players are *connected* if there is a path between them. For simplicity, we will use $ij \in g$, $g + ij$, and $g - ij$ as shorthand for $\{i, j\} \in g$, $g \cup \{i, j\}$, and $g \backslash \{i, j\}$, respectively. For each player i, i's neighborhood, denoted $L_i(g)$, is the set of players i is linked to, $L_i(g) = \{j | ij \in g\}$. The *degree* of a player i, denoted n_i, is the cardinality of i's neighborhood, $n_i = |L_i(g)|$.

Two particularly important networks are the star and regular networks. A *star* network is one where there exists one player, called the center, involved in every link. A *regular* network is one where all players have the same degree. Figure 1 provides an example of a complete, a 3-regular, and star network.

Given a utility function u, a network $g \in G$ is *efficient* if $\sum_i u_i(g) \geq \sum_i u_i(g')$ for all other networks $g' \in G$. A network $g \in G$ is *Pareto efficient* if there does not exist a $g' \in G$ such that $u_i(g') \geq u_i(g)$ for every $i \in \mathcal{N}$ with at least one inequality being strict. In this paper we focus on efficiency as a solution concept. When utility is transferable, as much of the analysis assumes, the two concepts coincide.

3 Degree based utility functions

The focus of this paper is on networks where an agent is harmed by her partner's other relationships. In this section we introduce and solve for a utility function that is rich enough to capture this trade-off but simple enough to be tractable. Specifically, we define a utility function to be *degree-based* if there exists a function ϕ such that

$$u_i(g) = \sum_{ij \in g} \phi(n_j) - c \cdot n_i.$$

As we are concerned with network formation under negative externalities, we assume ϕ is decreasing. The player receives a benefit from a direct relationship; however, this benefit diminishes if the partner maintains other relationships. Relationships are costly to maintain, and c is the cost per link. To simplify the algebra, we assume

$c \neq \phi(n)$ for any $n \in \mathbb{N}$. This assumption is without loss of generality since one can perturb c by an arbitrarily small ϵ.[8]

Degree-based utility functions are a natural counterpoint to distance-based utility functions that have been extensively studied in the literature. The *distance*, denoted $d(i, j)$, between i and j is the length of the shortest path between i and j ($d(i, j) = \infty$ if i and j are not connected). A utility function is *distance-based* if there exists a $c \geq 0$ and a nonincreasing function f such that:

$$u_i(g) = \sum_{j \neq i} f(d(i, j)) - c \cdot n_i$$

Under a distance-based payoff structure, externalities are only positive as any new link weakly decreases the distance between all other players. Bloch and Jackson (2007) demonstrate that for distance-based utility functions the unique, nontrivial, efficient network structure is the star network.

The star does not perform as well with a degree-based payoff structure. The perimeter vertices only get utility from their immediate neighbor, the center of the star. However, the center maintains so many links (the maximum possible) that the agents receive little utility from the relationship. Moreover, we do not expect the star to be an equilibrium of a network game as any two perimeter vertices would do better by dropping their connection to the center agent and forming a link to each other.

In this environment, a more symmetric graph does better socially and is more likely to be pairwise stable. A regular graph, where all agents have the same number of connections, is the natural place to look. Unfortunately, regular graphs do not always exist.[9] However, a regular graph always exists when there is an even number of vertices. We define a new class of graphs which exist regardless of the parity of the number of agents.

Definition 1 Let $\overline{n} = \max\{n_i : i \in \mathcal{N}\}$ and $\underline{n} = \min\{n_i : i \in \mathcal{N}\}$. Then:

1. A graph is **nearly-regular** if $(\overline{n} - \underline{n}) \leq 1$.
2. A graph is **nearly-*n*-regular** if $(n - 1) \leq \underline{n} \leq \overline{n} \leq n$.

The next proposition completely characterizes the set of socially optimal networks when there is an even number of vertices.

[8] There are at least two interesting ways the degree-based utility function could be extended. First, it would be interesting to allow for complementarities among neighbors. For example, in the co-authors model, it might matter whether or not an agent's co-authors are also co-authors. A second extension would be to combine the degree-based and distance-based models. In the co-author's model, an agent is harmed by vertices distance two away (her co-author's co-author), but she benefits from vertices distance three away. A vertex which is distance three away harms her co-author (reduces the time the co-author spends with the co-author's co-author). This increases the time the co-author has to spend with her. Such extensions, while interesting, are beyond the scope of this paper.

[9] For example, if we have an odd number of vertices, we can not have a $(2a + 1)$-regular graph. Since every edge contributes two to the sum of all vertex degrees, the total sum of degrees must be even. An odd-regular graph with an odd number of vertices would have an odd total degree sum which is not possible.

Proposition 1 *Suppose there is an even number of agents. A network $g \in G$ is socially optimal if and only if for every player i, $n_i \in \arg\max x(\phi(x) - c)$. In particular, for any $n \in \arg\max x(\phi(x) - c)$, all n-regular networks are socially optimal.*

Proof Each agent receives a payoff from her neighbors and contributes utility to her neighbors. As an accounting identity, the sum of what every agent receives must equal the sum of what every agent contributes. In particular

$$U(g) = \sum_{i=1}^{N} u_i(g) = \sum_{i=1}^{N} \sum_{ij \in g} \phi(n_j) - cn_i = \sum_{i=1}^{N} n_i(\phi(n_i) - c) \tag{1}$$

Let $n \in \arg\max x(\phi(x) - c)$. Since we have an even number of vertices, an n-regular graph exists.[10] Pick any n-regular graph $g \in G$. By Equation 1, g must be socially optimal. Moreover, if $g' \in G$ is a network with a node i such that $n_i \notin \arg\max x(\phi(x) - c)$, then $U(g') < U(g)$. □

This paper assumes that ϕ is decreasing, but note that Proposition 1 holds for any ϕ.

3.1 Pairwise stability

Pairwise stability is a standard equilibrium concept in network theory. Intuitively, it says no agent wishes to unilaterally drop one of her connections, and no two agents wish to bilaterally add a connection.

Definition 2 A network $g \in G$ is *pairwise stable* if:

1. If $ij \in g$, then $u_i(g) > u_i(g - ij)$ and $u_j(g) > u_j(G - ij)$.
2. If $ij \notin g$, then either $u_i(g) > u_i(g + ij)$ or $u_j(g) > u_j(g + ij)$.

This next structure appears several times so we explicitly define it.

Definition 3 $g \in G$ is a **maximal nearly-*n*-regular graph** if it is nearly-n-regular and there does not exist a nearly-n-regular graph $g' \in G$ such that $g \subset g'$.

In general, efficient networks are not pairwise stable for degree-based utility functions. For the purposes of social efficiency, the direct benefit of a link is weighed against the indirect costs to the other members of the network. However, players consider only their direct benefit from the link. Not surprisingly, the agents form more relationships than what is socially optimal.

Proposition 2 *Let $n = \max\{x \in \mathbb{N} | \phi(x) > c\}$ and assume $n < N$. All maximal nearly-n-regular graphs are pairwise stable.*

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[10] To see this, label the vertices 0 through $N - 1$. If N is even, then connect vertex i to vertices $i \pm j \mod (N)$, for $1 \leq j \leq \frac{N}{2}$. If N is odd, then connect vertex i to vertex $i + \frac{N}{2} \mod (N)$ and to vertices $i \pm j \mod (N)$, for $1 \leq j \leq \frac{N-1}{2}$.

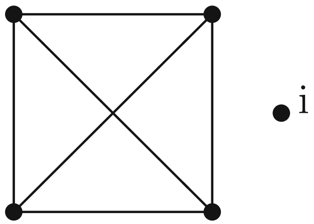

Fig. 2 An undesirable pairwise stable graph

Proof Let g be any maximal nearly-n-regular graph. Look at any players $i, j \in \mathcal{N}$ such that $ij \notin g$. The maximality of g implies either i or j must have degree n. Without loss of generality, assume $n_i = n$. Then $u_i(g + ij) - u_i(g) = \phi(n + 1) - c < 0$ by the maximality of n. Similarly, look at any players $i, j \in \mathcal{N}$ such that $ij \in g$. $u_i(g) - u_i(G - ij) \geq \phi(n) - c \geq 0$. Therefore G is pairwise stable. □

Pairwise stability is a fairly weak concept. Pairwise stability only allows a vertex to unilaterally drop a connection or to bilaterally add a connection but not both. For example, if $\max\{x \in \mathbb{N} | \phi(x) \geq c\} = 3$, then the graph in Fig. 2 is pairwise stable since i will be unwilling to add an edge with any vertex that already has degree 3. However, this is unsatisfying as any of the vertices in the 4-clique would be happy to *exchange* one of their edges for an edge with i. Similarly, as long as the vertex is willing to drop one of her edges, i would be happy to form an edge with any vertex in the 4-clique. This leads to a new solution concept.

Definition 4 A graph $g \in G$ is **Strongly Pairwise Stable** if

1. g is pairwise stable
2. There does not exist a i, j, and k such that $u_i(g') > u_i(g)$ and $u_j(g') > u_j(g)$ where $g' = g + ij - jk$.

With this stronger solution concept, we are able to completely characterize the set of strongly pairwise stable graphs.

Proposition 3 *Let* $n = \max\{x \in \mathbb{N} | \phi(x) > c\}$ *and assume* $n < N$. *For any* $g \in G$, g *is strongly pairwise stable if and only if* g *is a maximal nearly-*n*-regular graph.*

Proof It is straightforward to verify that a maximal nearly-n-regular graph is strongly pairwise stable. To prove the other direction, look at any strongly pairwise stable graph $g \in G$. First note that $\max\{n_i | i \in \mathcal{N}\} = n$ since otherwise G would not be pairwise stable.[11] Suppose for contradiction that g is not nearly-regular. Then there exists an i and j such that $n_i - n_j \geq 2$. Let $k \in L_i(g) \backslash L_j(g) \neq \emptyset$ and let $g' = g + jk - ik$. Note that $u_k(g') > u_k(g)$ since $n_j + 1 < n_i$. Moreover, $u_j(g') > u_j(g)$ since $n_k \leq n$ and $\phi(n) > c$. Therefore g is not strongly pairwise stable, a contradiction. Since g has max degree n and is nearly-regular, it must be nearly-n-regular. If g is not maximal, then there are two non-adjacent vertices of degree $n - 1$ and therefore g is not pairwise stable. □

[11] If the maximum degree is greater than n, some vertex would want to drop an edge. If the max degree is less than n, any unconnected vertices would be better off adding an edge.

What is striking about equilibrium networks, in the sense of strong pairwise stability, is that the network provides essentially no utility to any of the players. The players continue to form relationships until no other agent can provide them with any benefit. But in doing so, the value of each agent is diminished to the point where each player is contributing the minimum benefit possible. While this is a negative result, in the next section, we show that this is mitigated as long as agents are able to make transfers to their direct neighbors. In fact, we will show that the socially efficient network is the unique strongly pairwise stable network when transfers are allowed.

3.2 Equilibrium with transfers

The previous section demonstrated that for degree-based utility functions there is a divergence between the networks that are socially efficient and the networks that are pairwise stable. When each agent does not internalize the negative consequences of a relationship on her partners, she forms too many relationships. In this section, we show that as long as agents are able to make transfers to their immediate neighbors the stable networks will be socially efficient.

The game studied here is motivated by Bloch and Jackson (2007). They define several new network games which extend the traditional network games to allow players to make financial transfers. In their paper they make a distinction between who an agent is able to make a transfer to and on what an agent is able to condition this transfer payment. In the direct transfer game, an agent can only make a transfer for a link she is directly involved with. In the indirect transfer games she can only make demands on her own relationships, but she is free to subsidize any relationship. In their standard game, a transfer is conditional only on a link forming, but in their game with contingent transfers, an agent can condition a payment on the entire network structure.

More formally, the game consists of a graph g and a matrix of transfers T. In the transfer matrix, t_{ij} represents the transfer from agent i to agent j. An agent can only make a transfer to someone she has a direct relationship with. Therefore, $t_{ij} > 0$ only if $ij \in g$. To avoid ambiguity, we require that $t_{ij} = -t_{ji}$ as what is relevant is the net transfer. For this section, we normalize costs to be zero, $c = 0$.

Given a network $g \in G$ and transfers T, agent i receives a payoff of

$$\pi_i(g, T) = \sum_{ij \in g} [\phi(n_j) + t_{ji}] \tag{2}$$

Individually, an agent should be able to drop any of her edges and change the transfers she offers. Two agents should also be able to form a link if they so desire. Anytime an agent changes her edges or transfers, she alters the payoff of her neighbors and potentially jeopardizes these relationships. However, if two agents are able to move bilaterally to establish a mutually beneficial relationship and to adjust their transfers so that all of their neighbors are better off, then they do not jeopardize these relationships. Since such a change is within the power of the agents and preferred to the *status quo*, the *status quo* is not an equilibrium.

Definition 5 Given a network $g \in G$ with transfers T, agent i **blocks** $<g, T>$ if there exists an agent j,[12] subsets $A \subseteq L_i(g)$, $B \subseteq L_j(g)$, and transfers T' where $t'_{kl} = 0$ if $k, l \notin \{i, j, L_i(g) \backslash A, L_j(g) \backslash B\}$ such that:

$$\pi_x(g', T + T') > \pi_x(g, T)$$

for every $x \in \{i, j, L_i(g) \backslash A, L_j(g) \backslash B\}$ where $g' = g \cup \{i, j\} \backslash \{\{i, k\} : k \in A\} \backslash \{\{j, k\} : k \in B\}$.

As with strong pairwise stability, agents may unilaterally drop an edge, bilaterally add an edge, or do both simultaneously. Under strong pairwise stability with transfers, the agent may compensate her partners for any changes she makes.

Definition 6 A network $g \in G$ is **strongly pairwise stable with transfers** if there exists transfers T such that no agent blocks $<g, T>$. In such a case, we say the transfers T **support** g.[13]

When it is clear from context that it is a network game with transfers, we will just say strongly pairwise stable instead of strongly pairwise stable with transfers. With a mild regularity assumption on the payoff function ϕ, we will prove that the only network that is strongly pairwise stable with transfers is the socially optimal regular network.

Whenever an edge is added to a network, there is a social trade off. There is a direct benefit to the two agents forming the relationship but at a cost of a decreased payoff to all the agents they already share an edge with. As discussed in the proof of Proposition 1, the social contribution made by an agent with x many relationships is given by the function:

$$\Phi(x) = x\phi(x) \tag{3}$$

Therefore, the marginal contribution of her forming a new relationship is given by:

$$\Delta\Phi(x) = (x + 1)\phi(x + 1) - x\phi(x) \tag{4}$$

We know from Proposition 1 that a network is socially efficient if and only if each agent has degree n where $n \in \arg\max x\phi(x)$. We impose the following regularity condition to ensure there is a unique maximizer.

Definition 7 The social payoff function $\Phi(x)$ is **single-peaked** if either:

[12] We allow $j = \emptyset$ in which case we interpret $g + ij$ to be g.

[13] Strong pairwise stability with transfers is closely related to pairwise Nash equilibrium with unilateral status quo introduced by Hojman and Szeidl (2008). They consider a different network formation game than is considered in this paper, but the spirit of the two refinements is similar. They allow an agent to drop edges, add an edge by adjusting her transfer to that agent, and do both simultaneously. In addition to differences that result from the different games being considered, strong pairwise stability with transfers differs from pairwise Nash equilibrium with a unilateral status quo in that under strong pairwise stability with transfers, both players are able to drop edges and adjust transfers.

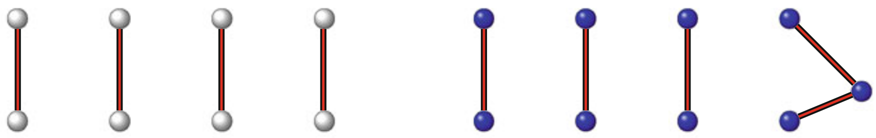

Fig. 3 Trivial graph for an even and odd number of vertices

1. $\Delta\Phi(n) > 0$, for all $n \in \{1, \ldots, N-1\}$.
2. $\Delta\Phi(n) < 0$, for all $n \in \{1, \ldots, N-1\}$.
3. There exists an integer M such that $\Delta\Phi(n) > 0$ for every $n < M$ and $\Delta\Phi(n) < 0$ for every $n > M$.

For each respective case, we define the **threshold** of $\Phi(x)$ to be:

1. $N-1$
2. 1
3. M

Single-peakedness serves the same purpose as a second-order condition but is a weaker assumption. In particular, if $\Delta\Phi$ is decreasing, then Φ is single-peaked. We make the additional assumption that if $\Phi(x)$ is single-peaked with threshold M with $1 < M < N-1$, then $\Delta\Phi(M) = 0$. This does not change the results in any significant way, but it does make the proofs cleaner. Moreover, if $\Delta\Phi(M) \neq 0$, then we can replace $\phi(x)$ by $\phi'(x)$ where $\phi'(x) = \phi(x) - \Delta\Phi(M)$. Now:

$$\begin{aligned}\Delta\Phi'(x) &= (x+1)\phi'(x+1) - x\phi'(x)\\ &= (x+1)\phi(x+1) - (x+1)\Delta\Phi(M) - x\phi(x) + x\Delta\Phi(M)\\ &= (x+1)\phi(x+1) - x\phi(x) - \Delta\Phi(M)\\ &= \Delta\Phi(x) - \Delta\Phi(M)\end{aligned}$$

and therefore $\Delta\Phi'(M) = 0$.

The next lemma is an immediate consequence of Proposition 1. We define the trivial network to consist of $\frac{N}{2}$ many pairs if N is even and $\frac{N-3}{2}$ many pairs plus the three remaining vertices connected as a path if N is odd (e.g. see Fig. 3). The proof is left to the Appendix as it simply involves verifying that $\Phi(x)$ is uniquely maximized at the threshold.

Lemma 1 *Suppose $\Phi(x)$ is single-peaked with threshold M.*

1. *When $1 < M < N-1$, then $g \in G$ is socially optimal if and only if g is nearly-$(M+1)$-regular.*
2. *When $M = N-1$, then the complete network is the unique socially optimal network.*
3. *When $M = 1$, then the unique socially optimal network is the trivial network.*

The main result of this paper is to demonstrate that for a well behaved utility function the unique strongly pairwise stable network with transfers is the socially efficient network. Note that this refinement differs from pairwise stability in only two ways: agents may transfer utility to an immediate neighbor and agents are allowed to drop and add an edge simultaneously. Moreover, the transfers need not be interpreted as a monetary transfer. For example, in the co-authors model, a transfer could consist of doing a greater share or the least desirable of the tasks.

It should be noted that we only need such a simple refinement because the degree-based utility function simplifies the interaction between members of the network. In particular, indirect transfers are not required to achieve efficiency because a link only imposes an externality on direct neighbors. Similarly, the utility function does not allow for complementarities among neighbors; otherwise, dropping and adding only one edge would not be sufficient to achieve efficiency. Such extensions, while interesting, are beyond the scope of this paper.

Proposition 4 *Suppose $\Phi(x)$ is single-peaked. Then $g \in G$ is strongly pairwise stable with transfers if and only if it is socially efficient.*

The trivial cases, when the threshold equals 1 or $N-1$, are proved in the Appendix. We prove the non-trivial case here with a sequence of lemmas. Throughout we assume that $\Phi(x)$ is single-peaked.

Lemma 2 *Suppose $1 < M \leq N-1$ and let $g \in G$ be strongly pairwise stable. If there exists a player i with degree less than M, then i is adjacent to every vertex with degree less than or equal to M.*

Proof Suppose for contradiction that g is supported by transfers T and has two non-adjacent vertices i and j with $n_i < M$ and $n_j \leq M$. Let

$$t'_{jx} = \begin{cases} \phi(n_j) - \phi(n_j+1) + \epsilon & x \in L_j(g) \\ \phi(n_i+1) - \phi(n_j+1) & x = i \end{cases}$$

$$t'_{ix} = \begin{cases} \phi(n_i) - \phi(n_i+1) + \epsilon & x \in L_i(g) \\ \phi(n_j+1) - \phi(n_i+1) & x = j \end{cases}$$

Then

$$\begin{aligned} \Delta\pi_i(g+ij, T+T') &= \phi(n_j+1) + t'_{ji} - \sum_{ik \in g} t'_{ik} \\ &= (n_i+1)\phi(n_i+1) - n_i\phi(n_i) - n_i * \epsilon \\ &= \Delta\Phi(n_i) - n_i * \epsilon \\ &> 0 \end{aligned}$$

for ϵ sufficiently small.

For $x \in L_i(g)$

$$\begin{aligned} \Delta\pi_x(g+ij, T+T') &= \phi(n_i+1) - \phi(n_i) + t'_{ix} \\ &= \phi(n_i+1) - \phi(n_i) + \phi(n_i) - \phi(n_i+1) + \epsilon \\ &= \epsilon \\ &> 0 \end{aligned}$$

Similarly, $\pi_j(g+ij, T+T')-\pi_j(g, T) > 0$ and $\pi_x(g+ij, T+T')-\pi_i(g, T) > 0$ for every $x \in L_j(g)$. Therefore, i and j block $<G, T>$, contradicting the assumption that T supports g. □

Lemma 3 *Suppose* $1 \leq M < N - 1$ *and let* g *be strongly pairwise stable. If there exists a vertex* i *with degree greater than* $M + 1$*, then* i *is not adjacent to any vertex with degree greater than or equal to* $M + 1$.

Proof Suppose for contradiction that g is supported by transfers T and has two adjacent vertices i and j with $n_i > M + 1$ and $n_j \geq M + 1$. Let $r_{xi} = \phi(n_i - 1) - \phi(n_i) - \epsilon$ for every $x \in L_i(g)\backslash\{j\}$. Similarly, let $s_{jx} = \phi(n_j) - \phi(n_j - 1) - \epsilon$ for every $x \in L_j(g)\backslash\{i\}$.

Then if i drops its edge with j in order to receive transfers R, it loses the benefit from j, $\phi(n_j)$, no longer makes the transfer t_{ij}, and gains the transfers R from each of its remaining neighbors. Specifically,

$$\Delta\pi_i(g - ij, T + R) = -\phi(n_j) + t_{ij} + (n_i - 1)(\phi(n_i - 1) - \phi(n_i) - \epsilon)$$

Similarly,

$$\Delta\pi_j(g - ij, T + S) = -\phi(n_i) + t_{ji} + (n_j - 1)(\phi(n_j - 1) - \phi(n_j) - \epsilon)$$

Adding these two equations yields

$$\begin{aligned} &\Delta\pi_i(g - ij, T + R) + \Delta\pi_j(g - ij, T + S) \\ &\quad = -\Delta\Phi(n_j - 1) - \Delta\Phi(n_i - 1) + t_{ij} + t_{ji} - \epsilon(n_i + n_j - 2) \\ &\quad = -\Delta\Phi(n_j - 1) - \Delta\Phi(n_i - 1) - \epsilon(n_i + n_j - 2) > 0 \end{aligned}$$

for sufficiently small ϵ. The first equality comes from rearranging terms. The second equality follows since $t_{ij} = -t_{ji}$ by definition. The final inequality follows since $n_i - 1 > M$ and $n_j - 1 \geq M$, so by the definition of the threshold, $\phi(n_i - 1) < 0$ and $\phi(n_j - 1) \leq 0$.

By construction, $\pi_x(g - ij, T + R) - \pi_x(g, T) > 0$ for every $x \in L_i(g)\backslash\{j\}$ and $\pi_x(g-ij, T+S)-\pi_x(g, T) > 0$ for every $x \in L_j(g)\backslash\{i\}$. Since $\Delta\pi_i(g-ij, T+R)+\Delta\pi_j(g - ij, T + S) > 0$, either $\Delta\pi_i(g - ij, T + R) > 0$ or $\Delta\pi_j(g - ij, T + S) > 0$. Whichever one is greater than zero blocks g, a contradiction. □

Next, we establish an upper bound on the size of a transfer.

Lemma 4 *Suppose* T *supports a network* g*. Then for every two agents* i *and* j *such that* $ij \in g$,

$$t_{ij} \leq \phi(n_j) - (n_i - 1)(\phi(n_i - 1) - \phi(n_i))$$

Proof i has $n_i - 1$ many neighbors who would be willing to pay up to $\phi(n_i - 1) - \phi(n_i)$ for i to sever her relationship with j. i receives a benefit of $\phi(n_j) - t_{ij}$ from her relationship with j, so if $\phi(n_j) - t_{ij} < (n_i - 1)(\phi(n_i - 1) - \phi(n_i))$ then i and all her remaining neighbors do strictly better if i drops her relationship with j and accepts a transfer of $\phi(n_i - 1) - \phi(n_i) - \epsilon$ from each of her remaining neighbors. □

Lemma 5 *Suppose* $1 \leq M < N - 1$ *and let* g *be strongly pairwise stable. If there exists a vertex* i *with* $n_i > M + 1$, *then all of* i*'s neighbors are adjacent.*

Proof Suppose not, and let i be such that $n_i > M + 1$, $j, k \in L_i(g)$, but $jk \notin g$. We know from Lemma 3 that $n_j \leq M$ and $n_k \leq M$. Since j and k are not adjacent, we know from Lemma 2 that neither j nor k has degree less than M. Therefore, $n_j = n_k = M$.

From Lemma 4

$$\begin{aligned} t_{ij} &\leq \phi(n_j) - (n_i - 1)(\phi(n_i - 1) - \phi(n_i)) \\ &= \phi(n_j) - (n_i - 1)\phi(n_i - 1) + (n_i - 1)\phi(n_i) + \phi(n_i) - \phi(n_i) \\ &= \phi(M) - \phi(n_i) + \Delta\Phi(n_i - 1) \end{aligned}$$

Similarly, $t_{ik} \leq \phi(M) - \phi(n_i) + \Delta\Phi(n_i - 1)$. Therefore

$$\begin{aligned} \Delta\pi_j(g + jk - ij - ik, T) &= \phi(n_k) - \phi(n_i) - t_{ij} \\ &= \phi(M) - \phi(n_i) - t_{ij} \\ &\geq \phi(M) - \phi(n_i) - (\phi(M) - \phi(n_i) + \Delta\Phi(n_i - 1)) \\ &= -\Delta\Phi(n_i - 1) \\ &> 0 \end{aligned}$$

where the last inequality follows from $n_i > M + 1$ and therefore, $\Delta\Phi(n_i - 1) < 0$.

Similarly, $\Delta\pi_k(g + jk - ij - ik, T) > 0$. Note that since the degree of j and k has not changed, all vertices in $L_j(g) \cup L_k(g) \setminus \{i\}$ are indifferent between $< g + jk - ij - ik, T >$ and $<g, T>$. Therefore agents j and k block $<G, T>$ contradicting the stability of g. □

Lemma 6 *Suppose* $1 \leq M < N - 1$ *and let* g *be strongly pairwise stable. No vertex in* g *has degree greater than* $M + 1$.

Proof This is a pigeonhole argument. Suppose for contradiction there is a vertex i with $n_i > M + 1$. By Lemma 3, every neighbor of i must have degree less than or equal to M. By Lemma 5, all neighbors of i must be adjacent. However, there are at least $M + 1$ neighbors of i. All are adjacent to the other neighbors of i (there are at least M other neighbors of i) plus i itself. Therefore, all neighbors of i must have degree at least $M + 1$, a contradiction. □

Lemma 7 *Suppose* $1 < M < N - 1$ *and let* g *be strongly pairwise stable. No vertex in* g *has degree less than* M.

Proof Suppose for contradiction there exists a vertex i with $n_i < M$. Since $M < N-1$, there exists a j not adjacent to i. By Lemma 2, $n_j > M$. Therefore, by Lemma 6, $n_j = M + 1$. Since j has $M + 1$ neighbors and i has less than M neighbors, there must exist a k which is adjacent to j but not adjacent to i. Repeating the above logic,

$n_k = M + 1$. We will demonstrate that i, k, and all of their neighbors can be made better off if i adds an edge with k and k drops it's edge with j.

From Lemma 4

$$\begin{aligned} t_{kj} &\leq \phi(n_j) - (n_k - 1)(\phi(n_k - 1) - \phi(n_k)) \\ &= \phi(M+1) - (M)(\phi(M) - \phi(M+1)) \\ &= \Delta\Phi(M) \\ &= 0 \end{aligned}$$

Similarly $t_{jk} \leq 0$, therefore $t_{kj} = t_{jk} = 0$. Let $g' = g + ik - jk$. Then

$$\begin{aligned} \Delta\pi_k(g', T) &= \phi(n_i + 1) - \phi(n_j) - t_{jk} \\ &= \phi(n_i + 1) - \phi(n_j) \\ \Delta\pi_i(g', T) &= \phi(n_k) \\ \Delta\pi_x(g', T) &= \phi(n_i + 1) - \phi(n_i) \quad \text{for every } x \in L_i(g) \\ \Delta\pi_x(g', T) &= 0 \quad \text{for every } x \in L_k(g) \end{aligned}$$

Let

$$\begin{aligned} t'_{ix} &= \begin{cases} \phi(n_i) - \phi(n_i + 1) + \epsilon & x \in L_i(g) \\ \phi(n_j) - \phi(n_i + 1) + \omega & x = k \end{cases} \\ t'_{kx} &= \delta \quad \text{for every } x \in L_k(g) \backslash \{j\}\,. \end{aligned}$$

Now

$$\begin{aligned} \Delta\pi_k(g', T + T') &= \omega - n_k * \delta \\ \Delta\pi_i(g', T + T') &= \phi(n_i + 1) - n_i(\phi(n_i) - \phi(n_i + 1)) - n_i * \epsilon - \omega \\ &= \Delta\Phi(n_i) - n_i * \epsilon - \omega \\ \Delta\pi_x(g', T + T') &= \epsilon \text{ for every } x \in L_i(g) \\ \Delta\pi_x(g', T + T') &= \delta \text{ for every } x \in L_k(g) \end{aligned}$$

Since $n_i < M$, $\Delta\Phi(n_i) > 0$, and therefore, i, k and all of their neighbors can be made better off in $g + ik - jk$. This contradicts the strong pairwise stability of G. □

Lemma 8 *Suppose* $1 < M < N - 1$. *If g is nearly-*$(M+1)$*-regular, then g is strongly pairwise stable with transfers.*

Proof Let G be any nearly-$(M + 1)$-regular graph. Define a set of transfers T by:

$$t_{ij} = \begin{cases} 0 & n_i = n_j \\ \phi(M) - \phi(M+1) & n_i = (M+1), n_j = M \\ \phi(M+1) - \phi(M) & n_i = M, n_j = M + 1 \end{cases}$$

Every agent with degree M receives a total payoff of $M\phi(M)$ and every agent with degree $M+1$ receives a payoff of $(M+1)\phi(M+1)$. Since $\Delta\Phi(M) = 0$, $M\phi(M) = (M+1)\phi(M+1)$.

A nearly-$(M+1)$-regular graph is optimal, so adding an edge cannot increase social payoff. Since all the benefits are captured by the two agents adding an edge and all costs are incurred by their neighbors, it is not possible for the two agents adding the edge to make all their neighbors better off. Similarly, an agent i has no wish to delete one of her edges. i's remaining neighbors receive all the benefit, while i incurs all the costs. Since the costs are greater than or equal to the benefits (the original graph was socially optimal), i's remaining neighbors will not be able to compensate i so that all are better off. Finally, two vertices i and j can not do better by each dropping an edge and creating an edge with each other. The new relationship is worth at most $\phi(M)$ which is exactly what they received from their previous relationship. □

Lemma 6 and Lemma 7 establishes the nearly-$(M+1)$-regularity is necessary for strong pairwise stability with transfers. Lemma 8 establishes that being nearly-$(M+1)$-regular is sufficient as well. This is a surprising and powerful result. In a network of relationships, an agent should be able to sever any ties it chooses and establish new ties when it is mutually desirable. Moreover, there should always be informal ways an agent can exert effort that is costly for herself but makes the relationship more beneficial for a partner. Proposition 4 establishes that if this is case, then the only network which will be an equilibrium is the socially optimal network.

The paper concludes by presenting the solution of a specific degree-based utility function. The solution follows as an immediate corollary to Proposition 4, and it is presented here to contrast with the solution to both the connections and co-authors model presented in Jackson and Wolinsky (1996). In the symmetric connections model, the utility player i receives the following being part of network g is:

$$u_i(g) = \sum_{j \neq i} \delta^{l_{ij}(g)} - n_i(g)c \tag{5}$$

where $l_{ij}(g)$ is the length of the shortest path between i and j.

Here we consider the following degree-based utility function which is a natural counterpoint to the connections model:

$$u_i(g) = \sum_{ij \in g} \gamma^{n_j} - n_i(g)c \tag{6}$$

where $0 \leq \gamma \leq 1$.

The motivation for this utility function is the same as for Jackson and Wolinsky (1996) coauthors model or any model of competition. Each academic receives a benefit from collaborating with a colleague; however, this benefit is diminished if the collaborator collaborates with other academics. The principal advantage of this new utility function is that it has a nontrivial solution space. The efficient network structure for the co-authors consists of $\frac{N}{2}$ pairs, and in pairwise stable networks, the authors divide into fully intraconnected components.

For convenience, we normalize $c = 0$ and assume $\gamma = \frac{\tau}{\tau+1}$ for some integer τ.

Corollary 5 *Suppose* $u_i(G) = \sum_{i \leftrightarrow j} \gamma^{d_{v_j}}$. *Then*

1. *G is socially optimal if and only if G is nearly-*$(\tau + 1)$*-regular.*
2. *G is strongly pairwise stable with transfers if and only if G is nearly-*$(\tau+1)$*-regular.*

Proof $\Delta\Phi(x) = (x + 1)\gamma^{(x+1)} - x\gamma^x$ is a decreasing function of x. Therefore $\Phi(x)$ is single-peaked. It is straightforward to show that since $\gamma = \frac{\tau}{\tau+1}$, the threshold is τ. Moreover:

$$\begin{aligned}\Delta\Phi(\tau) &= (\tau + 1)\gamma^{\tau+1} - \tau\gamma^{\tau} \\ &= \gamma^{\tau}\left((\tau + 1)\frac{\tau}{\tau + 1} - \tau\right) \\ &= 0\end{aligned}$$

Therefore all the assumptions of Proposition 4 are met. □

4 Conclusion

This paper studies social and business networks where an agent is harmed by any other relationships her partners maintain. Important examples include employment networks, buyers and sellers, and co-authors. We model this situation using degree-based utility functions and solve for the socially efficient, pairwise stable, and strongly pairwise stable networks. We show that in general, strongly pairwise stable networks are inefficient as agents maintain too many links.

We conclude with a positive result. By extending the networks game to allow agents to make transfers to their immediate neighbors, we demonstrate that the socially efficient network is the unique strongly pairwise stable network.

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Appendix—proofs

We define the trivial network to consists of $\frac{N}{2}$ many pairs if N is even and $\frac{|V(G)-3|}{2}$ many pairs plus the three remaining vertices connected as a path if N is odd. See Fig. 3 for an example.

Proposition 1 *Suppose* $\Phi(x)$ *is single-peaked with threshold* M.

1. *When* $1 < M < N - 1$, *then* $g \in G$ *is socially optimal if and only if g is nearly-*$(M + 1)$*-regular.*
2. *When* $M = N - 1$, *then the complete network is the unique socially optimal network.*
3. *When* $M = 1$, *then the unique socially optimal network is the trivial network.*

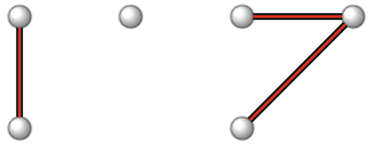

Fig. 4 The trade-off for the last three vertices when $M = 1$

Proof We know from the previous section that g is optimal if and only if every vertex has degree from the arg max $\Phi(x)$. Note that

$$\Phi(x) = \Phi(x+1) - \Delta\Phi(x) \tag{7}$$

If $M = N - 1$, then $\Delta\Phi(x) > 0$ for all $n \in \{1, \ldots, N-1\}$, and by Eq. 7, $\Phi(x)$ is maximized at $N - 1$. Therefore, the socially optimal network is the complete network.

Similarly, if $M = 1$, then $\Delta\Phi(x) < 0$ for all $n \in \{1, \ldots, N-1\}$, and $\Phi(x)$ is maximized at 1. Therefore, if a 1-regular graph exists, it must be optimal. A 1-regular graph exists when there is an even number of vertices (the trivial graph), but does not when the number of vertices is odd. It is never necessary to have more than one vertex with degree zero or two. When N is odd, the question is whether to leave one vertex disconnected or to create a path between three vertices (see Fig. 4). In the first case, the payoff to the three relevant players is $\phi(1) + \phi(1) + 0$. In the second case, the payoff is $\phi(2) + \phi(2) + 2\phi(1)$. Since $\phi(x)$ is a positive function, it must be optimal to connect the three vertices in a path.

If $1 < M < N-1$, then for $n < M$, $\Phi(n+1) > \Phi(n)$ and for $n > M$, $\Phi(n+1) < \Phi(n)$. Since $\Delta\Phi(M) = 0$, $\Phi(M) = \Phi(M+1)$ and therefore, $\arg\max \Phi(x) = \{M, M+1\}$. As mentioned previously, a nearly regular graph always exists. Therefore, a network is optimal if and only if it is nearly-$(M+1)$-regular. □

Proposition 4 *Suppose $\Phi(x)$ is single-peaked. Then $g \in G$ is strongly pairwise stable with transfers if and only if it is socially efficient.*

Proof The interesting case, where $1 < M < N - 1$, was proved in the main text. Let g be strongly pairwise stable with transfers. Suppose $M = N - 1$. By definition, $\Delta\Phi(n) > 0, \forall n \in \{1, \ldots, N-1\}$. By Lemma 2, if there exists a vertex i with degree less than M, then i is adjacent to every vertex j with $n_j \leq M = N - 1$. Since every vertex has degree less than or equal to $N - 1$, i is adjacent to all vertices, a contradiction. Therefore, if $M = N - 1$, every vertex has degree $N - 1$ and g must be the complete network.

Next, suppose $M = 1$. By definition, $\Delta\Phi(n) < 0, \forall n \in \{1, \ldots, N-1\}$. By Lemma 6, no vertex has degree greater than 2. First suppose there is a cycle, i.e. a sequence of vertices $i_1, \ldots, i_K$ such that $i_k i_{k+1} \in g$ for each $k < K$ and $i_K i_1 \in g$. Without loss of generality, $t_{1,2} \geq 0$ (if $t_{1,2} < 0$, then relabel the vertices with $i'_1 = i_2, i'_2 = i_1, i'_3 = i_K$, etc.). Label $i = i_K$, $j = i_1$, and $l = i_2$. Then consider the coalition i and j with transfer $t'_{ij} = \phi(2) + \epsilon$. $\Delta\pi_i(g - jl, T + T') = \phi(1) - \phi(2) - t'_{ij} = \phi(1) - 2\phi(2) - \epsilon = -\Delta\Phi(1) - \epsilon > 0$ for ϵ sufficiently small. $\Delta\pi_j(g - jl, T + T') = -\phi(2) + t'_{ij} + t_{jl} = \epsilon + t_{jl} > 0$. Therefore, i, j, and transfers T' block g, a contradiction. Therefore there is no cycle.

Suppose, for contradiction, there is a path of length greater than three. Label the first four vertices of the path i, j, k, and l. In particular, $n_i = 1$, $n_j = 2$, $n_k = 2$ and the degree of l is irrelevant. $\Delta\pi_j(g - jk) = \Delta\pi_l(g - jk) = \phi(1) - \phi(2)$. $\Delta\pi_j(g - jk) = \phi(2) - t_{kj}$. Therefore, if $\phi(1) - \phi(2) > \phi(2) - t_{kj}$, then a profitable deviation is for i to offer j a transfer of $\phi(2) - t_{kj} + \epsilon$ to drop j's edge with k. Therefore:

$$\begin{aligned} \phi(1) - \phi(2) &\leq \phi(2) - t_{kj} \\ t_{kj} &\leq 2\phi(2) - \phi(1) \\ &= \Delta\Phi(1) \\ &< 0 \end{aligned}$$

An identical argument shows that $t_{jk} < 0$, a contradiction.

Therefore, there is no i with $n_i > 2$, there is no cycle, and there is no path of length greater than three. If there is more than one vertex of degree 2, then there must be at least two paths of length exactly three. Note that if $u(g, T) < \phi(1)$ for two vertices, then they can profitably deviate by dropping all of their edges and forming an edge together. Therefore, at least five of the six vertices receive a payoff of at least $\phi(1)$, so in one of the paths of length three, all three vertices get a payoff of at least $\phi(1)$. However, the total surplus generated by the path is $2\phi(1) + 2\phi(2) = 3\phi(1) + 2\phi(2) - \phi(1) = 3\phi(1) + \Delta\Phi(1) < 3\phi(1)$. Since an agent can only make transfers to immediate neighbors, there is not enough total surplus for all three agents to receive $\phi(1)$, a contradiction.

Therefore, when $M = 1$, their can be at most one agent with degree two. Since each agent has degree at least one, the only possible strongly pairwise stable network is the trivial network. It is trivial to verify that the trivial network is in fact strongly pairwise stable. □

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